mathematics

Article

A Note on the Strong Predictable Representation Property and Enlargement of Filtration

Antonella Calzolari † and Barbara Torti *,†

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, I 00133 Roma, Italy; calzolar@mat.uniroma2.it
* Correspondence: torti@mat.uniroma2.it
† These authors contributed equally to this work.

Abstract: The strong predictable representation property of semi-martingales and the notion of enlargement of filtration meet naturally in modeling financial markets, and theoretical problems arise. Here, first, we illustrate some of them through classical examples. Then, we review recent results obtained by studying predictable martingale representations for filtrations enlarged by means of a full process, possibly with accessible components in its jump times. The emphasis is on the non-uniqueness of the martingale enjoying the strong predictable representation property with respect to the same enlarged filtration.

Keywords: predictable representations property; enlargement of filtration; completeness of a financial market

MSC: 60G48; 60G44; 60H05; 60H30

check for updates

Citation: Calzolari, A.; Torti, B. A Note on the Strong Predictable Representation Property and Enlargement of Filtration. *Mathematics* **2022**, *10*, 1783. https://doi.org/10.3390/math10101783

Academic Editor: Andreas C. Georgiou

Received: 30 March 2022
Accepted: 19 May 2022
Published: 23 May 2022

1. Introduction

Since the seminal papers by Harrison and Pliska (see [1,2]) fixed a filtration $\mathbb{F}$ on a probability space, the *strong predictable representation property* (from now on PRP) of a $\mathbb{F}$-semi-martingale $\mathbf{X}$ has become a fundamental topic in mathematical finance. In that framework, $\mathbf{X}$ models the discounted risky asset price and $\mathbb{F}$ the available information on the market. When the property holds, the payoff of any contingent claim can be written, up to an additive constant, as integral with respect to $\mathbf{X}$, and the market is *complete*, that is, a perfect replication of all options is possible. The PRP is a notion widely studied in stochastic analysis (see, e.g., [3–6]). As it is well-known, the PRP holds for a large class of diffusions with jumps. The incompleteness of a market typically occurs when the original complete model is altered, changing the information in one of two possible ways: by considering an *exogenous randomness* in the dynamic of $\mathbf{X}$, as in the case of *stochastic volatility models*, or by assuming the point of view of an *insider*, i.e., adding new information to that endowed by the asset price, as in the case of *defaultable markets*. In both cases, the information grows. In mathematical terms, it amounts to consider a new filtration $\mathbb{G}$ obtained by *enlarging* $\mathbb{F}$ (see [7,8], Chapter VI). In many cases, $\mathbb{G}$ turns out to be the union of $\mathbb{F}$ with a filtration $\mathbb{H}$. The filtration $\mathbb{H}$ models the additional information due to the new sources of uncertainty. Then, the first problem concerns the loss of the property of semi-martingale for $\mathbf{X}$ as $\mathbb{G}$-adapted process. If this is not the case, the second critical question is about the maintenance of the PRP of $\mathbf{X}$ with respect to $\mathbb{G}$. Finally, if the answer is negative, the last question is whether there exists a semi-martingale enjoying the PRP with respect to $\mathbb{G}$ and, eventually, which one it is. Translated into financial language, one asks if the market under full information may be completed by considering additional assets. There is a large literature, even recent, on these issues, which corresponds, respectively, to the so-called *hypothesis* $(\mathbf{H}')$ (see [9]), the *stability of the PRP* (see, e.g., [10,11]) and the *martingales representations* (see, e.g., [12,13]).

This note is a short introduction to the subject. Our purpose is twofold. On the one hand, we aim to present, briefly and without claiming to be exhaustive, the variety of situations that arise for a semi-martingale, in particular with respect to the PRP, when its reference filtration is enlarged. On the other hand, we highlight the role of probability measures in martingales representations for enlarged filtrations. To this end, we collect some results published in separate papers but as part of the same specific ongoing research (see [14–16]). The basic problem is the identification of a possibly multidimensional martingale enjoying the PRP with respect to an enlarged filtration or, equivalently, the identification of a martingale, *which drives* the martingales representation for that filtration. The filtration we refer to is the enlargement of the reference filtration $\mathbb{F}$ of a semi-martingale that enjoys the PRP with respect to $\mathbb{F}$ by means of a non-trivial process with a deterministic initial value. Therefore, this research does not deal with *initial enlargement*, which is the expansion at any time through a unique random variable, while it includes the case of *progressive enlargement*, which is expansion by means of the occurrence process of a random time. The novelty concerns the great generality of the processes involved in the definition of filtrations, which are not supposed to be *quasi-left continuous*, so that their jump times may have *accessible* components. In fact, until recent years, most of the authors dealing with martingale representations worked under hypotheses that exclude this possibility. In progressive enlargement, the usual assumptions, such as *Jacod's condition* with respect to a non-atomic measure, make the random time *totally inaccessible* (see, e.g., [9]). However, in mathematical finance, there is a growing interest in processes that jump on accessible random times (see, e.g., [17,18]). It is important to stress that the focus here is exclusively on the integrators of representations. More precisely, the problem of the explicit expressions of the predictable integrands is not addressed, and only that of their existence and uniqueness is studied. A key tool of the research is the *invariance principle* of PRP (see [12]). Roughly speaking, *by changing the probability measure into an equivalent one, a filtration is still represented, but the driving martingale changes*.

The paper is sectioned as follows. In Section 2 we introduce definitions and issues related to the enlargement of the reference filtration of a semi-martingale. In Section 3, we recall the notions necessary for dealing with martingale representations in the enlargement of filtrations and, among others, the invariance principle of the PRP. In Section 4, we state some auxiliary results of stochastic analysis central to the sequel. In Section 5, after giving the common framework of the reported results, we exhibit two driving square-integrable martingales with strongly orthogonal components, that is, two *bases* of square-integrable martingales. In Section 6, we introduce the notion of *martingale preserving measure* and set the problem of the explicit computation in our framework of a driving martingale under the original probability measure. Finally, the last section is devoted to some comments and to ideas for future research.

2. PRP and Enlargement of Filtration Trough Examples

Let $T > 0$ be a finite time horizon and let $\mathbf{X} = (\mathbf{X}_t)_{t\in[0,T]}$ be a special semi-martingale on a probability space $(\Omega, \mathbb{F}, \mathcal{F}, P)$ with the filtration $\mathbb{F}$ under standard conditions (when real, we denote that semi-martingale by X). Let us denote by $\mathbb{P}(\mathbf{X}, \mathbb{F})$ the set of local martingale measures for $\mathbf{X}$ on $(\Omega, \mathcal{F}_T)$ equivalent to $P|_{\mathcal{F}_T}$.

