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Stability of Quartic Functional Equation in Modular Spaces via Hyers and Fixed-Point Methods

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Abstract: In this work, we introduce a new type of generalised quartic functional equation and obtain the general solution. We then investigate the stability results by using the Hyers method in modular space for quartic functional equations without using the Fatou property, without using the Δ_b-condition and without using both the Δ_b-condition and the Fatou property. Moreover, we investigate the stability results for this functional equation with the help of a fixed-point technique involving the idea of the Fatou property in modular spaces. Furthermore, a suitable counter example is also demonstrated to prove the non-stability of a singular case.

Keywords: fixed-point method; quartic functional equation; Fatou property; Hyers-Ulam stability; Δ_b-condition

MSC: 39B52; 39B72; 47H09

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1. Introduction

Functional equations play a crucial role in the study of stability problems in several frameworks. Ulam was the first who questioned the stability of group homomorphisms and this opened the way to work on stability problems (see [1]). Using Banach spaces, Hyers [2] solved this stability problem by considering Cauchy's functional equation. Hyers' work was expanded upon by Aoki [3] by assuming an unbounded Cauchy difference. Rassias [4] presented work on additive mapping and these kinds of results are further presented by Găvruţa [5].

Nakmahachalasint [6], in 2007, provided the general answer and Hyers–Ulam–Rassias (H-U-R) stability of finite variable functional equations (see also Khodaei and Rassias [7]). Certain stability problems around additive functional equations were presented by Najati and Moghimi [8], Kenary [9], Gordji [10] and the references therein.

The concept of generalised Hyers–Ulam stability derives from historical contexts and this problem is found for different kinds of functional equations (FE). The functional equation

$$\phi(v_1 + v_2) + \phi(v_1 - v_2) = 2\phi(v_1) + 2\phi(v_2), \tag{1}$$

is connected to a biadditive symmetric function (see [11,12]). Each equation is naturally referred to as a quadratic FE.Any solution of Equation (1) is a quadratic function. A function

$\phi : E' \to E'$ (E': real vector space) is said to be quadratic if there is a unique symmetric biadditive function T satisfying $\phi(u) = T(u,u)$ for all u (see [11,12]).

The following functional equation was first presented by Jun and H. M. Kim [13]:

$$\phi(2u+v) + \phi(2u-v) = 2\phi(u+v) + 2\phi(u-v) + 12\phi(u), \tag{2}$$

which differs from Equation (1) in various ways. It is clear that the function $\phi(v) = cv^3$ is a solution to Equation (2). As a consequence, it is natural to say that Equation (2) is a cubic FE and so every solution of Equation (2) is a cubic function. In [14], Lee et al. presented the quartic FE as:

$$\phi(2u+v) + \phi(2u-v) = 4[\phi(u+v) + \phi(u-v)] + 24\phi(u) - 6\phi(v), \tag{3}$$

and found its solution and demonstrated the H-U-R stability. It is simple to demonstrate that $\phi(v) = cv^4$ satisfies Equation (3) so this equality is called quartic FE, and its solution is called quartic mapping (QM). Except for direct approaches, the fixed-point method is the most often used method for establishing the stability of FEs (see [15–17]). In [18], the authors proposed a generalised quartic FE and investigated Hyers–Ulam stability in modular spaces using a fixed-point method as well as the Fatou property. Many research papers on different generalisations and the generalised H-U stability's implications for various functional equations have been recently published (see [19–25]).

To obtain our results, we define the quartic FE by

$$\begin{aligned} 4\phi(v_1+v_2+v_3+v_4) + \sum_{i=1;i\neq j}^{4} \phi\left(-v_i + \sum_{1\le j\le 4} v_j\right) &= 4 \sum_{1\le i<j<k\le 4} \phi(v_i+v_j+v_k) \\ -2\left[\sum_{1\le i<j\le 4} \phi(v_i+v_j) - \phi(v_i-v_j)\right] &- \sum_{i=1}^{4}\phi(3v_i) + 77\sum_{i=1}^{4}\phi(v_i). \end{aligned} \tag{4}$$

We investigate certain stability results of the above quartic FE which will be based on Hyers and fixed-point methods involving the idea of the Fatou property and Δ_b-condition in the framework of modular spaces. Here, we consider the difference cases to obtain our results (i) with only the Fatou property, (ii) with only the Δ_b-condition, and (iii) without the Fatou property and the Δ_b-condition.

2. Preliminary

Nakano [26] conducted research on modular and modular spaces as generalisations of normed spaces. Many notable mathematicians [27–31] have worked on it intensively since the 1950s. In [30,32,33], interpolation theory and Orlicz spaces are two examples of uses for modular and modular spaces.

We begin by considering some fundamentally important concepts. Consider E to be a linear space over $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$). We call a functional $\rho : E \to [0,\infty)$ modular provided that for all $u, v \in E$,

(a) $\rho(u) = 0$ if and only if $u = 0$.
(b) $\rho(\beta u) = \rho(u)$ for all scalars β with $|\beta| = 1$.
(c) $\rho(\beta u + \gamma v) \le \rho(u) + \rho(v)$ for all scalars $\beta, \gamma \ge 0$ with $\beta + \gamma = 1$.
If the inequality in (c) is replaced by
(c′) $\rho(\beta u + \gamma v) \le \beta\rho(u) + \gamma\rho(v)$, then ρ is thus said to be convex modular.

If $\rho(\beta u) \le \beta\rho(u)$, then ρ is semi-convex modular. Clearly, every semi-convex modular is convex.

Note that ρ is the following vector space which defined by a modular ρ:

$$E_\rho := \{u \in E \mid \rho(\theta u) \to 0 \text{ as } \theta \to 0\},$$

and E_ρ is also known as a modular space.

Let E_ρ be a modular space and $\{u_n\} \in E_\rho$. One has

(1) If $\rho(u_n - u) \to 0$ as $n \to \infty$, $\{u_n\}$ is ρ-convergent to $u \in E_\rho$ and represented by $u_n \xrightarrow{\rho} u$.
(2) If for every $\varsigma > 0$ such that $\rho(u_n - u_m) < \varsigma$ as $m, n \to \infty$, then $\{u_n\}$ is ρ-Cauchy.
(3) If every ρ-Cauchy sequence is ρ-convergent in S, the subset $S \subseteq E_\rho$ is ρ-complete.

The modular ρ is said to have the Fatou property if and only if $\rho(u) \leq \lim_{n\to\infty} \inf \rho(u_n)$ when the sequence $\{u_n\}$ in modular space E_ρ is ρ-convergent to u.

Definition 1. *Let $b \geq 3$ be an integer. Then, ρ is said to satisfy the Δ_b-condition if there is $k_b > 0$ such that*

$$\rho(bx) \leq k_b \rho(x), \qquad \forall x \in X_\rho.$$

In this case, k_b is a Δ_b-constant related to Δ_b-condition.

Remark 1. *Consider ρ is a semi-convex which satisfies the Δ_b-condition with $k_b > 0$. If $k_b < b$, then*

$$\rho(u) \leq k_b \rho\left(\frac{u}{b}\right) \leq \frac{k_b}{b} \rho(u),$$

which implies $\rho = 0$. As a consequence, if ρ is semi-convex modular, we have the Δ_b-constant $k_b \geq b$.

