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On Restricted Cohomology of Modular Classical Lie Algebras and Their Applications

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Abstract: In this paper, we study the restricted cohomology of Lie algebras of semisimple and simply connected algebraic groups in positive characteristics with coefficients in simple restricted modules and their applications in studying the connections between these cohomology with the corresponding ordinary cohomology and cohomology of algebraic groups. Let G be a semisimple and simply connected algebraic group G over an algebraically closed field of characteristic $p > h$, where h is a Coxeter number. Denote the first Frobenius kernel and Lie algebra of G by G_1 and $\mathfrak{g}$, respectively. First, we calculate the restricted cohomology of $\mathfrak{g}$ with coefficients in simple modules for two families of restricted simple modules. Since in the restricted region the restricted cohomology of $\mathfrak{g}$ is equivalent to the corresponding cohomology of G_1, we describe them as the cohomology of G_1 in terms of the cohomology for G_1 with coefficients in dual Weyl modules. Then, we give a necessary and sufficient condition for the isomorphisms $H^n(G_1, V) \cong H^n(G, V)$ and $H^n(\mathfrak{g}, V) \cong H^n(G, V)$, and a necessary condition for the isomorphism $H^n(\mathfrak{g}, V) \cong H^n(G_1, V)$, where V is a simple module with highest restricted weight. Using these results, we obtain all non-trivial isomorphisms between the cohomology of G, G_1, and $\mathfrak{g}$ with coefficients in the considered simple modules.

Keywords: Lie algebra; algebraic group; cohomology; restricted cohomology; simple modules

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1. Introduction

The cohomology of modular classical Lie algebras can be viewed both in the category of all modules and so in the category of all restricted modules. Usually, the first one is called *ordinary cohomology*, and the second one is called *restricted cohomology*. Modular classical Lie algebras are Lie algebras of semisimple and simply connected algebraic groups in positive characteristics. They are restricted Lie algebras [1]. Hochschild introduced the restricted cohomology of a restricted Lie algebra with coefficients in a restricted module in [2]. In this paper, he obtained a long exact sequence establishing connections between ordinary and restricted cohomology and gave an explicit description of some of its initial terms. The described initial part of this sequence is called the *Hochschild five-term exact sequence*. The restricted cohomology of classical Lie algebras with coefficients in a trivial one-dimensional module was completely described by Friedlander and Parshall in [3]. For dual Weyl modules, a complete description of restricted cohomology is also obtained [4,5]. The spectral sequence relating the ordinary and restricted cohomology constructed by Friedlander, Parshall [6] (p. 1079), and Farnsteiner [7] (p. 114) make it possible to study the connections between ordinary and restricted cohomology of higher degrees. For instance, in [8] (Theorem 3.1), the five-term exact sequences for higher cohomology, generalizing the Hochschild five-term exact sequence, were obtained. To date, a cochain complex for restricted cohomology of a restricted Lie algebra has not been completely constructed. Evans and Fuchs obtained the spaces of cochains and differentials for the calculation of the first and second degrees restricted cohomology for non-abelian restricted Lie algebras [9].

Using these results, the authors of the papers [10–15] computed central extensions and deformations of some non-classical restricted Lie algebras.

As we know, investigations devoted to the study of the restricted cohomology of modular classical Lie algebras with coefficients in simple modules relate mainly with the cohomology of low degrees or the cohomology of Lie algebras of low rank. According to the Hochschild five-term exact sequence, for simple modules, the restricted first cohomology coincides with the ordinary first cohomology. Jantzen studied them in detail [16]. The restricted second cohomology of simple modules was described for the following classical Lie algebras: $\mathfrak{sl}_2$ [17] (p. 429); $\mathfrak{sl}_3$ [18] (p. 4706); $\mathfrak{sp}_4$ [19] (p. 1126); and G_2 [20] (p. 389). For restricted third cohomology, similar results were obtained for the simple classical Lie algebras of rank two [21] (pp. 52–53). In [22], the authors described the ordinary cohomology of two families of simple modules for modular classical Lie algebras. The aim of this paper is to study the restricted cohomology and the connections between the ordinary and restricted cohomology of these modules, as well as their connections between the cohomology of algebraic groups, related to the studied classical Lie algebras.

The interest in this problem is motivated by the fact that restricted cohomology is an important part of the cohomology theory of modular Lie algebras, in which there is no analogue in characteristic zero, and by the presence of unsolved problems that require deep research. Restricted cohomology also appears in the cohomology theory of semisimple and simply connected algebraic groups in positive characteristic as the cohomology of their infinitesimal subgroups. As is known, by homological tools, the study of the cohomology of algebraic groups can be reduced to the study of the cohomology of their infinitesimal subgroups. Consequently, our results on the restricted cohomology of modular classical Lie algebras with coefficients in restricted simple modules can be applied to the study of the corresponding cohomology of algebraic groups. In addition, known results on restricted cohomology of classical Lie algebras can be very useful in studying restricted cohomology of restricted Lie algebras related with classical Lie algebras, as well as in studying the usual cohomology of restricted Lie algebras.

The concept of cohomology has been defined for many other classes of algebras, such as Lie superalgebras [23,24], Leibniz algebras [25], Lie antialgebras [26], alternative algebras [27], non-associative algebras with metagroup relations [28–30], and n-Lie algebras with derivations [31]. In the modular case, among these classes of algebras, the restrictness of cohomology extends to Leibniz algebras [32] and to Lie superalgebras [33]. It would be very interesting if one developed the idea of restrictness for other classes of non-associative algebras.

In Section 2 below, we give the main notation, preliminary information, and a short presentation of the algorithm for computing restricted cohomology of classical modular Lie algebras with coefficients in simple modules. This algorithm is a well-known homology tool based on the use of the properties of long exact cohomology sequences. In Section 3, we formulate the main results and give their proofs. The restricted cohomology of simple modules related to Weyl modules with a simple radical is described in Theorem 1. A similar result for the restricted cohomology of simple modules, related to Weyl modules with a Janzen filtration of depth 2, is formulated in Theorems 2–4. Their proofs use the same arguments, so the detailed proof is given only for Theorem 3, which is the more variable among these theorems. In Theorem 5, formulated in Section 3.1, necessary and sufficient conditions are obtained for the isomrphisms $H^n(G_1, V) \cong H^n(G, V)$ and $H^n(\mathfrak{g}, V) \cong H^n(G, V)$, and a necessary condition for the isomrphism $H^n(\mathfrak{g}, V) \cong H^n(G_1, V)$. In Corollaries 1–6, formulated in Section 3.1, all non-trivial isomorphisms between the cohomologies G, G_1, and $\mathfrak{g}$ with coefficients in the considered simple modules are given. The proofs of the main results are given in Section 3.2.

2. Preliminaries

We will mainly use the standard notation and preliminary facts given in [22]. We summarize them and add some short information on restricted Lie algebras and restricted

Lie algebra cohomology. Let G be a semisimple and simply connected algebraic group over an algebraically closed field of characteristic $p > h$, where h is the Coxeter number. Denote the first Frobenius kernel and Lie algebra of G by G_1 and $\mathfrak{g}$, respectively. We denote the rank of $\mathfrak{g}$ by l. Let R be a root system of $\mathfrak{g}$ and assume that $R \subset \mathbb{R}^m$. On $\mathbb{R}^m$ there is the usual euclidian inner product $(\cdot,\cdot)$. This leads to the natural pairing $\langle\cdot,\cdot\rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ given by $\langle\lambda,\mu\rangle = (\lambda,\mu^\nu)$, where $\mu^\nu = \frac{2}{(\mu,\mu)}\mu$. Let R^+ be the set of positive roots and $\Delta = \{\alpha_1,\alpha_2,\cdots,\alpha_l\}$ be the set of simple roots.

Let $T \subset G$ be the maximal torus, and B be the Borel subgroup corresponding to the negative roots. We denote by U the unipotent radical of B. The set $X(T)$ of additive characters for T can be seen as a subset of $\mathbb{R}^m$ with basis $\{\omega_1,\omega_2,\cdots,\omega_l\}$ satisfying $\langle\omega_i,\alpha_j\rangle = \delta_{ij}$. The set $X(T)$ also has the following property:

$$X(T) = \{\lambda \in \mathbb{R}^m \,|\, \langle\lambda,\alpha\rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

Let $X(T)^+ = \{\lambda \in X(T) \,|\, \langle\lambda,\alpha\rangle \geq 0 \text{ for all } \alpha \in R^+\}$ be the set of dominant weights and let $X_1(T) = \left\{\lambda \in X(T)^+ \mid 0 \leq \langle\lambda,\alpha\rangle < p \text{ for all } \alpha \in \Delta\right\}$ be the set of restricted weights.

Let $\lambda \in X(T)^+$ and $H^0(\lambda) = \left\{f \in k[G] \mid f(gb) = \lambda(b)^{-1}f(g) \text{ for all } g \in G,\ b \in B\right\}$, where $k[G]$ is the algebra of all regular functions on G. The action of G on $H^0(\lambda)$ is defined by $gf(h) = f(g^1h)$, where $f \in H^0(\lambda)$, $g,h \in G$ [34] (p. 26). On the other hand, $H^0(\lambda) = \mathrm{Ind}_B^G(k_\lambda)$, where k_λ is a one dimensional B-module defined by $\lambda \in X(T(^+$ via the isomorphism $B/U \cong T$ [34] (p. 176). Let $L(\lambda)$ be a maximal semi-simple submodule (socle) of $H^0(\lambda)$. If $H^0(\lambda) \neq 0$, then $L(\lambda)$ is simple [34] (p. 177, II.2.3) and every simple G-module is isomorphic to $L(\lambda)$ for some $\lambda \in X(T)^+$ [34] (p. 177, II.2.4). Since $H^0(\lambda) \neq 0$ for all $\lambda \in X(T)^+$ [34] (p. 178, II.2.6), then, for all $\lambda \in X(T)^+$, there is a short exact sequence

$$0 \to L(\lambda) \to H^0(\lambda) \to H^0(\lambda)/L(\lambda) \to 0 \tag{1}$$

of G-modules. One of the effective ways to explicitly describe the structure of $H^0(\lambda)/L(\lambda)$ is to study the radical of the Weyl module $V(\lambda)$ with the highest weight $\lambda \in X(T)^+$. The Weyl module $V(\lambda)$ is isomorphic to $H^0(-w_0(\lambda)\,)^*$, where w_0 is the maximal element of the *Weyl group* W for R [34] (p. 182, II.2.13). So, for all $\lambda \in X(T)^+$, there is a short exact sequence

$$0 \to \mathrm{rad}\ V(\lambda) \to V(\lambda) \to L(\lambda) \to 0$$

of G-modules, where rad $V(\lambda)$ is the radical of $V(\lambda)$. For the Lie algebra $\mathfrak{g}$ of G, we will consider the corresponding differentials of the G-modules $H^0(\lambda)$, $V(\lambda)$, and $L(\lambda)$. We will keep these notations for the corresponding $\mathfrak{g}$-modules. In the restricted region, these three $\mathfrak{g}$-modules are restricted, moreover $L(\lambda)$ remains simple as a $\mathfrak{g}$-module.

For $\alpha \in R^+$ and $n \in \mathbb{Z}$, let us define the affine reflections $s_{\alpha,n}$ on $X(T)$ by

$$s_{\alpha,n}{\cdot}\lambda = \lambda - \langle\lambda+\rho,\alpha\rangle\alpha + np\alpha \text{ for all } \lambda \in X(T).$$

Denote by W_p the *affine Weyl group* generated by all $s_{\alpha,n}$ with $\alpha \in R^+$ and $n \in \mathbb{Z}$. The finite Weyl group W of R appears as the subgroup of W_p generated by the reflections $s_{\alpha,0}$ with $\alpha \in R^+$.

Let α_0 be the unique maximal short root of R. We will use the following short notation: $s_{\alpha_i,0}s_i$ for all $i \in \{1,2,\cdots,l\}$ and $s_0s_{\alpha_0,1}$. The set of simple reflections in W is $S = \{s_i \,|\, i = 1,2,\cdots,l\}$ and the set of simple affine reflections in W_p is $S_p = S \cup \{s_0\}$. Denote by $l(w)$ the *length of the element* $w \in W$ with respect to the simple reflections $s_1,\ s_2,\cdots,s_l$.

Denote by $L^{(1)}$ the *Frobenius twist* of the G-module L. Suppose V is a Frobenius twist of some rational G-module. Then, there is a unique rational G-module L such that $L^{(1)} = V$. Denote this module by $V^{(-1)}$.