Definition 1. $\mathbf{X}$ *enjoys the strong predictable representation property with respect to* $\mathbb{F}$ *(*$\mathbb{F}$*-PRP) if there exists* $Q \in \mathbb{P}(\mathbf{X}, \mathbb{F})$ *such that any real* $(Q, \mathbb{F})$*-local martingale can be written as* $m + \boldsymbol{\xi} \cdot \mathbf{X}$ *where* m *is an* $\mathcal{F}_0$*-measurable random variable,* $\boldsymbol{\xi}$ *is an* $\mathbb{F}$*-predictable process and* $\boldsymbol{\xi} \cdot \mathbf{X}$ *denotes the (vector) stochastic integral of* $\boldsymbol{\xi}$ *with respect to* $\mathbf{X}$ *(see [19,20]).*

One of the most celebrated results on PRP is the *Second Fundamental Asset Pricing Theorem*. It states that the PRP is equivalent to the existence of a unique, modulo $\mathcal{F}_0$, equivalent local martingale measure for $\mathbf{X}$ (see Proposition 3.1, page 34, in [10]). If the set $\mathbb{P}(\mathbf{X}, \mathbb{F})$ is a singleton, then $\mathcal{F}_0$ is the trivial σ-algebra (see Theorem, page 314, in [2]). This result is usefully applied to pricing problems of mathematical finance, where the existence of a local martingale measure for $\mathbf{X}$ equivalent to P is central to constructing optimal strategies of investment. In this setting, the condition $\mathbb{P}(\mathbf{X}, \mathbb{F}) \neq \emptyset$ guarantees that the market we refer to is free from arbitrage (see [21]).

Definition 2. *A filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, P)$ is an enlargement of the filtration $\mathbb{F}$ if*

$$\mathcal{F}_t \subset \mathcal{G}_t \text{ for all } t \in [0,T] \quad \text{and} \quad \mathcal{F}_t \subsetneq \mathcal{G}_t \text{ for some } t \in [0,T].$$

The most popular kinds of enlargement of filtration in the literature date back to the seminal paper of Ito and to the book of Jeulin (see [7,22]).

Definition 3. *Let $\mathbb{F}$, $\mathbb{H}$ and $\mathbb{G}$ be filtrations under standard conditions on a complete probability space $(\Omega, \mathcal{F}, P)$. Then, $\mathbb{G}$ is the enlargement of $\mathbb{F}$ by $\mathbb{H}$ when*

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \mathcal{H}_s, \quad t \in [0,T] \tag{1}$$

and
(i) if L is any random variable, $\mathbb{G}$ is called the initial enlargement of $\mathbb{F}$ by L when

$$\mathcal{H}_t := \sigma(L), \quad t \in [0,T];$$

(ii) if τ is any positive random variable, $\mathbb{G}$ is called the progressive enlargement of $\mathbb{F}$ by τ when

$$\mathcal{H}_t := \sigma(\tau \wedge t), \quad t \in [0,T].$$

Given a non trivial enlargement $\mathbb{G}$ of $\mathbb{F}$, it is natural to ask whether $\mathbf{X}$ enjoys the $\mathbb{G}$-PRP or, if not, whether it is possible to construct a possibly multidimensional $\mathbb{G}$-local martingale enjoying the $\mathbb{G}$-PRP. By definition, the first question above makes sense if $\mathbb{P}(\mathbf{X}, \mathbb{G}) \neq \emptyset$. However, it is not always the case, as the following example shows.

Example 1. *Let $B = (B_t)_{t\in[0,1]}$ be the restriction of a Brownian motion on the time interval $[0,1]$ and let $\mathbb{G}$ be the initial enlargement of the natural filtration $\mathbb{F}$ of B by B_1. Then, B is not a $\mathbb{G}$-martingale as easily follows by observing that, for any $t \in (0,1)$, $E(B_1 \mid \mathcal{G}_t) = B_1$, and since, the previous equality holds P-a.s., then $\mathbb{P}(\mathbf{X}, \mathbb{G}) = \emptyset$ (see Section 4.1.1 in [23]). However, B is a $\mathbb{G}$-semi-martingale (see Proposition 4.1, page 70, in [23]).*

The example suggests that, given an $\mathbb{F}$-semi-martingale $\mathbf{X}$, sometimes under enlargement at least the semi-martingale property for $\mathbf{X}$ is preserved, even if $\mathbb{P}(\mathbf{X}, \mathbb{G}) = \emptyset$. This fact is fundamental since as it is well-known the semi-martingale property is the core of the stochastic integration theory as the most general stochastic integrator (see, e.g., Chapter II in [8]). This is why the semi-martingale property is an essential feature of many processes in applied probability and, in particular, in mathematical finance, where the risky asset prices are modeled as semi-martingales. Nevertheless, under enlargement, the semi-martingale property can disappear, as shown by the next two examples.

Example 2. *Let B be a Brownian motion and let $g_t := \sup\{s \leq t, B_s = 0\}$. Then, the Azéma martingale ζ defined by $\zeta_t := sgn(B_t)\sqrt{t - g_t}$ is a martingale with respect to its natural filtration (see 4.3.8.1, page 243, in [24]). Moreover, it is not continuous at all, and therefore, it cannot be a martingale with respect to the natural filtration of B. Denoting by $\mathbb{F}$ and $\mathbb{H}$ the natural filtrations of ζ and B, respectively, since $\mathbb{F} \subset \mathbb{H}$ (therefore, trivially $\mathbb{H} = \mathbb{F} \vee \mathbb{H}$) then $\mathbb{H}$ can be interpreted as an enlargement of $\mathbb{F}$. However, $\mathbb{P}(\zeta, \mathbb{H}) = \emptyset$. In fact, under equivalent changes of measure $\mathbb{H}$-martingales remain continuous processes as well as ζ a discontinuous process. It can be proved that ζ is not even an $\mathbb{H}$-semi-martingale (see Example 9.4.2.3, page 531, in [24]).*

Example 3. *Let B be a Brownian motion and $\mathbb{F}$ its natural filtration. If $\mathbb{G}$ is the enlargement of $\mathbb{F}$ defined by $\mathcal{G}_t := \mathcal{F}_{t+\delta}$ with δ a strictly positive constant, then B is not a $\mathbb{G}$-semi-martingale (see Example 1.19 (c), page 10, in [23]).*

In financial applications, the stability of a semi-martingale property under enlargement, i.e., the condition

$(\mathbf{H}')$ any $(P, \mathbb{F})$-semi-martingale is a $(P, \mathbb{G})$-semi-martingale,

is a minimal requirement to work with. Assessing if $(\mathbf{H}')$ holds and, eventually, what is the $\mathbb{G}$-semi-martingale decomposition of an $\mathbb{F}$-semi-martingale, is a delicate matter in the literature (see, e.g., the seminal paper [9]). A sufficient condition for $(\mathbf{H}')$ is *hypothesis* $(\mathbf{H})$ or equivalently *immersion of* $\mathbb{F}$ *in* $\mathbb{G}$ *under* P, usually denoted by $\mathbb{F} \underset{P}{\hookrightarrow} \mathbb{G}$, which is the condition

$(\mathbf{H})$ any $(P, \mathbb{F})$-martingale is a $(P, \mathbb{G})$-martingale.

As proved by Bremaud and Yor, the immersion property strongly affects the relationship between the two involved filtrations. The following implication holds (see Theorem 3, page 284, in [25]).

