mathematics

Article

The Best Ulam Constant of the Fréchet Functional Equation

Irina Opraie †, Dorian Popa † and Liana Timboş *,†

Department of Mathematics, Technical University of Cluj-Napoca, G. Bariţiu No. 25, 400027 Cluj-Napoca, Romania; irina.opraie@math.utcluj.ro (I.O.); popa.dorian@math.utcluj.ro (D.P.)
* Correspondence: liana.timbos@math.utcluj.ro
† These authors contributed equally to this work.

Abstract: In this paper, we prove the Ulam stability of the Fréchet functional equation $f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x)$ arising from the characterization of inner product spaces and we determine its best Ulam constant. Using this result, we give a stability result for a pexiderized version of the Fréchet functional equation.

Keywords: Fréchet equation; Ulam stability; best constant

MSC: 39B52; 39B82

check for updates

Citation: Opraie, I.; Popa, D.; Timboş, L. The Best Ulam Constant of the Fréchet Functional Equation. *Mathematics* **2022**, *10*, 1769. https://doi.org/10.3390/math10101769

Academic Editors: Ana-Maria Acu and Ioannis K. Argyros

Received: 26 March 2022
Accepted: 20 May 2022
Published: 22 May 2022

1. Introduction and Preliminaries

An important problem in the study of normed spaces over the field $\mathbb{R}$ of real numbers or over the field $\mathbb{C}$ of complex numbers, is of deciding when the norm of the space is determined by an inner product. The best-known result on this direction is probably the Jordan and von Neumann theorem, which states that a normed space $(X, \|\cdot\|)$ is an inner product space if, and only if, for every $x, y \in X$ the following relation, called the parallelogram identity, holds:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1}$$

After Jordan and von Neumann characterization appeared a large variety of papers on this topic. Fréchet proved that a normed space $(X, \|\cdot\|)$ is an inner product space if, and only if, the relation

$$\|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x+y\|^2 + \|x+z\|^2 + \|y+z\|^2 \tag{2}$$

holds for every $x, y, z \in X$ (see [1]).

Many characterizations of inner product spaces among the normed spaces have been studied in the last years. We mention here the following papers [2–5].

All over this paper by $(G, +)$ we denote an Abelian group and by $(X, \|\cdot\|)$ a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The relations (1) and (2) lead to the study of the following functional equations

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{3}$$

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(z+x) \tag{4}$$

where $f : G \to X$, which are called the quadratic functional equation and the Fréchet functional equation, respectively. A function which satisfies the quadratic functional equation is called a quadratic function. Recall first some results on the set of solutions of the Equations (3) and (4).

Theorem 1 ([6], p. 222). *A function $f : G \to X$ satisfies the quadratic functional equation (3) if, and only if, there exists a symmetric and bi-additive function $B : G \times G \to X$, such that*

$$f(x) = B(x, x), \forall x \in G.$$

Remark that the previous theorem is formulated for functions $f : G \to K$, but the proof holds also for functions $f : G \to X$ (see also [7]).

Theorem 2 ([6], pp. 249–250). *A function $f : G \to X$ satisfies the Fréchet functional equation (4) if, and only if, there exists an additive function $A : G \to X$ and a symmetric and bi-additive function $B : G \times G \to X$, such that*

$$f(x) = A(x) + B(x, x), \forall x \in G.$$

The goal of this paper is to prove the Ulam stability of the Fréchet functional equation and to find its best Ulam constant.

The starting point of the stability theory for functional equations was a problem formulated by S.M. Ulam in a talk given at Madison University, Wisconsin, in 1940, and it concerns approximate homomorphisms of a metric group. The first result of Ulam's problem was given by D.H. Hyers in 1941 for the Cauchy functional equation and is contained in the next theorem.

Theorem 3 ([7]). *Let X be a normed space, Y a Banach space, and $f : X \to Y$, such that*

$$\|f(x + y) - f(x) - f(y)\| \le \delta$$

for some $\delta > 0$ and for all $x, y \in X$. Then, the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in X$, and $A : X \to Y$ is the unique additive function, such that

$$\|f(x) - A(x)\| \le \delta$$

for any $x \in X$.