Definition 2 ([34])**.** *Suppose the sequence $\{v_n\}$ in a modular space V_ρ. Then, we say that*

(D1) $v_n \xrightarrow{\rho} v$ if $\rho(v_n - v) \to 0$ $(n \to \infty)$.
(D2) $\{v_n\}$ is a ρ-Cauchy provided that $\rho(v_l - v_n) \to 0$ $(l, n \to \infty)$.
(D3) $A \subseteq V_\rho$ is ρ-complete iff every ρ-Cauchy sequence is ρ-convergent in the set A.

Suppose $A (\neq \emptyset) \subseteq V_\rho$. Then, a mapping $J : A \to A$ is a quasicontraction if $k < 1$ such that

$$\rho(Jl - Jm) \leq k \max\{\rho(l-m), \rho(l - Jm), \rho(m - Jl), \rho(l - Jl), \rho(m - Jm)\},$$

for any $l, m \in A$. The J orbit around a point u is

$$O(J) := \{u, Ju, J^2u, \cdots\}.$$

Then, the quantity

$$\Upsilon_\rho(J) := \sup\{\rho(p-q) | p, q \in O(J)\},$$

is known as the orbital diameter of J at u. If $\Upsilon_\rho(J) < \infty$ holds, J is said to has a bounded orbit at u (see [34]).

Proposition 1 ([35])**.** *In modular spaces,*

(1) If $u_n \xrightarrow{\rho} u$ and ϵ is a constant vector, then $u_n + \epsilon \xrightarrow{\rho} u + \epsilon$, and
(2) If $u_n \xrightarrow{\rho} u$ and $v_n \xrightarrow{\rho} v$, then $\beta u_n + \gamma v_n \xrightarrow{\rho} \beta u + \gamma v$, where $\beta + \gamma \leq 1$ and $\beta, \gamma \geq 0$.

It should be noted that if α is chosen from the equivalent scalar field with $|\alpha| > 1$ in modular spaces, the convergence of a sequence $\{u_n\}$ to u does not mean that $\{\alpha u_n\}$ converges to αu. Many mathematicians established additional criteria on modular spaces in order for the multiples of the convergent sequence $\{u_n\}$ in the modular spaces to naturally converge.

The modular ρ has the Fatou property if $\rho(v) \leq \lim_{m\to\infty} \inf \rho(v_m)$ whenever $\{v_m\} \xrightarrow{\rho} v$. Let $b \in \mathbb{N} - \{1\}$. A modular function satisfies Δ_b-condition if there is $k > 0$ such that

$$\rho(bv) \leq k\rho(v), \qquad \forall v \in V_\rho.$$

3. Main Results

3.1. Solution of the New Kind of Quartic FE

Theorem 1. *Let E and F be two vector spaces. If an even mapping $\phi : E \to F$ satisfies Equation (4) for all $v_1, v_2, v_3, v_4 \in E$, then ϕ is quartic.*

Proof. Suppose $\phi : E \to F$ is even. Then, ϕ satisfies

$$\phi(-v) = \phi(v),$$

for all $v \in E$. Letting $v_1 = v_2 = v_3 = v_4 = 0$ in Equation (4), we obtain $\phi(0) = 0$. Setting $v_1 = v$ and $v_2 = v_3 = v_4 = 0$ in Equation (4), we have

$$\phi(3v) = 3^4\phi(v), \qquad \forall v \in E. \tag{5}$$

It follows, by replacing v with $3v$ in Equation (5), that

$$\phi(3^2v) = 3^{4(2)}\phi(v), \qquad \forall v \in E. \tag{6}$$

Now, we obtain, by replacing v with $3v$ in Equation (6), that

$$\phi(3^3v) = 3^{4(3)}\phi(v), \qquad \forall v \in E.$$

In general, for any $n \in \mathbb{Z}_+$ (the set of positive integers), we have

$$\phi(3^nv) = 3^{4(n)}\phi(v), \qquad \forall v \in E.$$

Thus, the function ϕ is even and has a solution of quartic FE. Therefore, ϕ is quartic. Finally, by replacing (v_1, v_2, v_3, v_4) by $(u, u, v, 0)$ in Equation (4), we obtain the Equation (3). □

3.2. Stability of Quartic FE: Hyers Method

Consider a modular ρ as semi-convex. The Hyers–Ulam stability of Equation (4) in modular spaces is an important theorem in the absence of the Fatou condition.

For notational handiness, we define a mapping $\phi : E \to F_\rho$ (E: linear space; F_ρ: ρ-complete semi-convex modular space) by

$$\begin{aligned}
\Phi(v_1, v_2, v_3, v_4) \quad = \quad & 4\phi\left(\sum_{j=1}^{4} v_j\right) + \sum_{i=1; i\neq j}^{4} \phi\left(-v_i + \sum_{1\leq j\leq 4} v_j\right) \\
& -4 \sum_{1\leq i<j<k\leq 4} \phi(v_i + v_j + v_k) + \sum_{i=1}^{4} \phi(3v_i) \\
& +2\left[\sum_{1\leq i<j\leq 4} \phi(v_i + v_j) - \phi(v_i - v_j)\right] - 77\sum_{i=1}^{4} \phi(v_i),
\end{aligned}$$

for all $v_1, v_2, v_3, v_4 \in E$.

Theorem 2. *Let $b \geq 3$ be an integer. Suppose F_ρ satisfies the Δ_b-condition. If a mapping $\psi : E^4 \to [0, \infty)$ exists for which a mapping $\phi : E \to F_\rho$ satisfies all $v_1, v_2, v_3, v_4 \in E$,*

$$\rho(\Phi(v_1, v_2, v_3, v_4) \leq \psi(v_1, v_2, v_3, v_4), \tag{7}$$

$$\lim_{l\to\infty} k_b^{4l}\psi\Big(\frac{v_1}{b^l},\frac{v_2}{b^l},\frac{v_3}{b^l},\frac{v_4}{b^l}\Big)=0 \quad and \quad \sum_{j=1}^{\infty}\left(\frac{k_b^5}{b}\right)^j\psi\left(\frac{v}{b^j},0,0,0\right)<\infty,$$

then there is an unique QM $Q:E\to F_\rho$, defined by

$$Q(v)=\lim_{l\to\infty} b^{4l}\phi\Big(\frac{v}{b^l}\Big),$$

and

$$\rho(\phi(v)-Q(v))\le\frac{1}{bk_b^3}\sum_{j=1}^{\infty}\left(\frac{k_b^5}{b}\right)^j\psi\left(\frac{v}{b^j},0,0,0\right), \tag{8}$$

for all $v\in E$.