Proposition 1. *If $\mathbb{F} \underset{P}{\hookrightarrow} \mathbb{G}$ then $\mathcal{F}_t = \mathcal{G}_t \cap \mathcal{F}_T$ for any $t \in [0, T]$.*

Remark 1. *If $\mathbb{F} \underset{P}{\hookrightarrow} \mathbb{G}$ and $A \in \mathcal{G}_t$ but $A \notin \mathcal{F}_t$, for some $t \in [0, T]$, then $A \notin \mathcal{F}_T$.*

There is an easy relationship between $\mathbb{F}$-PRP of $\mathbf{X}$ and immersion property $(\mathbf{H})$. Fixed $Q \in \mathbb{P}(\mathbf{X}, \mathbb{G})$, if $\mathbf{X}$ enjoys the $\mathbb{F}$-PRP under P, then by the Second Fundamental Asset Pricing Theorem, it follows that $Q|_{\mathcal{F}_T} = P|_{\mathcal{F}_T}$. Therefore, any $(Q, \mathbb{F})$-local martingale is a $(P, \mathbb{F})$-local martingale and, in particular, a stochastic integral with respect to $\mathbf{X}$, and then, it is also a $(Q, \mathbb{G})$-local martingale.

Proposition 2. *If $\mathbf{X}$ enjoys the $\mathbb{F}$-PRP then $\mathbb{F} \underset{Q}{\hookrightarrow} \mathbb{G}$ for any $Q \in \mathbb{P}(\mathbf{X}, \mathbb{G})$.*

The next corollary follows by previous result joint with Proposition 1 and Remark 1. It allows checking directly from filtrations if $\mathbb{P}(\mathbf{X}, \mathbb{G})$ is a void set, and therefore, $\mathbb{G}$-PRP cannot hold.

Corollary 1. *Let $\mathbf{X}$ enjoy the $\mathbb{F}$-PRP. If there exists $t \in [0, T]$ such that $\mathcal{F}_t \neq \mathcal{G}_t \cap \mathcal{F}_T$ or if it holds $\mathcal{F}_T = \mathcal{G}_T$, then $\mathbb{P}(\mathbf{X}, \mathbb{G}) = \emptyset$.*

The second implication can be rephrased, saying that *if the reference market is complete, adding information without changing the payoff's set generates arbitrage opportunities*. This principle could be used for dealing quickly with Example 1, where $\mathcal{F}_1 = \mathcal{G}_1$ and also $\mathcal{G}_t \cap \mathcal{F}_1 = \mathcal{F}_t \vee \sigma(B_1) \neq \mathcal{F}_t$, for any $t \in [0, 1]$.

Now suppose that $\mathbf{X}$ enjoys the $\mathbb{F}$-PRP and $\mathbb{P}(\mathbf{X}, \mathbb{G}) \neq \emptyset$. To our knowledge, there are no general results in the literature to the question about maintenance of the PRP with respect to $\mathbb{G}$. The PRP stability, in the sense that $\mathbf{X}$ remains the process that drives the $\mathbb{G}$-martingale's representation, has been proved, under suitable assumptions, by Amendinger and by Grorud and Pontier in the case of initial enlargement (see Theorem 4.2, page 106, in [26], Proposition 5, page 1012, in [11]). In the same framework under weaker conditions and with a different methodology, Fontana in [27] showed a more general result identifying driving martingales possibly different from $\mathbf{X}$. Therefore, under initial enlargement, PRP stability could fail. Furthermore, when the enlargement starts from a trivial σ-algebra, the PRP is generally not preserved.

Proposition 3. *(Proposition 3.1, page 5, in [14])*

Assume that $\mathbf{X}$ *enjoys the* $\mathbb{F}$*-PRP and* $\mathcal{G}_0$ *is trivial. If the set*

$$\{t \in [0,T] : \mathcal{F}_t \subsetneq \mathcal{G}_t\}$$

has a minimum, then $\mathbf{X}$ *does not enjoy the* $\mathbb{G}$*-PRP.*

As proved by the following counterexample given in the setting of *weak Brownian filtrations*, the statement is false when the set $\{t \in [0,T] : \mathcal{F}_t \subsetneq \mathcal{G}_t\}$ does not have a minimum (see Section 6.1, page 104, in [28]).

Example 4. *Let B be a Brownian motion and let $\mathbb{G}$ be its natural filtration. Consider the processes $X := \int_0^{\cdot} sgn(B_s)dB_s$ with natural filtration $\mathbb{F}$ and $Z := sgn(B_{\cdot})$ with natural filtration $\mathbb{H}$. Then, $\mathbb{G}$ can be represented as an enlargement of $\mathbb{F}$ by $\mathbb{H}$. Moreover, X enjoys both the $\mathbb{F}$-PRP and the $\mathbb{G}$-PRP. Actually, $\mathbb{F}$ coincides with the natural filtration of the process $|B|$ and differs from $\mathbb{G}$ at any time different from zero, so that the set $\{t \in [0,T] : \mathcal{F}_t \subsetneq \mathcal{G}_t\}$ does not have a minimum.*

Our presentation suggests that with a fixed semi-martingale $\mathbf{X}$, there are no general conditions that guarantee the PRP of $\mathbf{X}$ with respect to some reference filtration $\mathbb{A}$. In any case, *when the PRP holds $\mathbb{A}$ it is not necessarily the natural filtration of $\mathbf{X}$ nor is it unique*. The last observation is reinforced by the last two examples of this section.

Example 5. *(Example 23.11, page 279, in [29])*

Let W, V be independent Brownian motions, then $X = \int_0^{\cdot} W_s dV_s$ does not enjoy the PRP with respect to its natural filtration $\mathbb{F}^X$. This is derived easily from the following observations. Since the quadratic variation process of X coincides with the process $\int_0^{\cdot} W_s^2 ds$, then W^2 turns out to be $\mathbb{F}^X$-adapted and so $(W_t^2 - t)_{t \geq 0}$ is not only an $\mathbb{F}^W$-martingale but also an $\mathbb{F}^X$-martingale. If X enjoyed the PRP with respect to $\mathbb{F}^X$, then a predictable process ϕ would exist such that at any time t

$$W_t^2 - t = \int_0^t \phi_s dX_s = \int_0^t \phi_s W_s dV_s.$$

This is impossible since $(W_t^2 - t)_{t \geq 0}$ and V are strongly orthogonal martingales.

Example 6. *(Section 5 in [30])*

Let us consider the space of continuous functions $C([0,T])$ with the Borel σ-algebra $\mathcal{B}(C([0,T]))$ endowed by the uniform distance and a probability space $(S, \mathcal{S}, \mu)$. Let Ω be the measurable space $C([0,T]) \times S$ with the product σ-algebra $\mathcal{F} := \mathcal{B}(C([0,T])) \otimes \mathcal{S}$ and general element $\omega := (\omega_0, \eta)$. Define on Ω, for any fixed t, the application $X_t(\omega) := \omega_0(t)$. Fix $\eta \in S$ and choose on $C([0,T])$ the functions $(\sigma_t(\cdot,\eta))_{t\in[0,T]}$ and $(\beta_t(\cdot))_{t\in[0,T]}$ such that the stochastic differential equation driven by the Brownian motion W

$$dX_t = \sigma_t(X,\eta)dW_t + \beta_t(X)dt, \quad X_0 = x \tag{2}$$

admits a unique weak solution. Let P_η be the corresponding distribution on $(C([0,T]), \mathcal{B}(C([0,T]))$. Set on $(\Omega, \mathcal{F})$ the probability measure P defined by the rule

$$P(d\omega_0, d\eta) := P_\eta(d\omega_0)\mu(d\eta).$$

The diffusion X enjoys the $\mathbb{F}^X \vee \sigma(G)$-PRP where G is the projection on S, that is, $G(\omega) := \eta$, but at the same time, $\mathbb{F}^X$-PRP fails.

