Supplementary Material of the paper
On robustness for spatio-temporal data

```
VOM+SAD approximation (10):

distribution<-function(e,n,a,va,g)
{
A<-1-pchisq(a*n/va,n)
B<-va/( sqrt(pi) *(a-va) )
C<-exp(-n*0.5*(  (a/va)-1-log(a/va) ))
D<-sqrt(va)/sqrt(a-a*g^2+(g^2)*va) -1
resul<-A+e*sqrt(n)*B*C*D
return(resul)
}

where

va = 2gamma(h,tau) = spatio-temporal variogram
n = n(h,tau)

################
"Exact" values by simulation

simu11<-function(n,B,e,va,g,a)
{
s<-NULL
for(j in 1:B){
s[j]<-mean((1-e)*va*rchisq(n,1)+e*g^2*va*rchisq(n,1))
}
P=ecdf(s)
1-P(a)
}

################
Table 1

Approximations:

> distribution(0.01,3,2.5,1.4,1.1)
[1] 0.1482988
> distribution(0.01,3,3,1.4,1.1)
[1] 0.09323288
```

```
> distribution(0.01,3,3.5,1.4,1.1)
[1] 0.05812367
> distribution(0.01,3,4,1.4,1.1)
[1] 0.03600603
> distribution(0.01,3,4.5,1.4,1.1)
[1] 0.02219628
> distribution(0.01,3,5,1.4,1.1)
[1] 0.01363265

Exact:

> simu11(3,100000,0.01,1.4,1.1,2.5)
[1]  0.14714
> simu11(3,100000,0.01,1.4,1.1,3)
[1] 0.09308
> simu11(3,100000,0.01,1.4,1.1,3.5)
[1] 0.05577
> simu11(3,100000,0.01,1.4,1.1,4)
[1] 0.03548
> simu11(3,100000,0.01,1.4,1.1,4.5)
[1] 0.02089
> simu11(3,100000,0.01,1.4,1.1,5)
[1] 0.01313

#########


Figure 1

dibu1<-function(n,B,e,va,g)
{
s<-NULL
for(j in 1:B){
s[j]<-mean((1-e)*va*rchisq(n,1)+e*g^2*va*rchisq(n,1))
}
P=ecdf(s)
t<-seq(0,2.1,len=100)
plot(t,1-P(t),type="l",lwd=2,ylab=" ", xlab="a")
lines(t,approx(e,n,t,va,g),type="l",lty=2,ylab=" ",lwd=2,col=2)
}

dibu1(3,100000,0.01,1.4,1.1)


############
```

Figure 2

```
dibu2<-function(n,B,e,va,g)
{
s<-NULL
for(j in 1:B){
s[j]<-mean((1-e)*va*rchisq(n,1)+e*g^2*va*rchisq(n,1))
}
P=ecdf(s)
t<-seq(3,5.5,len=100)
plot(t,1-P(t),type="l",lwd=2,ylab=" ",xlab="a")
}


dibu2(3,100000,0,1.4,1.1)

t<-seq(3,5.5,len=100)

lines(t,distribution(0.01,3,t,1.4,1.1),type="l",lty=2,ylab=" ",lwd=2,col=3)

lines(t,distribution(0.05,3,t,1.4,1.1),type="l",lty=3,ylab=" ",lwd=2,col=4)

lines(t,distribution(0.1,3,t,1.4,1.1),type="l",lty=4,ylab=" ",lwd=2,col=5)

lines(t,distribution(0.2,3,t,1.4,1.1),type="l",lty=5,ylab=" ",lwd=2,col=6)

legend(4.5, 0.06, legend=c("epsilon=0.01", "epsilon=0.05", "epsilon=0.1", "epsilon=0.2")
       col=c(3,4,5,6), lty=2:5, cex=0.8, lwd=2)

############
```

Example

If `vv` is the object (p. 72 in Wikle et al., 2019).

```
library("CCA")
library("dplyr")
library("ggplot2")
```

```
library("gstat")
library("sp")
library("spacetime")
library("STRbook")

data("STObj3", package = "STRbook")
STObj4 <- STObj3[, "1993-07-01::1993-07-31"]

vvv <-variogram(object = z~1 + lat,
data = STObj4,
cloud=T,
width = 80,
cutoff = 1000,
tlags = 0.01:6.01)
```