Proof. Note that $\phi(0)=0$ since $\psi(0,0,0,0)=0$ by the convergence of

$$\sum_{j=1}^{\infty}\left(\frac{k_b^5}{b}\right)^j\psi(0,0,0,0)<\infty.$$

We set $v_1=v$ and $v_2=v_3=v_4=0$ in inequality (7) to obtain

$$\rho\Big(\phi(bv)-b^4\phi(v)\Big)\le\psi(v,0,0,0),\qquad \forall v\in E.$$

Supposing the Δ_b-condition of ρ and $\sum_{j=1}^{l}\frac{1}{b^j}\le 1$, one can prove the equality

$$\begin{aligned}\rho\Big(\phi(v)-b^{4l}\phi\Big(\frac{v}{b^l}\Big)\Big) &= \rho\left(\sum_{j=1}^{l}\frac{1}{b^j}\left(b^{5j-4}\phi\Big(\frac{v}{b^{j-1}}\Big)-b^{5j}\phi\Big(\frac{v}{b^j}\Big)\right)\right)\\ &\le \frac{1}{k_b^4}\sum_{j=1}^{l}\left(\frac{k_b^5}{b}\right)^j\psi\left(\frac{v}{b^j},0,0,0\right),\end{aligned} \tag{9}$$

Now, replacing v by $b^{-p}v$ in Equation (9), we have

$$\begin{aligned}\rho\left(b^{4p}\phi\Big(\frac{v}{b^p}\Big)-b^{4(l+p)}\phi\Big(\frac{v}{b^{l+p}}\Big)\right) &\le k_b^{4p}\rho\left(\phi\Big(\frac{v}{b^p}\Big)-b^{4l}\phi\Big(\frac{v}{b^{l+p}}\Big)\right)\\ &\le k_b^{4p-4}\sum_{j=1}^{l}\left(\frac{k_b^5}{b}\right)^j\psi\left(\frac{v}{b^{j+p}},0,0,0\right)\\ &\le \frac{b^p}{k_b^{p+4}}\sum_{j=p+1}^{l+p}\left(\frac{k_b^5}{b}\right)^j\psi\left(\frac{v}{b^j},0,0,0\right),\end{aligned}$$

for all $v\in E$, which $\to 0$ $(p\to\infty)$, because $\frac{b}{k_b}\le 1$ and the inequality (7) converges.

As a result, the sequence $\{b^{4l}\phi\big(\frac{v}{b^l}\big)\}$ is ρ-Cauchy for all $v\in E$ and as a result, it is ρ-convergent in F_ρ since F_ρ is a ρ-complete. So, we can define $Q:E\to F_\rho$ as

$$Q(v):=\rho-\lim_{l\to\infty} b^{4l}\phi\Big(\frac{v}{b^l}\Big),\quad \text{i.e.,}\quad \lim_{l\to\infty}\rho\Big(b^{4l}\phi\Big(\frac{v}{b^l}\Big)-Q(v)\Big)=0,$$

for all $v\in E$. So, even without utilising the Fatou property, the Δ_b-condition shows that the inequality

$$\begin{aligned}
\rho(\phi(v)-Q(v)) &\leq \frac{1}{b}\rho\Big(b\phi(v)-b\cdot b^{4l}\phi\Big(\frac{v}{b^l}\Big)\Big)+\frac{1}{b}\rho\Big(b\cdot b^{4l}\phi\Big(\frac{v}{b^l}\Big)-b\cdot Q(v)\Big)\\
&\leq \frac{k_b}{b}\rho\Big(\phi(v)-b^{4l}\phi\Big(\frac{v}{b^l}\Big)\Big)+\frac{k_b}{b}\rho\Big(b^{4l}\phi\Big(\frac{v}{b^l}\Big)-Q(v)\Big)\\
&\leq \frac{1}{bk_b^3}\sum_{j=1}^{l}\left(\frac{k_b^5}{3}\right)^j\psi\left(\frac{v}{b^j},0,0,0\right)+\frac{k_b}{b}\rho\Big(b^{4l}\phi\Big(\frac{v}{b^l}\Big)-Q(v)\Big),
\end{aligned}$$

holds for an integer $l>1$ and for all $v\in E$. Taking $l\to\infty$, we have the inequality (8). Replacing (v_1,v_2,v_3,v_4) by $\left(b^{-l}v_1,b^{-l}v_2,b^{-l}v_3,b^{-l}v_4\right)$ in inequality (7), we see that

$$\rho\Big(b^{4l}\Phi\Big(b^{-l}v_1,b^{-l}v_2,b^{-l}v_3,b^{-l}v_4\Big)\Big)\leq k_b^{4l}\psi\left(\frac{v_1}{b^l},\frac{v_2}{b^l},\frac{b_3}{3^l},\frac{v_4}{b^l}\right)\to 0\ (l\to\infty),$$

for all $v_1,v_2,v_3,v_4\in E$. From the semi-convexity of ρ, it follows that

$$\begin{aligned}
&\rho\left(\frac{1}{361}Q(v_1,v_2,v_3,v_4)\right)\\
&\leq \frac{4}{361}\rho\left(Q\left(\sum_{j=1}^{4}v_j\right)-b^{4l}\phi\left(\sum_{j=1}^{4}b^{-l}v_j\right)\right)\\
&\quad+\frac{1}{361}\rho\left(\sum_{i=1;i\neq j}^{4}Q\left(-v_i+\sum_{1\leq j\leq 4}v_j\right)-b^{4l}\sum_{i=1;i\neq j}^{4}\phi\left(-b^{-l}v_i+\sum_{1\leq j\leq 4}b^{-l}v_j\right)\right)\\
&\quad+\frac{4}{361}\rho\left(\sum_{1\leq i<j<k\leq 4}Q(v_i+v_j+v_k)-b^{4l}\sum_{1\leq i<j<k\leq 4}\phi\Big(b^{-l}(v_i+v_j+v_k)\Big)\right)\\
&\quad+\frac{1}{361}\rho\left(\sum_{i=1}^{4}Q(3v_i)-b^{4l}\sum_{i=1}^{4}\phi\Big(b^{-l}(3v_i)\Big)\right)\\
&\quad+\frac{2}{361}\rho\left(\left[\sum_{1\leq i<j\leq 4}Q(v_i+v_j)-Q(v_i-v_j)\right]\right.\\
&\quad\left.-b^{4l}\left[\sum_{1\leq i<j\leq 4}\phi\Big(b^{-l}(v_i+v_j)\Big)-\phi\Big(b^{-l}(v_i-v_j)\Big)\right]\right)\\
&\quad+\frac{77}{361}\rho\left(\sum_{i=1}^{4}Q(v_i)-b^{4l}\sum_{i=1}^{4}\phi\Big(b^{-l}v_i\Big)\right)+\frac{1}{361}\rho\Big(b^{4l}\Phi\Big(b^{-l}v_1,b^{-l}v_2,b^{-l}v_3,b^{-l}v_4\Big)\Big),
\end{aligned}$$

for all $v_1,v_2,v_3,v_4\in E$ and all non-negative integers $l>1$. Taking the limit as $l\to\infty$, we can see that Q is quartic.

