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Impulsive Stabilization on Hyper-Chaotic Financial System under Neumann Boundary

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Abstract: This paper proposes a novel technique to obtain sufficient conditions for the existence and stabilization of positive solutions for a kind of hyper-chaotic financial model. Since some important economic indexes are heavily related to region, the authors consider a nonlinear chaotic financial system with diffusion, which leads to some mathematical difficulties in dealing with the infinite-dimension characteristic. In order to overcome these difficulties, novel analysis techniques have to be proposed on the basis of Laplacian semigroup and impulsive control. Sufficient conditions are provided for existence and global exponential stabilization of positive solution for the system. It is interesting to discover that the impulse strength can be larger than 1 in the newly obtained stability criterion. Numerical simulations show the effectiveness of theoretical analysis.

Keywords: chaotic dynamics; reaction diffusion; average profit margin (APM); impulse control

MSC: 34K45; 93D23

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1. Introduction

Nonlinear financial systems, which are related to the security sub-block, currency sub-block, labor sub-block, and production sub-block, have been attracting extensive attentions during recent decades [1–6]. For example, the authors of [7] introduce the APM into chaotic financial system, which simulates the chaotic complex dynamics of real finance markets. Of course, the new financial model brings greater complexity. Computer simulation shows that it is a hyperchaotic nonlinear system ([7]). In fact, various hyperchaotic systems or complex chaotic systems have already been studied by many researchers [7–9]. In fact, chaotic control is widely used in neural networks, financial systems and ecosystems. For example, chaotic control of chaotic oscillators with fractional order memristor and fuzzy predictive controller for chaotic flows are the important field in engineering technology ([10,11]). Chaotic control of fractional financial system is also a common macroeconomic means ([12–14]). However, reaction-diffusion financial systems are seldom investigated. Indeed, due to the imbalance and difference of regional economic development, many important economic indicators are related to region. Therefore, it is practical to consider financial systems by using nonlinear models with reaction-diffusion terms, which may better reflect the relationship between commodity demand and different regions in the real financial market. Financial systems with diffusion and Neumann boundary values imply that in the marginal areas of the economic circle, many important indicators do not change, precisely simulating the real financial market. However, the reaction-diffusion model with a Neumann boundary value leads to analysis difficulties in studying the the existence and global exponential stabilization of positive solutions since this kind of financial mathematical model is always considered in infinite-dimension space. Fortunately, some methods of the known literature give us useful enlightenment ([15–18]). For example, the authors in [15] successfully studied delayed model with a Neumann boundary value via adopting Sobolev space $H_0^1(\Xi)$. So we are willing to consider this Sobolev space in this paper. It should be pointed out that only the dynamics of hyperchaotic financial system was investigated in many research papers (see, e.g., [7]), and they did not reflect how the government or management actively control the economy. In particular, in the actual financial market, the unbalanced development of regional economies implies that the real financial system should be a reaction-diffusion chaotic financial system, so the dynamics of the system need to be considered in infinite dimensional space. Therefore, the methods and techniques in finite dimensional space in the previous literature are no longer applicable to the reaction-diffusion chaotic financial system in this article. We need to find a positive stationary solution in the reaction-diffusion financial model. In fact, the positive economic indicators of the financial system are in line with the national conditions of most countries. Positive interest rate is conducive to the stability of the banking industry, and a smaller positive interest rate is also conducive to promoting the increase of investment.

On the other hand, impulsive control is always an effective method in stabilizing nonlinear systems ([19–23]). Unlike intermittent control for a short period of time ([24]), impulsive control only takes place in a moment. It can be shown in [19,21–23] that sometimes impulsive control occurs at a fixed time, and sometimes at the critical time point under some designed event triggering mechanisms. Different from neural network whose activation function can be assumed to satisfy Lipschitz continuity, essentially linear, the so-called activation function of chaotic system can not be assumed, especially the activation function of financial system is truly nonlinear, which brings fundamental difficulties to controlling chaotic financial system. In fact, similarly as that of [20], some authors in [6] utilized impulsive control to make an equilibrium solution of the chaotic financial system stabilized globally. However, the equilibrium solution is not positive solution of the chaotic financial system. Similarly, some effective impulsive controls also used to study the stabilization of complex and chaotic financial models in [5,6]. However, any positive solution of the chaotic financial system has never studied before, let alone stability analysis on the positive solution. Such a case inspires our current study. Besides, the impulse was required to be less than 1 in [25], in which the active functions were assumed to be Lipscitz continuous. Although the "active functions" of chaotic financial system are truly nonlinear, we want to design the impulse control that is bigger than 1 in this paper. Moreover, such a bigger impulse will make the positive solution stabilized globally. This is another purpose of writing this paper. Finally, it is known from numerical simulations of [7] that the dynamic trajectory of hyper-chaotic financial model of this paper is more complex than that of chaotic financial system. By employing fixed point theorem and Laplacian semigroup, this paper gives the existence result of positive stationary solution (PSS) for the diffusion financial system. Moreover, the PSS is stabilized by designing impulsive control.

The main contributions are as follows.

- □ The newly obtained stability criterion admits big impulse, reducing some conservatism of criteria. Particularly, one of numerical examples shows that the new stability criterion allows the impulse strength to be bigger than 1. In fact, impulse was actually required to be less than 1 in some of the literature ([25,26]).
- □ The financial system is stable with positive interest rates and other economic indicators, which is in line with the national conditions of most countries. Particularly, in previous research papers ([5,6,9]), for example, in [6], the equilibrium point $P_1(\theta, \frac{k+ack}{c(k-d)}, -\frac{\theta}{c}, \frac{d\theta(1+ac)}{cd-ck})$ only involves positive interest rate $\theta > 0$, but other economic indicators are negative. However, positive stationary solution in this paper implies that all economic indicators are not negative. Particularly, Examples 1 and 2 in this article illustrate that all economic indicators are bigger than 0.01, which means newly derived criteria really have some advantages over those in previous articles.
- □ Diffusion leads to more complexity and chaos on the dynamic behavior of the system, and hence we have to adopt new methods, which are different from the known literature without diffusion, to solve the stabilization of reaction-diffusion financial system. In fact, it is the first article to employ Laplacian semigroup together with two

different fixed point theorems to get the existence of PSS and its global stabilization for the hyper-chaotic financial system.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries about the reaction-diffusion financial model, two fixed point theorems, Laplacian semigroup, some definitions and assumptions. In Section 3, by using variational methods and a fixed point theorem to derive the existence of PSS. Moreover, employing Laplacian semigroup, another fixed point theorem and impulsive control to get the stability of the PSS. In Section 4, two numerical example are provided to illustrate the effectiveness of newly obtained results. Finally, some conclusions are written in Section 5.

Comparison 1. *Figures (a)–(j) of Figure 2 in [7] are the computer simulations of chaos of financial system without diffusions. Now, in this paper, the following Figures 1–9 are the computer simulations of chaos of financial system with diffusion coefficients (0.01, 0.02, 0.03, 0.04). It is shown by Figures 1–9 that diffusion brings more complex to financial system, and the impact of tiny diffusions or regional economy affects the hyper-chaotic financial system continuously.*

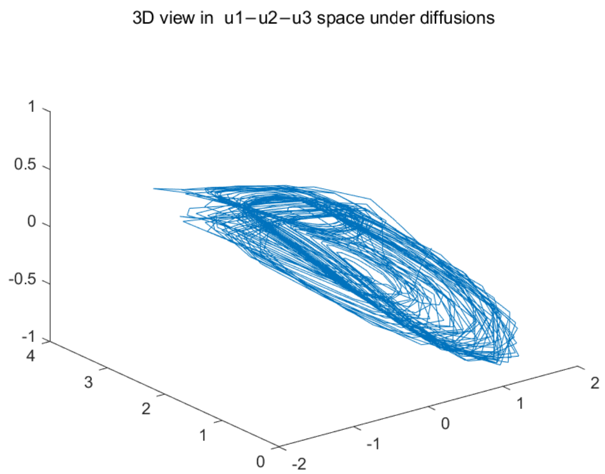

Figure 1. Computer simulations of chaos of financial system under diffusions.

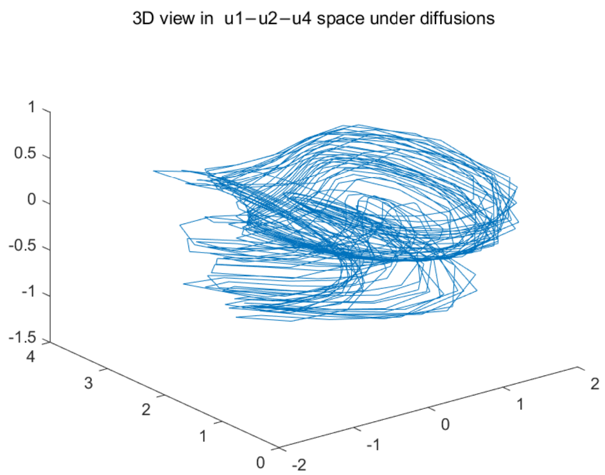

Figure 2. Computer simulations of chaos of financial system under diffusions.