TWO-DIMENSIONAL Huber's spatio-temporal semivariogram estimator (with tuning constant equal to 1.345) of daily Tmax from the NOAA data set during July 2003, computed using the estimator introduced in Section 6 of the paper

```
library(MASS)

 w1<-0
 w2<-huber(vvv[vvv$dist >     0 & vvv$dist< 80 & vvv$timelag== 0.01,]$gamma,1.345)$mu
 w3<-huber(vvv[vvv$dist >   80 & vvv$dist< 160 & vvv$timelag== 0.01,]$gamma,1.345)$mu
 w4<-huber(vvv[vvv$dist > 160 & vvv$dist< 240 & vvv$timelag== 0.01,]$gamma,1.345)$mu
 w5<-huber(vvv[vvv$dist > 240& vvv$dist< 320 & vvv$timelag== 0.01,]$gamma,1.345)$mu
 w6<-huber(vvv[vvv$dist > 320 & vvv$dist< 400 & vvv$timelag== 0.01,]$gamma,1.345)$mu
 w7<-huber(vvv[vvv$dist > 400 & vvv$dist< 480 & vvv$timelag== 0.01,]$gamma,1.345)$mu
 w8<-huber(vvv[vvv$dist > 480 & vvv$dist< 560 & vvv$timelag== 0.01,]$gamma,1.345)$mu
 w9<-huber(vvv[vvv$dist > 560 & vvv$dist< 640 & vvv$timelag== 0.01,]$gamma,1.345)$mu
w10<-huber(vvv[vvv$dist > 640 & vvv$dist< 720 & vvv$timelag== 0.01,]$gamma,1.345)$mu
w11<-huber(vvv[vvv$dist > 720 & vvv$dist< 800 & vvv$timelag== 0.01,]$gamma,1.345)$mu
w12<-huber(vvv[vvv$dist > 800 & vvv$dist< 880 & vvv$timelag== 0.01,]$gamma,1.345)$mu
w13<-huber(vvv[vvv$dist > 880 & vvv$timelag== 0.01,]$gamma,1.345)$mu

w14<-huber(vvv[vvv$dist == 0  & vvv$timelag== 1.01,]$gamma,1.345)$mu
w15<-huber(vvv[vvv$dist >     0 & vvv$dist< 80  & vvv$timelag== 1.01,]$gamma,1.345)$mu
w16<-huber(vvv[vvv$dist >   80 & vvv$dist< 160 & vvv$timelag== 1.01,]$gamma,1.345)$mu
w17<-huber(vvv[vvv$dist > 160 & vvv$dist< 240 & vvv$timelag== 1.01,]$gamma,1.345)$mu
w18<-huber(vvv[vvv$dist > 240& vvv$dist< 320 & vvv$timelag== 1.01,]$gamma,1.345)$mu
w19<-huber(vvv[vvv$dist > 320 & vvv$dist< 400 & vvv$timelag== 1.01,]$gamma,1.345)$mu
w20<-huber(vvv[vvv$dist > 400 & vvv$dist< 480 & vvv$timelag== 1.01,]$gamma,1.345)$mu
w21<-huber(vvv[vvv$dist > 480 & vvv$dist< 560 & vvv$timelag== 1.01,]$gamma,1.345)$mu
w22<-huber(vvv[vvv$dist > 560 & vvv$dist< 640 & vvv$timelag== 1.01,]$gamma,1.345)$mu
w23<-huber(vvv[vvv$dist > 640 & vvv$dist< 720 & vvv$timelag== 1.01,]$gamma,1.345)$mu
w24<-huber(vvv[vvv$dist > 720 & vvv$dist< 800 & vvv$timelag== 1.01,]$gamma,1.345)$mu
w25<-huber(vvv[vvv$dist > 800 & vvv$dist< 880 & vvv$timelag== 1.01,]$gamma,1.345)$mu
w26<-huber(vvv[vvv$dist > 880 & vvv$timelag== 1.01,]$gamma,1.345)$mu

w27<-huber(vvv[vvv$dist == 0    & vvv$timelag== 2.01,]$gamma,1.345)$mu
w28<-huber(vvv[vvv$dist >     0 & vvv$dist< 80  & vvv$timelag== 2.01,]$gamma,1.345)$mu
w29<-huber(vvv[vvv$dist >   80 & vvv$dist< 160 & vvv$timelag== 2.01,]$gamma,1.345)$mu
w30<-huber(vvv[vvv$dist > 160 & vvv$dist< 240 & vvv$timelag== 2.01,]$gamma,1.345)$mu
w31<-huber(vvv[vvv$dist > 240& vvv$dist< 320 & vvv$timelag== 2.01,]$gamma,1.345)$mu