We suppose a QM $Q':E\to F_\rho$ to demonstrate the uniqueness of Q. The function Q' satisfies the inequality

$$\rho\Big(\phi(v)-Q'(v)\Big)\leq\frac{1}{bk_b^3}\sum_{j=1}^{\infty}\left(\frac{k_b^5}{b}\right)^j\psi\left(\frac{v}{b^j},0,0,0\right),$$

for all $v\in E$. Then, we see from the inequality $Q(b^{-l}v)=b^{-4l}Q(v)$ and $Q'(b^{-l}v)=b^{-4l}Q'(v)$ that

$$\begin{aligned}
\rho\Big(Q(v)-Q'(v)\Big) &\leq \frac{1}{b}\rho\Big(b\cdot b^{4l}Q\Big(\frac{v}{b^l}\Big)-b\cdot b^{4l}\phi\Big(\frac{v}{b^l}\Big)\Big)\\
&\quad+\frac{1}{b}\rho\Big(b\cdot b^{4l}\phi\Big(\frac{v}{b^l}\Big)-b\cdot b^{4l}Q'\Big(\frac{v}{b^l}\Big)\Big)\\
&\leq \frac{k_b^{4l+1}}{b}\rho\Big(Q\Big(\frac{v}{b^l}\Big)-\phi\Big(\frac{v}{b^l}\Big)\Big)+\frac{k_b^{4l+1}}{b}\rho\Big(\phi\Big(\frac{v}{b^l}\Big)-Q'\Big(\frac{v}{b^l}\Big)\Big)
\end{aligned}$$

$$\leq \frac{2k_b^{4l-2}}{b^2}\sum_{j=1}^{\infty}\left(\frac{k_b^5}{b}\right)^j\psi\left(\frac{v}{b^{l+j}},0,0,0\right)$$
$$= \frac{2\cdot b^{l-2}}{k_b^{l+2}}\sum_{j=l+1}^{\infty}\left(\frac{k_b^5}{b}\right)^j\psi\left(\frac{v}{b^j},0,0,0\right),$$

for all $v \in E$. Taking $l \to \infty$, we finally find that Q is unique, which completes the proof. □

Corollary 1. *Let $b \geq 3$ be an integer. Suppose that a normed space E with $\|\cdot\|$ and F_ρ satisfies Δ_b-condition. If a mapping $\phi : E \to F_\rho$ such that*

$$\rho(\Phi(v_1, v_2, v_3, v_4)) \leq \lambda\left(\sum_{j=1}^{4}\|v_j\|^\alpha\right), \qquad \forall v_1, v_2, v_3, v_4 \in E,$$

then there is an unique QM $Q : E \to F_\rho$ satisfies

$$\rho(\phi(v) - Q(v)) \leq \frac{\lambda k_b^2}{b\left(b^{\alpha+1} - k_b^5\right)}\|v\|^\alpha, \qquad \forall v \in E,$$

where $\alpha > \log_b \frac{k_b^5}{b}$ and $\lambda > 0$.

Corollary 2. *Let $b \geq 3$ be an integer. Suppose that a normed space E with $\|\cdot\|$ and F_ρ satisfies Δ_b-condition. For any $\lambda > 0$ and $4\alpha > \log_b \frac{k_b^5}{b}$ are given real numbers, if a mapping $\phi : E \to F_\rho$ such that*

$$\rho(\Phi(v_1, v_2, v_3, v_4)) \leq \lambda\left(\sum_{j=1}^{4}\|v_j\|^{4\alpha} + \prod_{i=1}^{4}\|v_i\|^p\right), \qquad \forall v_1, v_2, v_3, v_4 \in E,$$

then there is an unique QM $Q : E \to F_\rho$ satisfying

$$\rho(\phi(v) - Q(v)) \leq \frac{\lambda k_b^2}{b\left(b^{4\alpha+1} - k_b^5\right)}\|v\|^{4\alpha},$$

for all $v \in E$.

An alternative stability theorem for Equation (4) in modular spaces will be proved without the Δ_b-condition, given below.

Theorem 3. *Let $b \geq 3$ be an integer. Let F_ρ satisfy the Fatou property. If a mapping $\phi : E \to F_\rho$ satisfies the inequality (7) and a mapping $\psi : E^4 \to [0, \infty)$ such that*

$$\lim_{l\to\infty}\frac{\psi\left(b^l v_1, b^l v_2, b^l v_3, b^l v_4\right)}{b^{4l}} = 0, \quad \text{and} \quad \sum_{j=0}^{\infty}\frac{\psi\left(b^j v, 0, 0, 0\right)}{b^{4j}} < \infty, \qquad \forall v, v_1, v_2, v_3, v_4 \in E,$$

then there is an unique QM $Q : E \to F_\rho$ having

$$\rho(\phi(v) - Q(v)) \leq \frac{1}{b^4}\sum_{j=0}^{\infty}\frac{\psi\left(b^j v, 0, 0, 0\right)}{b^{4j}}, \qquad \forall v \in E. \tag{10}$$

Proof. By replacing $v_1 = v$ and $v_2 = v_3 = v_4 = 0$ in Equation (7), we obtain

$$\rho\left(\phi(bv) - b^4\phi(v)\right) \leq \psi(v, 0, 0, 0), \qquad \forall v \in E.$$

Without using Δ_b-condition, the above inequality becomes

$$\begin{aligned}\rho\left(\frac{\phi(b^l v)}{b^{4l}}-\phi(v)\right) &= \rho\left(\sum_{j=0}^{l-1}\frac{1}{b^{4(j+1)}}\left(b^4\phi(b^j v)-\phi(b^{j+1}v)\right)\right)\\ &\leq \sum_{j=0}^{l-1}\frac{1}{b^{4(j+1)}}\rho\left(b^4\phi(b^j v)-\phi(b^{j+1}v)\right)\\ &\leq \frac{1}{b^4}\sum_{j=0}^{l-1}\frac{1}{b^{4j}}\psi\left(b^j v,0,0,0\right),\end{aligned}$$

for all $v \in E$ and for all integers $l > 1$. This yields

$$\begin{aligned}\rho\left(\frac{\phi(b^l v)}{b^{4l}}-\frac{\phi(b^p v)}{b^{4p}}\right) &= \frac{1}{b^{4p}}\rho\left(\frac{\phi(b^{l-p}\cdot b^p v)}{b^{4(l-p)}}-\phi(b^p v)\right)\\ &\leq \frac{1}{b^{4p}}\sum_{j=0}^{l-p-1}\frac{1}{b^{4(j+1)}}\psi\left(b^j\cdot b^p v,0,0,0\right)\\ &\leq \frac{1}{b^4}\sum_{j=p}^{l-1}\frac{1}{b^{4j}}\psi\left(b^j v,0,0,0\right),\end{aligned}$$

for all $v \in E$ and all $l, p \in \mathbb{N}$ with $l > p$. Thus, we see that the sequence $\{\frac{\phi(b^l v)}{b^{4l}}\}$ is a ρ-Cauchy on F_ρ. Since F_ρ is ρ-complete, there exists ρ-limit solution $Q : E \to F_\rho$ defined by

$$\rho-\lim_{l\to\infty}\frac{\phi(b^l v)}{b^{4l}} := Q(v),$$
$$\text{i.e., } \lim_{l\to\infty}\rho\left(\frac{\phi(b^l v)}{b^{4l}}-Q(v)\right)=0,$$

for all $v \in E$. Then, based on the Fatou property, it follows that the inequality

$$\begin{aligned}\rho(Q(v)-\phi(v)) &\leq \lim_{l\to\infty}\inf\rho\left(\frac{\phi(b^l v)}{b^{4l}}-\phi(v)\right)\\ &\leq \frac{1}{b^4}\sum_{j=0}^{\infty}\frac{1}{b^{4j}}\psi\left(b^j v,0,0,0\right), \qquad \forall v\in E.\end{aligned}$$