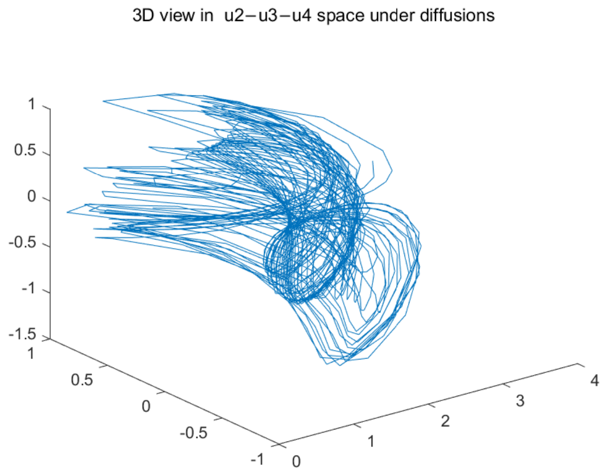

Figure 3. Computer simulations of chaos of financial system under diffusions.

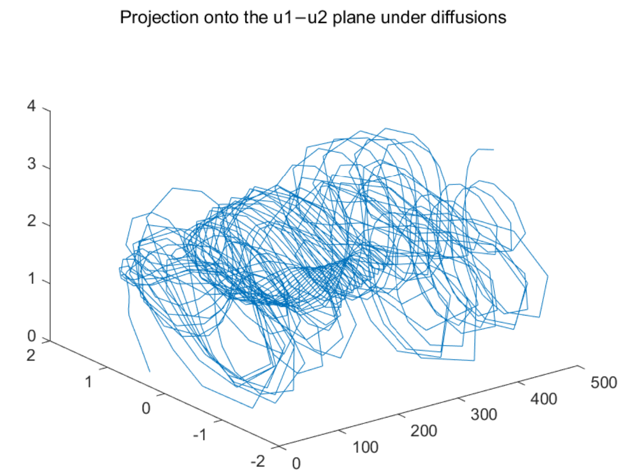

Figure 4. Computer simulations of chaos of financial system under diffusions.

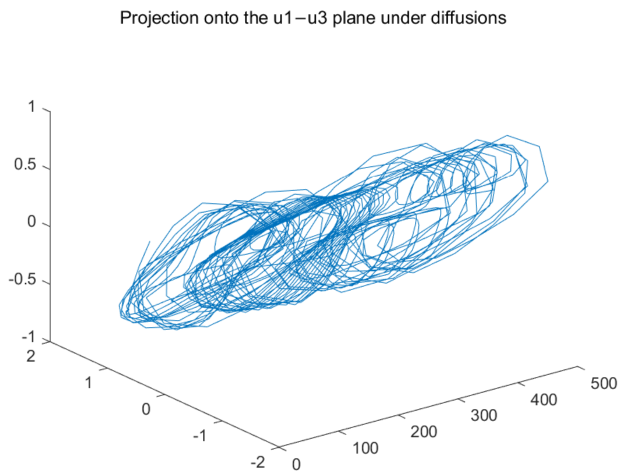

Figure 5. Computer simulations of chaos of financial system under diffusions.

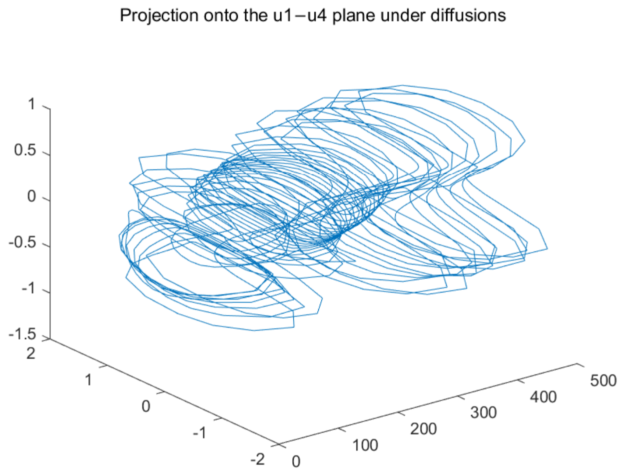

Figure 6. Computer simulations of chaos of financial system under diffusions.

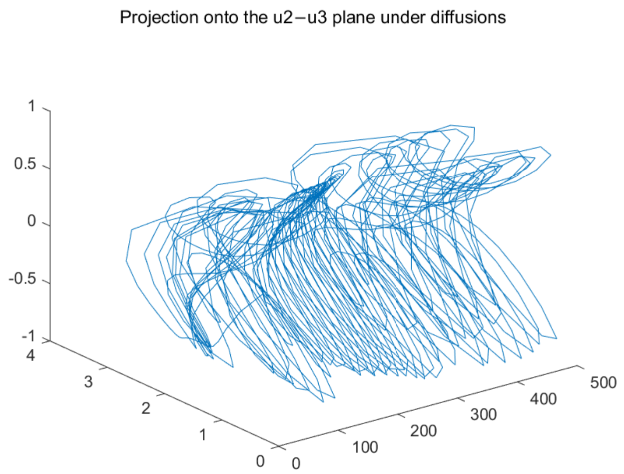

Figure 7. Computer simulations of chaos of financial system under diffusions.

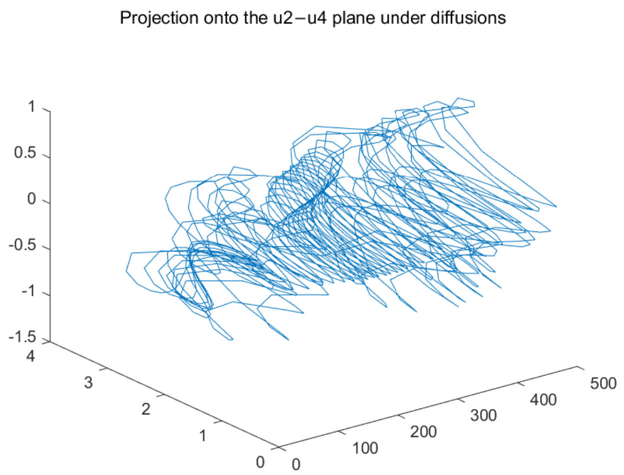

Figure 8. Computer simulations of chaos of financial system under diffusions.

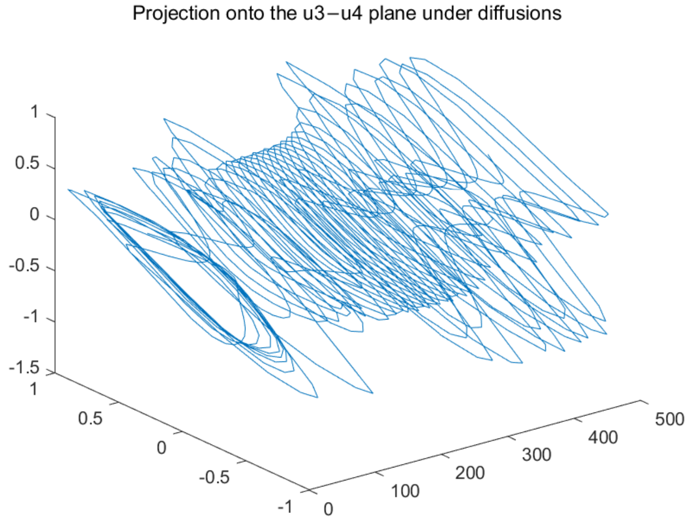

Figure 9. Computer simulations of chaos of financial system under diffusions.