3. Driving Martingales in Enlargement of Filtration: Preliminaries

The results presented in the rest of the paper are obtained by assuming in a sense the point of view of an insider of a financial market. In order to develop hedging strategies he must know which risky assets complete his market. In mathematical terms, he needs a driving local martingale under the risk neutral measure. In this regard, the first fundamental result is the general *principle of invariance* of the PRP by equivalent change of measures presented in this section (see Proposition 4 below).

For the ease of the reader, we recall basic notations and definitions on a general filtered standard probability space $(\Omega, \mathbb{A} = (\mathcal{A}_t)_{t\in[0,T]}, \mathcal{A}, R)$, where T is a finite time horizon. We denote by $\mathcal{M}(R, \mathbb{A})$ the set of $(R, \mathbb{A})$-martingales ($\mathcal{M}_0(R, \mathbb{A})$ when the initial value is null) and by $\mathcal{M}_{loc}(R, \mathbb{A})$ ($\mathcal{M}_{loc,0}(R, \mathbb{A})$) the set of $(R, \mathbb{A})$-local martingales. Analogously, $\mathcal{M}^2(R, \mathbb{A})$ ($\mathcal{M}^2_0(R, \mathbb{A})$) is the set of square-integrable $(R, \mathbb{A})$-martingales, and $\mathcal{M}^2_{loc}(R, \mathbb{A})$ ($\mathcal{M}^2_{loc,0}(R, \mathbb{A})$) is its localization. $\mathcal{M}^2(R, \mathbb{A})$ is a Hilbert space with the inner product $(Z^i, Z^j) \to E^R[Z^i_T Z^j_T]$ (see page 28 in [3]).

Given two local martingales Z and Z', we denote by $[Z, Z']$ their *quadratic covariation process* and by $\langle Z, Z'\rangle^{R,\mathbb{A}}$ their *sharp bracket process* (for definitions and existence's conditions, we refer to Chapter VII in [31]).

We refer to [19] for the definition of the vector stochastic integral and its relation with the componentwise stochastic integral. We recall that they coincide when the components of the integrator are pairwise strongly orthogonal local martingales (see Theorem 3.1, page 57, in [19]).

Following Chapter 4 in [3], we denote by $\mathcal{Z}^2(\boldsymbol{\mu})$ the *stable subspace* of $\mathcal{M}^2(R, \mathbb{A})$ generated by the multidimensional locally square-integrable local martingale $\boldsymbol{\mu}$, that is

$$\mathcal{Z}^2(\boldsymbol{\mu}) = \{\boldsymbol{\xi} \bullet \boldsymbol{\mu},\ \boldsymbol{\xi} \in \mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A})\},$$

where $\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A})$ is the set of $\mathbb{A}$-predictable r-dimensional processes $\boldsymbol{\xi}$ such that

$$E^R\left[\int_0^{+\infty} \boldsymbol{\xi}^*_t\, C_t\, \boldsymbol{\xi}_t\, dB_t\right] < +\infty,$$

with $B_t := \sum_{i=1}^r \langle \mu^i, \mu^i\rangle_t$ and $(C_t)_{ij} := \frac{d\langle \mu^i, \mu^j\rangle_t}{dB_t}$, $i, j \in (1, \ldots, r)$.

Finally, we refer to Definition 1 and the subsequent remark for the notion of PRP of a semi-martingale $\mathbf{S}$. From now on, when $\mathbb{P}(\mathbf{S}, \mathbb{A}) = \{Q\}$, we will take the memory of the measure Q saying, more precisely, that $\mathbf{S}$ enjoys the $(Q, \mathbb{A})$-PRP.

Let $\mathbf{S}$ enjoy the $(Q, \mathbb{A})$-PRP. For a probability measure $\tilde{Q}$ locally equivalent to Q, set

$$\eta_t := \frac{d\tilde{Q}}{dQ}\Big|_{\mathcal{A}_t},\quad t \le T.$$

Then, $\eta = (\eta_t)_{t\in[0,T]}$ is a positive $(Q, \mathbb{A})$-martingale and the following invariance principle of the PRP holds.

Proposition 4. *(Lemma 2.5, page 4247, in [12])*

Assume that for any component S^i of $\mathbf{S}$ there exists $\langle \eta, S^i\rangle^{Q,\mathbb{A}}$. Set

$$\tilde{S}^i_t := S^i_t - \int_0^t \frac{1}{\eta_{s^-}}\, d\langle \eta, S^i\rangle^{Q,\mathbb{A}}_s.$$

Then, $\tilde{\mathbf{S}}$ enjoys the $(\tilde{Q}, \mathbb{A})$-PRP.

As a consequence, the following proposition holds.

Proposition 5. *(Proposition 2.2, page 6, in [15])*

Let $\mathbf{S} = \mathbf{S}_0 + \boldsymbol{\mu} + \mathbf{V}$ be a square-integrable special semi-martingale on $(\Omega, \mathbb{A}, \mathcal{A}, R)$. If $\mathbb{P}(\mathbf{S}, \mathbb{A}) = \{P^{\mathbf{S}}\}$, so that $\mathcal{A}_0$ is trivial and $\mathbf{S}$ enjoys the $(P^{\mathbf{S}}, \mathbb{A})$-PRP, then $\boldsymbol{\mu}$ enjoys the $(R, \mathbb{A})$-PRP.

Let us introduce the definition of *strong orthogonality* of local martingales.

Definition 4. *The $\mathbb{A}$-local martingales $\mathbf{Z} = (Z^1, \ldots, Z^r)$ and $\mathbf{Z'} = (Z'^1, \ldots, Z'^s)$ are strongly orthogonal if and only if, for all $i = 1, \ldots, r$, and $j = 1, \ldots, s$, the process $Z^i Z'^j$ is an $\mathbb{A}$-local martingale with null initial value.*

Starting from the previous definition, we present finally the notion of *basis* of a filtration.

Definition 5. *An $(R, \mathbb{A})$-basis of local martingales is a vector of pairwise orthogonal local martingales which enjoys the $(R, \mathbb{A})$-PRP.*

4. PRP of Locally Square-Integrable Local Martingales: Strong Orthogonality, Stable Subspaces and Independence

Strong orthogonality of locally square-integrable local martingales and stable subspaces are related notions as explained by the following result.

Proposition 6. *(Lemma 3.2, page 7, and Remark 3.2, page 8, in [15])*

(i) If $\boldsymbol{\mu}$ and $\boldsymbol{\mu'}$ are strongly orthogonal locally square integrable $\mathbb{A}$-local martingales, then, for any $\boldsymbol{\xi}$ in $\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A})$ and any $\boldsymbol{\eta}$ in $\mathcal{L}^2(\boldsymbol{\mu'}, R, \mathbb{A})$, the real $\mathbb{A}$-local martingales $\boldsymbol{\xi} \bullet \boldsymbol{\mu}$ and $\boldsymbol{\eta} \bullet \boldsymbol{\mu'}$ are strongly orthogonal.