Recall also the result on the stability of the quadratic functional equation which will be used in our proofs.

Theorem 4 ([8]). *If $f : G \to X$ is a function which satisfies the relation*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \le \delta, \forall x, y \in G,$$

for some $\delta > 0$, then there exists a unique quadratic mapping $q : G \to X$, such that

$$\|f(x) - q(x)\| \le \frac{\delta}{2}, \forall x \in G.$$

Due to Ulam's problem and Hyers' result, the stability of the functional equations is called today Ulam stability or Hyers–Ulam stability.

Definition 5. *The Equation (4) is called Ulam stable if there exists a constant $K \ge 0$ such that for every $\varepsilon > 0$ and every function $f : G \to X$ satisfying the relation*

$$\| f(x + y + z) + f(x) + f(y) + f(z) - f(x + y) - f(y + z) - f(z + x) \| \le \varepsilon \tag{5}$$

for all $x, y, z \in G$*, there exists a solution* $\widetilde{f} : G \to X$ *of the Equation (4), such that*

$$\| f(x) - \widetilde{f}(x) \| \leq K\varepsilon, \forall x \in G. \tag{6}$$

A number $K \geq 0$ which satisfies (6) for some positive ε is called an Ulam constant of the Equation (4). Denote by K_F the infimum of all Ulam constants of the Equation (4). Generally, the infimum of all Ulam constants of an equation is not an Ulam constant of that equation (see [9]), but if it is, then will call it the best Ulam constant of that equation. Interesting results on Ulam stability and hyperstability of the Fréchet equation and some generalizations of it were obtained in the papers [10–15]

In [10] the authors consider a very general equation that extends Cauchy, Jensen, Fréchet, and Popoviciu functional equation. Using the fixed point technique they obtain results on generalized Ulam stability for this equation. A result on the hyperstability of the classical Fréchet equation and, characterization of inner product spaces are given in [11]. Ulam stability of a Fréchet type functional equation with constant coefficients is studied in [13] by using the fixed point technique. In [15], Sikorska et al. give a result on Ulam stability for a functional equation in a single variable which leads to the stability of Drygas, Fréchet and some equations of Jensen type. Recent results on the best Ulam constant for some difference and differential equations are obtained by D.R. Anderson, M. Onitsuka [16–18].

In the second part of our work, we deal with Ulam stability of the generalized Fréchet equation (in Pexider sense)

$$f(x+y+z) + f(x) + f(y) + f(z) = g(x+y) + h(y+z) + k(z+x) \tag{7}$$

where $f, g, h, k : G \to X$.

A generalization of the Equation (4) was also considered by Kannappan in [4] where the author obtain a characterization of its solutions.

The Equation (7) is called Ulam stable if there exists some non-negative constants L_i, $1 \leq i \leq 4$, such that for every $\varepsilon > 0$ and every $f, g, h, k : G \to X$ satisfying

$$\|f(x+y+z) + f(x) + f(y) + f(z) - g(x+y) - h(y+z) - k(z+x)\| \leq \varepsilon, \forall x, y, z \in G \tag{8}$$

there exists some functions $f_0, g_0, h_0, k_0 : G \to X$ satisfying Equation (7), such that

$$\begin{matrix} \|f(x) - f_0(x)\| \leq L_1\varepsilon \\ \|g(x) - g_0(x)\| \leq L_2\varepsilon \\ \|h(x) - h_0(x)\| \leq L_3\varepsilon \\ \|k(x) - k_0(x)\| \leq L_4\varepsilon \end{matrix}, \forall x \in G. \tag{9}$$

The functions f, g, h and k which satisfy (8) for some positive ε are called approximative solutions of the Equation (7), and L_1, L_2, L_3, L_4 are called Ulam constants of (7).

2. The Stability of the Fréchet Equation

The main result on Ulam stability of the Fréchet Equation (4) is contained in the next theorem.