2. Preliminaries

In [6], a hyper-chaotic financial model with APM was proposed as follows:

$$\begin{cases} \dot{x} = -ax + (xy + u + z), \\ \dot{y} = 1 - x^2 - b \cdot y, \\ \dot{z} = -cz - x, \\ \dot{u} = -ku - dxy, \end{cases} \tag{1}$$

which can be written in compact form

$$\dot{X}(t) - f(X(t)) = AX(t), \quad t \geqslant 0, \tag{2}$$

where $X = (x, y, z, u)^T$,

$$A = \begin{pmatrix} -a & 0 & 1 & 1 \\ 0 & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ 0 & 0 & 0 & -k \end{pmatrix}, \quad f(X) = (f_1(X), f_2(X), f_3(X), f_4(X))^T = \begin{pmatrix} xy \\ -x^2 + 1 \\ 0 \\ -dxy \end{pmatrix}. \tag{3}$$

In consideration of the impact of regional economy on important indicators of the financial system, this paper extend the system (1) to reaction-diffusion model as follows:

$$\begin{cases} \partial_t U(x,t) = D\Delta U(x,t) + f(U(x,t)) + AU(x,t), \quad t \geqslant 0, \, x \in \Xi, \\ \partial_\nu U(x,t) = 0, \quad t \geqslant 0, \ x \in \partial\Xi, \end{cases} \tag{4}$$

where $\partial_t U = \partial U / \partial t$, $\partial_\nu U = \frac{\partial U}{\partial \nu}$, $x \in \Xi \subset R^n (n \leqslant 2)$ is a bounded domain, and its boundary $\partial\Xi$ is smooth, ν represents the external normal direction of $\partial\Xi$. Denote $U = (U_1, U_2, U_3, U_4)^T$ and $\Delta U = (\Delta U_1, \Delta U_2, \Delta U_3, \Delta U_4)^T$. Besides, $D = \text{diag}(d_1, d_2, d_3, d_4) \in R^{4\times 4}$ is a positive diagonal matrix. Below, it can be shown that the reaction-diffusion system (4) owns a positive equilibrium point $U_*(x)$. Set $W(t,x) = U(t,x) - U_*(x)$, then it is obtained from (4) that

$$\begin{cases} \partial_t W(x,t) = F(W(x,t)) + D\Delta W(x,t) + AW(x,t), \quad x \in \Xi, \, t \geqslant 0, \\ \partial_\nu W(x,t) = 0, \quad x \in \partial\Xi, \, t \geqslant 0, \end{cases} \tag{5}$$

where $f(U(x,t)) - f(U_*(x)) = F(W(x,t))$. Considering the impulsive effects in (5), it is obtained that

$$\begin{cases} \partial_t W(x,t) = AW(x,t) + D\Delta W(x,t) + F(W(x,t)), & t \neq t_k,\ t \geqslant 0,\ x \in \Xi, \\ H_k W(t_k^-, x) = W(t_k^+, x), & k \in \mathbb{Z}_+,\ x \in \Xi, \\ \partial_\nu W(x,t) = 0, & x \in \partial\Xi,\ t \geqslant 0, \\ \phi(x) = W(0,x), & x \in \Xi, \end{cases} \tag{6}$$

where $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$, $H_k = \mathrm{diag}(h_{k1}, h_{k2}, h_{k3}, h_{k4})$, $0 < t_1 < t_2 < \cdots$, and each $t_k (k \in \mathbb{Z}_+)$ represents a fixed impulsive instant, $W(x, t_k^-) = \lim_{t \to t_k^-} W(x,t) = W(x, t_k)$, $W(x, t_k^+) = \lim_{t \to t_k^+} W(x,t)$.

As for system (6), the definition of mild solution is given below.

Definition 1. *A $L^2(\Xi)$-valued function $W = \{W(t)\}_{[0,T]}$ is said to be a mild solution of (6) provided that $W_i(t,x) \in \mathcal{C}([0,T]; L^2(\Xi))$ satisfies $\int_0^t \|W_i(s)\|^p ds < \infty,\ i = 1,2,3,4$, and for any $x \in \Xi$ and $t \in [0,T]$,*

$$\begin{cases} W_1(t,x) = e^{d_1 t\Delta} \sum\limits_{t > t_k > 0} e^{-d_1 t_k \Delta}(h_{k1} - 1) W_1(t_k, x) + e^{d_1 t\Delta}\phi_1(x) + \int_0^t e^{d_1(t-r)\Delta}[-aW_1(x,r) + W_3(x,r) + W_4(x,r) + F_1(W(x,r))]dr, \\ W_2(t,x) = e^{d_2 t\Delta}\phi_2(x) + e^{d_2 t\Delta} \sum\limits_{t > t_k > 0} e^{-d_2 t_k \Delta}(h_{k2} - 1) W_2(t_k, x) + \int_0^t e^{d_2(t-r)\Delta}[-bW_2(x,r) + F_2(W(x,r))]dr, \\ W_3(t,x) = e^{d_3 t\Delta} \sum\limits_{t > t_k > 0} e^{-d_3 t_k \Delta}(h_{k3} - 1) W_3(t_k, x) + e^{d_3 t\Delta}\phi_3(x) + \int_0^t e^{d_3(t-r)\Delta}[-W_1(r,t) - cW_3(r,t) + F_3(W(x,r))]dr, \\ W_4(t,x) = e^{d_4 t\Delta}\phi_4(x) + e^{d_4 t\Delta} \sum\limits_{t > t_k > 0} e^{-d_4 t_k \Delta}(h_{k4} - 1) W_4(t_k, x) + \int_0^t e^{d_4(t-r)\Delta}[-kW_4(x,r) + F_4(W(x,r))]dr, \end{cases} \tag{7}$$

and

$$\partial W(t,x) / \partial \nu = 0, \quad x \in \partial\Xi,\ t \geqslant 0.$$

Before processing our study, the following preconditions are assumed.

(A1) $\|e^{t\Delta}\| \leqslant e^{-t\gamma} M$, where constants $\gamma, M > 0$.
(A2) there are positive constants $M_i > 0$ such that

$$M_i \geqslant U_i \geqslant 0, \quad x \in \Xi,\ t \geqslant 0, \forall i.$$

In fact, due to the limitation of natural and social resources, economic indicators are limited in the real economic market, and so (A2) is a suitable assumption (see, e.g., [6]).

Definition 2. *The stationary solution $U_*(x)$ is globally exponentially stable (GES) if $W = 0$ in (6) is GES.*

Lemma 1 ([27])**.** *Suppose $\mathfrak{Q}$ is a B space while $\mathfrak{S}$ is a set of $\mathfrak{Q}$, which is closed and convex. Letting the mapping $\mathfrak{E} : \mathfrak{S} \to \mathfrak{S}$ be compact so that for $\mathfrak{T} \in \mathfrak{S}$ with $M = \|\mathfrak{T}\|$, $\mathfrak{T} \neq r\mathfrak{E}(\mathfrak{T})$ holds for each $0 \leqslant r \leqslant 1$, then a fixed point $\mathfrak{E}$ exits for the mapping such that $\|\mathfrak{T}\| \leqslant M$ with $\mathfrak{T} \in \mathfrak{S}$.*

Lemma 2 ([28])**.** *If the mapping $\mathcal{G}$ is contractive on a metric space $\mathcal{S}$ that is complete, then $u \in \mathcal{S}$ with $u = \mathcal{G}(u)$.*

3. Main Results

Firstly, a PSS of (4) should be proved to exist, which actually involves in the existence of positive solution for the corresponding elliptic equations. Furthermore, so variational methods are always employed to solve such a problem.

Theorem 1. *Assume (A2) holds, in addition, there is $c_0 > 0$ such that*

$$c_0 DE \geqslant f(U) + AU \geqslant 0, \tag{8}$$

then (4) owns a PSS $U_*(x)$, or equivalently, (5) owns a zero solution. where $E = (1,1,1,1)^T \in R^4$.

Proof. Firstly, set $H = H_0^1(\Xi)$ with the norm $\|v_i\|_H = \sqrt{\int_\Xi |\nabla v_i|^2 dx}$(See, Appendix A), and $\|v\|_H = \sum_{i=1}^{4} \|v_i\|_H$ for $v = (v_1, v_2, v_3, v_4)^T$. If the PSS of (4) exists, it may be denote as $U_*(x)$.

Let $\mathcal{W} : [C(\overline{\Xi})]^n \to [C(\overline{\Xi})]^n$ be the operator as follows,

$$\mathcal{W} = \begin{pmatrix} -\Delta & 0 & 0 & 0 \\ 0 & -\Delta & 0 & 0 \\ 0 & 0 & -\Delta & 0 \\ 0 & 0 & 0 & -\Delta \end{pmatrix}. \tag{9}$$

Then its inverse operator $\mathcal{W}^{-1}$ is denoted as follows,

$$\mathcal{W}^{-1} = \begin{pmatrix} (-\Delta)^{-1} & 0 & 0 & 0 \\ 0 & (-\Delta)^{-1} & 0 & 0 \\ 0 & 0 & (-\Delta)^{-1} & 0 \\ 0 & 0 & 0 & (-\Delta)^{-1} \end{pmatrix}, \tag{10}$$

obviously the operator $\mathcal{W}^{-1} : [C(\overline{\Xi})]^4 \to [C(\overline{\Xi})]^4$ is positive, compact, and linear ([29]), and

$$\mathcal{W}U(x) = D^{-1}[f(U(x)) + AU(x)], \quad x \in \Xi.$$

Define

$$\mathfrak{S} = \{\varpi(x) \in [C(\overline{\Xi})]^4 : \varpi(x) \geqslant 0,\, x \in \Xi;\, \frac{\partial \varpi(x)}{\partial \nu}|_{\partial\Xi} = 0; \|\varpi(x)\|_H < +\infty\},$$

and so the cone $\mathfrak{S}$ is positive and closed, and is a convex subset for $[C(\overline{\Xi})]^n$. Let $\mathfrak{E} : \mathfrak{S} \to \mathfrak{S}$ be an operator such that

$$\mathfrak{E}\varphi = \mathcal{W}^{-1}\Big(D^{-1}f(U(x)) + D^{-1}AU(x)\Big), \quad \varphi \in \mathfrak{S}.$$

Due to the fact that the operator $\mathcal{W}^{-1}$ is positive, linear and compact ([29]), and the item $D^{-1}[f(U) + AU]$ is continuous, and positive, so $\mathfrak{E} : \mathfrak{S} \to \mathfrak{S}$ is positive and compact .