For Lie algebras, the concept of a restricted Lie algebra (also a p-Lie algebra) was first introduced by Jacobson in [35] (p. 210). A Lie algebra $\mathfrak{g}$ over a field k of characteristic

p is called *restricted* if it admits an additional unary operation $x \mapsto x^{[p]}$ that satisfies the following conditions:

- $(ax)^{[p]} = a^p x^{[p]}$, for all $a \in k$ and for all $x \in \mathfrak{g}$;
- $ad\left(x^{[p]}\right) = (ad\ x)^{[p]}$ for all $x \in \mathfrak{g}$;
- $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$for all $x, y \in \mathfrak{g}$,

where $s_i(x,y)$ is the coefficient of a^{i-1} in $(ad\ (ax+y))^{p-1}(y)$. As noted above, all classical Lie algebras are restricted; as an unary operation, we can take the Frobenius mapping. Another example of a restricted Lie algebra is the general linear Lie algebra $\mathfrak{gl}_n$. Lie algebras of Cartan type of height one are also restricted Lie algebras.

Let $\mathfrak{g}$ be a restricted Lie algebra over a field k of characteristic p, denote by $I_{\mathfrak{g}}$ the ideal of $U(\mathfrak{g})$ which generated by the elements of the form $x^p - x^{[p]}$ with $x \in \mathfrak{g}$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. The quotient algebra $U_0(\mathfrak{g}) = U(\mathfrak{g})/I_{\mathfrak{g}}$ is called the restricted universal enveloping algebra of the Lie algebra $\mathfrak{g}$. Following Hochschild, we define the *restricted cohomology* $H^n_*(\mathfrak{g}, V)$ with coefficients in a restricted module V as the Cartan-Eilenberg extension $\mathrm{Ext}^n_{U_0(\mathfrak{g})}(k, V)$ [2] (p. 561). As is known, the cohomology $H^n(G_1, V)$ is equivalent to the restricted cohomology $H^n_*(\mathfrak{g}, V)$ [34] (p. 129). Since our main research method is to use the tools of the representations theory of algebraic groups in positive characteristic, for restricted cohomology below, we will use the notation $H^n(G_1, V)$.

There is the long exact cohomological sequence

$$\cdots \to H^n(G_1, L(\lambda)) \to H^n(G_1, H^0(\lambda)) \to H^n(G_1, H^0(\lambda)/L(\lambda)) \to \cdots \tag{2}$$

corresponding to the short exact sequence (1) and the following Andersen-Janzen formula on cohomology of G_1 with coefficients in $H^0(\lambda)$ [4]: let $p > h$, and $\lambda = w \cdot 0 + p\nu$, then

$$H^n(G_1, H^0(\lambda))^{(-1)} \cong \begin{cases} \mathrm{Ind}_B^G\left(S^{\frac{n-l(w)}{2}}(\mathfrak{n}^*) \otimes k_\nu\right) & \text{if } n - l(w) \text{ } even, \\ 0 & \text{if } n - l(w) \text{ } odd, \end{cases} \tag{3}$$

where $\mathfrak{n}$ is the maximal nilpotent subalgebra of $\mathfrak{g}$, corresponding to the negative roots. The Lie algebra $\mathfrak{n}$ is the Lie algebra of the unipotent radical U of B.

Let $\lambda \in X(T)^+$ and $\lambda = (w \cdot 0 + p\nu)$. According to (3), the cohomology of $H^n(G_1, H^0(\lambda))^{(-1)}$ is trivial if $n < l(w)$ or $n - l(w)$ is odd. Then, according to the exactness of (2),

$$H^n(G_1, L(\lambda)) \cong H^{n-1}(G_1, H^0(\lambda)/L(\lambda)) \tag{4}$$

if $1 \leq n < l(w)$, and the short sequence

$$0 \to H^{n-1}(G_1, H^0(\lambda)/L(\lambda)) \to H^n(G_1, L(\lambda)) \to H^n(G_1, H^0(\lambda)) \to 0 \tag{5}$$

is exact if $n \geq l(w)$.

Formula (4) or the short exact sequence (5) allows us to describe the cohomology $H^n(G_1, L(\lambda))$. So, if $\lambda = w \cdot 0 + p\nu \in X_1(T)$, then we get the following algorithm for calculating the cohomology $H^n(G_1, L(\lambda))$:

- Calculate ν and $l(w)$.
- Describe the structure of $H^0(\lambda)/L(\lambda)$ as a G_1-module.
- Describe the cohomology $H^{n-1}(G_1, H^0(\lambda)/L(\lambda))$.
- Calculate the cohomology $H^n(G_1, L(\lambda))$ using Formula (4) if $n < l(w)$, the short exact sequence (5) otherwise.

3. Results

3.1. Formulation of Results

To use long exact cohomological exact sequences to describe the cohomology of classical modular Lie algebras with coefficients in simple modules, one needs complete information on the structures of the Weyl modules associated with these simple modules. As is known, in the general case, the structures of Weyl modules are well studied for affine dominant alcoves along the walls of the dominant Weyl chambers [36] and for affine dominant alcoves close to them [37]. Their highest weights are denoted by $\lambda_1, \lambda_2, \cdots, \lambda_s$, $\mu_1, \mu_2, \cdots, \mu_{s-1}$. These highest weights can be obtained by using the action of the Weyl group and translation to zero weight. Such descriptions of them were obtained in [22] (pp. 9, 13). For convenience, we will list them in Tables 1 and 2. For Tables 1 and 2, the following notation are used: α_0 is the maximal short root; $w_{i,j} = s_i s_{i+1} \cdots s_j$, where $i < j$. Note that $w_{ij}^{-1} = s_j s_{j-1} \cdots s_i$.

Table 1. Descriptions of $\lambda_i, i = 1, 2, \cdots, s$.

Root System	i	λ_i
A_l	$1, 2, \cdots, s$	$w_{1,l} w_{i,l-1}^{-1} \cdot 0 + p\alpha_0$
B_l	$1, 2, \cdots, l$ $l+1, l+2, \cdots, s$	$w_{1,l} w_{i,l-1}^{-1} \cdot 0 + p\alpha_0$ $w_{1,2l-i} \cdot 0 + p\alpha_0$
C_l	$1, 2, \cdots, l-1$ $l, l+1, \cdots, s$	$w_{2,l} w_{1,l-1}^{-1} w_{2,l} w_{i+1,l-1}^{-1} \cdot 0 + p\alpha_0$ $w_{2,l} w_{1,l-1}^{-1} w_{2,2l-i-1} w_{1,l-1}^{-1} \cdot 0 + p\alpha_0$
D_l	$1, 2, \cdots, s$	$w_{2,l-2} w_{1,l}^{-1} w_{2,l-2} w_{i+1,l}^{-1} \cdot 0 + p\alpha_0$
E_6	1 2 3 4 5	$s_2 w_{4,6}^{-1} w_{3,5} w_{1,2}^{-1} w_{3,4}^{-1} s_1 w_{4,6} s_2 w_{4,5} w_{3,\,4} s_2 \cdot 0 + p\alpha_0$ $s_2 w_{4,6}^{-1} w_{3,5} w_{1,2}^{-1} w_{3,4}^{-1} s_1 w_{4,6} s_2 w_{4,5} w_{3,4} \cdot 0 + p\alpha_0$ $s_2 w_{4,6}^{-1} w_{3,5} w_{1,2}^{-1} w_{3,4}^{-1} s_1 w_{4,6} s_2 w_{4,5} s_3 \cdot 0 + p\alpha_0$ $s_2 w_{4,6}^{-1} w_{3,5} w_{1,2}^{-1} w_{3,4}^{-1} s_1 w_{4,6} s_2 w_{3,4}^{-1} \cdot 0 + p\alpha_0$ $s_2 w_{4,6}^{-1} w_{3,5} w_{1,2}^{-1} w_{3,4}^{-1} s_1 w_{4,5} s_2 w_{3,4}^{-1} \cdot 0 + p\alpha_0$
E_7	1 2, 3, 4, 5 6	$s_1 w_{2,4}^{-1} w_{4,5}^{-1} w_{5,6}^{-1} w_{3,4} w_{1,3} w_{1,7}^{-1} w_{3,5}^{-1} w_{4,6}^{-1} s_2 w_{3,7}^{-1} s_1 \cdot 0 + p\alpha_0$ $s_1 w_{2,4}^{-1} w_{4,5}^{-1} w_{5,6}^{-1} w_{3,4} w_{1,3} w_{1,7}^{-1} w_{3,5}^{-1} w_{4,6}^{-1} s_2 w_{i+1,7}^{-1} \cdot 0 + p\alpha_0$ $s_1 w_{2,4}^{-1} w_{4,5}^{-1} w_{5,6}^{-1} w_{3,4} w_{1,3} w_{1,7}^{-1} w_{3,5}^{-1} w_{4,6}^{-1} s_2 s_7 \cdot 0 + p\alpha_0$
E_8	$1, 2, \cdots, 7$	$w_{1,8}^{-1} w_{3,4}^{-1} w_{4,5}^{-1} s_2 w_{3,6}^{-1} s_1 w_{2,7}^{-1} w_{1,8}^{-1} w_{3,5}^{-1} w_{4,6}^{-1} s_2 w_{3,7}^{-1} w_{4,8}^{-1} w_{1,9-i} \cdot 0 + p\alpha_0$
F_4	1 2 3	$w_{1,4}^{-1} w_{2,3}^{-1} w_{1,4}^{-1} w_{2,4}^{-1} w_{3,4} \cdot 0 + p\alpha_0$ $w_{1,4}^{-1} w_{2,3}^{-1} w_{1,4}^{-1} w_{2,4}^{-1} s_3 \cdot 0 + p\alpha_0$ $w_{1,4}^{-1} w_{2,3}^{-1} w_{1,4}^{-1} w_{2,4}^{-1} \cdot 0 + p\alpha_0$
G_2	1 2	$w_{1,2} w_{1,2} s_1 \cdot 0 + p\alpha_0$ $w_{1,2} w_{1,2} \cdot 0 + p\alpha_0$

Table 2. Descriptions of $\mu_i, j = 1, 2, \cdots, s-1$.

Root System	j	μ_j
A_l	$1, 2, \cdots, s-1$	$w_{1,l-2} w_{j,l}^{-1} \cdot 0 + p\alpha_0$
B_l	$1, 2, \cdots, l-1$ $l, l+1, \cdots, s-1$	$w_{1,l} w_{j+1,l-1}^{-1} w_{1,l} w_{1,l-1}^{-1} \cdot 0 + p\omega_2$ $w_{1,2l-j-1} w_{1,l} w_{1,l-1}^{-1} \cdot 0 + p\omega_2$
C_l	$1, 2, \cdots, l-2$ $l-1, l, \cdots, s-2$ $2l-2$	$w_{2,l} w_{3,l-1}^{-1} w_{1,l} w_{j+2,l}^{-1} \cdot 0 + p\alpha_0$ $w_{2,l} w_{3,l-1}^{-1} w_{2,2l-j-2} \cdot 0 + p\alpha_0$ $w_{2,l} w_{2,l-1}^{-1} \cdot 0 + p\alpha_0$
D_l	$1, 2, \cdots, s-1$	$w_{2,l-2} w_{3,l}^{-1} w_{1,l-2} w_{j+2,l}^{-1} \cdot 0 + p\alpha_0$

In [22], the authors computed the ordinary cohomology of simple modules with dominant highest weights $\lambda_1, \lambda_2, \cdots, \lambda_s, \mu_1, \mu_2, \cdots, \mu_{s-1}$. For the corresponding restricted cohomology, the following results hold.