w32<-huber(vvv[vvv$dist > 320 & vvv$dist< 400 & vvv$timelag== 2.01,]$gamma,1.345)$mu
w33<-huber(vvv[vvv$dist > 400 & vvv$dist< 480 & vvv$timelag== 2.01,]$gamma,1.345)$mu
w34<-huber(vvv[vvv$dist > 480 & vvv$dist< 560 & vvv$timelag== 2.01,]$gamma,1.345)$mu
w35<-huber(vvv[vvv$dist > 560 & vvv$dist< 640 & vvv$timelag== 2.01,]$gamma,1.345)$mu
w36<-huber(vvv[vvv$dist > 640 & vvv$dist< 720 & vvv$timelag== 2.01,]$gamma,1.345)$mu
w37<-huber(vvv[vvv$dist > 720 & vvv$dist< 800 & vvv$timelag== 2.01,]$gamma,1.345)$mu
w38<-huber(vvv[vvv$dist > 800 & vvv$dist< 880 & vvv$timelag== 2.01,]$gamma,1.345)$mu
w39<-huber(vvv[vvv$dist > 880 & vvv$timelag== 2.01,]$gamma,1.345)$mu
```

```
w40<-huber(vvv[vvv$dist == 0  & vvv$timelag== 3.01,]$gamma,1.345)$mu
w41<-huber(vvv[vvv$dist >     0 & vvv$dist< 80  & vvv$timelag== 3.01,]$gamma,1.345)$mu
w42<-huber(vvv[vvv$dist >   80 & vvv$dist< 160 & vvv$timelag== 3.01,]$gamma,1.345)$mu
w43<-huber(vvv[vvv$dist > 160 & vvv$dist< 240 & vvv$timelag== 3.01,]$gamma,1.345)$mu
w44<-huber(vvv[vvv$dist > 240& vvv$dist< 320 & vvv$timelag== 3.01,]$gamma,1.345)$mu
w45<-huber(vvv[vvv$dist > 320 & vvv$dist< 400 & vvv$timelag== 3.01,]$gamma,1.345)$mu
w46<-huber(vvv[vvv$dist > 400 & vvv$dist< 480 & vvv$timelag== 3.01,]$gamma,1.345)$mu
w47<-huber(vvv[vvv$dist > 480 & vvv$dist< 560 & vvv$timelag== 3.01,]$gamma,1.345)$mu
w48<-huber(vvv[vvv$dist > 560 & vvv$dist< 640 & vvv$timelag== 3.01,]$gamma,1.345)$mu
w49<-huber(vvv[vvv$dist > 640 & vvv$dist< 720 & vvv$timelag== 3.01,]$gamma,1.345)$mu
w50<-huber(vvv[vvv$dist > 720 & vvv$dist< 800 & vvv$timelag== 3.01,]$gamma,1.345)$mu
w51<-huber(vvv[vvv$dist > 800 & vvv$dist< 880 & vvv$timelag== 3.01,]$gamma,1.345)$mu
w52<-huber(vvv[vvv$dist > 880 & vvv$timelag== 3.01,]$gamma,1.345)$mu

w53<-huber(vvv[vvv$dist == 0  & vvv$timelag== 4.01,]$gamma,1.345)$mu
w54<-huber(vvv[vvv$dist >     0 & vvv$dist< 80  & vvv$timelag== 4.01,]$gamma,1.345)$mu
w55<-huber(vvv[vvv$dist >   80 & vvv$dist< 160 & vvv$timelag== 4.01,]$gamma,1.345)$mu
w56<-huber(vvv[vvv$dist > 160 & vvv$dist< 240 & vvv$timelag== 4.01,]$gamma,1.345)$mu
w57<-huber(vvv[vvv$dist > 240& vvv$dist< 320  & vvv$timelag== 4.01,]$gamma,1.345)$mu
w58<-huber(vvv[vvv$dist > 320 & vvv$dist< 400 & vvv$timelag== 4.01,]$gamma,1.345)$mu
w59<-huber(vvv[vvv$dist > 400 & vvv$dist< 480 & vvv$timelag== 4.01,]$gamma,1.345)$mu
w60<-huber(vvv[vvv$dist > 480 & vvv$dist< 560 & vvv$timelag== 4.01,]$gamma,1.345)$mu
w61<-huber(vvv[vvv$dist > 560 & vvv$dist< 640 & vvv$timelag== 4.01,]$gamma,1.345)$mu
w62<-huber(vvv[vvv$dist > 640 & vvv$dist< 720 & vvv$timelag== 4.01,]$gamma,1.345)$mu
w63<-huber(vvv[vvv$dist > 720 & vvv$dist< 800 & vvv$timelag== 4.01,]$gamma,1.345)$mu