Next, it will be proved that $\mathfrak{E}$ owns at least one fixed point in $\mathfrak{S}$.

In fact, if it does not hold, then there is $\{p_n\} \subset [0,1]$ and $\{\Psi_n\} \subset \mathfrak{S}$ with

$$\Psi_n = p_n \mathfrak{E}(\Psi_n) = p_n \mathcal{W}^{-1}\Big(D^{-1}f(\Psi_n) + D^{-1}A\Psi_n\Big) \tag{11}$$

and

$$\|\Psi_n\|_H = M_n \to +\infty, \quad n \to +\infty.$$

Then there exists a subsequence of $\{p_n\}$, say, $\{p_n\}$ such that $\lim_{n\to\infty} p_n = p_0$.

Let

$$\mathfrak{A}_n = \frac{\Psi_n}{\|\Psi_n\|_H},$$

then (11) and (8) yields that if $p_n \to p_0 \in [0,1]$,

$$\mathfrak{A}_n \to \mathfrak{A}_0 \in \mathfrak{S}, \quad \|\mathfrak{A}_0\|_H = 1.$$

Indeed, if $n \to \infty$,

$$\mathfrak{A}_n = p_n \mathcal{W}^{-1}\left(\frac{D^{-1}f(\Psi_n) + D^{-1}A\Psi_n}{\|\Psi_n\|_H}\right) \to 0 \in R^n.$$

Note that $\mathfrak{A}_0 = 0$ means $\|\mathfrak{A}_0\|_H = 0$ while $\mathfrak{A}_n \to \mathfrak{A}_0$ and $\|\mathfrak{A}_n\|_H = 1$ yield that $\|\mathfrak{A}_0\|_H = 1$, is contradict with $\|\mathfrak{A}_0\|_H = 0$, and now Lemma 1 yields, there must exist $U(x) \in \mathfrak{S}$ such that $U(x)$ is a positive and bounded solution of (4). □

Remark 1. *It is the first paper to use the variational methods and fixed point theory (Lemma 1) to get the existence of PSS for hyper-chaotic financial system (4) with diffusion under Neumann boundary. In fact, even in the case of ordinary differential equations, the existence of positive solution for chaotic financial system was not derived in many research papers ([5–9]). For example, although the equilibrium solution of [9] is globally stable under impulsive control, the stable equilibrium solution is not a positive solution. Similarly, the equilibrium solutions in [5,6] are also globally stable under some suitable impulse control, the stable equilibrium solutions are still not positive. However, in this paper, by employing a fixed point theorem and variational methods, we overcome the mathematical difficulty, obtaining originally the existence of positive solution that will be proved to be globally stable under impulsive control in the next theorem.*

Theorem 2. *Assume (A1), (A2) and (12) hold, besides,*

$$0 < \omega < 1, \tag{12}$$

then, in model (6), $W = 0$ is GES in the pth moment ($p \geqslant 1$), or equivalently, the PSS $U_*(x)$ is GES in the pth moment, where the variable $W(t,x) = U(t,x) - U_*(x)$ in (6), $\omega = \max\limits_{i\in\{1,2,3,4\}} \omega_i$, $\mu = \inf\limits_{k\in\mathbb{Z}_+}(t_{k+1} - t_k)$, and

$$\omega_1 = 5^{p-1}\left[(|a|^p + 2)(\frac{M}{d_1\gamma})^p + 2^{p-1}(\frac{M}{d_1\gamma})^p\left(M_2^p + M_1^p\right) + M^{2p}(\sup_k |h_{k1} - 1|)^p\left(1 + \frac{1}{d_1\mu\gamma}\right)^p\right], \tag{13}$$

$$\omega_2 = 3^{p-1}\left[(\frac{M|b|}{d_2\gamma})^p + (\frac{2M_1M}{d_2\gamma})^p + M^{2p}(\sup_k |h_{k2} - 1|)^p\left(1 + \frac{1}{d_2\mu\gamma}\right)^p\right], \tag{14}$$

$$\omega_3 = 4^{p-1}\left[(1 + |c|^p)(\frac{M}{d_3\gamma})^p + M^{2p}(\sup_k |h_{k3} - 1|)^p\left(1 + \frac{1}{d_3\mu\gamma}\right)^p\right], \tag{15}$$

$$\omega_4 = 3^{p-1}\left[(\frac{M|k|}{d_4\gamma})^p + 2^{p-1}(\frac{M|d|}{d_4\gamma})^p\left(M_2^p + M_1^p\right) + M^{2p}(\sup_k |h_{k4} - 1|)^p\left(1 + \frac{1}{d_4\mu\gamma}\right)^p\right], \tag{16}$$

Proof. Let the normed space $\mathcal{H}$ be such a functions space that all pth moment continuous processes are consisting of $W(t,x)$ at $t \geqslant 0$ with $t \neq t_k$ so that $\|W_i(t)\|^p e^{t\alpha} \to 0$ for $i = 1,2,3,4$ and $t \to +\infty$, where $0 < \alpha < \min\{d_1\gamma, d_2\gamma, d_3\gamma, d_4\gamma\}$. Besides, $W = U - U_*(x)$ with $0 \leqslant U_i \leqslant M_i$. Furthermore, for any given $x \in \Xi$, $\lim\limits_{t\to t_k^-} W(t,x)$ and $\lim\limits_{t\to t_k^+} W(t,x)$ exist, and $\lim\limits_{t\to t_k^-} W(t,x) = W(t_k,x)$. In addition, $W(0,x) = \phi(x)$.

Obviouly the metric space $\mathcal{H}$ is complete with the distance: for any $W, V \in \mathcal{H}$,

$$\mathrm{dist}\Big(W,V\Big) = \left[\max\left\{\sup_{t\geqslant 0}\|W_1(t) - V_1(t)\|^p, \sup_{t\geqslant 0}\|W_2(t) - V_2(t)\|^p, \sup_{t\geqslant 0}\|W_3(t) - V_3(t)\|^p, \sup_{t\geqslant 0}\|W_4(t) - V_4(t)\|^p\right\}\right]^{\frac{1}{p}}. \tag{17}$$

Construct an operator $\Theta = (\Theta_1, \Theta_2, \Theta_3, \Theta_4)$ such that for any given $W = (W_1, W_2, W_3, W_4)^T \in \mathcal{H}$,

$$\begin{cases} \Theta_1(W_1)(t,x) = e^{d_1 t\Delta}\phi_1(x) + \int_0^t e^{d_1(t-s)\Delta}[-aW_1(s,x) + W_3(s,x) + W_4(s,x) + F_1(W(s,x))]ds \\ \qquad\qquad + e^{d_1 t\Delta} \sum_{t>t_k>0} e^{-d_1 t_k \Delta}(h_{k1}-1)W_1(t_k,x),\ t \geqslant 0 \\ \Theta_2(W_2)(t,x) = e^{d_2 t\Delta}\phi_2(x) + \int_0^t e^{d_2(t-s)\Delta}[-bW_2(s,x) + F_2(W(s,x))]ds + e^{d_2 t\Delta} \sum_{t>t_k>0} e^{-d_2 t_k \Delta}(h_{k2}-1)W_2(t_k,x),\ t \geqslant 0 \\ \Theta_3(t,x) = e^{d_3 t\Delta}\phi_3(x) + \int_0^t e^{d_3(t-s)\Delta}[-W_1(s,t) - cW_3(s,t) + F_3(W(s,x))]ds + e^{d_3 t\Delta} \sum_{t>t_k>0} e^{-d_3 t_k \Delta}(h_{k3}-1)W_3(t_k,x),\ t \geqslant 0 \\ \Theta_4(t,x) = e^{d_4 t\Delta}\phi_4(x) + \int_0^t e^{d_4(t-s)\Delta}[-kW_4(s,x) + F_4(W(s,x))]ds + e^{d_4 t\Delta} \sum_{t>t_k>0} e^{-d_4 t_k \Delta}(h_{k4}-1)W_4(t_k,x),\ t \geqslant 0 \\ \frac{\partial \Theta(W)}{\partial \nu} = 0, \quad x \in \partial\Xi,\ t \geqslant 0, \end{cases} \tag{18}$$

Below, we claim that $\Theta : \mathcal{H} \to \mathcal{H}$.