(ii) Moreover, let $\boldsymbol{\mu^i} = (\mu^i_1, \ldots, \mu^i_{r_i})$, $i = 1, \ldots, d$, be d pairwise strongly orthogonal locally square integrable $\mathbb{A}$-local martingales. Then, denoting by $\boldsymbol{\nu}$ the vector $(\boldsymbol{\mu^1}, \ldots \boldsymbol{\mu^d})$, it holds (the symbol $\oplus$ denotes the direct sum of orthogonal subspaces)

$$\mathcal{Z}^2(\boldsymbol{\nu}) = \oplus_{i=1}^d \mathcal{Z}^2(\boldsymbol{\mu^i}).$$

Remark 2. *By the previous result, joint with the comment to Corollary 11.4, page 340, in [3], it follows that $\boldsymbol{\nu} = (\boldsymbol{\mu^1}, \ldots \boldsymbol{\mu^d})$ is a $(R, \mathbb{A})$-basis of locally square-integrable local martingales if and only if*

$$\mathcal{M}^2_0(R, \mathbb{A}) = \oplus_{i=1}^d \mathcal{Z}^2(\boldsymbol{\mu^i}).$$

The next result points out the equivalence between independence and strong orthogonality of locally square-integrable local martingales enjoying the PRP, extending in this way a well-known property of the Brownian setting.

Proposition 7. *(Lemma 3.3, page 8, in [15])*

Let $\mathbb{B}$ and $\mathbb{B}'$ standard filtrations with $\mathcal{B}_T, \mathcal{B}'_T \subset \mathcal{A}$. Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_r)$ be a locally square-integrable $(R, \mathbb{B})$-local martingale and let $\boldsymbol{\nu'} = (\nu'_1, \ldots, \nu'_s)$ be a locally square-integrable $(R, \mathbb{B}')$-local martingale, with ${}^T\boldsymbol{\nu}_0 \boldsymbol{\nu'}_0 \equiv 0$.
Denote by

$[\boldsymbol{\nu}, \boldsymbol{\nu'}]^{\mathbf{V}}$ *any sort order of* $\Big([\nu_i, \nu'_j], i = 1, \ldots, r, j = 1, \ldots, s\Big)$.

(i) If $\mathbb{B}$ and $\mathbb{B}'$ are independent, then $\boldsymbol{\nu}$ and $\boldsymbol{\nu'}$ are strongly orthogonal and the rs-dimensional $\mathbb{B} \vee \mathbb{B}'$-local martingale $[\boldsymbol{\nu}, \boldsymbol{\nu'}]^{\mathbf{V}}$ has locally square-integrable components.

(ii) If $\boldsymbol{\nu}$ and $\boldsymbol{\nu'}$ are strongly orthogonal, $\boldsymbol{\nu}$ enjoys the $(R|_{\mathcal{B}_T}, \mathbb{B})$-PRP and $\boldsymbol{\nu'}$ enjoys the $(R|_{\mathcal{B}'_T}, \mathbb{B}')$-PRP, then $\mathbb{B}$ and $\mathbb{B}'$ are independent if and only if $\boldsymbol{\nu}$ and $\boldsymbol{\nu'}$ are strongly orthogonal $\mathbb{B} \vee \mathbb{B}'$-local martingales.

In fact, the PRP for both $\boldsymbol{\nu}$ and $\boldsymbol{\nu'}$ is needed to state the previous proposition, as the following example shows.

Example 7. *Let W, V be mutually independent Brownian motions, and let $\mathbb{F}^W$ and $\mathbb{F}^V$ be their natural filtrations. Set $X = \int_0^\cdot W_s dV_s$. Then, X and W are $\mathbb{G}$-strongly orthogonal martingales, with $\mathbb{G} := \mathbb{F}^W \vee \mathbb{F}^V$, but their natural filtrations are not independent. This could be expected since, in Example 5, the martingale X does not enjoy PRP with respect to its natural filtration.*

5. Bases of Square-Integrable Martingales

In the light of Remark 2, bases of (locally) square-integrable (local) martingales play an important role, first of all because they guarantee the existence and uniqueness of the predictable integrands in the representation of any fixed local martingale. Now, in great generality, we present two bases of this kind for the same enlarged filtration.

On a probability space $(\Omega, \mathcal{F}, P)$ let $\mathbb{F}$ and $\mathbb{H}$ be two filtrations under standard conditions. Consider two càdlàg processes $\mathbf{X} = (\mathbf{X}_t)_{t\in[0,T]}$ and $\mathbf{Y} = (\mathbf{Y}_t)_{t\in[0,T]}$ such that $\mathbf{X}$ is a special square-integrable $\mathbb{F}$-semi-martingale in $\mathbb{R}^m$ and $\mathbf{Y}$ a special square-integrable $\mathbb{H}$-semi-martingale in $\mathbb{R}^n$ with standard decompositions

$$\mathbf{X} = \mathbf{M} + \mathbf{A}, \quad \mathbf{Y} = \mathbf{N} + \mathbf{B}.$$

Denote by $[\mathbf{M}, \mathbf{N}]^{\mathbf{V}}$ any sort order of $\left([M^i, N^j], i = 1, \dots, m, j = 1, \dots, n\right)$ and by $[\mathbf{X}, \mathbf{Y}]^{\mathbf{V}}$ any sort order of $\left([X^i, Y^j], i = 1, \dots, m, j = 1, \dots, n\right)$. Let us introduce the following condition.

(A1) $\mathbb{P}(\mathbf{X}, \mathbb{F}) = \{P^{\mathbf{X}}\}, \quad \mathbb{P}(\mathbf{Y}, \mathbb{H}) = \{P^{\mathbf{Y}}\}$.

Proposition 5 implies that under previous conditions, $\mathbf{M}$ enjoys the $(P, \mathbb{F})$-PRP and $\mathbf{N}$ enjoys the $(P, \mathbb{H})$-PRP.

Let $\mathbb{G}$ be the filtration defined as in (1). The heuristic of the first result relies on Ito's formula. If $J \in L^2(\Omega, \mathcal{F}_T, P)$, $K \in L^2(\Omega, \mathcal{H}_T, P)$, $E[J] = E[K] = 0$ and P-a.s.

$$J = \int_0^T \Phi_s \, dM_s \quad K = \int_0^T \Psi_s \, dN_s,$$

then $JK \in L^2(\Omega, \mathcal{G}_T, P)$ and P-a.s.

$$JK = \int_0^\infty K_{s^-} \Phi_s \, dM_s + \int_0^\infty J_{s^-} \Psi_s \, dN_s + \int_0^\infty K_{s^-} J_{s^-} \Phi_s \Psi_s \, d[M, N]_s.$$

Theorem 1. *(Theorem 3.5, page 12, in [15])*

*Let assumption **(A1)** be in force. If $\mathbf{M}$ and $\mathbf{N}$ are strongly orthogonal $\mathbb{G}$-martingales, then*

(i) $\mathcal{F}_T$ and $\mathcal{H}_T$ are P-independent;

(ii) $\mathbb{G}$ satisfies usual conditions;

(iii) the triplet $(\mathbf{M}, \mathbf{N}, [\mathbf{M}, \mathbf{N}]^{\mathbf{V}})$ is a $(P, \mathbb{G})$-basis of martingales.