w64<-huber(vvv[vvv$dist > 800 & vvv$dist< 880 & vvv$timelag== 4.01,]$gamma,1.345)$mu
w65<-huber(vvv[vvv$dist > 880 & vvv$timelag== 4.01,]$gamma,1.345)$mu

w66<-huber(vvv[vvv$dist == 0  & vvv$timelag== 5.01,]$gamma,1.345)$mu
w67<-huber(vvv[vvv$dist >     0 & vvv$dist< 80  & vvv$timelag== 5.01,]$gamma,1.345)$mu
w68<-huber(vvv[vvv$dist >   80 & vvv$dist< 160 & vvv$timelag== 5.01,]$gamma,1.345)$mu
w69<-huber(vvv[vvv$dist > 160 & vvv$dist< 240 & vvv$timelag== 5.01,]$gamma,1.345)$mu
w70<-huber(vvv[vvv$dist > 240& vvv$dist< 320  & vvv$timelag== 5.01,]$gamma,1.345)$mu
w71<-huber(vvv[vvv$dist > 320 & vvv$dist< 400 & vvv$timelag== 5.01,]$gamma,1.345)$mu
w72<-huber(vvv[vvv$dist > 400 & vvv$dist< 480 & vvv$timelag== 5.01,]$gamma,1.345)$mu
w73<-huber(vvv[vvv$dist > 480 & vvv$dist< 560 & vvv$timelag== 5.01,]$gamma,1.345)$mu
w74<-huber(vvv[vvv$dist > 560 & vvv$dist< 640 & vvv$timelag== 5.01,]$gamma,1.345)$mu
w75<-huber(vvv[vvv$dist > 640 & vvv$dist< 720 & vvv$timelag== 5.01,]$gamma,1.345)$mu
w76<-huber(vvv[vvv$dist > 720 & vvv$dist< 800 & vvv$timelag== 5.01,]$gamma,1.345)$mu
w77<-huber(vvv[vvv$dist > 800 & vvv$dist< 880 & vvv$timelag== 5.01,]$gamma,1.345)$mu
w78<-huber(vvv[vvv$dist > 880 & vvv$timelag== 5.01,]$gamma,1.345)$mu

w79<-huber(vvv[vvv$dist == 0  & vvv$timelag== 6.01,]$gamma,1.345)$mu
w80<-huber(vvv[vvv$dist >     0 & vvv$dist< 80  & vvv$timelag== 6.01,]$gamma,1.345)$mu
w81<-huber(vvv[vvv$dist >   80 & vvv$dist< 160 & vvv$timelag== 6.01,]$gamma,1.345)$mu
w82<-huber(vvv[vvv$dist > 160 & vvv$dist< 240 & vvv$timelag== 6.01,]$gamma,1.345)$mu
w83<-huber(vvv[vvv$dist > 240& vvv$dist< 320  & vvv$timelag== 6.01,]$gamma,1.345)$mu
w84<-huber(vvv[vvv$dist > 320 & vvv$dist< 400 & vvv$timelag== 6.01,]$gamma,1.345)$mu
w85<-huber(vvv[vvv$dist > 400 & vvv$dist< 480 & vvv$timelag== 6.01,]$gamma,1.345)$mu
w86<-huber(vvv[vvv$dist > 480 & vvv$dist< 560 & vvv$timelag== 6.01,]$gamma,1.345)$mu
```

```
w87<-huber(vvv[vvv$dist > 560 & vvv$dist< 640 & vvv$timelag== 6.01,]$gamma,1.345)$mu
w88<-huber(vvv[vvv$dist > 640 & vvv$dist< 720 & vvv$timelag== 6.01,]$gamma,1.345)$mu
w89<-huber(vvv[vvv$dist > 720 & vvv$dist< 800 & vvv$timelag== 6.01,]$gamma,1.345)$mu
w90<-huber(vvv[vvv$dist > 800 & vvv$dist< 880 & vvv$timelag== 6.01,]$gamma,1.345)$mu
w91<-huber(vvv[vvv$dist > 880 & vvv$timelag== 6.01,]$gamma,1.345)$mu


w<-c(w1,w2,w3,w4,w5,w6,w7,w8,w9,w10,w11,w12,w13,w14,w15,w16,w17,w18,w19,w20,w21,
w22,w23,w24,w25,w26,w27,w28,w29,w30,w31,w32,w33,w34,w35,w36,w37,w38,w39,w40,w41,
w42,w43,w44,w45,w46,w47,w48,w49,w50,w51,w52,w53,w54,w55,w56,w57,w58,w59,w60,w61,
w62,w63,w64,w65,w66,w67,w68,w69,w70,w71,w72,w73,w74,w75,w76,w77,w78,w79,w80,w81,
w82,w83,w84,w85,w86,w87,w88,w89,w90,w91)

huberst<-vv
huberst$gamma<-w
```