Theorem 1. *Let G be a semisimple and simply connected algebraic group over an algebraically closed field of characteristic $p > h$, where h is the Coxeter number, and G_1 be the first Frobenius kernel of G. Consider simple G_1-modules with highest weights $\lambda_1, \lambda_2, \cdots, \lambda_s$ and write them in the following form, as described in* Table 1*: $\lambda_i = w_i \cdot 0 + p\nu_i$, where $w_i \in W$ and $\nu_i \in X(T)^+$. Then $H^n(G_1, L(\lambda_i)) = 0$, except in the following cases:*

(a) *if $i \in \{1, \cdots, t_\lambda\}$, then*
- *(i)* $H^n(G_1, L(\lambda_i)) \cong k$, *where* $n = i$,
- *(ii)* $H^n(G_1, L(\lambda_i)) \cong H^{n-i}(G_1, k)$, *where $n - i$ is even and* $i < n < l(w_i)$,
- *(iii)* $H^n(G_1, L(\lambda_i)) \cong H^{n-i}(G_1, k) + H^n\big(G_1, H^0(\lambda_i)\big)$, *where* $n = l(w_i)$,
- *(iv)* $H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{i} H^{n-i+j}\big(G_1, H^0(\lambda_j)\big)$, *where $n > l(w_i)$ and $n - l(w_i)$ is even;*

(b) *if $R = B_l$ and $i = t_\lambda + 1$, then*
- *(i)* $H^i(G_1, L(\lambda_i)) \cong k \oplus L(\alpha_0)^{(1)}$, *where* $n = i$,
- *(ii)* $H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{i} H^{n-i+j}\big(G_1, H^0(\lambda_j)\big)$, *where $n > i$ and $n - i$ is even;*

(c) *if $R = B_l$ and $i \in \{t_\lambda + 2, t_\lambda + 3, \cdots, s\}$, then*
- *(i)* $H^n(G_1, L(\lambda_i)) \cong L(\alpha_0)^{(1)}$, *where* $n = l(w_i)$,
- *(ii)* $H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{n} H^j\big(G_1, H^0(\lambda_{j+2})\big)$, *where $n - l(w_i)$ is even and* $l(w_i) < n < i$,
- *(iii)* $H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{i} H^j\big(G_1, H^0(\lambda_j)\big)$, *where* $n = i$,
- *(iv)* $H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{i} H^{n-i+j}\big(G_1, H^0(\lambda_j)\big)$, *where $n > i$ and $n - i$ is even;*

(d) *if $R = C_l$ and $i = s$, then*
- *(i)* $H^n(G_1, L(\lambda_i)) \cong k \oplus L(\alpha_0)^{(1)}$, *where* $n = i$,
- *(ii)* $H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{i} H^{n-i+j}\big(G_1, H^0(\lambda_j)\big)$, *where $n > i$ and $n - i$ is even.*

Here $\lambda_0 = 0$ and

$$t_\lambda = \begin{cases} s & \text{if } R = A_l, D_l, E_6, E_7, E_8, F_4, G_2, \\ s - l + 1 & \text{if } R = B_l, \\ s - 1 & \text{if } R = C_l. \end{cases}$$

In the cases of the classical Lie algebras of types A_l and B_l, the variability of the restricted cohomology cases for the simple modules with highest weights $\mu_1, \mu_2, \cdots, \mu_{s-1}$ is slightly different from the general case. Therefore, below the results for them are formulated separately.

Theorem 2. *Let G be a semisimple and simply connected algebraic group of type A_l $(l \geq 2)$ over an algebraically closed field of characteristic $p > h$, where h is the Coxeter number, and G_1 be the first Frobenius kernel of G. Consider simple G_1-modules with highest weights $\mu_1, \mu_2, \cdots, \mu_{s-1}$ and write them in the following form, as described in* Table 2*: $\mu_j = u_j \cdot 0 + p\delta_j$, where $u_j \in W$ and $\delta_j \in X(T)^+$. Then $H^n\big(G_1, L(\mu_j)\big) = 0$, except in the following cases:*

(a) *if $j = 1$, then*
- *(i)* $H^n\big(G_1, L(\mu_j)\big) \cong k$, *where* $n = 2$,
- *(ii)* $H^n\big(G_1, L(\mu_j)\big) \cong H^{n-2}(G_1, k)$, *where $n - 2$ is even and*$2 < n < l\big(u_j\big)$,
- *(iii)* $H^n\big(G_1, L(\mu_j)\big) \cong H^{n-2}(G_1, k) + H^n\big(G_1, H^0\big(\mu_j\big)\big)$, *where* $n = l\big(u_j\big)$,
- *(iv)* $H^n\big(G_1, L(\mu_j)\big) \cong H^{n-2}(G_1, k) + H^{n-1}\big(G_1, H^0(\lambda_1)\big) + H^n\big(G_1, H^0\big(\mu_j\big)\big)$, *where $n > l\big(u_j\big)$ with$n - l\big(u_j\big)$ is even;*

(b) *if $j \in \{2, 3, \cdots, s-1\}$, then*

(i) $H^n(G_1, L(\mu_j)) \cong k$, *where* $n = j - 1$,

(ii) $H^n(G_1, L(\mu_j)) \cong H^{n-j+1}(G_1, k) + H^{n-j-1}(G_1, k)$, *where* $l(u_j) - n$ *is even and* $j < n < l(u_j)$,

(iii) *for* $n = l(u_j)$, *where* $n - l(u_j)$ *is even,*

$$H^n(G_1, L(\mu_j)) \cong H^{n-j+1}(G_1, k) + \sum_{i=0}^{j} H^{n-j-1+i}(G_1, H^0(\lambda_i)) + \sum_{i=2}^{j} H^{n-j+i}(G_1, H^0(\mu_i)).$$

(iv) *for all* $n > l(u_j)$, *where* $n - l(u_j)$ *is even,*

$$H^n(G_1, L(\mu_j)) \cong H^{n-j+1}(G_1, k) + \sum_{i=0}^{j} H^{n-j-1+i}(G_1, H^0(\lambda_i)) + \sum_{i=1}^{j} H^{n-j+i}(G_1, H^0(\mu_i)).$$

Here $\lambda_0 = 0$.

Theorem 3. *Let* G *be a semisimple and simply connected algebraic group of type* B_l $(l \geq 2)$ *over an algebraically closed field of characteristic* $p > h$, *where* h *is the Coxeter number, and* G_1 *be the first Frobenius kernel of* G. *Consider simple* G_1*-modules with highest weights* $\mu_1, \mu_2, \cdots, \mu_{s-1}$ *and write them in the following form, as described in* Table 2: $\mu_i = u_j \cdot 0 + p\delta_j$, *where* $u_j \in W$ *and* $\delta_j \in X(T)^+$. *Then* $H^n(G_1, L(\mu_j)) = 0$, *except in the following cases:*

(a) *if* $j = 1$ *and* $l = 2$, *then*

(i) $H^n(G_1, L(\mu_j)) \cong k \oplus H^2(G_1, H^0(\lambda_2))$, *where* $n = 3$,

(ii) $H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{2} H^{n-3+i}(G_1, H^0(\lambda_i)) + H^n(G_1, H^0(\mu_j))$, *where* $n > 3$ *and* $n - 3$ *is even;*

(b) *if* $j = 1$ *and* $l > 2$, *then*

(i) $H^n(G_1, L(\mu_j)) \cong k$, *where* $n = 3$,

(ii) $H^n(G_1, L(\mu_j)) \cong H^{n-3}(G_1, k)$, *where* $n - 3$ *is even and* $3 < n < l(w_2)$,

(iii) $H^n(G_1, L(\mu_j)) \cong H^{n-3}(G_1, k) + H^{n-1}(G_1, H^0(\lambda_2))$, *where* $n = l(w_2) + 1$,

(iv) $H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{2} H^{n-3+i}(G_1, H^0(\lambda_i))$, *where* $l(u_j) - n$ *is even and* $l(w_2) + 1 < n < l(u_j)$;

(ii) $H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{2} H^{n-3+i}(G_1, H^0(\lambda_i)) + H^n(G_1, H^0(\mu_j))$, *where* $n \geq l(u_j)$ *and* $n - l(u_j)$ *is even;*

(c) *if* $j \in \{2, 3, \cdots, t_\lambda - 1\}$ *and* $l > 2$, *then*

(i) $H^n(G_1, L(\mu_j)) \cong k$, *where* $n = j$,

(ii) $H^n(G_1, L(\mu_j)) \cong H^{n-j}(G_1, k) \oplus H^{n-j-2}(G_1, k)$, *where* $j < n < l(w_j)$ *and* $l(w_j) - n$ *is even,*

(iii) $H^n(G_1, L(\mu_j)) \cong H^{n-j}(G_1, k) + H^{n-j-2}(G_1, k) + H^{n-1}(G_1, H^0(\lambda_{j+1}))$, *where* $n = l(w_j)$,

(iv) $H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{1} H^{n-j+i}(G_1, H^0(\lambda_i)) + \sum_{i=0}^{j+1} H^{n-j-2+i}(G_1, H^0(\lambda_i))$, *where* $l(w_j) < n < l(u_j)$ *and* $l(u_j) - n$ *is even,*

(v) *for all* $n \geq l(u_j)$, *where* $n - l(u_j)$ *is even,*

$$H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{1} H^{n-j+i}(G_1, H^0(\lambda_i)) + \sum_{i=0}^{j+1} H^{n-j-2+i}(G_1, H^0(\lambda_i)) + \sum_{i=m}^{j} H^{n-j+i}(G_1, H^0(\mu_i)),$$

$$\text{where } m = \begin{cases} 2 \text{ if } n = l(u_j), \\ 1 \text{ if } n = l(u_j); \end{cases}$$

(d) *if* $j = t_\lambda$ *and* $l > 2$, *then*

(i) $H^n(G_1, L(\mu_j)) \cong k$, *where* $n = j$,

(ii) $H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{1} H^{n-j+i}(G_1, H^0(\lambda_i)) + \sum_{i=0}^{j+1} H^{n-j-2+i}(G_1, H^0(\lambda_i))$, *where* $j < n < l(u_j)$ *and* $l(u_j) - n$ *is even,*

(iii) *for all* $n \geq l(u_j)$, *where* $n - l(u_j)$ *is even,*

$$H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{1} H^{n-j+i}(G_1, H^0(\lambda_i)) + \sum_{i=0}^{j+1} H^{n-j-2+i}(G_1, H^0(\lambda_i)) + \sum_{i=1}^{j} H^{n-j+i}(G_1, H^0(\mu_i));$$

(e) *if* $j \in \{t_\lambda + 1, t_\lambda + 2, \cdots, s-1\}$ *and* $l > 2$, *then*

(i) $H^n(G_1, L(\mu_j)) \cong k \oplus H^{n-1}(G_1, H^0(\lambda_{j+1}))$, *where* $n = j$,

(ii) $H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{1} H^{n-j+i}(G_1, H^0(\lambda_i)) + \sum_{i=0}^{j+1} H^{n-j-2+i}(G_1, H^0(\lambda_i))$, *where* $j < n < l(u_j)$ *and* $l(u_j) - n$ *is even*,

(iii) *for all* $n \geq l(u_j)$, *where* $n - l(u_j)$ *is even*,

$$H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{1} H^{n-j+i}(G_1, H^0(\lambda_i)) + \sum_{i=0}^{j+1} H^{n-j-2+i}(G_1, H^0(\lambda_i)) + \sum_{i=1}^{j} H^{n-j+i}(G_1, H^0(\mu_i)).$$

Here $\lambda_0 = 0$ *and* $t_\lambda = s - l + 1$.

Theorem 4. *Let* G *be a semisimple and simply connected algebraic group of type* C_l $(l \geq 3)$ *or* D_l $(l \geq 4)$ *over an algebraically closed field of characteristic* $p > h$, *where* h *is the Coxeter number, and* G_1 *be the first Frobenius kernel of* G. *Consider simple* G_1*-modules with highest weights* $\mu_1, \mu_2, \cdots, \mu_{s-1}$ *and write them in the following form, as described in* Table 1: $\mu_i = u_j \cdot 0 + p\delta_j$, *where* $u_j \in W$ *and* $\delta_j \in X(T)^+$. *Then* $H^n(G_1, L(\mu_j)) = 0$, *except in the following cases:*

(a) *if* $j = 1$, *then*

(i) $H^n(G_1, L(\mu_j)) \cong k$, *where* $n = 3$,

(ii) $H^n(G_1, L(\mu_j)) \cong H^{n-3}(G_1, k)$, *where* $n - 3$ *is even and* $3 < n < l(u_j)$,

(iii) $H^n(G_1, L(\mu_j)) \cong H^{n-3}(G_1, k) + H^n(G_1, H^0(\mu_j))$, *where* $n = l(u_j)$,

(iv) $H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{2} H^{n-3+i}(G_1, H^0(\lambda_i)) + H^n(G_1, H^0(\mu_j))$, *where* $n > l(u_j)$ *and* $n - l(u_j)$ *is even;*

(b) *if* $j \in \{2, 3, \cdots, t_\mu\}$, *then*

(i) $H^n(G_1, L(\mu_j)) \cong k$, *where* $n = j$,

(ii) $H^n(G_1, L(\mu_j)) \cong H^{n-j}(G_1, k) \oplus H^{n-j-2}(G_1, k)$, *where* $j < n < l(u_j)$ *and* $n - l(u_j)$ *is even*,

(iii) $H^n(G_1, L(\mu_j)) \cong H^{n-j}(G_1, k) + H^{n-j-2}(G_1, k) + H^n(G_1, H^0(\mu_j))$, *where* $n = l(u_j)$, *for all* $n > l(u_j)$, *where* $n - l(u_j)$ *is even*,

$$H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{1} H^{n-j+i}(G_1, H^0(\lambda_i)) + \sum_{i=0}^{j+1} H^{n-j-2+i}(G_1, H^0(\lambda_i)) + \sum_{i=1}^{j} H^{n-j+i}(G_1, H^0(\mu_i));$$

(c) *if* $j = s - 1$ *and* $R = C_l$, *then*

(i) $H^n(G_1, L(\mu_j)) \cong k \oplus H^n(G_1, H^0(\mu_j))$, *where* $n = l(u_j)$,

(ii) *for all* $n > l(u_j)$, *where* $n - l(u_j)$ *is even*,

$$H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{1} H^{n-j+i}(G_1, H^0(\lambda_i)) + \sum_{i=0}^{j+1} H^{n-j-2+i}(G_1, H^0(\lambda_i)) + \sum_{i=1}^{j} H^{n-j+i}(G_1, H^0(\mu_i)).$$

Here $\lambda_0 = 0$ *and*

$$t_\mu = \begin{cases} s-1 \text{ if } R = D_l, \\ s-2 \text{ if } R = C_l. \end{cases}$$

The following general result shows the importance of the cohomology of G_1 (restricted cohomology) in studying the connections between the cohomology of $\mathfrak{g}$, G_1, and G with coefficients in simple restricted modules.