Indeed, for any $W = (W_1, W_2, W_3, W_4)^T \in \mathcal{H}$, it can be obtained by a routine proof that $e^{\alpha t}\|\Theta_i(W_i)(t)\|^p \to 0$ for $i = 1,2,3,4$ and $t \to +\infty$. Furthermore, for any given $x \in \Xi$, $\lim_{t\to t_k^-} \Theta(W)(t,x)$ and $\lim_{t\to t_k^+} \Theta(W)(t,x)$ exist, and $\lim_{t\to t_k^-} \Theta(W)(t,x) = \Theta(W)(t_k,x)$. In addition, $\Theta(W)(0,x) = \phi(x)$. That is, $\Theta(\mathcal{H}) \subset \mathcal{H}$.

Below, it will be true that the mapping Θ is contractive on $\mathcal{H}$.

Indeed, for any $W, \widetilde{W} \in \mathcal{H}$ with $W = (W_1, W_2, W_3, W_4)^T$ and $\widetilde{W} = (\widetilde{W}_1, \widetilde{W}_2, \widetilde{W}_3, \widetilde{W}_4)^T$, the conditions (A1), (A2) and Holder inequality yield

$$\begin{aligned} &\sup_{t\geqslant 0} \| \int_0^t e^{d_1(t-s)\Delta}[F_1(W(s,x)) - F_1(\widetilde{W}(s,x))]ds\|^p \\ =&\sup_{t\geqslant 0} \| \int_0^t e^{d_1(t-s)\Delta}[f_1(U(s,x)) - f_1(\widetilde{U}(s,x))]ds\|^p \\ \leqslant& M^p \sup_{t\geqslant 0}\left[2^{p-1}M_2^p\left([\frac{1}{d_1\gamma}]^{\frac{p-1}{p}} \cdot [\int_0^t e^{-d_1\gamma(t-s)}\|W_1-\widetilde{W}_1\|^p ds]^{\frac{1}{p}}\right)^p + 2^{p-1}M_1^p\left(\int_0^t e^{-d_1\gamma(t-s)}\|W_2-\widetilde{W}_2\|ds\right)^p\right] \\ \leqslant& M^p\left(M_2^p(\frac{2}{d_1\gamma})^{p-1}\cdot\int_0^t e^{-d_1\gamma(t-s)}ds + M_1^p(\frac{2}{d_1\gamma})^{p-1}\cdot\int_0^t e^{-d_1\gamma(t-s)}\right)\cdot[\mathrm{dist}(W,\widetilde{W})]^p \\ =&2^{p-1}(\frac{M}{d_1\gamma})^p\left(M_2^p + M_1^p\right)\cdot[\mathrm{dist}(W,\widetilde{W})]^p, \end{aligned} \tag{19}$$

$$\begin{aligned} &\sup_{t\geqslant 0} \| \int_0^t e^{d_1(t-s)\Delta}(-a)[W_1(s,x) - \widetilde{W}_1(s,x)]ds\|^p \\ \leqslant&|a|^p M^p \sup_{t\geqslant 0}\left[\int_0^t e^{-d_1\gamma(t-s)}\|W(s,x)-\widetilde{W}(s,x)\|ds\right]^p \\ \leqslant&|a|^p M^p \sup_{t\geqslant 0}\left([\frac{1}{d_1\gamma}]^{\frac{p-1}{p}}\cdot[\int_0^t e^{-d_1\gamma(t-s)}\|W_1-\widetilde{W}_1\|^p ds]^{\frac{1}{p}}\right)^p \\ \leqslant&(\frac{|a|M}{d_1\gamma})^p\cdot[\mathrm{dist}(W,\widetilde{W})]^p, \end{aligned} \tag{20}$$

$$\sup_{t\geqslant 0} \| \int_0^t e^{d_1(t-s)\Delta}[W_3(s,x) - \widetilde{W}_3(s,x)]ds\|^p \leqslant \quad (\frac{M}{d_1\gamma})^p\cdot[\mathrm{dist}(W,\widetilde{W})]^p, \tag{21}$$

and

$$\sup_{t\geqslant 0} \| \int_0^t e^{d_1(t-s)\Delta}[W_4(s,x) - \widetilde{W}_4(s,x)]ds\|^p \leqslant \quad (\frac{M}{d_1\gamma})^p\cdot[\mathrm{dist}(W,\widetilde{W})]^p \tag{22}$$

Let $t_h \geqslant t > t_{h-1}$, then

$$
\begin{aligned}
&\sup_{t\geqslant 0}\left[\|e^{d_1 t\Delta}\sum_{t>t_k>0}e^{-d_1 t_k\Delta}(h_{k1}-1)[W_1(x,t_k)-\widetilde{W}_1(x,t_k)]\|^p\right]\\
\leqslant & M^{2p}(\sup_k|h_{k1}-1|)^p\left[e^{-d_1\gamma t}\sum_{t>t_k>0}e^{d_1\gamma t_k}[\sup_{t\geqslant 0}\|W_1(x,t)-\widetilde{W}_1(x,t)\|^p]^{\frac{1}{p}}\right]^p\\
\leqslant & M^{2p}(\sup_k|h_{k1}-1|)^p\left[e^{-d_1\gamma t}\left(e^{d_1\gamma t_{h-1}}+\frac{1}{\mu}\sum_{0<t_k\leqslant t_{h-2}}e^{d_1\gamma t_k}(t_{k+1}-t_k)\right)\right]^p\cdot[\mathrm{dist}(W,\widetilde{W})]^p\\
\leqslant & M^{2p}\left(1+\frac{1}{d_1\mu\gamma}\right)^p(\sup_k|h_{k1}-1|)^p\cdot[\mathrm{dist}(W,\widetilde{W})]^p
\end{aligned}
\tag{23}
$$

It follows from (23)–(28) that

$$
\begin{aligned}
&\sup_{t\geqslant 0}\|\Theta_1(W_1)(t,x)-\Theta_1(\widetilde{W}_1)(t,x)\|^p\\
\leqslant & 5^{p-1}\Bigg[\sup_{t\geqslant 0}\|\int_0^t e^{d_1(t-s)\Delta}(-a)[W_1(s,x)-\widetilde{W}_1(s,x)]ds\|^p+\sup_{t\geqslant 0}\|\int_0^t e^{d_1(t-s)\Delta}[W_3(s,x)-\widetilde{W}_3(s,x)]ds\|^p\\
&+\sup_{t\geqslant 0}\|\int_0^t e^{d_1(t-s)\Delta}[W_4(s,x)-\widetilde{W}_4(s,x)]ds\|^p\\
&+\sup_{t\geqslant 0}\|\int_0^t e^{d_1(t-s)\Delta}[F_1(W(s,x))-F_1(\widetilde{W}(s,x))]ds\|^p+\sup_{t\geqslant 0}\|e^{d_1 t\Delta}\sum_{t>t_k>0}e^{-d_1 t_k\Delta}(h_{k1}-1)[W_1(t_k,x)-\widetilde{W}_1(t_k,x)]\|^p\Bigg]\\
\leqslant & 5^{p-1}\left[(|a|^p+2)(\frac{M}{d_1\gamma})^p+2^{p-1}(\frac{M}{d_1\gamma})^p\left(M_2^p+M_1^p\right)+M^{2p}(\sup_k|h_{k1}-1|)^p\left(1+\frac{1}{d_1\mu\gamma}\right)^p\right]\cdot[\mathrm{dist}(W,\widetilde{W})]^p\\
\leqslant & \omega\cdot[\mathrm{dist}(W,\widetilde{W})]^p
\end{aligned}
\tag{24}
$$