Figure 3 of the paper

```
plot(huberst,main="Huber's  spatio-temporal semivariogram estimator")
```

THREE-DIMENSIONAL Classical and Hurber's spatio-temporal semivariogram estimators

Classical

Figure 4 of the paper

```
plot(vv,  all=T, wireframe=T, zlim=c(0,30),zlab=NULL,main="Classical spatio-temporal
semivariogram estimator of daily maximun temperatures", xlab=list("distance (km)",
rot=30),ylab=list("time lag (days)", rot=-35),scales=list(arrows=F, z = list(distance = 5)))
```

Robust

Figure 5 of the paper

```
plot(huberst, all=T, wireframe=T, zlim=c(0,30),zlab=NULL,
main="Huber's  spatio-temporal semivariogram estimator",
xlab=list("distance (km)", rot=30),ylab=list("time lag (days)", rot=-35),
scales=list(arrows=F, z = list(distance = 5)))
```

Figure 6 of the paper

```
plot(vv,map=F)
```

Figure 7 of the paper

```
plot(huberst,map=F)
```

```
va= 2gamma(h,t)    initial t
a= 2gamma(h,t+tau)    posterior t+tau

valuetime<-function(n,va,a)
{
A<-1-pchisq(a*n/va,n)
B<-va/( sqrt(pi) *(a-va) )
C<-exp(-n*0.5*(  (a/va)-1-log(a/va) ))
D<-sqrt(a/va)*exp( -a/(2*va)+0.5 )
resul<-A+sqrt(n)*B*C*(D-1)
return(resul)
}

 y5<-mean(vvv[vvv$dist > 240& vvv$dist< 320 & vvv$timelag== 0.01,]$gamma)

y18<-mean(vvv[vvv$dist > 240& vvv$dist< 320 & vvv$timelag== 1.01,]$gamma)

y31<-mean(vvv[vvv$dist > 240& vvv$dist< 320 & vvv$timelag== 2.01,]$gamma)

y57<-mean(vvv[vvv$dist > 240& vvv$dist< 320  & vvv$timelag== 4.01,]$gamma)

y70<-mean(vvv[vvv$dist > 240& vvv$dist< 320  & vvv$timelag== 5.01,]$gamma)

y83<-mean(vvv[vvv$dist > 240& vvv$dist< 320  & vvv$timelag== 6.01,]$gamma)

n=806 in all of them
```

Significant difference between the initial moment and the first temporal moment:

```
> valuetime(806,y5,y18)
[1] -1.053956e-12
```

or between the first and second moment:

```
> valuetime(806,y18,y31)
[1] -3.526265e-07
```

No significant differences between the fourth and fifth temporal lags

```
> valuetime(806,y57,y70)
[1] 0.9427521
```

or between the fifth and sixth temporal lags

```
> valuetime(806,y70,y83)
[1] 0.9737844
```

Identification of spatio-temporal outliers. Example 1.