Theorem 5. *Let G be a semisimple and simply connected algebraic group over an algebraically closed field of characteristic $p > h$, where h is the Coxeter number, G_1 be the first Frobenius kernel of G, and $\mathfrak{g}$ be the Lie algebra of G. Suppose that V is a simple module with the restricted highest weight. Then, for all $n > 0$,*

(a) $H^n(G_1, V) \cong H^n(G, V)$ *if and only if* $H^n(G_1, V) \cong \mathrm{Hom}_G(k, H^n(G_1, V)^{(-1)})$;
(b) $H^n(\mathfrak{g}, V) \cong H^n(G, V)$ *if and only if* $H^n(\mathfrak{g}, V) \cong \mathrm{Hom}_G(k, H^n(G_1, V)^{(-1)})$;
(c) $H^n(G, V) \cong \mathrm{Hom}_G(k, H^n(\mathfrak{g}, V)^{(-1)})$ *if* $H^n(\mathfrak{g}, V) \cong H^n(G_1, V)$.

Theorems 1–5 allow us to compare the structures of ordinary cohomology (cohomology for $\mathfrak{g}$), restricted cohomology (cohomology for G_1), and cohomology of the algebraic group associated with a given Lie algebra (cohomology for G). For example, comparison of Theorem 1 with the results on cohomology $H^n(\mathfrak{g}, L(\lambda_i))$ obtained in [22] yields the following result:

Corollary 1. *Let G be a semisimple and simply connected algebraic group over an algebraically closed field k of characteristic $p > h$, where h is the Coxeter number, G_1 be the first Frobenius kernel of G, and $\mathfrak{g}$ be the Lie algebra of G. Then, the nontrivial isomorphism*

$$H^n(\mathfrak{g}, L(\lambda_i)) \cong H^n(G_1, L(\lambda_i))$$

of G -modules holds only in the following cases:

(a) $n = i$ *and* $i \in \{1, \cdots, t_\lambda\}$;
(b) $\mathfrak{g} = B_l$, $n = i$, *and* $i = t_\lambda + 1$;
(c) $\mathfrak{g} = B_l$, $n = l(w_i)$, *and* $i \in \{t_\lambda + 2, t_\lambda + 3, \cdots, s\}$;
(d) $\mathfrak{g} = C_l$, $n = i$, *and* $i = s$.

Comparison of Theorems 2–4 with the results on cohomology $H^n(\mathfrak{g}, L(\mu_j))$ obtained in [22] yields the following result:

Corollary 2. *Let G be a semisimple and simply connected algebraic group over an algebraically closed field k of characteristic $p > h$, where h is the Coxeter number, G_1 be the first Frobenius kernel of G, and $\mathfrak{g}$ be the Lie algebra of G. Then, the non-trivial isomorphism*

$$H^n(\mathfrak{g}, L(\mu_j)) \cong H^n(G_1, L(\mu_j))$$

of G -modules holds only in the following cases:

(a) $\mathfrak{g} = A_l$, $n = j + 1$, *and* $j = 1$;
(b) $\mathfrak{g} = B_l,\ C_l,\ D_l$, $n = j + 2$, *and* $j = 1$;
(c) $\mathfrak{g} = A_l$, $n = j - 1$, *and* $j \in \{2, 3, \cdots, s - 1\}$;
(d) $\mathfrak{g} = B_l,\ C_l,\ D_l$, $n = j$, *and* $j \in \{2, 3, \cdots, s - 1\}$.

Using Theorem 1 and Statement (*a*) of Theorem 5, we obtain the following result on the connection between cohomology $H^n(G_1, L(\lambda_i))$ and $H^n(G, L(\lambda_i))$:

Corollary 3. *Let G be a semisimple and simply connected algebraic group over an algebraically closed field k of characteristic $p > h$, where h is the Coxeter number, and G_1 be the first Frobenius kernel of G. Then, the non-trivial isomorphism $H^n(G_1, L(\lambda_i)) \cong H^n(G, L(\lambda_i))$ of G-modules holds only for $n = i$ and $i \in \{1, \cdots, t_\lambda\}$.*

Similarly, using Theorems 2–4 and Statement (*a*) of Theorem 5, we obtain the following result on the connection between cohomology $H^n(G_1, L(\mu_j))$ and $H^n(G, L(\mu_j))$:

Corollary 4. *Let G be a semisimple and simply connected algebraic group over an algebraically closed field k of characteristic $p > h$, where h is the Coxeter number, and G_1 be the first Frobenius* *kernel of G. Then, the nontrivial isomorphisms $H^n(G_1, L(\mu_j)) \cong H^n(G, L(\mu_j))$ of G-modules holds only in the following cases:*

(a) $\mathfrak{g} = A_l$, $n = j+1$, *and* $j = 1$;
(b) $\mathfrak{g} = B_l,\ C_l,\ D_l$, $n = j+2$, *and* $j = 1$;
(c) $\mathfrak{g} = A_l$, $n = j-1$, *and* $j \in \{2, 3, \cdots, s-1\}$;
(d) $\mathfrak{g} = B_l$, $n = j$, *and* $j \in \{2, 3, \cdots, t_\lambda\}$;
(a) $\mathfrak{g} = C_l,\ D_{l,}$ $n = j$, *and* $j \in \{2, 3, \cdots, t_\mu\}$.

Corollaries 1 and 3 immediately imply the following result:

Corollary 5. *Let G be a semisimple and simply connected algebraic group over an algebraically closed field k of characteristic $p > h$, where h is the Coxeter number, $\mathfrak{g}$ be the Lie algebra of G, and G_1 be the first Frobenius kernel of G. Then, the non-trivial isomorphisms $H^n(\mathfrak{g}, L(\lambda_i)) \cong H^n(G_1, L(\lambda_i)) \cong H^n(G, L(\lambda_i))$ of G-modules hold only for $n = i$ and $i \in \{1, \cdots, t_\lambda\}$.*

The existence of these isomorphisms was previously established in [22] (p. 7). Corollary 5 establishes that for simple modules $L(\lambda_1), L(\lambda_2), \cdots, L(\lambda_s)$, there are no other such non-trivial isomorphisms.

Similarly, the following result immediately follows from Corollaries 2 and 4:

Corollary 6. *Let G be a semisimple and simply connected algebraic group over an algebraically closed field k of chacteristic $p > h$, where h. is the Coxeter number, $\mathfrak{g}$ be the Lie algebra of G, and G_1 be the first Frobenius kernel of G. Then, the non-trivial isomorphisms $H^n(\mathfrak{g}, L(\mu_j)) \cong H^n(G, L(\mu_j)) \cong H^n(G_1, L(\mu_j))$ of G-modules hold only in the following cases:*

(a) $\mathfrak{g} = A_l$, $n = j+1$, *and* $j = 1$;
(b) $\mathfrak{g} = B_l,\ C_l,\ D_l$, $n = j+2$, *and* $j = 1$;
(c) $\mathfrak{g} = A_l$, $n = j-1$, *and* $j \in \{2, 3, \cdots, s-1\}$;
(d) $\mathfrak{g} = B_l$, $n = j$, *and* $j \in \{2, 3, \cdots, t_\lambda\}$;
(e) $\mathfrak{g} = C_l,\ D_{l,}$ $n = j$, *and* $j \in \{2, 3, \cdots, t_\mu\}$.

3.2. Proof of the Results

Proof of Theorem 1. We will use the algorithm for calculating restricted cohomology with coefficients in simple modules given at the end of Section 2 for $\lambda = \lambda_i$ with $i \in \{1, 2, \cdots, s\}$. Let us calculate ν_i and $l(w_i)$. According to Table 1, for all $i \in \{1, 2, \cdots, s\}$, $\nu_i = \alpha_0$ and

$$l(w_i) = \begin{cases} 2l - i & \text{for } g = A_l,\ B_l, \\ 4l - i - 4 & \text{for } g = C_l, \\ 4l - i - 6 & \text{for } g = D_l, \\ 22 - i & \text{for } g = E_6, \\ 34 - i & \text{for } g = E_7, \\ 58 - i & \text{for } g = E_8, \\ 16 - i & \text{for } g = F_4, \\ 6 - i & \text{for } g = G_2. \end{cases} \tag{6}$$

Now, let us give the structure of $H^0(\lambda_i)/L(\lambda_i)$ as a G_1-module. Since in the restricted region the representation theories G and G_1 are equivalent, then $H^0(\lambda_i)/L(\lambda_i)$ has the same structure as a G-module and a G_1-module. Therefore, by Statement (*a*) of Lemma 4.1 in [22] (p. 3870),

$$H^0(\lambda_i)/L(\lambda_i) \cong L(\lambda_{i-1}) \tag{7}$$

for all $i \in \{1, 2, \cdots, s\}$ as G_1-module.

The next two steps of the algorithm for calculating the cohomology $H^n(G_1, L(\lambda_i))$ will be done separately in the corresponding statements of the theorem. $\square$

Proof of Statement (*a*) of Theorem 1. We will calculate $H^{n-1}(G_1, H^0(\lambda_i)/L(\lambda_i))$ and $H^n(G_1, L(\lambda_i))$ simultaneously. By (7), $H^{n-1}(G_1, H^0(\lambda_i)/L(\lambda_i)) \cong H^{n-1}(G_1, L(\lambda_{i-1}))$ for all $i \in \{1, 2, \cdots, t_\lambda\}$. By (6), $i < l(w_i)$ for all $i \in \{1, 2, \cdots, t_\lambda\}$. Then, by (4) and (5), for all $i \in \{1, 2, \cdots, t_\lambda\}$,

$$H^n(G_1, L(\lambda_i)) \cong \begin{cases} H^{n-1}(G_1, L(\lambda_{i-1})) & \text{if } n < l(w_i), \\ H^{n-1}(G_1, L(\lambda_{i-1})) + H^n(G_1, H^0(\lambda_i)) & \text{if } n \geq l(w_i). \end{cases} \tag{8}$$

Let $n < i$. Then, using the induction on i, from (8), we obtain

$$H^n(G_1, L(\lambda_i)) \cong H^{n-1}(G_1, L(\lambda_{i-1})) \tag{9}$$

for all $i \in \{1, 2, \cdots, t_\lambda\}$. We use the induction on i. If $i = 1$, then, by (9),

$$H^n(G_1, L(\lambda_1)) \cong H^{n-1}(G_1, L(\lambda_0)) = H^{n-1}(G_1, k) = 0,$$

since $n < 1$. Suppose that $H^n(G_1, L(\lambda_i)) = 0$ for all $i < i_0$, where $i_0 \leq t_\lambda$, and prove that $H^n(G_1, L(\lambda_{i_0})) = 0$. By (9), $H^n(G_1, L(\lambda_{i_0})) \cong H^{n-1}(G_1, L(\lambda_{i_0-1}))$ for all $n < i_0$. By the induction hypothesis, $H^{n-1}(G_1, L(\lambda_{i_0-1})) = 0$ for all $n < i_0$. Therefore, $H^n(G_1, L(\lambda_{i_0})) = 0$ for all $n < i_0$. Thus, $H^n(G_1, L(\lambda_i)) = 0$ for all $i \in \{1, 2, \cdots, t_\lambda\}$ and for all $n < i$. Now, let $n = i$. Then, using (8), we see that in this case also (9) holds. If $i = 1$, then, by (9), $H^n(G_1, L(\lambda_1)) \cong H^0(G_1, L(\lambda_0)) = H^0(G_1, k) = k$, since $n = 1$. Suppose that $H^n(G_1, L(\lambda_i)) = k$ for all $i < i_0$, where $i_0 \leq t_\lambda$, and $n = i$. We prove that $H^n(G_1, L(\lambda_{i_0})) = k$ if $n = i_0$. Using (9), we get that $H^n(G_1, L(\lambda_{i_0})) \cong H^{n-1}(G_1, L(\lambda_{i_0-1}))$ if $n = i_0$. By the induction hypothesis, $H^{n-1}(G_1, L(\lambda_{i_0-1})) = k$ if $n = i_0$. Therefore, $H^n(G_1, L(\lambda_{i_0})) = k$ if $n = i_0$. Thus, $H^n(G_1, L(\lambda_i)) = k$ for all $i \in \{1, 2, \cdots, t_\lambda\}$ and for $n = i$. This proves the sub-statement (*i*).