Theorem 6. *Let G be an abelian group,* $(X, \|\cdot\|)$ *a Banach space,* $\varepsilon > 0$*, and a function* $f : G \to X$ *satisfying (5). Then, there exists a unique solution* $\widetilde{f}$ *of the Fréchet functional Equation (4), such that*

$$\| f(x) - \widetilde{f}(x) \| \leq \varepsilon, \forall x \in G.$$

Proof. Suppose that $f : G \to X$ satisfies (5) and denote

$$\varphi(x,y,z) := f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x), \forall x,y,z \in G. \tag{10}$$

Then,

$$\| \varphi(x,y,z) \| \leq \varepsilon, \forall x,y,z \in G. \tag{11}$$

Let $z = -x - y, x, y \in G$ in (10). Then,

$$f(0) + f(x) + f(y) + f(-x-y) - f(x+y) - f(-y) - f(-x) = \varphi(x,y,-x-y)$$

and denoting

$$a(x) := f(x) - f(-x), x \in G, \tag{12}$$

it follows $a(x) + a(y) - a(x+y) + f(0) = \varphi(x,y,-x-y)$.

Therefore, if $A(x) = a(x) + f(0), x \in G$, we obtain

$$A(x+y) - A(x) - A(y) = -\varphi(x,y,-x-y), x,y \in G.$$

Taking into account (11) we obtain

$$\| A(x+y) - A(x) - A(y) \| \leq \varepsilon, x,y \in G.$$

Using Theorem 3 we conclude that there exists a unique additive function $\widetilde{A} : G \to X$, such that

$$\| A(x) - \widetilde{A}(x) \| \leq \varepsilon, \forall x \in G. \tag{13}$$

Let now $z = -y$ in (10). Then,

$$2f(x) + f(y) + f(-y) - f(x+y) - f(0) - f(x-y) = \varphi(x,y,-y)$$

Replace x by $-x$ in previous relation to obtain

$$2f(-x) + f(y) + f(-y) - f(-x+y) - f(0) - f(-x-y) = \varphi(-x,y,-y).$$

Adding the last two relations we obtain

$$\begin{aligned} 2f(x) + 2f(-x) + 2f(y) + 2f(-y) - f(x+y) - f(-x-y) - \\ - f(x-y) - f(-x+y) - 2f(0) = \varphi(x,y,-y) + \varphi(-x,y,-y). \end{aligned} \tag{14}$$

We denote

$$q(x) = f(x) + f(-x), \forall x \in G. \tag{15}$$

Then, from (14) and (15) we obtain

$$2q(x) + 2q(y) - q(x+y) - q(x-y) - 2f(0) = \varphi(x,y,-y) + \varphi(-x,y,-y).$$

Let $Q(x) = q(x) - f(0), x \in G$. It follows

$$2Q(x) + 2Q(y) - Q(x+y) - Q(x-y) = \varphi(x,y,-y) + \varphi(-x,y,-y), \forall x,y \in G.$$

which leads to

$$\| 2Q(x) + 2Q(y) - Q(x+y) - Q(x-y) \| = \| \varphi(x,y,-y) + \varphi(-x,y,-y) \| \leq 2\varepsilon,$$

for all $x, y \in G$.

Then, according to Theorem 4, we conclude that there exists a unique quadratic mapping $\widetilde{Q} : G \to X$, such that

$$\| Q(x) - \widetilde{Q}(x) \| \leq \varepsilon, \forall x \in G. \tag{16}$$

From (25) and (15) we obtain

$$f(x) = \frac{a(x) + q(x)}{2} = \frac{A(x) + Q(x)}{2}.$$

Let $\widetilde{f}(x) = \dfrac{\widetilde{A}(x) + \widetilde{Q}(x)}{2}, \forall x \in G.$

Then, $\widetilde{f}$ is a solution of Fréchet functional equation, since $\widetilde{A}$ and $\widetilde{Q}$ are solutions of Fréchet functional equation and

$$\begin{aligned} \|f(x) - \widetilde{f}(x)\| &= \left\| \frac{Q(x) - \widetilde{Q}(x)}{2} + \frac{A(x) - \widetilde{A}(x)}{2} \right\| \\ &\leq \frac{1}{2} \parallel Q(x) - \widetilde{Q}(x) \parallel + \frac{1}{2} \parallel A(x) - \widetilde{A}(x) \parallel \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall x \in G. \end{aligned}$$

So, the existence of the function $\widetilde{f}$ is proved.