Similarly,

$$
\begin{aligned}
&\sup_{t\geqslant 0}\|\Theta_2(W_2)(t,x)-\Theta_2(\widetilde{W}_2)(t,x)\|^p\\
\leqslant & 3^{p-1}\Bigg[\sup_{t\geqslant 0}\|\int_0^t e^{d_2(t-s)\Delta}(-b)[W_2(s,x)-\widetilde{W}_2(s,x)]ds\|^p+\sup_{t\geqslant 0}\|\int_0^t e^{d_2(t-s)\Delta}[F_2(W(s,x))-F_2(\widetilde{W}(s,x))]ds\|^p\\
&+\sup_{t\geqslant 0}\|e^{d_2 t\Delta}\sum_{t>t_k>0}e^{-d_2 t_k\Delta}(h_{k2}-1)[W_2(t_k,x)-\widetilde{W}_2(t_k,x)]\|^p\Bigg]\\
\leqslant & 3^{p-1}\left[(\frac{M|b|}{d_2\gamma})^p+(\frac{2M_1M}{d_2\gamma})^p+M^{2p}(\sup_k|h_{k2}-1|)^p\left(1+\frac{1}{d_2\mu\gamma}\right)^p\right]\cdot[\mathrm{dist}(W,\widetilde{W})]^p\\
\leqslant & \omega\cdot[\mathrm{dist}(W,\widetilde{W})]^p
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&\sup_{t\geqslant 0}\|\Theta_3(W_3)(t,x)-\Theta_3(\widetilde{W}_3)(t,x)\|^p\\
\leqslant & 4^{p-1}\Bigg[\sup_{t\geqslant 0}\|\int_0^t e^{d_3(t-s)\Delta}(-1)[W_1(s,x)-\widetilde{W}_1(s,x)]ds\|^p+\sup_{t\geqslant 0}\|\int_0^t e^{d_3(t-s)\Delta}(-c)[W_3(s,x)-\widetilde{W}_3(s,x)]ds\|^p\\
&+\sup_{t\geqslant 0}\|\int_0^t e^{d_3(t-s)\Delta}[F_3(W(s,x))-F_3(\widetilde{W}(s,x))]ds\|^p+\sup_{t\geqslant 0}\|e^{d_3 t\Delta}\sum_{t>t_k>0}e^{-d_3 t_k\Delta}(h_{k3}-1)[W_3(t_k,x)-\widetilde{W}_3(t_k,x)]\|^p\Bigg]\\
\leqslant & 4^{p-1}\left[(1+|c|^p)(\frac{M}{d_3\gamma})^p+M^{2p}(\sup_k|h_{k3}-1|)^p\left(1+\frac{1}{d_3\mu\gamma}\right)^p\right]\cdot[\mathrm{dist}(W,\widetilde{W})]^p\\
\leqslant & \omega\cdot[\mathrm{dist}(W,\widetilde{W})]^p
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&\sup_{t\geqslant 0}\|\Theta_4(W_4)(t,x)-\Theta_4(\widetilde{W}_4)(t,x)\|^p\\
\leqslant & [\mathrm{dist}(W,\widetilde{W})]^p 3^{p-1}\left[(\frac{M|k|}{d_4\gamma})^p+2^{p-1}(\frac{M|d|}{d_4\gamma})^p\left(M_2^p+M_1^p\right)+(\sup_k|h_{k4}-1|)^pM^{2p}\left(\frac{1}{d_4\mu\gamma}+1\right)^p\right]\\
\leqslant & \omega\cdot[\mathrm{dist}(W,\widetilde{W})]^p
\end{aligned}
\tag{27}
$$

Combining (21) and (28)–(31) results in

$$[\mathrm{dist}(\Theta(W),\Theta(\widetilde{W}))]^p \leqslant \omega\cdot[\mathrm{dist}(W,\widetilde{W})]^p$$

or

$$\mathrm{dist}(\Theta(W),\Theta(\widetilde{W})) \leqslant \omega^{\frac{1}{p}}\cdot\mathrm{dist}(W,\widetilde{W}),$$

which together with (16) implies that $\Theta:\mathcal{H}\to\mathcal{H}$ is contractive so that there is the fixed point W of Θ in $\mathcal{H}$, satisfying $e^{\alpha t}\|U(t,x)-U_*(x)\|^p\to 0$ or $e^{\alpha t}\|W(t,x)\|^p\to 0$, and the proof is over. □

Remark 2. *Theorem 2 does not propose that the limiting pulse must be less than 1, which reduces the conservatism of the algorithm. In fact, impulse was actually required to be less than 1 in some of the literature ([25,26]).*

In sum, Theorem 1 is actually a new result, because Theorem 1 shows an existence of positive solution while the locally stable solution is non-positive in [12]. Similarly, in [5,9,13,14], the so-called stable equilibrium solutions all are not truly positive solution. In fact, below Examples 1 and 2 will give a true positive solution that is globally stable, and this implies Theorems 1 and 2 both are truly new results. Of course, some ideas or methods come from the above-mentioned literature.

4. Numerical Examples

The conditions of Theorem 1 are relaxed and easy to be satisfied, and now a numerical example is designed to illuminate its feasibility as follows.

Example 1. *Let* $0.15\leqslant U_1\leqslant 0.25$, $0.05\leqslant U_2\leqslant 0.15$, $0.3\leqslant U_3\leqslant 0.425$, $0.01\leqslant U_4\leqslant 0.05$. *In addition, set* c_0 = *10,000,* $D=\mathrm{diag}(d_1,d_2,d_3,d_4)=\mathrm{diag}(0.27,0.05,0.21,0.03)$ *, then* $M_1=0.25$, $M_2=0.15$, $M_3=0.425$, $M_4=0.05$, *and the condition (A2) is satisfied. Let* $\Xi=[0,\pi]$, *then* $\|e^{t\Delta}\|\leqslant e^{-\pi^2 t}$ *with* $\gamma=\pi^2$ *and* $M=1$. *Set* $(a,b,c,d,k)=(0.05,0.25,-0.85,-0.25,0.075)$, *then (3.1) holds. Now, it shows by Theorem 3, the considered system (4) owns the PSS* $u_*(x)$.

Below, another numerical example is given to show the stability of the above PSS $u_*(x)$.

Example 2. *Adopt all the data of the above example, and suppose* $p=1.001,\mu=2$,

Case 1: $h_{ki}=0.999,\forall\, i=1,2,3,4.$

$$\omega_1=5^{p-1}\left[(|a|^p+2)(\frac{M}{d_1\gamma})^p+2^{p-1}(\frac{M}{d_1\gamma})^p\left(M_2^p+M_1^p\right)+M^{2p}(\sup_k|h_{k1}-1|)^p\left(1+\frac{1}{d_1\mu\gamma}\right)^p\right]=0.9210, \tag{28}$$

$$\omega_2=3^{p-1}\left[(\frac{M|b|}{d_2\gamma})^p+(\frac{2M_1M}{d_2\gamma})^p+M^{2p}(\sup_k|h_{k2}-1|)^p\left(1+\frac{1}{d_2\mu\gamma}\right)^p\right]=0.6607, \tag{29}$$

$$\omega_3=4^{p-1}\left[(1+|c|^p)(\frac{M}{d_3\gamma})^p+M^{2p}(\sup_k|h_{k3}-1|)^p\left(1+\frac{1}{d_3\mu\gamma}\right)^p\right]=0.8943, \tag{30}$$

$$\omega_4=3^{p-1}\left[(\frac{M|k|}{d_4\gamma})^p+2^{p-1}(\frac{M|d|}{d_4\gamma})^p\left(M_2^p+M_1^p\right)+M^{2p}(\sup_k|h_{k4}-1|)^p\left(1+\frac{1}{d_4\mu\gamma}\right)^p\right]=0.5937, \tag{31}$$

Case 2: $h_{ki}=1.001,\forall\, i=1,2,3,4.$

$$\omega_1=5^{p-1}\left[(|a|^p+2)(\frac{M}{d_1\gamma})^p+2^{p-1}(\frac{M}{d_1\gamma})^p\left(M_2^p+M_1^p\right)+M^{2p}(\sup_k|h_{k1}-1|)^p\left(1+\frac{1}{d_1\mu\gamma}\right)^p\right]=0.9210, \tag{32}$$

$$\omega_2=3^{p-1}\left[(\frac{M|b|}{d_2\gamma})^p+(\frac{2M_1M}{d_2\gamma})^p+M^{2p}(\sup_k|h_{k2}-1|)^p\left(1+\frac{1}{d_2\mu\gamma}\right)^p\right]=0.6607, \tag{33}$$

$$\omega_3 = 4^{p-1}\left[(1+|c|^p)(\frac{M}{d_3\gamma})^p + M^{2p}(\sup_k |h_{k3}-1|)^p\left(1+\frac{1}{d_3\mu\gamma}\right)^p\right] = 0.8943, \tag{34}$$

$$\omega_4 = 3^{p-1}\left[(\frac{M|k|}{d_4\gamma})^p + 2^{p-1}(\frac{M|d|}{d_4\gamma})^p\left(M_2^p + M_1^p\right) + M^{2p}(\sup_k |h_{k4}-1|)^p\left(1+\frac{1}{d_4\mu\gamma}\right)^p\right] = 0.5937, \tag{35}$$

Whether Case 1 or Case 2, we have the same conclusion: $\omega = 0.9210 \in (0,1)$. Therefore, Theorem 2 yields that the positive steady state solution $U_*(x)$ is GES in the pth moment with $p = 1.001$.