If $i < n < l(w_i)$, then, by (8), in this case also the Formula (9) holds. Let $i = 1$, then, by (9), $H^n(G_1, L(\lambda_1)) \cong H^{n-1}(G_1, k)$ if $1 < n < l(w_1)$. If $n - 1$ is even, then, by (3), $H^{n-1}(G_1, k)$ is non-trivial, otherwise $H^{n-1}(G_1, k) = 0$. Suppose that $H^n(G_1, L(\lambda_i)) \cong H^{n-i}(G_1, k)$ for all $i < i_0$ if $i < n < l(w_i)$. Prove that $H^n(G_1, L(\lambda_{i_0})) = H^{n-i_0}(G_1, k)$ if $i_0 < n < l(w_{i_0})$. Using (9), we get that

$$H^n(G_1, L(\lambda_{i_0})) \cong H^{n-1}(G_1, L(\lambda_{i_0-1}))$$

if $i_0 < n < l(w_{i_0})$. By the induction hypothesis, $H^{n-1}(G_1, L(\lambda_{i_0-1})) = H^{n-i_0}(G_1, k)$ if $i_0 < n < l(w_{i_0})$. Therefore, $H^n(G_1, L(\lambda_{i_0})) = H^{n-i_0}(G_1, k)$ if $i_0 < n < l(w_{i_0})$. Thus, $H^n(G_1, L(\lambda_i)) = H^{n-i}(G_1, k)$ for all $i \in \{1, 2, \cdots, t_\lambda\}$ if $t_\lambda < n < l(w_{t_\lambda})$. If $n - i$ is even, then, by (3), $H^{n-i}(G_1, k)$ is non-trivial, otherwise $H^{n-i}(G_1, k) = 0$. So, we get the sub-statement (*ii*).

If $n = l(w_i)$, then (8),

$$H^n(G_1, L(\lambda_i)) \cong H^{n-1}(G_1, L(\lambda_{i-1})) + H^n(G_1, H^0(\lambda_i)) \tag{10}$$

for all $i \in \{1, 2, \cdots, t_\lambda\}$. By the sub-statement (*ii*) of this Statement (*a*), $H^{n-1}(G_1, L(\lambda_{i-1})) \cong H^{n-i}(G_1, k)$ if $n = l(w_i)$. Then, by (10),

$$H^n(G_1, L(\lambda_i)) \cong H^{n-i}(G_1, k) + H^n(G_1, H^0(\lambda_i))$$

for all $i \in \{1, 2, \cdots, t_\lambda\}$. By (3), $H^{n-i}(G_1, k)$ and $H^n(G_1, H^0(\lambda_i))$ are non-trivial. Therefore, the sub-statements (*iii*) hold.

Finally, let $n > l(w_i)$. Then, using (8), we see that in this case (10) also holds. Let $i = 1$, then, by (10),

$$H^n(G_1, L(\lambda_1)) \cong H^{n-1}(G_1, k) + H^n(G_1, H^0(\lambda_1))$$

if $n > l(w_1)$. If $n - l(w_i)$ is even, then, by (3), both summands of the sum of the left-hand side of the last isomorphism are nontrivial, otherwise both of them are trivial. Suppose that

$$H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{i} H^{n-i+j}(G_1, H^0(\lambda_j))$$

for all $i < i_0$ if $n > l(w_i)$. Prove that

$$H^n(G_1, L(\lambda_{i_0})) \cong \sum_{j=0}^{i_0} H^{n-i_0+j}(G_1, H^0(\lambda_j))$$

if $n > l(w_{i_0})$. Using (10), we get that

$$H^n(G_1, L(\lambda_{i_0})) \cong H^{n-1}(G_1, L(\lambda_{i_0-1})) + H^n(G_1, H^0(\lambda_{i_0}))$$

if $n > l(w_{i_0})$. By the induction hypothesis,

$$H^{n-1}(G_1, L(\lambda_{i_0-1})) = \sum_{j=0}^{i_0-1} H^{n-i_0+j}(G_1, H^0(\lambda_j))$$

if $n > l(w_{i_0})$. Therefore,

$$H^n(G_1, L(\lambda_{i_0})) = \sum_{j=0}^{i_0-1} H^{n-i_0+j}(G_1, H^0(\lambda_j)) + H^n(G_1, H^0(\lambda_{i_0})) = \sum_{j=0}^{i_0} H^{n-i_0+j}(G_1, H^0(\lambda_j))$$

if $n > l(w_{i_0})$. Thus,

$$H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{i} H^{n-i+j}(G_1, H^0(\lambda_j))$$

for all $i \in \{1, 2, \cdots, t_\lambda\}$ if $n > l(w_{t_\lambda})$. If $n - l(w_i)$ is even, then, by (3), all summands of the sum of the left-hand side of the last isomorphism are non-trivial, otherwise they are all trivial. Therefore, the sub-statement (*iv*) is true. The proof of the statement (*a*) is complete.

If $i > t_\lambda$, then the situation is slightly different from the previous case. The following statements cover them. □

Proof of Statement (*b*) of Theorem 1. In this case, $R = B_l$ and $i = t_\lambda + 1$. Note that $i = l(w_i) = l$.

Let $n < i$. Since $i < l(w_i)$, we will use the Formula (4) for $\lambda = \lambda_i$. Using (4) and (7), we get $H^n(G_1, L(\lambda_i)) \cong H^{n-1}(G_1, L(\lambda_{i-1}))$. By the Statement (*a*), $H^{n-1}(G_1, L(\lambda_{i-1})) = 0$. Therefore, $H^n(G_1, L(\lambda_i)) = 0$.

Let $n = i$. By (7), $H^{n-1}(G_1, H^0(\lambda_i)/L(\lambda_i)) \cong H^{n-1}(G_1, L(\lambda_{i-1}))$ and by the Statement (*a*), $H^{n-1}(G_1, L(\lambda_{i-1})) \cong k$. Moreover, according to (3), $H^n(G_1, H^0(\lambda_i)) = L(\alpha_0)^{(1)}$. Because $\mathrm{Ext}^1_G\left(k, L(\alpha_0)^{(1)}\right) = 0$, then, for $\lambda = \lambda_i$, the short exact sequence (5) splits. So, we get an isomorphism

$$H^n(G_1, L(\lambda_i)) \cong k \oplus L(\alpha_0)^{(1)}.$$

This is the sub-statement (*i*).

If $n > i$, then by (5) for $\lambda = \lambda_i$, $H^n(G_1, L(\lambda_i)) \cong H^{n-1}(G_1, L(\lambda_{i-1})) + H^n(G_1, H^0(\lambda_i))$. To avoid repetition, here and in what follows we will omit all the details of the induction process on i. Using the sub-statement (*i*) of this Statement (*b*) and the induction on i, we get $H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{i} H^{n-i+j}(G_1, H^0(\lambda_j))$. By (3), all summands of this sum are non-trivial if $n - i$ is even, otherwise they are all trivial. Hence, the sub-statement (*ii*) is true. □

Proof of Statement (*c*) of Theorem 1. Note that $R = B_l$, $i = \{t_\lambda + 1,\ t_\lambda + 2, \cdots, s\}$,. and $i > l(w_i)$.

Let $n < l(w_i)$. Using (4) and (7), we get $H^n(G_1, L(\lambda_i)) \cong H^{n-1}(G_1, L(\lambda_{i-1}))$.. By the Statement (*b*), $H^{n-1}(G_1, L(\lambda_{i-1})) = 0$. Therefore, $H^n(G_1, L(\lambda_i)) = 0$.

Let $n = l(w_i)$. We will use the short exact sequence (5) for $\lambda = \lambda_i$. By (7), $H^{n-1}\big(G_1, H^0(\lambda_i)/L(\lambda_i)\big) \cong H^{n-1}(G_1, L(\lambda_{i-1}))$. Since $i > l(w_i)$, then, by the Statement (*b*), $H^{n-1}(G_1, L(\lambda_{i-1})) = 0$. Moreover according to (3), $H^n\big(G_1, H^0(\lambda_i)\big) \cong L(\alpha_0)^{(1)}$. Then, using the short exact sequence (5), we get an isomorphism

$$H^n(G_1, L(\lambda_i)) \cong L(\alpha_0)^{(1)}.$$

This is the sub-statement (*i*).

If $l(w_i) < n < i$,. then by (5) for $\lambda = \lambda_i$,. $H^n(G_1, L(\lambda_i)) \cong H^{n-1}(G_1, L(\lambda_{i-1})) + H^n\big(G_1, H^0(\lambda_i)\big)$. Using the sub-statement (*i*) of this Statement (*c*) and the induction on i, we get $H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{n} H^j\big(G_1, H^0(\lambda_{j+2})\big)$. By (3), all summands of this sum are non-trivial if $n - l(w_i)$ is even, otherwise they are all trivial. Hence, the sub-statement (*ii*) is true.

Now let $n = i$. Then by (5) for $\lambda = \lambda_i$, $H^n(G_1, L(\lambda_i)) \cong H^{n-1}(G_1, L(\lambda_{i-1})) + H^n\big(G_1, H^0(\lambda_i)\big)$.. Using the sub-statement (*ii*) of this Statement (*c*) and the induction on i,. we get $H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{i} H^j\big(G_1, H^0(\lambda_j)\big)$. By (3), all summands of this sum are non-trivial. So, we get the sub-statement (*iii*).

Finally let $n > i$. Then, by (5) for $\lambda = \lambda_i$, $H^n(G_1, L(\lambda_i)) \cong H^{n-1}(G_1, L(\lambda_{i-1})) + H^n\big(G_1, H^0(\lambda_i)\big)$. Using the sub-statement (*iii*) of this Statement (*c*) and the induction on i, we get $H^n(G_1, L(\lambda_i)) \cong \sum_{j=0}^{i} H^{n-i+j}\big(G_1, H^0(\lambda_j)\big)$. By (3), all summands of this sum are non-trivial if $n - i$ is even, otherwise they are all trivial. Hence, we have obtained sub-statement (*iv*). □

Proof of Statement (*d*) of Theorem 1. Is similar to Proof of Statement (*b*) of Theorem 1. We only note that, in this case, $R = C_l$, $i = t_\lambda + 1$ and $i = l(w_i) = 2l - 2$. □

Proof of Theorems 2–4. First, we will calculate δ_j and According to Table 2, for all $\in \{1, 2, \cdots, s-1\}$, $\delta_j = \alpha_0$, except in the case where $R = B_l$ and $\delta_j = \omega_1$, and

$$l(u_j) = \begin{cases} 2l - j - 1 \text{ if } g = A_l, \\ 4l - j - 2 \text{ if } g = B_l, \\ 4l - j - 6 \text{ if } g = C_l, \\ 4l - j - 8 \text{ if } g = D_l. \end{cases} \tag{11}$$

Now, let us give the structure of $H^0(\mu_j)/L(\mu_j)$ as a G_1-module. Since in the restricted region the representation theories G and G_1 are equivalent, then $H^0(\mu_j)/L(\mu_j)$ has the same structure as an G. -module and a G_1-module. Therefore, by the statements (*b*)–(*d*) of Lemma 4.1 in [22] (p. 3870),

$$H^0(\mu_1)/L(\mu_1) \cong L(\lambda_2), \tag{12}$$

and there exist the following short exact sequences:

$$0 \to L(\mu_1) \oplus L(\lambda_3) \oplus L(\lambda_1) \to H^0(\mu_2)/L(\mu_2) \to L(\lambda_2) \to 0, \tag{13}$$

and

$$0 \to L(\mu_{j-1}) \oplus L(\lambda_{j+1}) \to H^0(\mu_j)/L(\mu_j) \to L(\lambda_j) \to 0 \tag{14}$$

for all $j \in \{3, 4, \cdots, s-1\}$.

The next two steps of the algorithm for calculating the cohomology $H^n(G_1, L(\mu_j))$ will be done separately in the corresponding statements. Since the proofs of Theorems 2–4 are similar, we will only illustrate in more detail the proofs of Statements of Theorem 3, which is more variable. □

Proof of Statement (*a*) of Theorem 3. In this case, $\mathfrak{g} = B_2$ and $j = 1$. By (12), $H^{n-1}(G_1, H^0(\mu_j)/L(\mu_j)) \cong H^{n-1}(G_1, L(\lambda_2))$. By (6), $l(w_2) = 2$, and by (11), $l(u_1) = 5$.

Let $n < 3$. Since $H^{n-1}(G_1, L(\lambda_2)) = 0$ for all $n < 3$, then $H^n(G_1, L(\mu_j)) \cong H^{n-1}(G_1, L(\lambda_2)) = 0$.

If $n = 3$, then, by Statement (*b*) of Theorem 1, $H^{n-1}(G_1, L(\lambda_2)) \cong k \oplus L(\alpha_0)^{(1)}$. So, by (4),

$$H^n(G_1, L(\mu_j)) \cong H^{n-1}(G_1, L(\lambda_2)) = k \oplus L(\alpha_0)^{(1)}.$$

This is the sub-statement (*i*).