Finally, we prove the uniqueness of $\widetilde{f}$. Suppose that for a function $f : G \to X$ satisfying (5) there exist two solutions $f_1, f_2 : G \to X$ of the Fréchet functional equation, $f_1 \neq f_2$, such that

$$\|f(x) - f_i(x)\| \leq \varepsilon, x \in G, i \in \{1, 2\}.$$

Then,

$$\|f_1(x) - f_2(x)\| \leq \|f_1(x) - f(x)\| + \|f(x) - f_2(x)\| \leq 2\varepsilon, \forall x \in G.$$

The function $f_1 - f_2$ is also a solution of the Fréchet functional equation, so there exists an additive function $A : G \to X$ and a bi-additive and symmetric function $B : G \times G \to X$, such that

$$f_1(x) - f_2(x) = A(x) + B(x, x), \forall x \in G.$$

Let $x_0 \in G$ be such that $f_1(x_0) \neq f_2(x_0)$. Then, for every $n \in \mathbb{N}$ we obtain

$$f_1(nx_0) - f_2(nx_0) = nA(x_0) + n^2 B(x_0, x_0)$$

and

$$\|nA(x_0) + n^2 B(x_0, x_0)\| \leq 2\varepsilon. \tag{17}$$

If $B(x_0, x_0) \neq 0$ then (17) can be rewritten as

$$n^2 \|\frac{1}{n} A(x_0) + B(x_0, x_0)\| \leq 2\varepsilon. \tag{18}$$

Therefore letting $n \longrightarrow \infty$ in (18) it follows $+\infty \leq 2\varepsilon$, a contradiction.

If $B(x_0, x_0) = 0$, then $A(x_0) \neq 0$, and the relation (17) becomes

$$n\|A(x_0)\| \leq 2\varepsilon$$

and similarly, if $n \longrightarrow \infty$ we obtain $+\infty \leq 2\varepsilon$, a contradiction. We conclude that $f_1 = f_2$ and the theorem is proved.

□

Example 7. *Let $\varepsilon > 0$ and $u \in X$, $||u|| \leq 1$. Then, $f : G \to X$, $f(x) = \varepsilon u$, $\forall x \in G$, is an approximate solution of the Equation (4). The unique solution $\widetilde{f} : G \to X$ of the Equation (4) satisfying $||f(x) - \widetilde{f}(x)|| \leq \varepsilon$, $\forall x \in G$, is $\widetilde{f}(x) = 0$, $\forall x \in G$, according to Theorem 6.*

Theorem 8. *The best Ulam constant of the Fréchet functional equation is $K_F = 1$.*

Proof. From Theorem 6 it follows that the Fréchet equation is stable with the Ulam constant $K = 1$. Suppose that (4) admits an Ulam constant $K_0 < 1$. Let $\varepsilon > 0$, $u \in X$, $\|u\| = 1$ and consider the constant function $f : G \to X$, $f(x) = \varepsilon u$, $x \in G$. Then