Now, we assert that Q satisfies the quartic FE. It should be noted that:

$$\rho\left(\frac{1}{b^{4l}}\Phi\left(b^l v_1, b^l v_2, b^l v_3, b^l v_4\right)\right) \leq \frac{1}{b^{4l}}\psi\left(b^l v_1, b^l v_2, b^l v_3, b^l v_4\right),$$

for all $v_1, v_2, v_3, v_4 \in E$, and all $l \in \mathbb{N}$. As a result of the semi-convexity of ρ, we can see that

$$\begin{aligned}&\rho\left(\frac{1}{361}Q(v_1,v_2,v_3,v_4)\right)\\ &\leq \frac{4}{361}\rho\left(Q\left(\sum_{j=1}v_j\right)-b^{-4l}\phi\left(\sum_{j=1}b^l v_j\right)\right)\\ &\quad+\frac{1}{361}\rho\left(\sum_{i=1;i\neq j}^{4}Q\left(-v_i+\sum_{1\leq j\leq 4}v_j\right)-b^{-4l}\sum_{i=1;i\neq j}^{4}\phi\left(-b^l v_i+\sum_{1\leq j\leq 4}b^l v_j\right)\right)\\ &\quad+\frac{4}{361}\rho\left(\sum_{1\leq i<j<k\leq 4}Q(v_i+v_j+v_k)-b^{-4l}\sum_{1\leq i<j<k\leq 4}\phi\left(b^l(v_i+v_j+v_k)\right)\right)\end{aligned}$$

$$+\frac{1}{361}\rho\left(\sum_{i=1}^{4}Q(3v_i)-b^{-4l}\sum_{i=1}^{4}\phi\Big(b^l(3v_i)\Big)\right)$$
$$+\frac{2}{361}\rho\left(\left[\sum_{1\le i<j\le 4}Q(v_i+v_j)-Q(v_i-v_j)\right]\right.$$
$$\left.-b^{-4l}\left[\sum_{1\le i<j\le 4}\phi\Big(b^l(v_i+v_j)\Big)-\phi\Big(b^l(v_i-v_j)\Big)\right]\right)$$
$$+\frac{77}{361}\rho\left(\sum_{i=1}^{4}Q(v_i)-b^{-4l}\sum_{i=1}^{4}\phi\Big(b^lv_i\Big)\right)+\frac{1}{361}\rho\Big(b^{-4l}\Phi\Big(b^lv_1,b^lv_2,b^lv_3,b^lv_4\Big)\Big),$$

holds for all $v_1,v_2,v_3,v_4\in E$, and then taking $l\to\infty$, we obtain $\rho\Big(\frac{1}{361}Q(v_1,v_2,v_3,v_4)\Big)=0$. As a result, Q must be quartic.

To demonstrate that the function Q is unique, we consider that $Q^{'}:E\to F_\rho$ is an another quartic function which satisfies the inequality (10). As Q and $Q^{'}$ are quartic, as evidenced by the previous equality, $Q(b^lv)=b^{4l}Q(v)$ and $Q^{'}(b^lv)=b^{4l}Q^{'}(v)$, so that

$$\begin{aligned}
\rho\left(\frac{1}{2}Q(v)-\frac{1}{2}Q^{'}(v)\right) &\le \frac{1}{2}\rho\left(\frac{Q(b^lv)}{b^{4l}}-\frac{\phi(b^lv)}{b^{4l}}\right)+\frac{1}{2}\rho\left(\frac{\phi(b^lv)}{b^{4l}}-\frac{Q^{'}(b^lv)}{b^{4l}}\right)\\
&\le \frac{1}{2}\cdot\frac{1}{b^{4l}}\rho\Big(Q(b^lv)-\phi(b^lv)\Big)+\frac{1}{2}\cdot\frac{1}{b^{4l}}\rho\Big(\phi(b^lv)-Q^{'}(b^lv)\Big)\\
&\le \frac{1}{b^{4l}}\sum_{j=0}^{\infty}\frac{1}{b^{4(j+1)}}\psi\Big(b^j\cdot b^lv,0,0,0\Big)\\
&= \sum_{j=l}^{\infty}\frac{1}{b^{4(j+1)}}\psi\Big(b^jv,0,0,0\Big),
\end{aligned}$$

for all $v\in E$. Taking $l\to\infty$, we conclude that $Q=Q^{'}$. Hence, Q is the only quartic mapping near ϕ that satisfies the inequality (10). □

Corollary 3. *Let $b\ge 3$ be an integer. Suppose that a normed space E with $\|\cdot\|$ and F_ρ satisfy the Fatou property. For any $\lambda>0$ and $\alpha\in(-\infty,4)$ are real numbers, if a mapping $\phi:E\to F_\rho$ such that*

$$\rho(\Phi(v_1,v_2,v_3,v_4))\le\lambda\left(\sum_{j=1}^{4}\|v_j\|^\alpha\right),\qquad \forall v_1,v_2,v_3,v_4\in E,$$

then there is an unique QM $Q:E\to F_\rho$ having

$$\rho(\phi(v)-Q(v))\le\frac{\lambda\|v\|^\alpha}{b^4-b^\alpha},$$

for all $v\in E$, where $v\neq 0$ if $\alpha<0$.

Corollary 4. *Let $b\ge 3$ be an integer. Suppose that a normed space E with $\|\cdot\|$ and F_ρ satisfy the Fatou property. For any $\lambda>0$ and $4\alpha\in(-\infty,4)$ are given real numbers, if a mapping $\phi:E\to F_\rho$ such that*

$$\rho(\Phi(v_1,v_2,v_3,v_4))\le\lambda\left(\sum_{j=1}^{4}\|v_j\|^{4\alpha}+\prod_{j=1}^{4}\|v_j\|^\alpha\right),\qquad \forall v_1,v_2,v_3,v_4\in E,$$

then there is an unique QM $Q:E\to F_\rho$ having

$$\rho(\phi(v) - Q(v)) \leq \frac{\lambda \|v\|^{4\alpha}}{b^4 - b^{4\alpha}},$$

for all $v \in E$, where $v \neq 0$ if $\alpha < 0$.

The upcoming proposition is a revised version of modular stability results of Theorem 3 in [36], which does not need the Δ_b-condition of ρ, which is given below.

Proposition 2. *Let F_ρ satisfy the Fatou property. If a mapping $\phi : E \to F_\rho$ satisfy the inequality (7) and a mapping $\psi : E^4 \to [0, \infty)$ such that*

$$\lim_{l\to\infty} \frac{\psi(b^l v_1, b^l v_2, b^l v_3, b^l v_4)}{b^{4l}} = 0, \text{ and } \psi(bv, 0, 0, 0) \leq b^4 L \psi(v, 0, 0, 0), \forall v, v_1, v_2, v_3, v_4 \in E,$$

then there is an unique QM $Q : E \to F_\rho$ satisfying

$$\rho(\phi(v) - Q(v)) \leq \frac{1}{b^4(1-L)} \psi(v, 0, 0, 0), \qquad \forall v \in E.$$

Now, in modular spaces, we present an alternative stability Theorem 2 that does not utilise both the Fatou property and the Δ_b-condition.