Let $3 < n < l(u_1) = 5$. In this case, there is only one value $n = 4$. Then, by Statement (*b*) of Theorem 1, $H^{n-1}(G_1, L(\lambda_2)) = 0$. Therefore, $H^n(G_1, L(\mu_j)) = 0$.

If $n \geq l(u_1) = 5$, Then, by Statement (*b*) of Theorem 1,

$$H^{n-1}(G_1, L(\lambda_2)) \cong \sum_{i=0}^{2} H^{n-3-i}(G_1, H^0(\lambda_i)).$$

Then, by (5),

$$H^n(G_1, L(\mu_j)) \cong H^{n-1}(G_1, L(\lambda_2)) + H^n(G_1, H^0(\mu_j)) = \sum_{i=0}^{2} H^{n-3-i}(G_1, H^0(\lambda_i)) + H^n(G_1, H^0(\mu_j)).$$

By (3), all summands of this sum are non-trivial if $n - 3$ is even, otherwise they are trivial. Hence, we have obtained sub-statement (*ii*). □

Proof of Statement (*b*) of Theorem 3. In this case, $\mathfrak{g} = B_l$ $(l > 2)$ and $j = 1$. By (12),

$$H^{n-1}(G_1, H^0(\mu_j)/L(\mu_j)) \cong H^{n-1}(G_1, L(\lambda_2)).$$

By (6), $l(w_2) = 2l - 2$, and by (11), $l(u_1) = 4l - 3$.

Let $n < 3$. Since $H^{n-1}(G_1, L(\lambda_2)) = 0$ for all $n < 3$, then $H^n(G_1, L(\mu_j)) \cong H^{n-1}(G_1, L(\lambda_2)) = 0$.

If $n = 3$, then, by Statement (*a*) of Theorem 1, $H^{n-1}(G_1, L(\lambda_2)) \cong k$. So, by (4),

$$H^n(G_1, L(\mu_j)) \cong H^{n-1}(G_1, L(\lambda_2)) = k.$$

This is the sub-statement (*i*).

Let $3 < n < l(w_2) = 2l - 2$. Then, by Statement (*a*) of Theorem 1, $H^{n-1}(G_1, L(\lambda_2)) \cong H^{n-3}(G_1, k)$. Therefore, by (4), $H^n(G_1, L(\mu_j)) \cong H^{n-1}(G_1, L(\lambda_2)) \cong H^{n-3}(G_1, k)$. This cohomology is non-trivial if $n - 3$ is even, and is trivial otherwise. So, we get the sub-statement (*ii*).

If $n = l(w_2) = 2l - 2$, then, by Statement (*a*) of Theorem 1, $H^{n-1}(G_1, L(\lambda_2)) = 0$. Therefore, by (4), $H^n(G_1, L(\mu_j)) \cong H^{n-1}(G_1, L(\lambda_2)) = 0$.

If $n = l(w_2) + 1 = 2l - 1$, then, by Statement (*a*) of Theorem 1, $H^{n-1}(G_1, L(\lambda_2)) \cong H^{n-3}(G_1, k) + H^{n-1}(G_1, H^0(\lambda_2))$. Therefore, by (4),

$$H^n(G_1, L(\mu_j)) \cong H^{n-1}(G_1, L(\lambda_2)) = H^{n-3}(G_1, k) + H^{n-1}(G_1, H^0(\lambda_2)).$$

This is the sub-statement (*iii*).

If $l(w_2) + 1 < n < l(u_1) = 4l - 3$, then, by Statement (*b*) of Theorem 1,

$$H^{n-1}(G_1, L(\lambda_2)) \cong \sum_{i=0}^{2} H^{n-3-i}(G_1, H^0(\lambda_i)).$$

Using (4), we get

$$H^n\big(G_1, L(\mu_j)\big) \cong H^{n-1}(G_1, L(\lambda_2)) = \sum_{i=0}^{2} H^{n-3-i}(G_1, H^0(\lambda_i)).$$

If $l(u_i) - n$ is even, then, by (3), all summands of the sum of the left-hand side of the last isomorphism are non-trivial, otherwise they are all trivial. Therefore, the sub-statement (*iv*) is true.

Finally, let $n \geq l(u_1) = 4l - 3$. then, by Statement (*b*) of Theorem 1,

$$H^{n-1}(G_1, L(\lambda_2)) \cong \sum_{i=0}^{2} H^{n-3-i}(G_1, H^0(\lambda_i)).$$

Using (5), we get

$$H^n\big(G_1, L(\mu_j)\big) \cong H^{n-1}(G_1, L(\lambda_2)) + H^n(G_1, H^0(\mu_j)) = \sum_{i=0}^{2} H^{n-3-i}(G_1, H^0(\lambda_i)) + H^n(G_1, H^0(\mu_j)).$$

If $n - l(u_i)$ is even, then, by (3), all summands of the sum of the left-hand side of the last isomorphism are non-trivial, otherwise they are all trivial. Therefore, the sub-statement (*v*) is true. $\square$

Proof of Statement (*c*) of Theorem 3. In this case, $\mathfrak{g} = B_l$ $(l > 2)$ and $j \in \{2, 3, \cdots, t_\lambda\}$. The long cohomological sequences corresponding to the short exact sequences (13) and (14) yield the exact sequences

$$0 \to H^{n-2}(G_1, L(\lambda_2)) \to H^{n-1}(G_1, L(\mu_1) \oplus L(\lambda_3) \oplus L(\lambda_1)) \to H^{n-1}(G_1, H^0(\mu_2)/L(\mu_2)) \to 0. \tag{15}$$

$$0 \to H^{n-2}\big(G_1, L(\lambda_j)\big) \to H^{n-1}\big(G_1, L(\mu_{j-1}) \oplus L(\lambda_{j+1})\big) \to H^{n-1}(G_1, H^0(\mu_j)/L(\mu_j)) \to 0,\ j > 2, \tag{16}$$

respectively.

Let $n < j$. By Statement (*a*) of Theorem 1 and Statement (*b*) of this Theorem 3, $H^{n-2}\big(G_1, L(\lambda_j)\big) = 0$, $H^{n-1}(G_1, L(\mu_1) \oplus L(\lambda_3) \oplus L(\lambda_1)) = 0$, and

$$H^{n-1}\big(G_1, L(\mu_{j-1}) \oplus L(\lambda_{j+1})\big) = 0.$$

Then it follows from the exactness of the sequences (15) and (16) that $H^{n-1}\big(G_1, H^0(\mu_2)/L(\mu_2)\big) = 0$. Therefore, by (4),

$$H^n\big(G_1, L(\mu_j)\big) = H^{n-1}(G_1, H^0(\mu_2)/L(\mu_2)) = 0.$$

Let $n = j$. We use the induction on j. If $j = 2$, then $H^{n-2}(G_1, L(\lambda_2)) = 0$, and, by Statement (*a*) of Theorem 1 and Statement (*b*) of this Theorem 3,

$$H^{n-1}(G_1, L(\mu_1) \oplus L(\lambda_3) \oplus L(\lambda_1)) \cong H^{n-1}(G_1, L(\lambda_1)) \cong k.$$

Then it follows from the exactness of the sequence (15) that $H^{n-1}\big(G_1, H^0(\mu_2)/L(\mu_2)\big) = k$. Therefore, by (4),

$$H^n(G_1, L(\mu_2)) \cong H^{n-1}(G_1, H^0(\mu_2)/L(\mu_2)) \cong k.$$

Now suppose that $H^n(G_1, L(\mu_j)) \cong k$ for all $j < j_0$. By Statement (*a*) of Theorem 1, $H^{n-2}(G_1, L(\lambda_{j_0})) \cong 0$ and $H^{n-1}(G_1, L(\lambda_{j_0+1})) \cong 0$. By the induction hypothesis, $H^{n-1}(G_1, L(\mu_{j_0-1})) \cong k$. Then the exactness of the sequence (16) yields

$$H^{n-1}\Big(G_1, H^0(\mu_{j_0})/L(\mu_{j_0})\Big) = k.$$

Hence, by (4), $H^n(G_1, L(\mu_{j_0})) \cong H^{n-1}(G_1, H^0(\mu_{j_0})/L(\mu_{j_0})) \cong k$. So, $H^n(G_1, L(\mu_j)) \cong k$ for all $j \in \{2, 3, \cdots, t_\lambda\}$, which proves the sub-statement (*i*).

Let $j < n < l(w_2) = 2l - 2$. We will use induction on j. If $j = 2$, then by Statement (*a*) of Theorem 1, $H^{n-1}(G_1, L(\lambda_1)) \cong H^{n-2}(G_1, k)$, $H^{n-2}(G_1, L(\lambda_2)) \cong H^{n-4}(G_1, k)$, and $H^{n-1}(G_1, L(\lambda_3)) \cong H^{n-4}(G_1, k)$. By Statement (*b*) of this Theorem 3, $H^{n-1}(G_1, L(\mu_1)) \cong H^{n-4}(G_1, k)$. Then it follows from the exactness of the sequence (15) that

$$H^{n-1}\Big(G_1, H^0(\mu_2)/L(\mu_2)\Big) = H^{n-2}(G_1, k) + H^{n-4}(G_1, k).$$

Now suppose that $H^n(G_1, L(\mu_j)) \cong H^{n-j}(G_1, k) + H^{n-j-2}(G_1, k)$ for all $j < j_0$. By Statement (*a*) of Theorem 1, $H^{n-2}(G_1, L(\lambda_{j_0})) \cong H^{n-j_0-2}(G_1, k)$ and $H^{n-1}(G_1, L(\lambda_{j_0+1})) \cong H^{n-j_0-2}(G_1, k)$. By the induction hypothesis, $H^{n-1}(G_1, L(\mu_{j_0-1})) \cong H^{n-j_0}(G_1, k) + H^{n-j_0-2}(G_1, k)$. Then the exactness of the sequence (16) yields

$$H^{n-1}\Big(G_1, H^0(\mu_{j_0})/L(\mu_{j_0})\Big) = H^{n-j_0}(G_1, k) + H^{n-j_0-2}(G_1, k).$$

Hence, by (4),

$$H^n(G_1, L(\mu_{j_0})) \cong H^{n-1}\Big(G_1, H^0(\mu_{j_0})/L(\mu_{j_0})\Big) \cong H^{n-j_0}(G_1, k) + H^{n-j_0-2}(G_1, k).$$

So, $H^n(G_1, L(\mu_j)) \cong H^{n-j}(G_1, k) + H^{n-j-2}(G_1, k)$ for all $j \in \{2, 3, \cdots, t_\lambda\}$, which proves the sub-statement (*ii*).

Let $n = l(w_j) = 2l - j$. If $j = 2$, then by Statement (*a*) of Theorem 1, $H^{n-1}(G_1, L(\lambda_1)) \cong H^{n-2}(G_1, k)$, $H^{n-2}(G_1, L(\lambda_2)) \cong H^{n-4}(G_1, k)$, and $H^{n-1}(G_1, L(\lambda_3)) \cong H^{n-4}(G_1, k) + H^{n-1}(G_1, H^0(\lambda_3))$. By Statement (*b*) of this Theorem 3, $H^{n-1}(G_1, L(\mu_1)) \cong H^{n-4}(G_1, k)$. Then it follows from the exactness of the sequence (15) that

$$H^{n-1}\Big(G_1, H^0(\mu_2)/L(\mu_2)\Big) = H^{n-2}(G_1, k) + H^{n-4}(G_1, k) + H^{n-1}\Big(G_1, H^0(\lambda_3)\Big).$$

Now suppose that

$$H^n(G_1, L(\mu_j)) \cong H^{n-j}(G_1, k) + H^{n-j-2}(G_1, k) + H^{n-1}\Big(G_1, H^0(\lambda_{j+1})\Big)$$

for all $j < j_0$. By Statement (*a*) of Theorem 1, $H^{n-2}(G_1, L(\lambda_{j_0})) \cong H^{n-j_0-2}(G_1, k)$ and $H^{n-1}(G_1, L(\lambda_{j_0+1})) \cong H^{n-j_0-2}(G_1, k) + H^{n-1}(G_1, H^0(\lambda_{j_0+1}))$. By the induction hypothesis,

$$H^{n-1}(G_1, L(\mu_{j_0-1})) \cong H^{n-j_0}(G_1, k) + H^{n-j_0-2}(G_1, k) + H^{n-1}\Big(G_1, H^0(\lambda_{j_0+1})\Big).$$

Then the exactness of the sequence (16) yields

$$H^{n-1}\Big(G_1, H^0(\mu_{j_0})/L(\mu_{j_0})\Big) = H^{n-j_0}(G_1, k) + H^{n-j_0-2}(G_1, k) + H^{n-1}\Big(G_1, H^0(\lambda_{j_0+1})\Big).$$

Hence, by (4),

$$H^n(G_1, L(\mu_{j_0})) \cong H^{n-j_0}(G_1, k) + H^{n-j_0-2}(G_1, k) + H^{n-1}\Big(G_1, H^0(\lambda_{j_0+1})\Big).$$

So, $H^n\big(G_1, L(\mu_j)\big) \cong H^{n-j}(G_1,k) + H^{n-j-2}(G_1,k) + H^{n-1}\big(G_1, H^0(\lambda_{j+1})\big)$ for all $j \in \{2,3,\cdots,t_\lambda\}$, which proves the sub-statement (*iii*).