Remark 3. *Example 2 illuminates the effectiveness of Theorem 2. Indeed, Table 1 shows that Theorem 2 admits the impulse is bigger than 1, and Figures 10–13 show the boundedness and stability of U, where* $W(t,x) = U(t,x) - U_*(x)$ *in (6). i.e.,* $W_1(t,x) = U_1(t,x) - U_{*1}$, $W_2(t,x) = U_2(t,x) - U_{*2}$, $W_3(t,x) = U_3(t,x) - U_{*3}$, *and* $W_4(t,x) = U_4(t,x) - U_{*4}$. *Speaking specifically, it is shown from Figures 1–4 that* $0.15 \leqslant U_1(t,x) \leqslant 0.25$, $0.05 \leqslant U_2(t,x) \leqslant 0.15$, $0.3 \leqslant U_3(t,x) \leqslant 0.425$, $0.01 \leqslant U_4(t,x) \leqslant 0.05$, *and* $0.15 \leqslant U_{*1} \leqslant 0.25$, $0.05 \leqslant U_{*2} \leqslant 0.15$, $0.3 \leqslant U_{*3} \leqslant 0.425$, $0.01 \leqslant U_{*4} \leqslant 0.05$. *This implies that the globally stable stationary solution* U_* *is positive and bounded.*

Table 1. Comparisons the influences between big impulse and small impulse when other data are unchanged.

	Case 1: $h_{ki} = 0.999$	**Case 2:** $h_{ki} = 1.001$
ω_1	0.9210	0.9210
ω_2	0.6607	0.6607
ω_3	0.8943	0.8943
ω_4	0.5937	0.5937
ω	$0.6665 \in (0,1)$	$0.6665 \in (0,1)$

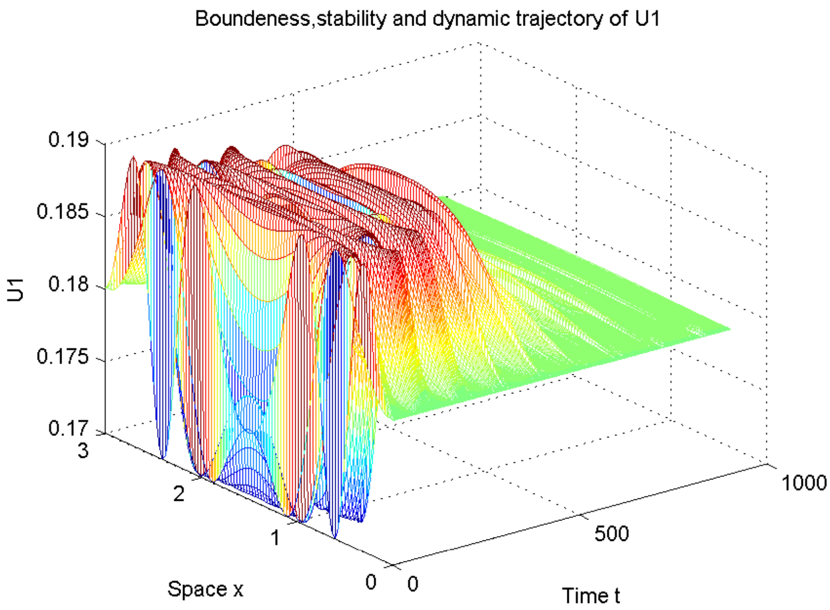

Figure 10. Computer simulation of dynamics of the interest rate U_1.

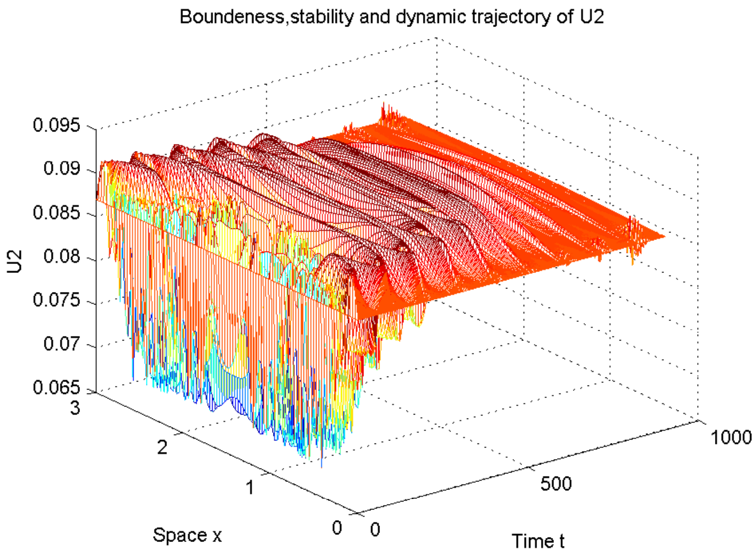

Figure 11. Computer simulation of dynamics of the investment demand U_2.

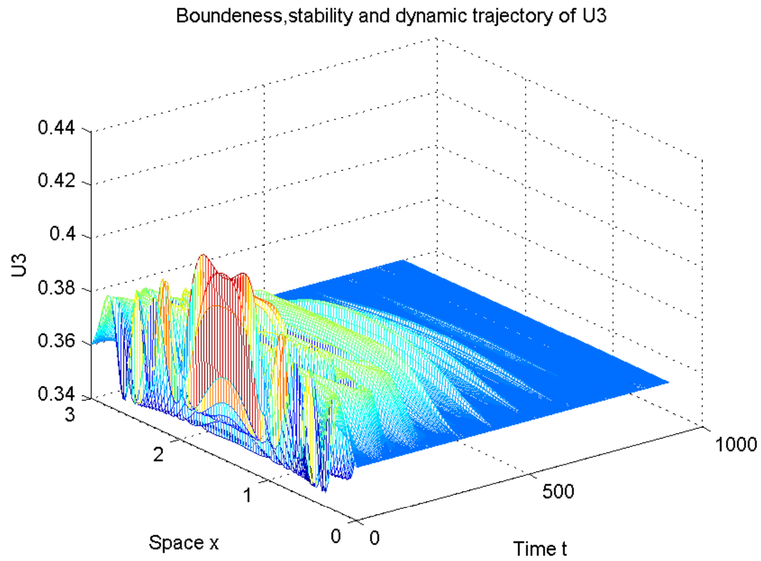

Figure 12. Computer simulation of dynamics of the price exponent U_3.

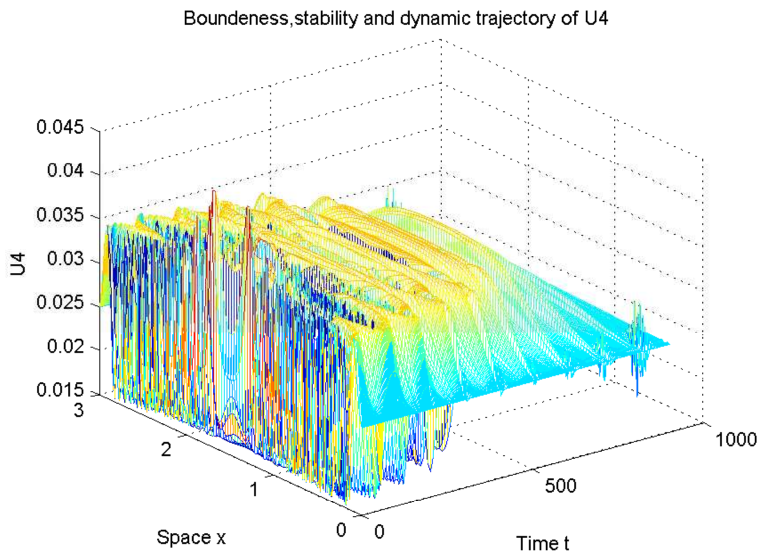

Figure 13. Computer simulation of dynamics of the average profit margin U_4.

Finally, we consider the mean square stability in the case of $p = 2$:

Example 3. *Let* $0.0001 \leqslant U_1 \leqslant 3$, $0.48 \leqslant U_2 \leqslant 5$, $0.5 \leqslant U_3 \leqslant 5$, $0.015 \leqslant U_4 \leqslant 5$. *In addition, set* $c_0 = 10{,}000$, $D = \mathrm{diag}(d_1, d_2, d_3, d_4) = \mathrm{diag}(2.5, 2.3, 2.2, 0.03)$ *, then* $M_1 = 3; M_2 = M_3 = M_4 = 5$, *and the condition (A2) is satisfied. Let* $\Xi = [0, \pi]$, *then* $\|e^{t\Delta}\| \leqslant e^{-\pi^2 t}$ *with* $\gamma = \pi^2$ *and* $M = 1$. *Set* $(a, b, c, d, k) = (0.1, -10, -10, 0.01, -10)$, *then (3.1) holds. Now, it shows by Theorem 1, the considered system (4) owns the PSS* $u_*(x)$. *Let* $p = 2$, $h_{k1} \equiv 1.000001$, $\mu = 2$, *then direct calculation leads to the following data:*

$$\omega_1 = 5^{p-1}\left[(|a|^p + 2)(\frac{M}{d_1\gamma})^p + 2^{p-1}(\frac{M}{d_1\gamma})^p\left(M_2^p + M_1^p\right) + M^{2p}(\sup_k |h_{k1} - 1|)^p\left(1 + \frac{1}{d_1\mu\gamma}\right)^p\right] = 0.5750, \quad (36)$$

$$\omega_2 = 3^{p-1}\left[(\frac{M|b|}{d_2\gamma})^p + (\frac{2M_1M}{d_2\gamma})^p + M^{2p}(\sup_k |h_{k2} - 1|)^p\left(1 + \frac{1}{d_2\mu\gamma}\right)^p\right] = 0.7919, \quad (37)$$

$$\omega_3 = 4^{p-1}\left[(1 + |c|^p)(\frac{M}{d_3\gamma})^p + M^{2p}(\sup_k |h_{k3} - 1|)^p\left(1 + \frac{1}{d_3\mu\gamma}\right)^p\right] = 0.8570, \quad (38)$$

$$\omega_4 = 3^{p-1}\left[(\frac{M|k|}{d_4\gamma})^p + 2^{p-1}(\frac{M|d|}{d_4\gamma})^p\left(M_2^p + M_1^p\right) + M^{2p}(\sup_k |h_{k4} - 1|)^p\left(1 + \frac{1}{d_4\mu\gamma}\right)^p\right] = 0.8533, \quad (39)$$