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x)\| = \varepsilon,\ x,y,z \in G,$$

so, in view of Ulam stability of (4) with the constant K_0, there exists a solution $\widetilde{f} : G \to X$ of the Equation (4) and

$$\|f(x) - \widetilde{f}(x)\| \le K_0\varepsilon,\ \forall x \in G.$$

Then,

$$\|\widetilde{f}(x)\| \le \|\widetilde{f}(x) - f(x)\| + \|f(x)\| \le (K_0 + 1)\varepsilon,\ \forall x \in G, \tag{19}$$

hence $\widetilde{f}$ is bounded. On the other hand $\widetilde{f}$ has the form

$$\widetilde{f}(x) = A(x) + B(x,x),\ \forall x \in G,$$

where $A : G \to X$ is additive and $B : G \times G \to X$ is symmetric and bi-additive. The following relation holds:

$$\widetilde{f}(nx) = nA(x) + n^2 B(x,x),\ \forall n \in \mathbb{N},\ \forall x \in G. \tag{20}$$

Then, from (19) and (20) we obtain

$$||nA(x) + n^2 B(x,x)|| \le (K_0 + 1)\varepsilon,\ \forall n \in \mathbb{N}, \forall x \in G,$$

which leads to $A(x) = 0$ and $B(x,x) = 0$ for all $x \in G$. (see the proof of uniqueness in Theorem 6.)

It follows that $\widetilde{f} = 0$, so (19) becomes $\varepsilon \le K_0\varepsilon \iff 1 \le K_0$, which is a contradiction. □

3. Stability of a Generalized Fréchet Functional Equation

In this section, we prove the Ulam stability of the generalized Fréchet Equation (7). First, we deal with the inhomogeneous generalized Fréchet equation

$$f(x+y+z) + f(x) + f(y) + f(z) = g(x+y) + h(y+z) + k(z+x) + U(x,y,z) \tag{21}$$

where $U : G \times G \times G \to X$ is a given function.

We introduce the functions $V : G \to X$ and $W : G \times G \times G \to X$ given by

$$V(x) = \frac{1}{2}\Big(U(0,0,0) - U(x,0,0) - U(0,x,0) - U(0,0,x)\Big),\ \forall x \in G$$

and

$$\begin{aligned} W(x,y,z) = \frac{1}{2}\Big(&U(y+z,0,0) + U(0,z+x,0) + U(0,0,x+y) \\ &- U(x+y,0,0) - U(z+x,0,0) - U(0,y+z,0) \\ &- U(0,x+y,0) - U(0,0,z+x) - U(0,0,y+z)\Big),\ \forall x,y,z \in G. \end{aligned}$$

In the next lemma we give a representation for the solutions of the Equation (21).

Lemma 9. *Suppose that $f, g, h, k : G \to X$ satisfy the Equation (21). Then, f is the solution of the functional equation*

$$\begin{aligned} &f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x) \\ &\qquad = U(x,y,z) + W(x,y,z) + f(0) + \frac{1}{2}U(0,0,0) \end{aligned} \tag{22}$$

and

$$\begin{aligned} g(x) &= f(x) + V(x) + U(0,0,x) + g(0) - f(0) \\ h(x) &= f(x) + V(x) + U(x,0,0) + h(0) - f(0), \forall x \in G. \\ k(x) &= f(x) + V(x) + U(0,x,0) + k(0) - f(0) \end{aligned} \tag{23}$$

Proof. Putting $x = y = 0$ in Equation (21) we obtain

$$2f(z) + 2f(0) - g(0) - h(z) - k(z) = U(0,0,z), \forall z \in G. \tag{24}$$

Taking $z = x$ in (24) we obtain

$$2f(x) + 2f(0) - g(0) - h(x) - k(x) = U(0,0,x),\ x \in G. \tag{25}$$

Analogously we obtain

$$\begin{aligned} 2f(x) + 2f(0) - g(x) - h(x) - k(0) = U(0,x,0) \\ 2f(x) + 2f(0) - g(x) - h(0) - k(x) = U(x,0,0) \end{aligned}, x \in G \tag{26}$$

Adding the relations (25) and (26) and dividing by 2 we obtain

$$\begin{aligned} 3f(x) + 3f(0) - g(x) - h(x) - k(x) - \frac{g(0)+h(0)+k(0)}{2} \\ = \frac{U(x,0,0)+U(0,x,0)+U(0,0,x)}{2} \end{aligned} \tag{27}$$

for all $x \in G$.