Let $l(w_j) < n < l(u_j) = 4l-j-2$. If $j = 2$, then by Statement (*a*) of Theorem 1, $H^{n-1}(G_1, L(\lambda_1)) \cong \sum_{i=0}^{1} H^{n-2+i}\big(G_1, H^0(\lambda_i)\big)$, $H^{n-2}(G_1, L(\lambda_2)) \cong \sum_{i=0}^{2} H^{n-4+i}\big(G_1, H^0(\lambda_i)\big)$, and $H^{n-1}(G_1, L(\lambda_3)) \cong \sum_{i=0}^{3} H^{n-4+i}\big(G_1, H^0(\lambda_i)\big)$. By Statement (*b*) of this Theorem 3, $H^{n-1}(G_1, L(\mu_1)) \cong \sum_{i=0}^{2} H^{n-4+i}\big(G_1, H^0(\lambda_i)\big)$. Then it follows from the exactness of the sequence (15) that

$$H^{n-1}\Big(G_1, H^0(\mu_2)/L(\mu_2)\Big) = \sum_{i=0}^{1} H^{n-2+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{3} H^{n-4+i}\Big(G_1, H^0(\lambda_i)\Big).$$

Now suppose that

$$H^n\big(G_1, L(\mu_j)\big) \cong \sum_{i=0}^{1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j+1} H^{n-j-2+i}\Big(G_1, H^0(\lambda_i)\Big)$$

for all $j < j_0$. By Statement (*a*) of Theorem 1, $H^{n-2}\big(G_1, L(\lambda_{j_0})\big) \cong \sum_{i=0}^{j_0} H^{n-2-j_0+i}\big(G_1, H^0(\lambda_i)\big)$ and $H^{n-1}\big(G_1, L(\lambda_{j_0+1})\big) \cong \sum_{i=0}^{j_0+1} H^{n-2-j_0+i}\big(G_1, H^0(\lambda_i)\big)$. By the induction hypothesis,

$$H^{n-1}\big(G_1, L(\mu_{j_0-1})\big) \cong \sum_{i=0}^{1} H^{n-j_0+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j_0} H^{n-j_0-2+i}\Big(G_1, H^0(\lambda_i)\Big).$$

Then the exactness of the sequence (16) yields

$$H^{n-1}\Big(G_1, H^0(\mu_{j_0})/L(\mu_{j_0})\Big) = \sum_{i=0}^{1} H^{n-j_0+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j_0+1} H^{n-2-j_0+i}\Big(G_1, H^0(\lambda_i)\Big).$$

Hence, by (4),

$$H^n\big(G_1, L(\mu_{j_0})\big) \cong \sum_{i=0}^{1} H^{n-j_0+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j_0+1} H^{n-2-j_0+i}\Big(G_1, H^0(\lambda_i)\Big).$$

So,

$$H^n\big(G_1, L(\mu_j)\big) \cong \sum_{i=0}^{1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j+1} H^{n-j-2+i}\Big(G_1, H^0(\lambda_i)\Big)$$

for all $j \in \{2,3,\cdots,t_\lambda - 1\}$, which proves the sub-statement (*iv*).

Let $n \geq l(u_j) = 4l-j-2$. If $j = 2$, then by Statement (*a*) of Theorem 1, $H^{n-1}(G_1, L(\lambda_1)) \cong \sum_{i=0}^{1} H^{n-2+i}\big(G_1, H^0(\lambda_i)\big)$, $H^{n-2}(G_1, L(\lambda_2)) \cong \sum_{i=0}^{2} H^{n-4+i}\big(G_1, H^0(\lambda_i)\big)$, and $H^{n-1}(G_1, L(\lambda_3)) \cong \sum_{i=0}^{3} H^{n-4+i}\big(G_1, H^0(\lambda_i)\big)$. By Statement (*b*) of this Theorem 3,

$$H^{n-1}(G_1, L(\mu_1)) \cong \begin{cases} \sum\limits_{i=0}^{2} H^{n-4+i}(G_1, H^0(\lambda_i)) & \text{if } n = l(u_j), \\ \sum\limits_{i=0}^{2} H^{n-4+i}(G_1, H^0(\lambda_i)) + H^{n-1}(G_1, L(\mu_1)) & \text{if } n > l(u_j). \end{cases}$$

Then it follows from the exactness of the sequence (15) that

$$H^{n-1}\Big(G_1, H^0(\mu_2)/L(\mu_2)\Big) = \begin{cases} \sum\limits_{i=0}^{1} H^{n-2+i}(G_1, H^0(\lambda_i)) + \sum\limits_{i=0}^{3} H^{n-4+i}(G_1, H^0(\lambda_i)) & \text{if } n = l(u_j), \\ \sum\limits_{i=0}^{1} H^{n-2+i}(G_1, H^0(\lambda_i)) + \sum\limits_{i=0}^{3} H^{n-4+i}(G_1, H^0(\lambda_i)) + H^{n-1}(G_1, L(\mu_1)) & \text{if } n > l(u_j). \end{cases}$$

Since $n \geq l(u_j)$, using (5), we get

$$H^n(G_1, L(\mu_j)) \cong H^{n-1}\Big(G_1, H^0(\mu_2)/L(\mu_2)\Big) + H^n\Big(G_1, H^0(\mu_j)\Big) \cong$$

$$\sum_{i=0}^{1} H^{n-2+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{3} H^{n-4+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=m}^{j} H^{n-j+i}\Big(G_1, H^0(\mu_i)\Big),$$

where $m = \begin{cases} 2 \text{ if } n = l(u_j), \\ 1 \text{ if } n = l(u_j). \end{cases}$

Now suppose that

$$H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j+1} H^{n-j-2+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=m}^{j} H^{n-j+i}\Big(G_1, H^0(\mu_i)\Big).$$

for all $j < j_0$, where $j_0 \leq t_\lambda - 1$. By Statement (*a*) of Theorem 1, $H^{n-2}(G_1, L(\lambda_{j_0})) \cong \sum_{i=0}^{j_0} H^{n-2-j_0+i}(G_1, H^0(\lambda_i))$ and $H^{n-1}(G_1, L(\lambda_{j_0+1})) \cong \sum_{i=0}^{j_0+1} H^{n-2-j_0+i}(G_1, H^0(\lambda_i))$. By the induction hypothesis,

$$H^{n-1}(G_1, L(\mu_{j_0-1})) \cong \sum_{i=0}^{1} H^{n-j_0+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j_0} H^{n-j_0-2+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=m}^{j_0-1} H^{n-j_0+i}\Big(G_1, H^0(\mu_i)\Big).$$

Then the exactness of the sequence (16) yields

$$H^{n-1}\Big(G_1, H^0(\mu_{j_0})/L(\mu_{j_0})\Big) = \sum_{i=0}^{1} H^{n-j_0+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j_0+1} H^{n-2-j_0+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=m}^{j_0-1} H^{n-j_0+i}\Big(G_1, H^0(\mu_i)\Big).$$

Hence, by (5),

$$H^n(G_1, L(\mu_{j_0})) \cong \sum_{i=0}^{1} H^{n-j_0+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j_0+1} H^{n-2-j_0+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=m}^{j_0} H^{n-j_0+i}\Big(G_1, H^0(\mu_i)\Big).$$

So,

$$H^n(G_1, L(\mu_j)) \cong \sum_{i=0}^{1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j+1} H^{n-j-2+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=m}^{j} H^{n-j+i}\Big(G_1, H^0(\mu_i)\Big),$$

for all $j \in \{2, 3, \cdots, t_\lambda - 1\}$, which proves the sub-statement (v). $\square$

Proof of Statement (*d*) of Theorem 3. In this case, $\mathfrak{g} = B_l$ ($l > 2$) and $j = t_\lambda$. Let $n < j$. By Statement (*a*) of Theorem 1 and Statement (*b*) of this Theorem 3, $H^{n-2}(G_1, L(\lambda_j)) = 0$, $H^{n-1}(G_1, L(\lambda_{j+1})) = 0$, and $H^{n-1}(G_1, L(\mu_{j-1})) = 0$. Then it follows from the exactness of the sequence (16) that $H^{n-1}(G_1, H^0(\mu_j)/L(\mu_j)) = 0$. Therefore, by (4),

$$H^n(G_1, L(\mu_j)) = H^{n-1}\Big(G_1, H^0(\mu_j)/L(\mu_j)\Big) = 0$$

.

Let $n = j = t_\lambda$. Then, by Statement (*a*) of Theorem 1 and Statement (*b*) of this Theorem 3, $H^{n-2}(G_1, L(\lambda_j)) = 0$, $H^{n-1}(G_1, L(\lambda_{j+1})) = 0$, and $H^{n-1}(G_1, L(\mu_{j-1})) = k$. Then it follows from the exactness of the sequence (16) that $H^{n-1}(G_1, H^0(\mu_j)/L(\mu_j)) = k$. Therefore, by (4), $H^n(G_1, L(\mu_j)) = H^{n-1}(G_1, H^0(\mu_j)/L(\mu_j)) = k$. We get the sub-statement (*i*).

Let $j < n < l(u_j)$. Then, by Statement (*a*) of Theorem 1, $H^{n-2}(G_1, L(\lambda_j)) = \sum_{i=0}^{j} H^{n-2-j+i}(G_1, H^0(\lambda_i))$, $H^{n-1}(G_1, L(\lambda_{j+1})) = \sum_{i=0}^{j+1} H^{n-2-j+i}(G_1, H^0(\lambda_i))$, and by Statement (*b*) of this Theorem 3,

$$H^{n-1}(G_1, L(\mu_{j-1})) = \sum_{i=0}^{1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big).$$

Then it follows from the exactness of the sequence (16) that

$$H^{n-1}\Big(G_1, H^0(\mu_j)/L(\mu_j)\Big) = \sum_{i=0}^{1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j+1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big).$$

Therefore, by (4),

$$H^n(G_1, L(\mu_j)) = H^{n-1}\Big(G_1, H^0(\mu_j)/L(\mu_j)\Big) = \sum_{i=0}^{1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j+1} H^{n-2-j+i}\Big(G_1, H^0(\lambda_i)\Big).$$

So, we get the sub-statement (*ii*).

Let $n \geq l(u_j)$. Then, by Statement (*a*) of Theorem 1, $H^{n-2}(G_1, L(\lambda_j)) = \sum_{i=0}^{j} H^{n-2-j+i}(G_1, H^0(\lambda_i))$, $H^{n-1}(G_1, L(\lambda_{j+1})) = \sum_{i=0}^{j+1} H^{n-2-j+i}(G_1, H^0(\lambda_i))$, and by Statement (*b*) of this Theorem 3,

$$H^{n-1}(G_1, L(\mu_{j-1})) = \sum_{i=0}^{1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=1}^{j-1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big).$$

Then it follows from the exactness of the sequence (16) that

$$H^{n-1}\Big(G_1, H^0(\mu_j)/L(\mu_j)\Big) = \sum_{i=0}^{1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j+1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=1}^{j-1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big).$$

Therefore, by (5),

$$H^n(G_1, L(\mu_j)) = \sum_{i=0}^{1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=0}^{j+1} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big) + \sum_{i=1}^{j} H^{n-j+i}\Big(G_1, H^0(\lambda_i)\Big).$$

So, we get the sub-statement (*iii*). □

Proof of Statement (*e*) of Theorem 3. In this case, $\mathfrak{g} = B_l$ $(l > 2)$ and $j = \{t_\lambda + 1, t_\lambda + 2, \cdots, s-1\}$. Let $n < j$. By Statement (*a*) of Theorem 1 and Statement (*b*) of this Theorem 3, $H^{n-2}(G_1, L(\lambda_j)) = 0$, $H^{n-1}(G_1, L(\lambda_{j+1})) = 0$, and $H^{n-1}(G_1, L(\mu_{j-1})) = 0$. Then it follows from the exactness of the sequence (16) that $H^{n-1}(G_1, H^0(\mu_j)/L(\mu_j)) = 0$. Therefore, by (4),

$$H^n(G_1, L(\mu_j)) = H^{n-1}\Big(G_1, H^0(\mu_j)/L(\mu_j)\Big) = 0.$$

Let $n = j$. If $j = t_\lambda + 1$, then by Statement (*b*) of Theorem 1, $H^{n-2}(G_1, L(\lambda_j)) = 0$ and $H^{n-1}(G_1, L(\lambda_{j+1})) \cong H^{n-1}(G_1, H^0(\lambda_{j+1}))$. By Statement (*d*) of this Theorem 3, $H^{n-1}(G_1, L(\mu_{j-1})) \cong k$. Then it follows from the exactness of the sequence (15) that $H^{n-1}(G_1, H^0(\mu_j)/L(\mu_j)) = k + H^{n-1}(G_1, H^0(\lambda_{j+1}))$. Therefore, by (4),

$$H^n(G_1, L(\mu_j)) \cong H^{n-1}\Big(G_1, H^0(\mu_j)/L(\mu_j)\Big) \cong k + H^{n-1}\Big(G_1, H^0(\lambda_{j+1})\Big).$$