Theorem 4. *If a mapping $\phi : E \to F_\rho$ satisfy the inequality (7) and a mapping $\psi : E^4 \to [0, \infty)$ such that*

$$\lim_{l\to\infty} \frac{\psi\left(b^l v_1, b^l v_2, b^l v_3, b^l v_4\right)}{b^{4l}} = 0, \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{\psi\left(b^j v, 0, 0, 0\right)}{b^{4j}} < \infty, \qquad \forall v, v_1, v_2, v_3, v_4 \in E,$$

then there is an unique QM $Q : E \to F_\rho$ having

$$\rho(\phi(v) - Q(v)) \leq \frac{1}{b^4} \sum_{j=0}^{\infty} \frac{\psi\left(b^j v, 0, 0, 0\right)}{b^{4j}}, \tag{11}$$

for all $v \in E$.

Proof. Letting $v_1 = v$ and $v_2 = v_3 = v_4 = 0$ in inequality (7), one has

$$\rho\left(\phi(bv) - b^4\phi(v)\right) \leq \psi(v, 0, 0, 0),$$

and then the semi-convexity of ρ and $\sum_{j=0}^{l-1} \frac{1}{b^{4(j+1)}} \leq 1$ provide us with the result

$$\begin{aligned}
\rho\left(\phi(v) - \frac{\phi(b^l v)}{b^{4l}}\right) &\leq \rho\left(\sum_{j=0}^{l-1}\left(\frac{b^4\phi(b^j v) - \phi(b^{j+1}v)}{b^{4(j+1)}}\right)\right) \\
&\leq \sum_{j=0}^{l-1} \frac{\rho\left(b^4\phi(b^j v) - \phi(b^{j+1}v)\right)}{b^{4(j+1)}} \\
&\leq \frac{1}{b^4} \sum_{j=0}^{l-1} \frac{\psi\left(b^j v, 0, 0, 0\right)}{b^{4j}},
\end{aligned}$$

for all $v \in E$ and all $l > 0$. By the similar argument of the proof of Theorem 3, we have a ρ-Cauchy sequence $\{\frac{\phi(b^l v)}{b^{4l}}\}$ and the limit of function $Q : E \to F_\rho$ defined as

$$\rho - \lim_{l\to\infty} \frac{\phi(b^l v)}{b^{4l}} = Q(v),$$

$$\text{i.e.,}\quad \lim_{l\to\infty}\rho\left(\frac{\phi(b^l v)}{b^{4l}}-Q(v)\right)=0,$$

for all $v \in E$ without employing the Fatou property and the Δ_b-condition. Furthermore, as in the proof of Theorem 2, one may show that Q satisfies Equation (4).

Now, without invoking the Fatou property and the Δ_b-condition, we verify the inequality (11) of ϕ by Q. By utilizing the semi-convexity of ρ and $\sum_{j=0}^{l-1}\frac{1}{b^{4(j+1)}}+\frac{1}{b^4}\le 1$, we obtain

$$\begin{aligned}
\rho(\phi(v)-Q(v)) &\le \rho\left(\sum_{j=0}^{l-1}\left(\frac{b^4\phi(b^j v)-\phi(b^{j+1}v)}{b^{4(j+1)}}\right)+\frac{\phi(b^l v)}{b^{4l}}-\frac{Q(bv)}{b^4}\right)\\
&\le \sum_{j=0}^{l-1}\frac{1}{b^{4(j+1)}}\rho\left(b^4\phi(b^j v)-\phi(b^{j+1}v)\right)+\frac{1}{b^4}\rho\left(\frac{\phi(b^{l-1}\cdot bv)}{b^{4(l-1)}}-Q(bv)\right)\\
&\le \frac{1}{b^4}\sum_{j=0}^{l-1}\frac{1}{b^{4j}}\psi(b^j v,0,0,0)+\frac{1}{b^4}\rho\left(\frac{\phi(b^{l-1}\cdot bv)}{b^{4(l-1)}}-Q(bv)\right),
\end{aligned}$$

for all integer $l > 1$ and for all $v \in E$. We arrive to the conclusion by using $l \to \infty$. □

Corollary 5. *Let $b \ge 3$ be an integer. Suppose that a normed space E with $\|\cdot\|$. Any $\lambda > 0$ and $\alpha \in (-\infty, 4)$ are real numbers if a mapping $\phi : E \to F_\rho$, such that*

$$\rho(\Phi(v_1,v_2,v_3,v_4))\le\lambda\left(\sum_{j=1}^{4}\|v_j\|^\alpha\right),\qquad \forall v_1,v_2,v_3,v_4\in E,$$

then there is an unique QM $Q : E \to F_\rho$ having

$$\rho(\phi(v)-Q(v))\le\frac{\lambda\|v\|^\alpha}{b^4-b^\alpha},\qquad \forall v\in E,$$

where $v \ne 0$ if $\alpha < 0$.

Corollary 6. *Let $b \ge 3$ be an integer. Suppose that a normed space E with $\|\cdot\|$. Any $\lambda > 0$ and $4\alpha \in (-\infty, 4)$ are real numbers, if a mapping $\phi : E \to F_\rho$ such that*

$$\rho(\Phi(v_1,v_2,v_3,v_4))\le\lambda\left(\sum_{j=1}^{4}\|v_j\|^{4\alpha}+\prod_{j=1}^{4}\|v_j\|^\alpha\right),\qquad \forall v_1,v_2,v_3,v_4\in E,$$

then there is an unique QM $Q : E \to F_\rho$ having

$$\rho(\phi(v)-Q(v))\le\frac{\lambda\|v\|^{4\alpha}}{b^4-b^{4\alpha}},\qquad \forall v\in E,$$

where $v \ne 0$ if $\alpha < 0$.

Proposition 3. *Let a mapping $\psi : E^4 \to [0,\infty)$ satisfy*

$$\lim_{l\to\infty}\frac{\psi(b^l v_1,b^l v_2,b^l v_3,b^l v_4)}{b^{4l}}=0,\quad \textit{and}\quad \psi(bv,0,0,0)\le b^4L\psi(v,0,0,0),$$

for all $v, v_1, v_2, v_3, v_4 \in E$ and for some $L \in (0,4)$. If a mapping $\phi : E \to F_\rho$ satisfies Equation (7), then there is an unique QM $Q : E \to F_\rho$ having

$$\rho(\phi(v)-Q(v))\le\frac{1}{b^4(1-L)}\psi(v,0,0,0),\qquad \forall v\in E.$$

3.3. Stability of Quartic FE: Fixed-Point Method

Theorem 5. *Let $b \geq 3$ be an integer and a mapping $\psi : E^4 \to [0, +\infty)$ such that*

$$\lim_{m\to\infty} \frac{1}{b^{4m}} \psi(b^m v_1, b^m v_2, b^m v_3, b^m v_4) = 0,$$

and

$$\psi(bv_1, bv_2, bv_3, bv_4) \leq b^4 L\psi(v_1, v_2, v_3, v_4), \tag{12}$$

for all $v_i \in E$; $i = 1, 2, 3, 4$, with $0 < L < 1$. If an even mapping $\phi : E \to F_\rho$ with $\phi(0) = 0$ satisfies

$$\rho(\Phi(v_1, v_2, v_3, v_4)) \leq \psi(v_1, v_2, v_3, v_4), \tag{13}$$

for all $v_i \in E$; $i = 1, 2, 3, 4$, then there is an unique QM $Q_4 : E \to F_\rho$ having

$$\rho(Q_4(v) - \phi(v)) \leq \frac{1}{b^4(1-L)} \psi(v, 0, 0, 0), \tag{14}$$

for all $v \in E$.