From (25) and (27) we obtain

$$g(x) = f(x) + f(0) + \frac{g(0)-h(0)-k(0)}{2} - \frac{U(x,0,0)+U(0,x,0)-U(0,0,x)}{2}.$$

Putting $z = 0$ in (24) we obtain

$$4f(0) = g(0) + h(0) + k(0) + U(0,0,0). \tag{28}$$

Taking into account (28) and the definition of the function V we obtain

$$g(x) = f(x) + V(x) + U(0,0,x) + g(0) - f(0).$$

Similarly, we obtain

$$h(x) = f(x) + V(x) + U(x,0,0) + h(0) - f(0),$$

$$k(x) = f(x) + V(x) + U(0,x,0) + k(0) - f(0),$$

for all $x \in G$.

Replacing g, h, k in (21) we obtain

$$\begin{aligned} f(x+y+z) + f(x) + f(z) =& f(x+y) + f(y+z) + f(z+x) + V(x+y) + V(y+z) \\ &+ V(z+x) + U(0,0,x+y) + U(y+z,0,0) + U(0,z+x,0) \\ &+ g(0) + h(0) + k(0) - 3f(0). \end{aligned}$$

Taking into account the definition of W and V, and the relation (28) we obtain

$$\begin{aligned} f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x) \\ = U(x,y,z) + W(x,y,z) + F(0) + \frac{1}{2}U(0,0,0), \forall x,y,z \in G. \end{aligned}$$

□

From Lemma 9 we obtain a representation of the solutions of the Equation (7).

Theorem 10. *Let $f, g, h, k : G \to X$. Then, f, g, h, k satisfy the generalized Fréchet Equation (7) if and only if there exists $a, b, c \in X$ and a solution $\widetilde{f} : G \to X$ of the Fréchet functional equation, i.e.,*

$$\widetilde{f}(x+y+z) + \widetilde{f}(x) + \widetilde{f}(y) + \widetilde{f}(z) = \widetilde{f}(x+y) + \widetilde{f}(y+z) + \widetilde{f}(z+x),\ x, y, z \in G \tag{29}$$

such that

$$\begin{aligned} f(x) &= \widetilde{f}(x) + \frac{a+b+c}{4} \\ g(x) &= \widetilde{f}(x) + a \\ h(x) &= \widetilde{f}(x) + b \\ k(x) &= \widetilde{f}(x) + c \end{aligned}\quad , \forall x \in G. \tag{30}$$

Proof. "⇒" Suppose that f, g, h, k satisfy (7). For $U(x, y, z) = 0, \forall x, y, z \in G$ in Lemma 9 it follows that f is a solution of the equation

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(z+x) + f(0).$$

Define $\widetilde{f}(x) = f(x) - f(0), \forall x \in G$. Then $\widetilde{f}$ satisfies (29) and the relations (30) are obtained from (23) if we consider $a = g(0), b = h(0), c = k(0)$.

"⇐" Suppose now that f, g, h, k are defined by (30), where $\widetilde{f}$ is a solution of the Fréchet functional Equation (4). Then, f, g, h, k satisfy the Equation (7).

□

The main result on Ulam stability for Equation (7) is given in the next theorem.

Theorem 11. *Let $\varepsilon > 0$ and suppose $f, g, h, k : G \to X$ satisfy (8), i.e.,*

$$\|f(x+y+z) + f(x) + f(y) + f(z) - g(x+y) - h(y+z) - k(z+x)\| \le \varepsilon, \forall x, y, z \in G.$$

Then, there exists $f_0, g_0, h_0, k_0 : G \to X$ satisfying (7), such that

$$\begin{aligned} \|f(x) - f_0(x)\| &\le 6\varepsilon \\ \|g(x) - g_0(x)\| &\le 9\varepsilon \\ \|h(x) - h_0(x)\| &\le 9\varepsilon \\ \|k(x) - k_0(x)\| &\le 9\varepsilon \end{aligned}, \forall x \in G.$$

Proof. Suppose that $f, g, h, k : G \to X$ satisfy (8) and let

$$U(x, y, z) := f(x+y+z) + f(x) + f(y) + f(z) - g(x+y) - h(y+z) - k(z+x)$$

for all $x, y, z \in G$.