2*vv= classical spatio-temporal estimator.
2*huberst= Huber's spatio-temporal estimator.

With

```
> 2*vv$gamma-2*huberst$gamma
[1] 0.00000000 0.11815214 0.05988491 0.04634051 0.27104432 0.30030481
 [7] 0.64103610 0.55708326 1.41763057 1.72616658 2.09372284 2.63389680
[13] 2.76044786 1.08615712 3.08811040 1.63682044 1.72247747 1.22742633
[19] 0.88295277 0.79972846 0.45464791 0.81244694 1.23359746 1.07838803
[25] 1.17471658 1.53057483 2.59012665 6.46893813 3.82553145 4.46647336
[31] 2.65997242 2.87686600 2.70273487 1.78644411 1.30851738 1.30007439
[37] 0.87995595 0.69687875 0.32447860 2.06136807 6.69595187 3.41120232
[43] 3.79548627 2.70105019 2.32587473 2.52065924 1.86054958 1.44473634
[49] 1.61053058 1.11151622 1.22499377 0.73239271 1.91869460 6.37045254
[55] 2.20228578 2.73033814 2.04338732 2.00811742 1.44673245 1.15043167
[61] 0.84946158 0.86724293 0.60570998 0.64352349 0.41799575 1.35191287
[67] 5.75647785 2.22999570 4.06605990 2.68697489 2.10983544 3.41917519
[73] 2.31835873 2.99840272 3.77751597 1.66521984 1.20132517 1.19066048
[79] 2.84824909 6.55972115 4.11515046 4.97258728 4.04917717 3.12657133
[85] 3.21807028 2.64135431 2.37752224 1.86357447 1.43937821 2.04703953
[91] 1.53870648
```

we obtain the differences (always positive) of these two M-estimators. Some are very small: 0.00000000, 0.05988491, ... Others are somewhat large: 3.79548627, 6.55972115, ... Which ones are significant?

Because the sample sizes are large, we shall use the normal approximation as we say in Remark 1,

$$P_F\{T_n > a\} \simeq 1 - \Phi((a - E[T_n])/\sigma_{T_n})$$

where $T_n = \frac{1}{n}\sum_{i=1}^{n(\mathbf{h},\tau)} (X_i - \psi_b(X_i))$. In fact we can use the Central Limit Theorem to obtain the previous approximation.

Huber's spatio-temporal variogram estimator is biased for the spatio-temporal variogram $2\,\gamma_z(\mathbf{h};\tau)$ because it is based on the squares X_u. Then, although the classical one is not biased, the M-estimator difference will be also biased being its expectation under F,

$$F = (1-\epsilon)\,2\,\gamma_z(\mathbf{h};\tau)\,\chi^2_1 + \epsilon\, g^2\, 2\,\gamma_z(\mathbf{h};\tau)\,\chi^2_1$$

$$E\left[X_i - \psi_b(X_i)\right] = (1-\epsilon)\left(2\,\gamma_z(\mathbf{h};\tau)\left(1 - F_{\chi^2_3}(b/(2\,\gamma_z(\mathbf{h};\tau)))\right) - b\left(1 - F_{\chi^2_1}(b/(2\,\gamma_z(\mathbf{h};\tau)))\right)\right)$$

$$+\epsilon \left(g^2\, 2\, \gamma_z(\mathbf{h};\tau) \left(1 - F_{\chi_3^2}(b/(g^2\, 2\, \gamma_z(\mathbf{h};\tau))) \right) - b \left(1 - F_{\chi_1^2}(b/(g^2\, 2\, \gamma_z(\mathbf{h};\tau))) \right) \right).$$