Now we have the conclusion: $\omega = 0.8570 \in (0, 1)$. Therefore, Theorem 2 yields that the positive steady state solution $U_*(x)$ is GES in the mean square ($p = 2$) (see Figures 14–17).

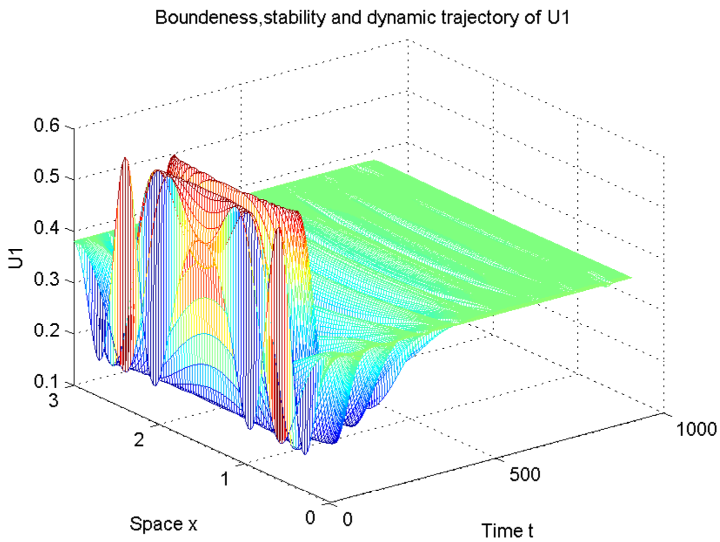

Figure 14. Computer simulation of dynamics of the interest rate U_1.

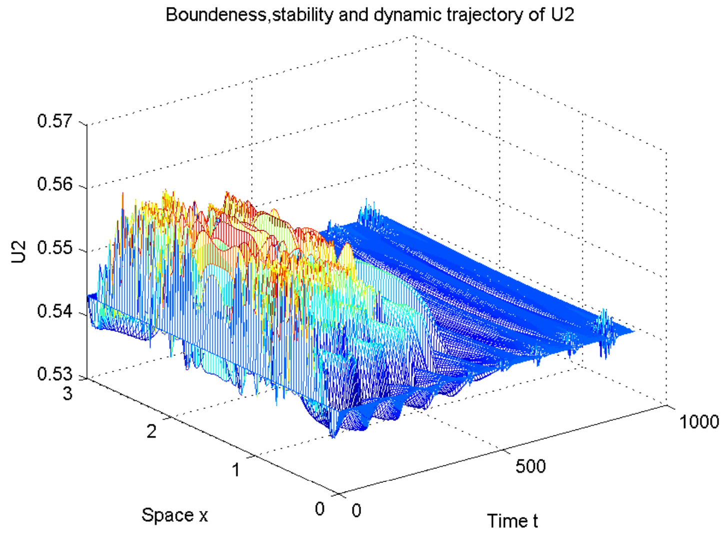

Figure 15. Computer simulation of dynamics of the investment demand U_2.

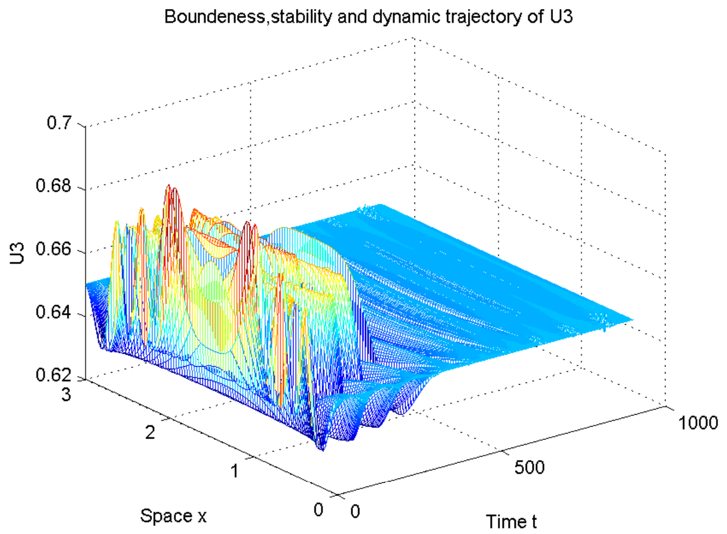

Figure 16. Computer simulation of dynamics of the price exponent U_3.

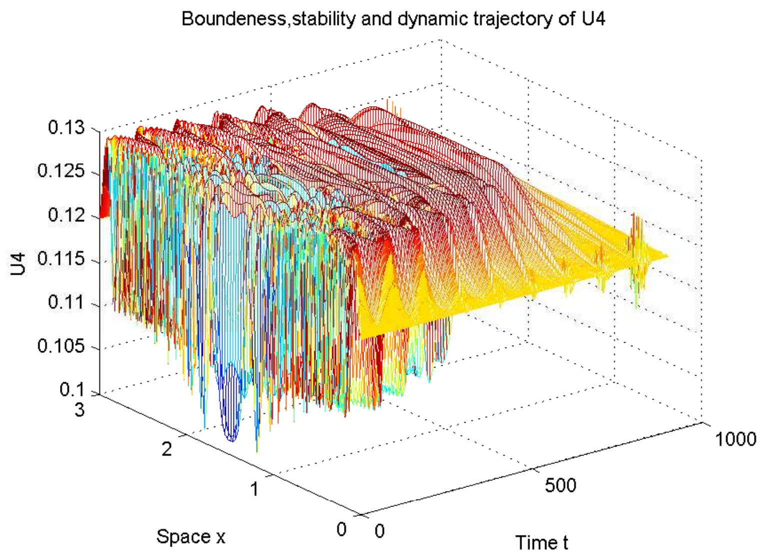

Figure 17. Computer simulation of dynamics of the average profit margin U_4.

5. Conclusions

Different from much of the previous literature, this paper has studied the stabilization of PSS in hyper-chaotic financial system. In fact, the financial system is stable at positive interest rates and other economic indicators, which is in line with the national conditions of most countries while many previous only involved in positive interest rates. In addition, numerical examples show that the new theorem has a larger allowable range of pulses and does not limit that the pulses must be less than 1, by which the conservatism is reduced. Particularly, in this article, Examples 1 and 2 illuminate that all economic indicators are bigger than 0.01, which means newly derived criteria really have advantage over those of previous research papers. Indeed, in [5,6,9], all the stable equilibrium solutions are not positive, because only the interest rate of equilibrium solutions is positive, but other economic indicators are not all positive. Many important economic indicators, such as investment demand, price index, and so on, are actually related to the regional economy. So we choose reaction-diffusion model in this paper. Of course, we may consider a discrete network of the system (1) in designing our next paper.

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Appendix A

For convenience, some marks and symbols are listed below:

- ◇ Denote by Ξ a bounded domain in $R^n (n \leqslant 2)$ with smooth boundary $\partial\Xi$;
- ◇ Denote by ν the external normal direction of $\partial\Xi$;
- ◇ $\partial_\nu U = \frac{\partial U}{\partial \nu}$;
- ◇ $\partial_t U = \partial U \big/ \partial t$;
- ◇ Denote by $\|\varphi\|_H = \sqrt{\int_\Omega |\nabla\varphi(x)|^2 dx}$ the norm of the Sobolev space $H_0^1(\Xi)$;
- ◇ Denote by $\Delta = \sum\limits_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}$ Laplace operator, and by $e^{t\Delta}$ the Laplacian semigroup ;
- ◇ $\overline{\Xi} = \Xi \bigcup \partial\Xi$

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