Then, f, g, h, k satisfy (22) and (23).

Let $\widetilde{f} : G \to X$, $\widetilde{f}(x) = f(x) - \alpha$, $\forall x \in G$, where $\alpha = f(0) + \frac{1}{2}U(0, 0, 0)$. Then $\widetilde{f}$ satisfies the equation

$$\widetilde{f}(x+y+z) + \widetilde{f}(x) + \widetilde{f}(y) + \widetilde{f}(z) - \widetilde{f}(x+y) - \widetilde{f}(y+z) - \widetilde{f}(z+x) = U(x, y, z) + W(x, y, z)$$

according to (22), therefore

$$\begin{aligned}\|\widetilde{f}(x+y+z)+\widetilde{f}(x)+\widetilde{f}(y)+\widetilde{f}(z)-\widetilde{f}(x+y)-\widetilde{f}(y+z)-\widetilde{f}(z+x)\| \\ \leq \|U(x,y,z)\|+\|W(x,y,z)\| \\ \leq \varepsilon+\frac{9\varepsilon}{2}=\frac{11\varepsilon}{2}, \forall x,y,z\in G.\end{aligned}$$

Taking into account the Ulam stability of Fréchet functional Equation (4), it follows that there exists a solution $\widetilde{f_0}$ of the Fréchet equation, such that

$$\|\widetilde{f}(x)-\widetilde{f_0}(x)\|\leq\frac{11\varepsilon}{2}, x\in G.$$

Let

$$\left.\begin{aligned}f_0(x)&=\widetilde{f_0}(x)+f(0)\\ g_0(x)&=\widetilde{f_0}(x)+g(0)\\ h_0(x)&=\widetilde{f_0}(x)+h(0)\\ k_0(x)&=\widetilde{f_0}(x)+k(0)\end{aligned}\right., \forall x\in G.$$

We obtain

$$\begin{aligned}\|f(x)-f_0(x)\|&=\|\widetilde{f}(x)+\alpha-\widetilde{f_0}(x)-f(0)\|\\ &=\|\widetilde{f}(x)-\widetilde{f_0}(x)+\frac{1}{2}U(0,0,0)\|\\ &\leq\|\widetilde{f}(x)-\widetilde{f_0}(x)\|+\frac{1}{2}\|U(0,0,0)\|\\ &\leq\frac{11}{2}\varepsilon+\frac{\varepsilon}{2}=6\varepsilon, \forall x\in G.\end{aligned}$$

and

$$\begin{aligned}\|g(x)-g_0(x)\|&=\|\widetilde{f}(x)+\alpha+V(x)+U(0,0,x)+g(0)-f(0)-\widetilde{f_0}(x)-g(0)\|\\ &=\|\widetilde{f}(x)-\widetilde{f_0}(x)\|+\|V(x)\|+\|U(0,0,x)\|+\|\frac{1}{2}U(0,0,0)\|\\ &\leq\frac{11}{2}\varepsilon+\frac{1}{2}\cdot 4\varepsilon+\varepsilon+\frac{\varepsilon}{2}\\ &=9\varepsilon, \forall x\in G.\end{aligned}$$

Analogously

$$\|h(x)-h_0(x)\|\leq 9\varepsilon,$$

$$\|k(x)-k_0(x)\|\leq 9\varepsilon,$$

for all $x\in G$. The theorem is proved. $\square$

4. Conclusions

In this paper, we give a stability result in Ulam sense for the Fréchet functional equation and we obtain the best Ulam constant. This result is applied to obtain the Ulam stability for a generalized Fréchet functional equation. Even there are many results on Ulam stability or generalized Ulam stability for equations of Fréchet type, there is no result on the best Ulam constant for the classical Fréchet equation, as far as we know. We think that the result on the stability for the Pexider version of the Fréchet equation is also new in the literature. Since we use the best Ulam constant of the Fréchet equation to obtain the stability of its

Pexider version we obtain sharp estimates of the difference between the approximate and the exact solution.