Also it is

$$E\left[(X_i - \psi_b(X_i))^2\right] = (1-\epsilon) \left(3(2\, \gamma_z(\mathbf{h};\tau))^2 \left(1 - F_{\chi_5^2}(b/(2\, \gamma_z(\mathbf{h};\tau))) \right) \right.$$

$$-2b(2\, \gamma_z(\mathbf{h};\tau)) \left(1 - F_{\chi_3^2}(b/(2\, \gamma_z(\mathbf{h};\tau))) \right)$$

$$\left. +b^2 \left(1 - F_{\chi_1^2}(b/(2\, \gamma_z(\mathbf{h};\tau))) \right) \right) + \epsilon \left(3(g^2\, 2\, \gamma_z(\mathbf{h};\tau))^2 \left(1 - F_{\chi_5^2}(b/(g^2\, 2\, \gamma_z(\mathbf{h};\tau))) \right) \right.$$

$$-2b(g^2\, 2\, \gamma_z(\mathbf{h};\tau)) \left(1 - F_{\chi_3^2}(b/(g^2\, 2\, \gamma_z(\mathbf{h};\tau))) \right)$$

$$\left. +b^2 \left(1 - F_{\chi_1^2}(b/(g^2\, 2\, \gamma_z(\mathbf{h};\tau))) \right) \right)$$