Now suppose that $H^n(G_1, L(\mu_j)) \cong k + H^{n-1}(G_1, H^0(\lambda_{j+1}))$ for all $t_\lambda < j < j_0$, where $j_0 \leq s-1$. By Statement (*c*) of Theorem 1, $H^{n-2}(G_1, L(\lambda_{j_0})) \cong \sum_{i=1}^{n-2} H^i(G_1, H^0(\lambda_{i+2}))$ and $H^{n-1}(G_1, L(\lambda_{j_0+1})) \cong \sum_{i=1}^{n-1} H^i(G_1, H^0(\lambda_{i+2}))$. By the induction hypothesis, $H^{n-1}(G_1, L(\mu_{j_0-1})) \cong k$. Then the exactness of the sequence (16) yields

$$H^{n-1}\Big(G_1, H^0(\mu_{j_0})/L(\mu_{j_0})\Big) = k + H^{n-1}\Big(G_1, H^0(\lambda_{j_0+1})\Big).$$

Hence, by (4), $H^n(G_1, L(\mu_{j_0})) \cong H^{n-1}(G_1, H^0(\mu_{j_0})/L(\mu_{j_0})) \cong k + H^{n-1}(G_1, H^0(\lambda_{j_0+1}))$. So, $H^n(G_1, L(\mu_j)) \cong k + H^{n-1}(G_1, H^0(\lambda_{j+1}))$ for all $j \in \{t_\lambda+1, t_\lambda+2, \cdots, s-1\}$, which proves the sub-statement (*i*).

The proofs of the sub-statements *(ii)* and *(iii)* are similar to the proofs of the sub-statements *(iv)* and (*v*) of Statement (*c*) of this Theorem 3. □

Proof of Theorem 5. By Theorem 1 in [38] (p. 38), for all $n > 0$ there is an isomorphism

$$H^n(G, V) \cong \mathrm{Hom}_G\Big(k, H^n(G_1, V)^{(-1)}\Big) \tag{17}$$

of G-modules, where V is a simple G-modules with the restricted highest weight. □

Proof of Statement (*a*) of Theorem 5. *Necessity.* If $H^n(G_1, V) \cong H^n(G, V)$, then, by (17),

$$H^n(G_1, V) \cong \mathrm{Hom}_G\Big(k, H^n(G_1, V)^{(-1)}\Big).$$

Sufficiency. If $H^n(G_1, V) \cong \mathrm{Hom}_G\Big(k, H^n(G_1, V)^{(-1)}\Big)$, then, by (17), $H^n(G_1, V) \cong H^n(G, V)$. □

Proof of Statement (*b*) of Theorem 5. *Necessity.* If $H^n(\mathfrak{g}, V) \cong H^n(G, V)$, then, by (17),

$$H^n(\mathfrak{g}, V) \cong \mathrm{Hom}_G\Big(k, H^n(G_1, V)^{(-1)}\Big).$$

Sufficiency. If $H^n(\mathfrak{g}, V) \cong \mathrm{Hom}_G\Big(k, H^n(G_1, V)^{(-1)}\Big)$, then, by (17), $H^n(\mathfrak{g}, V) \cong H^n(G, V)$. □

Proof of Statement (*c*) of Theorem 5. If $H^n(\mathfrak{g}, V) \cong H^n(G_1, V)$, then, by (17).

$$H^n(G, V) \cong \mathrm{Hom}_G\Big(k, H^n(\mathfrak{g}, V)^{(-1)}\Big). \tag{18}$$

□

Proof of Corollary 1. If $n < i$, then by Theorem 1, $H^n(G_1, L(\lambda_i)) = 0$, and, by Theorem 1 in [22] (p. 6), $H^n(\mathfrak{g}, L(\lambda_i)) = 0$. Therefore, in this case, there is no non-trivial isomorphism $H^n(\mathfrak{g}, L(\lambda_i)) \cong H^n(G_1, L(\lambda_i))$.

Let $i \in \{1, 2, \cdots, t_\lambda\}$ and $i < n < l(w_i)$. If the cohomology $H^n(\mathfrak{g}, L(\lambda_i))$ and $H^n(G_1, L(\lambda_i))$ are non-trivial, then, according to Theorem 1 in [22] (p. 6), the cohomology $H^n(\mathfrak{g}, L(\lambda_i))$ is isomorphic to the cohomology $H^{n-i}(\mathfrak{g})$, but, by Theorem 1 of this paper, the cohomology $H^n(G_1, L(\lambda_i))$ is isomorphic to the cohomology $H^{n-i}(G_1, k)$. It is known that the cohomology $H^{n-i}(\mathfrak{g})$ is a G-module with a trivial action of G [39] (pp. 173–174). According to (3), the cohomology $H^{n-i}(G_1, k)$ is not a trivial as G-module. Consequently, in this case, too, a non-trivial isomorphism $H^n(\mathfrak{g}, L(\lambda_i)) \cong H^n(G_1, L(\lambda_i))$ does not exist.

If $i \in \{1, 2, \cdots, t_\lambda\}$ and $n \geq l(w_i)$, then arguing as in the previous case, we obtain that, in the non-trivial cases, the cohomology $H^n(\mathfrak{g}, L(\lambda_i))$ and $H^n(G_1, L(\lambda_i))$ are not isomorphic.

Now, let $i \in \{1, 2, \cdots, t_\lambda\}$ and $n = i$. Then, by Theorem 1, $H^n(G_1, L(\lambda_i)) \cong k$, and, by Theorem 1 in [22] (p. 6), $H^n(\mathfrak{g}, L(\lambda_i)) \cong k$. Therefore, we get the non-trivial isomorphism $H^n(\mathfrak{g}, L(\lambda_i)) \cong H^n(G_1, L(\lambda_i))$. Thus, we have Statement (*a*).

Let $\mathfrak{g} = B_l$ and $i = t_\lambda$. If $n < i$, then by Theorem 1, $H^n(G_1, L(\lambda_i)) = 0$, and, by Theorem 1 in [22] (p. 6), $H^n(\mathfrak{g}, L(\lambda_i)) = 0$. If $n = i$, then, by Theorem 1, $H^n(G_1, L(\lambda_i)) \cong k \oplus L(\alpha_0)^{(1)}$, and, by Theorem 1 in [22] (p. 7), $H^n(\mathfrak{g}, L(\lambda_i)) \cong k \oplus L(\alpha_0)^{(1)}$. Therefore, we get the non-trivial isomorphism $H^n(\mathfrak{g}, L(\lambda_i)) \cong H^n(G_1, L(\lambda_i))$. Thus, we have Statement (*b*). If $n > i$, then arguing as in the proof of Statement (*a*), we obtain that the required non-trivial isomorphism does not exist.

Let $\mathfrak{g} = B_l$ and $i \in \{t_\lambda + 1,\ t_\lambda + 2, \cdots, s\}$. If $n < l(w_i) = 2l - i$, then by Theorem 1, $H^n(G_1, L(\lambda_i)) = 0$, and, by Theorem 1 in [22] (p. 7), $H^n(\mathfrak{g}, L(\lambda_i)) = 0$. If $n = l(w_i)$, then, by Theorem 1, $H^n(G_1, L(\lambda_i)) \cong L(\alpha_0)^{(1)}$, and, by Theorem 1 in [22] (p. 7), $H^n(\mathfrak{g}, L(\lambda_i)) \cong L(\alpha_0)^{(1)}$. Therefore, we get the non-trivial isomorphism $H^n(\mathfrak{g}, L(\lambda_i)) \cong H^n(G_1, L(\lambda_i))$. Thus, we have Statement (*c*). In other cases, there are no non-trivial isomorphisms, since $H^n(\mathfrak{g})$ and $H^n(G_1, k)$ have different G-module structures.

Let $\mathfrak{g} = B_l$ and $i = s$. If $n < s = 2l - 2$, then by Theorem 1, $H^n(G_1, L(\lambda_i)) = 0$, and, by Theorem 1 in [22] (p. 7), $H^n(\mathfrak{g}, L(\lambda_i)) = 0$. If $n = s$, then, by Theorem 1, $H^n(G_1, L(\lambda_i)) \cong k \oplus L(\alpha_0)^{(1)}$, and, by Theorem 1 in [22] (p. 7), $H^n(\mathfrak{g}, L(\lambda_i)) \cong k \oplus L(\alpha_0)^{(1)}$. Therefore, we get the non-trivial isomorphism $H^n(\mathfrak{g}, L(\lambda_i)) \cong H^n(G_1, L(\lambda_i))$. Thus, we have Statement (*d*). Since $H^n(\mathfrak{g})$ and $H^n(G_1, k)$ have different G-module structures, no other nontrivial isomorphisms appear. □

Proof of Corollary 2. Is similar to that of Corollary 1. □

Proof of Corollary 3. Follows from Theorem 1 and Statement (*a*) of Theorem 5. □

Proof of Corollary 4. Follows from Theorems 2–4 and Statement (*a*) of Theorem 5. □

4. Discussion

The results of this paper relate to the following topical cohomology problem for semisimple and simply connected algebraic groups in positive characteristic and their Lie algebras:

- examine the cohomology of simple modules for $\mathfrak{g}$;
- examine the cohomology of simple modules for G_1;
- determine the connection between the cohomology of simple modules for $\mathfrak{g}$, G_1, and G.

It was formulated in [22] and the authors studied its first part in detail for the simple modules with highest weights $\lambda_1, \lambda_2, \cdots, \lambda_s$; $\mu_1, \mu_2, \cdots, \mu_s$. Our paper is a continuation of this work. We have completely solved the second and third parts of this problem for the considered simple modules. In addition, we have obtained a necessary and sufficient condition for the isomorphisms $H^n(G_1, V) \cong H^n(G, V)$ and $H^n(\mathfrak{g}, V) \cong H^n(G, V)$, and a necessary condition for the isomorphism $H^n(\mathfrak{g}, V) \cong H^n(G_1, V)$, where V is a simple restricted module . To obtain the isomorphisms $H^n(\mathfrak{g}, V) \cong H^n(G_1, V)$, we did not use Theorem 5, since the isomorphism $H^n(G, V) \cong \mathrm{Hom}_G\left(k, H^n(\mathfrak{g}, V)^{(-1)}\right)$ is not a sufficient condition. Although all non-trivial isomorphisms of Corollaries 1 and 2 satisfy this condition, the question on a sufficiency condition for the isomorphism $H^n(\mathfrak{g}, V) \cong H^n(G_1, V)$ remains open.

Statement (*b*) of Theorem 5 generalizes the results of the papers [40,41] in which necessary and sufficient conditions of the isomorphism $H^n(\mathfrak{g}, V) \cong H^n(G, V)$ are obtained for $n = 1$ and $n = 2$, respectively. In the case $n = 1$, the necessary and sufficient condition of the paper [40] (p. 492) coincides with the condition of Statement (*b*) of Theorem 5, since $H^1(\mathfrak{g}, V) \cong H^1(G_1, V)$. In the case the necessary and sufficient condition in Theorem 5 simplifies the two conditions $H^2(G_1, V) \cong \mathrm{Hom}_G\left(k, H^2(G_1, V)^{(-1)}\right)$ and $\mathrm{Im}\, f = 0$ in [41]

(p. 843) to one condition $H^2(\mathfrak{g}, V) \cong \mathrm{Hom}_G\left(k, H^2(G_1, V)^{(-1)}\right)$. The results of this paper can be used in the study of restricted cohomology of restricted Lie algebras related to the modular classical Lie algebras. Such Lie algebras, for example, include the restricted Lie algebras of Cartan type and the general linear Lie algebra $\mathfrak{gl}_n$. The restricted cohomology of the Lie algebras of Cartan type, as noted above, were computed only for the trivial one-dimensional and adjoint modules. In other cases, the restricted cohomology of the Lie algebras of Cartan type with coefficients in simple modules has not yet been studied. The restricted cohomology of the Lie algebra $\mathfrak{gl}_n$ with coefficients in simple modules has also not yet been calculated.

Analysis of the obtained results shows that the restricted cohomology plays an important role in the study of ordinary cohomology for modular restricted Lie algebras. Therefore, applying the idea of restrictness of an algebra, module, and cohomology to other classes of algebras can give a new motivation to their development.

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