Following Remark 2, the last point may be expressed by

$$\mathcal{M}_0^2(P, \mathbb{G}) = \mathcal{Z}^2(\mathbf{M}) \oplus \mathcal{Z}^2(\mathbf{N}) \oplus \mathcal{Z}^2([\mathbf{M}, \mathbf{N}]^{\mathbf{V}}). \tag{3}$$

A recursive application of previous equality allows to derive a martingales representation result for $\mathbb{F}^1 \vee \ldots \vee \mathbb{F}^d$, where $\mathbb{F}^i$, $i = 1, \dots, d$, are filtrations contained in $\mathcal{F}$ satisfying usual conditions, and there exist $M^1, \dots, M^d$ càdlàg real processes such that M^i, for all $i = 1, \dots, d$, is a square-integrable $\mathbb{F}^i$-martingale. The filtration $\vee_{i=1}^d \mathbb{F}^i$ may be considered without loss of generality as the enlargement of $\mathbb{F}^1$ by means of the union of remaining filtrations.

Corollary 2. *(Theorem 3.5, page 12, in [15])*

Assume that $\mathbb{P}(M^i, \mathbb{F}^i) = \{P|_{\mathcal{F}_T^i}\}$, $i = 1, \dots, d$.

If, for any $k \in (2, \dots, d)$ and $i_1 < i_2 < \ldots < i_k$, the iterated covariation process is $M^{i_1, \dots i_k}$ recursively defined by

$$M^{i_1, \dots i_k} := [M^{i_1, \dots i_{k-1}}, M^{i_k}]$$

is a $\vee_{i=1}^d \mathbb{F}^i$-martingale, then

(i) $\mathcal{F}_T^1, \dots, \mathcal{F}_T^d$ are P-independent;

(ii) $\vee_{i=1}^d \mathbb{F}^i$ satisfies the usual conditions;

(iii) if $\mathcal{Z}^{2,i_1,\dots i_k}$ denotes the stable subspace generated by $M^{i_1,\dots,i_k}$, then

$$\mathcal{M}_0^2\left(P, \vee_{i=1}^d \mathbb{F}^i\right) = \oplus_{k=1}^d \oplus_{i_1<\dots<i_k} \mathcal{Z}^{2,i_1,\dots i_k}.$$

Remark 3. *It may be worth noting that some of the subspaces in the direct sum above may be void or, equivalently, that some of the covariation processes may be null. Analogously, in the setting of the theorem, it may be $[\mathbf{M},\mathbf{N}] \equiv 0$. Regarding this, we recall the formula*

$$[\mathbf{M},\mathbf{N}]_t = \langle \mathbf{M}^c, \mathbf{N}^c \rangle_t + \sum_{s\le t} \Delta\mathbf{M}_s \Delta\mathbf{N}_s,$$

where $\mathbf{M}^c$ and $\mathbf{N}^c$ are the continuous local martingale part of $\mathbf{M}$ and $\mathbf{N}$, respectively, (see Proposition 9.3.4.6, page 524, in [24]). Therefore, the covariation process in our framework vanishes if and only if $\mathbf{M}$ and $\mathbf{N}$ have no accessible common jump times.

The drawback of Theorem 1 is that $\mathbf{M}$ and $\mathbf{N}$ in many situations of interest are not observable quantities, while $\mathbf{X}$ and $\mathbf{Y}$ are. Therefore, it is more convenient using Proposition 4 to derive a driving process in terms of $\mathbf{X}$ and $\mathbf{Y}$. Let us introduce the Radon–Nikodym derivatives

$$L_t^{\mathbf{X}} := \frac{dP^{\mathbf{X}}}{dP}\Big|_{\mathcal{F}_t}, \; L_t^{\mathbf{Y}} := \frac{dP^{\mathbf{Y}}}{dP}\Big|_{\mathcal{H}_t}, \; t \in [0,T] \tag{4}$$

and consider the following condition.

(A2) $L^{\mathbf{X}}$ and $L^{\mathbf{Y}}$ are locally bounded processes.

Theorem 2. *(Theorem 3.5, page 12, in [15])*

*Let assumptions **(A1)** and **(A2)** be in force. If $\mathbf{M}$ and $\mathbf{N}$ are strongly orthogonal $\mathbb{G}$-martingales and Q is the probability measure on $(\Omega, \mathcal{G}_T)$ defined by*

$$\frac{dQ}{dP} := L^{\mathbf{X}} \cdot L^{\mathbf{Y}}, \tag{5}$$

then

(i) $\mathcal{F}_T$ and $\mathcal{H}_T$ are Q-independent;

(ii) the triplet $(\mathbf{X}, \mathbf{Y}, [\mathbf{X},\mathbf{Y}]^{\mathbf{V}})$ is a $(Q,\mathbb{G})$-basis of martingales.

The last point can also be rephrased saying that every element $Z \in \mathcal{M}^2(Q,\mathbb{G})$ can be represented Q-a.s. as

$$Z = Z_0 + \boldsymbol{\eta}^Z \bullet \mathbf{X} + \boldsymbol{\theta}^Z \bullet \mathbf{Y} + \boldsymbol{\zeta}^Z \bullet [\mathbf{X},\mathbf{Y}]^{\mathbf{V}},$$

where $\boldsymbol{\eta}^Z \in \mathcal{L}^2(\mathbf{X},Q,\mathbb{G})$, $\boldsymbol{\theta}^Z \in \mathcal{L}^2(\mathbf{Y},Q,\mathbb{G})$, $\boldsymbol{\zeta}^Z \in \mathcal{L}^2([\mathbf{X},\mathbf{Y}]^{\mathbf{V}},Q,\mathbb{G})$ are uniquely defined.

6. One More Driving Martingale

The hypothesis of $(P,\mathbb{G})$-strong orthogonality for $\mathbf{M}$ and $\mathbf{N}$, in force above, can be weakened by assuming that there exists a *martingale preserving decoupling measure*, which is a probability measure P^* on $(\Omega, \mathcal{G}_T)$ such that

(i) P^* is equivalent to $P|_{\mathcal{G}_T}$;

(ii) $\mathcal{F}_T$ and $\mathcal{H}_T$ are P^*-independent;

(iii) $P|_{\mathcal{F}_T} = P^*|_{\mathcal{F}_T}$ and $P|_{\mathcal{H}_T} = P^*|_{\mathcal{H}_T}$.

In this case, Theorem 1 holds replacing P with P^*. Analogously, Theorem 2 holds replacing Q with Q^* defined by

$$(L^{\mathbf{X}})_t^* := \frac{dP^{\mathbf{X}}}{dP^*}\Big|_{\mathcal{F}_t}, \; (L^{\mathbf{Y}})_t^* := \frac{dP^{\mathbf{Y}}}{dP^*}\Big|_{\mathcal{H}_t}, \; t \in [0,T], \qquad \frac{dQ^*}{dP^*} := (L^{\mathbf{X}})^* \cdot (L^{\mathbf{Y}})^*.$$

Remark 4. *The measure P^* has been introduced by different authors working with initial and progressive enlargement (see, e.g., [11,26,32,33]). The existence of P^* is guaranteed by the decoupling condition*

***(D)** there exists a probability measure equivalent to $P|_{\mathcal{G}_T}$ under which $\mathcal{F}_T$ and $\mathcal{H}_T$ are independent.*

In the case of progressive enlargement, that is, when $\mathcal{H}_t := \sigma(\tau \wedge t)$ for all $t \in [0, T]$ (see Definition 3) and the martingale $\mathbf{N}$ coincides with the $\mathbb{H}$-compensation of the sub-martingale $\mathbb{I}_{\tau \le \cdot}$, the condition is verified under the so-called density hypothesis for τ, that is $P(\tau \in dx \mid \mathcal{F}_t) \ll dx$ for all t, joint with hypothesis $(\mathbf{H})$ (see Theorem 6.1, page 1030, [34]). For the reader's convenience, we recall that the $\mathbb{H}$-compensation of $\mathbb{I}_{\tau \le \cdot}$ enjoys the $\mathbb{H}$-PRP (see Theorem 7.2.5.1, page 416, in [24]).