then, we can compute the tail probabilities

$$P_F\{T_n > a\} \simeq 1 - \Phi((a - E[T_n])/\sigma_{T_n})$$

for the points a, considering here that $\epsilon = 0.01$ and $g = 1.1$,

```
vario = 2*gamma

media<-function(b,vario)
{
A1<-0.99*(vario*(1-pchisq(b/vario,3))-b*(1-pchisq(b/vario,1)))+
0.01*((1.1)^2*vario*(1-pchisq(b/((1.1)^2*vario),3))-b*(1-pchisq(b/((1.1)^2*vario),1)))
return(A1)
}


variance<-function(b,vario)
{
A1<-0.99*(vario*(1-pchisq(b/vario,3))-b*(1-pchisq(b/vario,1)))+
0.01*((1.1)^2*vario*(1-pchisq(b/((1.1)^2*vario),3))-b*(1-pchisq(b/((1.1)^2*vario),1)))
A2<-0.99*( 3*((vario)^2)*(1-pchisq(b/vario,5))-(2*b*vario)*
(1-pchisq(b/vario,3))+(b^2)*(1-pchisq(b/vario,1)) ) + 0.01*
(3*((vario*(1.1)^2)^2)*(1-pchisq(b/(vario*(1.1)^2),5))
-(2*b*vario*(1.1)^2)*(1-pchisq(b/(vario*(1.1)^2),3))+(b^2)*(1-pchisq(b/(vario*(1.1)^2),1))    )
solu<-A2-(A1)^2
return(solu)
}
```

```
a = 2*vv$gamma-2*huberst$gamma    |    Tail probability
 [1] 0.00000000                         1
 [2] 0.11815214                         0.9696010
 [3] 0.05988491                         0.9780117
 [4] 0.04634051                         0.9790160
 [5] 0.27104432                         0.9604991
 [6] 0.30030481                         0.9578807
 [7] 0.64103610                         0.9090644
 [8] 0.55708326                         0.9205907
 [9] 1.41763057                         0.6816292
[10]1.72616658                          0.5576323
[11] 2.09372284                         0.4016904
[12] 2.63389680                         0.2052651
[13] 2.76044786                         0.1657873
[14] 1.08615712                         0.7490757
[15] 3.08811040                         0.0649727
[16] 1.63682044                         0.5550302
[17] 1.72247747                         0.5352132
[18] 1.22742633                         0.7489740
[19] 0.88295277                         0.8538907
[20] 0.79972846                         0.8692775
[21] 0.45464791                         0.9342956
[22] 0.81244694                         0.8671335
[23] 1.23359746                         0.7525291
[24] 1.07838803                         0.7960837
[25] 1.17471658                         0.7630998
[26] 1.53057483                         0.6353079
[27] 2.59012665                         0.1613022
[28] 6.46893813                         1.8235e-07          <------------
[29] 3.82553145                         0.0139424           <------------
[30] 4.46647336                         0.0022001           <------------
[31] 2.65997242                         0.1725364
[32] 2.87686600                         0.1283164
[33] 2.70273487                         0.1740595
[34] 1.78644411                         0.5210816
[35] 1.30851738                         0.7077834
[36] 1.30007439                         0.7139480
[37] 0.87995595                         0.8430800
[38] 0.69687875                         0.8840712
[39] 0.32447860                         0.9412339
[40] 2.06136807                         0.3682308
[41] 6.69595187                         8.9457e-08          <------------
[42] 3.41120232                         0.0423737           <------------
[43] 3.79548627                         0.0179283           <------------
[44] 2.70105019                         0.1767490
[45] 2.32587473                         0.2995524
[46] 2.52065924                         0.2341492
[47] 1.86054958                         0.4950422
[48] 1.44473634                         0.6594336
```

```
[49] 1.61053058                        0.5978700
[50] 1.11151622                        0.7796753
[51] 1.22499377                        0.7412523
[52] 0.73239271                        0.8740286
[53] 1.91869460                        0.4081195
[54] 6.37045254                        4.3693e-07          <------------
[55] 2.20228578                        0.2916914
[56] 2.73033814                        0.1435263
[57] 2.04338732                        0.3807738
[58] 2.00811742                        0.4066451
[59] 1.44673245                        0.6285479
[60] 1.15043167                        0.7494908
[61] 0.84946158                        0.8359542
[62] 0.86724293                        0.8288259
[63] 0.60570998                        0.8905084
[64] 0.64352349                        0.8805237
[65] 0.41799575                        0.9191782
[66] 1.35191287                        0.6235374
[67] 5.75647785                        1.2212e-05          <------------
[68] 2.22999570                        0.2857393
[69] 4.06605990                        0.0079754           <------------
[70] 2.68697489                        0.1608218
[71] 2.10983544                        0.3477347
[72] 3.41917519                        0.0431910           <------------
[73] 2.31835873                        0.2808030
[74] 2.99840272                        0.1028271
[75] 3.77751597                        0.0202341           <------------
[76] 1.66521984                        0.5364928
[77] 1.20132517                        0.7115017
[78] 1.19066048                        0.7216741
[79] 2.84824909                        0.1096408
[80] 6.55972115                        1.5528e-07          <------------
[81] 4.11515046                        0.0059802           <------------
[82] 4.97258728                        0.0003689           <------------
[83] 4.04917717                        0.0079846           <------------
[84] 3.12657133                        0.0722094
[85] 3.21807028                        0.0584898
[86] 2.64135431                        0.1709064
[87] 2.37752224                        0.2434866
[88] 1.86357447                        0.4240598
[89] 1.43937821                        0.6023280
[90] 2.04703953                        0.3630105
[91] 1.53870648                        0.5682633
```

For instance,

```
> 1-pnorm(   (2*vv$gamma-2*huberst$gamma)[52] - media(b52, 2*vv$gamma[52]) /
+ (sqrt(   variance(b52 ,  2*vv$gamma[52]) / length(datos52)) )   )
[1] 0.8740286
```

Figure 9 of the paper

```
x0<-c(0,40,120,200,280,360,440,520,600,680,760,840,920)
x<-(rep(x0,7))
y<-c( rep(0,13), rep(1,13),rep(2,13),rep(3,13),rep(4,13),rep(5,13),rep(6,13))
plot(x,y,xlab="spatial lags",ylab="temporal lags")
points(40,2,pch=16,col=2)
points(120,2,pch=16,col=2)
points(200,2,pch=16,col=2)
points(40,3,pch=16,col=2)
points(200,3,pch=16,col=2)
points(40,4,pch=16,col=2)
points(40,5,pch=16,col=2)
points(200,5,pch=16,col=2)
points(40,6,pch=16,col=2)
points(120,6,pch=16,col=2)
points(200,6,pch=16,col=2)
points(280,6,pch=16,col=2)
points(40,1,pch=16,col=4)
points(120,3,pch=16,col=4)
points(440,5,pch=16,col=4)
points(680,5,pch=16,col=4)
points(360,6,pch=16,col=4)
points(440,6,pch=16,col=4)
```