Proof. We define the set

$$\mathrm{Y} = \{p : E \to F_\rho\},$$

and $\overline{\rho}$ is a function on Y as

$$\overline{\rho}(p) =: \inf\{\lambda > 0 : \rho(p(v)) \leq \lambda\psi(v, 0, 0, 0),\ \ \forall\, v \in E\}.$$

Now, we need to demonstrate that the function $\overline{\rho}$ is a semi-convex modular on Y. Clearly, $\overline{\rho}$ holds conditions (a) and (b). So, it is enough to verify that $\overline{\rho}$ is semi-convex modular. Given $\varepsilon > 0$, $\exists\, \lambda_1 > 0$ such that

$$\overline{\rho}(p) \leq \lambda_1 < \overline{\rho}(p) + \varepsilon.$$

Additionally,

$$\rho(p(v)) \leq \lambda_1 \psi(v, 0, 0, 0),$$

for all $v \in E$. For any $\beta \geq 0$, we have

$$\begin{aligned} \rho(\beta p(v)) &\leq \beta\rho(p(v)) \\ &\leq \beta\lambda_1\psi(v, 0, 0, 0), \end{aligned}$$

so we obtain

$$\overline{\rho}(\beta p) < \beta\overline{\rho}(p) + \beta\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, from above, we find that $\overline{\rho}$ is semi-convex modular on Y. Next, we want to verify that $\mathrm{Y}_{\overline{\rho}}$ is $\overline{\rho}$-complete.

Suppose a sequence $\{p_n\}$ is $\overline{\rho}$-Cauchy in $\mathrm{Y}_{\overline{\rho}}$. Given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ satisfies

$$\overline{\rho}(p_n - p_m) < \varepsilon,$$

for all $n, m \geq n_0$. Thus, we have

$$\rho(p_n(v) - p_m(v)) \leq \varepsilon\psi(v, 0, 0, 0), \tag{15}$$

for all $v \in E$, and $n, m \geq n_0$. Therefore, a ρ-Cauchy sequence $\{p_n(v)\}$ in F_ρ. As F_ρ is ρ-complete, $\{p_n(v)\}$ is convergent in F_ρ $\forall\, v \in E$.

Now, let us define a mapping $p : E \to F_\rho$ by

$$p(v) := \lim_{n\to\infty} p_n(v), \qquad \forall v \in E.$$

We arrive by taking into account Equation (15) that

$$\rho(p_n(v) - p(v)) \leq \lim_{m\to\infty} \inf \rho(p_n(v) - p_m(v)) \leq \varepsilon\psi(v,0,0,0),$$

so

$$\overline{\rho}(p_n - p) \leq \varepsilon, \qquad \forall n \geq n_0,$$

since ρ holds the Fatou property. Thus, $\{p_n\}$ $\overline{\rho}$-converges and so $Y_{\overline{\rho}}$ is $\overline{\rho}$-complete.

We now want to prove that $\overline{\rho}$ holds Fatou property. Suppose $\{p_n\}$ is $\overline{\rho}$-convergent to $p \in Y_{\overline{\rho}}$.

For all $\varepsilon > 0$, consider a constant λ_n ($n \in \mathbb{N}$) which is real such that

$$\overline{\rho}(p_n) \leq \lambda_n < \overline{\rho}(p_n) + \varepsilon.$$

So

$$\rho(p_n(v)) \leq \lambda_n \psi(v,0,0,0),$$

for all $v \in E$. We know that ρ holds the Fatou property, so we obtain

$$\begin{aligned} \rho(p(v)) &\leq \lim_{n\to\infty} \inf \rho(p_n(v)) \\ &\leq \lim_{n\to\infty} \inf \lambda_n \psi(v,0,0,0) \\ &< \left[\lim_{n\to\infty} \inf \overline{\rho}(p_n) + \varepsilon\right] \psi(v,0,0,0). \end{aligned}$$

Thus, we obtain

$$\overline{\rho}(p) \leq \lim_{n\to\infty} \inf \overline{\rho}(p_n),$$

since $\varepsilon > 0$ was arbitrary. Hence, $\overline{\rho}$ also holds the Fatou property.

Let us define a mapping $\chi : Y_{\overline{\rho}} \to Y_{\overline{\rho}}$ by

$$\chi p(v) = \frac{1}{b^4} p(bv) \qquad (\forall v \in E, p \in Y_{\overline{\rho}}). \tag{16}$$

Suppose $p, q \in Y_{\overline{\rho}}$ and $\lambda \in [0,1]$ with $\overline{\rho}(p - q) < \lambda$ (λ is an arbitrary constant). Employing the definition of $\overline{\rho}$, we write

$$\rho(p(v) - q(v)) \leq \lambda\psi(v,0,0,0), \qquad \forall v \in E.$$

Using Equations (12) and (16), we have

$$\begin{aligned} \rho\left(\frac{p(bv)}{b^4} - \frac{q(bv)}{b^4}\right) &\leq \frac{1}{b^4}\rho(p(bv) - q(bv)) \\ &\leq \frac{1}{b^4}\lambda\psi(bv,0,0,0) \\ &\leq \lambda L\psi(v,0,0,0), \end{aligned}$$

for all $v \in E$. Hence,

$$\overline{\rho}(\chi p - \chi q) \leq L\overline{\rho}(p - q), \qquad \forall p, q \in Y_{\overline{\rho}}$$

which means that χ is a $\overline{\rho}$-contraction. Now, we will show that χ has a ϕ bounded orbit. In Equation (13), we replace (v_1, v_2, v_3, v_4) with $(v, 0, 0, 0)$ so that

$$\begin{aligned} \rho\left(\phi(bv) - b^4\phi(v)\right) &\leq \psi(v,0,0,0), \\ \Rightarrow \rho\left(\frac{\phi(bv)}{b^4} - \phi(v)\right) &\leq \frac{1}{b^4}\psi(v,0,0,0), \qquad \forall v \in E. \end{aligned} \tag{17}$$

Replacing v with bv in inequality (17), we obtain

$$\begin{aligned} \rho\left(\frac{\phi(b^2v)}{b^4} - \phi(bv)\right) &\leq \frac{1}{b^4}\psi(bv,0,0,0), \\ \Rightarrow \rho\left(\frac{\phi(b^2v)}{b^{4(2)}} - \frac{\phi(bv)}{b^4}\right) &\leq \frac{1}{b^{4(2)}}\psi(bv,0,0,0), \qquad \forall v \in E. \end{aligned} \tag{18}$$

By using Equations (17) and (18), we obtain

$$\begin{aligned} \rho\left(\frac{\phi(b^2v)}{b^{4(2)}} - \phi(v)\right) &\leq \rho\left(\frac{\phi(b^2v)}{b^{4(2)}} - \frac{\phi(bv)}{b^4} + \frac{\phi(bv)}{b^4} - \phi(v)\right) \\ &\leq \rho\left(\frac{\phi(b^2v)}{b^{4(2)}} - \frac{\phi(bv)}{b^4}\right) + \rho\left(\frac{\phi(bv)}{b^4} - \phi(v)\right) \\ &\leq \frac{1}{b^{4(2)}}\psi(bv,0,0,0) + \frac{1}{b^4}\psi(v,0,0,0), \qquad \forall v \in E. \end{aligned}$$