Summarizing the above discussion, if the measure P^* exists, or equivalently condition **(D)** holds, then $(\mathbf{M}, \mathbf{N}, [\mathbf{M}, \mathbf{N}]^{\mathbf{V}})$ and $(\mathbf{X}, \mathbf{Y}, [\mathbf{X}, \mathbf{Y}]^{\mathbf{V}})$ are two driving martingales, respectively, under P^* and Q^*.

In general, P^* is different from $P|_{\mathcal{G}_T}$ and, since the two measures are equivalent, Proposition 4 suggests how to compute a third driving martingale for $\mathbb{G}$, this time under P. When $[\mathbf{M}, \mathbf{N}]^{\mathbf{V}} \equiv 0$, a possible solution is the pair of $\mathbb{G}$-martingales whose components are defined by

$$M^i_t - \int_0^t \frac{1}{\eta_{s^-}}\, d\langle \eta, M^i \rangle_s^{P^*, \mathbb{G}}, \;\; i = 1, \ldots, m,$$
$$N^j_t - \int_0^t \frac{1}{\eta_{s^-}}\, d\langle \eta, N^j \rangle_s^{P^*, \mathbb{G}}, \;\; j = 1, \ldots, n$$

where η stays for $\frac{P|_{\mathcal{G}_T}}{dP^*}$. Then, this result easily follows.

Theorem 3. *Assume **(A1)**, **(D)** and hypothesis $(\mathbf{H})$. Let $P|_{\mathcal{G}_T} \neq P^*$ and $[\mathbf{M}, \mathbf{N}]^{\mathbf{V}} \equiv 0$. Denote by $\mathbf{N}'$ the martingale part of the Doob–Meyer decomposition of the $(P, \mathbb{G})$-semi-martingale $\mathbf{N}$. Then, the pair $(\mathbf{M}, \mathbf{N}')$ enjoys the $(P, \mathbb{G})$-PRP.*

We conclude this note with a toy example.

Let $M := W + H^\eta$ where W is a standard Brownian motion, and H^η, the compensated occurrence process of a random variable η with values in the set $\{1, 2, 3\}$. Let τ be a random time with values in the set $\{2, 4\}$, such that W is independent of (η, τ), and the joint law $p_{\eta,\tau}$ of (η, τ) is strictly positive on the set $\{1, 2, 3\} \times \{2, 4\}$. Assume also

$$P(\tau = 2 \mid \eta = 2) = P(\tau = 2 \mid \eta \neq 1) = P(\tau = 2 \mid \eta = 3).$$

Let $\mathbb{H}$ and N be defined as in Remark 4. If we denote by $\mathbb{F}$ the natural filtration of M, then we are in the framework of previous results. In this case, the covariation process is not null, and more precisely,

$$[M, N]_t = \Delta M_2 \Delta N_2 \mathbb{I}_{2 \le t}.$$

We refer to [16] for the explicit computation of a driving martingale for $\mathbb{G}$ under P. The result is the triplet of square-integrable martingales defined at time t by

$$W_t + \mathbb{I}_{\eta \le t} - P(\eta = 1)\mathbb{I}_{1 \le t} - P(\eta = 2 \mid \sigma\{\eta = 1\})\mathbb{I}_{2 \le t} - \mathbb{I}_{\eta = 3}\mathbb{I}_{3 \le t},$$
$$\mathbb{I}_{\tau \le t} - P(\tau = 2 \mid \sigma\{\eta = 1\})\mathbb{I}_{2 \le t} - \mathbb{I}_{\tau = 4}\mathbb{I}_{4 \le t},$$
$$\big(\mathbb{I}_{\tau = 2, \eta = 2} - P(\tau = 2)\mathbb{I}_{\eta = 2} - \mathbb{I}_{\tau = 2} P(\eta = 2 | \sigma\{\eta = 1\}) + P(\tau = 2) P(\eta = 2 | \sigma\{\eta = 1\})\big)\mathbb{I}_{2 \le t}.$$

7. Conclusions

A problem arising in the enlargement of filtration is the identification of a martingale that drives martingale representations. There are no general rules providing a solution, but the answer depends both on the considered enlargement and on the probability measure one works with. The hypothesis of the existence of a decoupling measure for the reference filtration and the one that generates the enlargement (here, hypothesis **(D)**) allows solving the problem in many cases not considered so far in the literature. In this work, after collecting some results obtained by the authors on this subject, a new theorem devoted to martingales representation under the original measure P is stated (see Theorem 3).

Theorem 3 of this note gives immediately the integrators in the classic *Kusuoka's martingales representation for the progressively enlarged Brownian filtration* (see [35] and Theorem 7.5.5.1, page 432, in [24]). Kusuoka's result has been the guiding example for many contributions to martingale representations in the case of progressive enlargement by a random time with the avoidance property (see, e.g., [32–34,36,37]).

The objects of ongoing research are martingale representations results under assumptions weaker than those in Theorem 3 and their applications to problems of stochastic optimal control arising in finance, in the same line, among others, of [38,39].