Author Contributions: Conceptualization, D.P.; Formal analysis, L.T.; Investigation, I.O.; Supervision, D.P.; Writing—original draft, L.T.; Writing—review & editing, I.O. These authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Fréchet, M. Sur la définition axiomatique d'une classe d'espaces vectoriel distances applicables vectoriellement sur l'espace de Hilbert. *Ann. Math.* **1935**, *36*, 705–718. [CrossRef]
2. Alsina, C.; Sikorska, J.; Tomas, M.S. *Norm Derivatives and Characterizations of Inner Product Spaces*; World Scientific Publishing Co.: Singapore, 2010.
3. Amir, D. *Characterizations of Inner Product Spaces, Operator Theory: Advances and Applications*; Birkhauser: Basel, Switzerland, 1986; Volume 20.
4. Kannappan, P.L. Quadratic Functional equation and Inner Product Spaces. *Results. Math.* **1995**, *27*, 368–372. [CrossRef]
5. Rassias, Th M. New characterizations of inner product spaces. *Bull. Sci. Math.* **1984**, *108*, 95–99.
6. Kannappan, P.L. *Functional equations and Inequalities with Application*; Springer Monographs in Mathematics; Springer: Berlin/Heidelberg, Germany, 2009.
7. Hyers, D.H.; Isac, G.; Rassias, T.M. *Stability of Functional Equations in Several Variables*; Birkhauser: Boston, MA, USA, 1998.
8. Choleva, P.W. Remarks on the stability of functional equations. *Aequat. Math.* **1984**, *27*, 76–86. [CrossRef]
9. Brzdek, J.; Popa, D.; Raşa, I.; Xu, B. *Ulam Stability of Operators*; Academic Press: Oxford, UK, 2018.
10. Bahyrycz, A.; Brzdek, J.; Jablonska, E.; Malejki, R. Ulam's stability of a generalization of the Fréchet functional equation. *J. Math. Anal. Appl.* **2016**, *442*, 537–553. [CrossRef]
11. Bahyrycz, A.; Brzdek, J.; Piszczek, M.; Sikorska, J. Hyperstability of the Fréchet equation and a characterization of inner product spaces. *J. Funct. Spaces Appl.* **2013**, *2013*, 496361-1–496361-6. [CrossRef]
12. Brzdek, J.; Jablonska, E.; Moslehian, M.S.; Pacho, P. On stability of a functional equation of quadratic type. *Acta Math. Hung.* **2016**, *149*, 160–169. [CrossRef]
13. Brzdek, J.; Lesniak, Z.; Malejki, R. On the generalized Fréchet functional equation with constant coefficients and its stability. *Aequat. Math.* **2018**, *92*, 355–373. [CrossRef]
14. Popa, D.; Raşa, I. The Fréchet functional equation with application to the stability of certain operators. *J. Approx. Theory* **2012**, *164*, 138–144. [CrossRef]
15. Sikorska, J. On a direct method for proving the Hyers-Ulam stability of functional equations. *J. Math. Anal. Appl.* **2010**, *372*, 99–109. [CrossRef]
16. Anderson, D.R.; Onitsuka M. Best constant for Hyers–Ulam stability of second-order h-difference equations with constant coefficients. *Results Math.* **2019**, *74*, 151. [CrossRef]
17. Anderson, D.R.; Onitsuka M. Hyers–Ulam stability for a discrete time scale with two step sizes. *Appl. Math. Comput.* **2019**, *344–345*, 128–140. [CrossRef]
18. Anderson, D.R.; Onitsuka, M.; Rassias, J.M. Best constant for Ulam stability of first-order h-difference equations with periodic coefficient. *J. Math. Anal. Appl.* **2020**, *491*, 124363. [CrossRef]