Author Contributions: Conceptualization, A.C. and B.T.; methodology, A.C. and B.T.; software, A.C. and B.T.; validation, A.C. and B.T.; formal analysis, A.C. and B.T.; investigation, A.C. and B.T.; data curation, A.C. and B.T.; writing—original draft preparation, A.C. and B.T.; writing—review and editing, A.C. and B.T.; visualization, A.C. and B.T.; supervision, A.C. and B.T.; project administration, A.C. and B.T.; funding acquisition, A.C. and B.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially funded by the Dipartimenti di Eccellenza 2018-2022 grant by the Italian Ministry of Education and Research.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors acknowledge the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Harrison, J.M.; Pliska, S.R. Martingales and stochastic integrals in the theory of continuous trading. *Stoch. Process. Appl.* **1981**, *11*, 215–260. [CrossRef]
2. Harrison, J.M.; Pliska, S.R. A stochastic calculus model of continuous trading: Complete markets. *Stoch. Process. Appl.* **1983**, *15*, 313–316. [CrossRef]
3. Jacod, J. *Calcul Stochastique et Problèmes de Martingales*; *Lecture Notes in Mathematics*; Springer: Berlin, Germany, 1979; Volume 714, pp. x+539.
4. Jacod, J. A general theorem of representation for martingales. In *Proceedings of Symposia in Pure Mathematics*; American Mathematical Society: Providence, RI, USA, 1977; Volume 31, pp. 37–53.
5. He, S.W.; Wang, J.G.; Yan, J.A. *Semimartingale Theory and Stochastic Calculus; Kexue Chubanshe*; Science Press: Beijing, China; CRC Press: Boca Raton, FL, USA, 1992; pp. xiv+546.
6. Jacod, J.; Shiryaev, A.N. *Limit Theorems for Stochastic Processes*, 2nd ed.; Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]; Springer: Berlin, Germany, 2003; Volume 288, pp. xx+661.
7. Jeulin, T. *Semi-Martingales et Grossissement d'une Filtration*; Lecture Notes in Mathematics; Springer: Berlin, Germany, 1980; Volume 833, pp. ix+142.
8. Protter, P.E. *Stochastic Integration and Differential Equations*, 2nd ed.; Stochastic Modelling and Applied Probability; Springer: Berlin, Germany, 2005; Volume 21, pp. xiv+419.
9. Jacod, J. Grossissement initial, hypothèse (H'), et théorème de Girsanov. In *Grossissements de Filtrations: Exemples et Applications*; Jeulin, T., Yor, M., Eds.; Lecture Notes in Math; Springer: Berlin, Germany, 1985; Volume 1118, pp. 15–35.
10. Amendinger, J.; Becherer, D.; Schweizer, M. A monetary value for initial information in portfolio optimization. *Financ. Stoch.* **2003**, *7*, 29–46. [CrossRef]
11. Grorud, A.; Pontier, M. Probabilités neutres au risque et asymétrie d'information. *C. R. Acad. Sci. Paris Sér. I Math.* **1999**, *329*, 1009–1014. [CrossRef]

12. Jeanblanc, M.; Song, S. Martingale representation property in progressively enlarged filtrations. *Stoch. Process. Appl.* **2015**, *125*, 4242–4271. [CrossRef]
13. Aksamit, A.; Jeanblanc, M.; Rutkowski, M. Integral representations of martingales for progressive enlargements of filtrations. *Stoch. Process. Appl.* **2019**, *129*, 1229–1258. [CrossRef]
14. Calzolari, A.; Torti, B. Enlargement of filtration and predictable representation property for semi-martingales. *Stochastics* **2016**, *88*, 680–698. [CrossRef]
15. Calzolari, A.; Torti, B. Martingale representations in progressive enlargement by the reference filtration of a semi-martingale: A note on the multidimensional case. *Stochastics* **2019**, *91*, 265–287. [CrossRef]
16. Calzolari, A.; Torti, B. An example of martingale representation in progressive enlargement by an accessible random time. In *Frontiers in Stochastic Analysis—BSDEs, SPDEs and Their Applications*; Springer: Cham, Switzerland, 2019; Volume 289, pp. 109–121.
17. Jiao, Y.; Li, S. The generalized density approach in progressive enlargement of filtrations. *Electron. J. Probab.* **2015**, *20*, 21. [CrossRef]
18. Fontana, C.; Schmidt, T. General dynamic term structures under default risk. *Stoch. Process. Appl.* **2018**, *128*, 3353–3386. [CrossRef]
19. Chatelain, M.; Stricker, C. On componentwise and vector stochastic integration. *Math. Financ.* **1994**, *4*, 57–65. [CrossRef]
20. Shiryaev, A.N.; Chernyĭ, A.S. A vector stochastic integral and the fundamental theorem of asset pricing. *Tr. Mat. Inst. Steklova* **2002**, *237*, 12–56.
21. Delbaen, F.; Schachermayer, W. Non-arbitrage and the fundamental theorem of asset pricing: Summary of main results. In *Introduction to Mathematical Finance*; American Mathematical Society: Providence, RI, USA, 1999; Volume 57, pp. 49–58.
22. Itô, K. Extension of stochastic integrals. In Proceedings of the International Symposium on Stochastic Differential Equations, Kyoto, Japan, 22–29 March 1976; Wiley: New York, NY, USA; Chichester, UK; Brisbane, Australia, 1978; pp. 95–109.
23. Aksamit, A.; Jeanblanc, M. *Enlargement of Filtration with Finance in View*; SpringerBriefs in Quantitative Finance; Springer: Cham, Switzerland, 2017; pp. x+150.
24. Jeanblanc, M.; Yor, M.; Chesney, M. *Mathematical Methods for Financial Markets*; Springer: London, UK, 2009; pp. xxvi+732.
25. Brémaud, P.; Yor, M. Changes of filtrations and of probability measures. *Z. Wahrsch. Verw. Geb.* **1978**, *45*, 269–295. [CrossRef]
26. Amendinger, J. Martingale representation theorems for initially enlarged filtrations. *Stoch. Process. Appl.* **2000**, *89*, 101–116. [CrossRef]
27. Fontana, C. The strong predictable representation property in initially enlarged filtrations under the density hypothesis. *Stoch. Process. Appl.* **2018**, *128*, 1007–1033. [CrossRef]
28. Mansuy, R.; Yor, M. *Aspects of Brownian Motion*; Springer: Berlin, Germany, 2008; pp. xiv+195.
29. Stoyanov, J.M. *Counterexamples in Probability*, 2nd ed.; Wiley Series in Probability and Statistics; John Wiley & Sons, Ltd.: Chichester, UK, 1997; pp. xxviii+342.
30. Föllmer, H.; Schweizer, M. Hedging of contingent claims under incomplete information. In *Applied Stochastic Analysis*; Gordon and Breach: New York, NY, USA, 1991; Volume 5, pp. 389–414.
31. Dellacherie, C.; Meyer, P.A. *Probabilities and Potential. B*; North-Holland Mathematics Studies; Theory of Martingales, Translated from the French by J. P. Wilson; North-Holland Publishing Co.: Amsterdam, The Netherlands, 1982; Volume 72, pp. xvii+463.
32. Jeanblanc, M.; Le Cam, Y. Progressive enlargement of filtrations with initial times. *Stoch. Process. Appl.* **2009**, *119*, 2523–2543. [CrossRef]
33. Callegaro, G.; Jeanblanc, M.; Zargari, B. Carthaginian enlargement of filtrations. *ESAIM Probab. Stat.* **2013**, *17*, 550–566. [CrossRef]
34. El Karoui, N.; Jeanblanc, M.; Jiao, Y. What happens after a default: The conditional density approach. *Stoch. Process. Appl.* **2010**, *120*, 1011–1032. [CrossRef]
35. Kusuoka, S. A remark on default risk models. In *Advances in Mathematical Economics*; Springer: Tokyo, Japan, 1999; Volume 1, pp. 69–82.
36. Di Tella, P.; Jeanblanc, M. Martingale representation in the enlargement of the filtration generated by a point process. *Stoch. Process. Appl.* **2021**, *131*, 103–121. [CrossRef]
37. Di Tella, P.; Engelbert, H.J. Martingale representation in progressively enlarged Lévy filtrations. *Stochastics* **2022**, *94*, 311–333. [CrossRef]
38. Ceci, C.; Colaneri, K.; Cretarola, A. Indifference pricing of pure endowments via BSDEs under partial information. *Scand. Actuar. J.* **2020**, *2020*, 904–933. [CrossRef]
39. Bandini, E.; Confortola, F.; Di Tella, P. Progressive Enlargement of Filtrations and Control Problems for Step Processes. Technical report. *arXiv* **2021**, arXiv:2112.12884v1.