Clearly, by induction,

$$\begin{aligned} \rho\left(\frac{\phi(b^nv)}{b^{4n}} - \phi(v)\right) &\leq \sum_{i=1}^{n}\frac{1}{b^{4i}}\psi\left(3^{i-1}v,0,0,0\right) \\ &\leq \frac{1}{Lb^4}\psi(v,0,0,0)\sum_{i=1}^{n}L^i \\ &\leq \frac{1}{b^4(1-L)}\psi(v,0,0,0), \qquad \forall v \in E. \end{aligned} \tag{19}$$

It follows from Equation (19) that

$$\begin{aligned} \rho\left(\frac{\phi(b^nv)}{b^{4n}} - \frac{\phi(b^mv)}{b^{4m}}\right) &\leq \frac{1}{2}\rho\left(2\frac{\phi(b^nv)}{b^{4n}} - 2\phi(v)\right) + \frac{1}{2}\rho\left(2\frac{\phi(b^mv)}{b^{4m}} - 2\phi(v)\right) \\ &\leq \frac{k}{2}\rho\left(\frac{\phi(b^nv)}{b^{4n}} - \phi(v)\right) + \frac{k}{2}\rho\left(\frac{\phi(b^mv)}{b^{4m}} - \phi(v)\right) \\ &\leq \frac{k}{2}\frac{1}{b^4(1-L)}\psi(v,0,0,0) + \frac{k}{2}\frac{1}{b^4(1-L)}\psi(v,0,0,0) \\ &\leq \frac{k}{b^4(1-L)}\psi(v,0,0,0), \end{aligned}$$

for $n, m \in \mathbb{N}$ and all $v \in E$. We conclude that by defining $\overline{\rho}$,

$$\overline{\rho}(\chi^n\phi - \chi^m\phi) \leq \frac{k}{b^4(1-L)}.$$

This means that an orbit of χ at ϕ is bounded. The sequence of $\{\chi^n\phi\}$ $\overline{\rho}$-converges into $Q_4 \in Y_{\overline{\rho}}$, according to Theorem 1.5 in [34]. Now, we have the $\overline{\rho}$-contractivity of χ, where

$$\overline{\rho}(\chi^n\phi - \chi Q_4) \leq L\overline{\rho}\left(\chi^{n-1}\phi - Q_4\right).$$

Taking the limit $n \to \infty$ and apply $\overline{\rho}$ Fatou property, we get

$$\begin{aligned} \overline{\rho}(\chi Q_4 - Q_4) &\leq \lim_{n\to\infty}\inf\overline{\rho}(\chi Q_4 - \chi^n\phi) \\ &\leq L\lim_{n\to\infty}\inf\overline{\rho}\left(Q_4 - \chi^{n-1}\phi\right) = 0. \end{aligned}$$

Thus, Q_4 is a fixed point of χ. Replacing (v_1, v_2, v_3, v_4) by $\left(b^l v_1, b^l v_2, b^l v_3, b^l v_4\right)$ in (13), we obtain

$$\rho\left(\Phi\left(b^l v_1, b^l v_2, b^l v_3, b^l v_4\right)\right) \leq \psi\left(b^l v_1, b^l v_2, b^l v_3, b^l v_4\right), \qquad \forall v_1, v_2, v_3, v_4 \in E.$$

Thus, we have

$$\rho\left(\frac{1}{b^{4l}}\Phi\left(b^l v_1, b^l v_2, b^l v_3, b^l v_4\right)\right) \leq \frac{1}{b^{4l}}\psi\left(b^l v_1, b^l v_2, b^l v_3, b^l v_4\right).$$

Letting $l \to \infty$, we obtain

$$Q_4(v_1, v_2, v_3, v_4) = 0, \qquad \forall v_1, v_2, v_3, v_4 \in E.$$

Theorem 1, Q_4 is quartic. So, the inequality (19) gives (14).

Let $Q_4^{'} : E \to F_\rho$ be an another QM that meets inequality (14) to prove the uniqueness of Q_4. Thus, $Q_4^{'}$ is a fixed point of χ, so

$$\overline{\rho}\left(Q_4 - Q_4^{'}\right) = \overline{\rho}\left(\chi Q_4 - \chi Q_4^{'}\right) \leq L\overline{\rho}\left(Q_4 - Q_4^{'}\right).$$

This yields $\overline{\rho}\left(Q_4 - Q_4^{'}\right) = 0$. Consequently, $Q_4 = Q_4^{'}$. which proves the uniqueness of function Q_4. □

Corollary 7. *Let $b \geq 3$ be an integer and a mapping $\psi : E^4 \to [0, +\infty)$ such that*

$$\lim_{l\to\infty} \frac{1}{b^{4l}}\psi\left(b^l v_1, b^l v_2, b^l v_3, b^l v_4\right) = 0,$$

and

$$\psi(bv_1, bv_2, bv_3, bv_4) \leq Lb^4\psi(v_1, v_2, v_3, v_4), \qquad \forall v_1, v_2, v_3, v_4 \in E,$$

with $0 < L < 1$. If $\phi : E \to F$ is an even mapping with $\phi(0) = 0$ such that

$$\|\Phi(v_1, v_2, v_3, v_4)\| \leq \psi(v_1, v_2, v_3, v_4),$$

for all $v_i \in E$; $i = 1, 2, 3, 4$, so there is an unique QM $Q_4 : E \to F$ having

$$\|Q_4(v) - \phi(v)\| \leq \frac{1}{b^4(1-L)}\psi(v, 0, 0, 0), \qquad \forall v \in E.$$

Remark 2. *If we replace $\psi(v_1, v_2, v_3, v_4)$ with $\alpha\left(\sum_{i=1}^4 \|v_i\|^p\right)$ and taking $L = b^{p-4}$ in the last corollary, then we arrive at the stability result for the sum of norms as*

$$\|Q_4(v) - \phi(v)\| \leq \frac{\alpha\|v\|^p}{(b^4 - b^p)}, \qquad \forall v \in E.$$

where p $(p < 4)$ and α are constants.

Theorem 6. *Let $b \geq 3$ be an integer. Suppose a mapping $\psi : E^4 \to [0, +\infty)$ satisfies*

$$\lim_{m\to\infty} b^{4m}\psi\left(\frac{v_1}{b^m}, \frac{v_2}{b^m}, \frac{v_3}{b^m}, \frac{v_4}{b^m}\right) = 0$$

and

$$\psi\left(\frac{v_1}{b}, \frac{v_2}{b}, \frac{v_3}{b}, \frac{v_4}{b}\right) \leq \frac{L}{b^4}\psi(v_1, v_2, v_3, v_4), \qquad \forall v_1, v_2, v_3, v_4 \in E,$$

with $0 < L < 1$. *If a mapping* $\phi : E \to F_\rho$ *is even with* $\phi(0) = 0$ *such that the inequality* (13) *holds, then there is an unique QM* $Q_4 : E \to F_\rho$ *having*

$$\rho(Q_4(v) - \phi(v)) \leq \frac{L}{b^4(1-L)}\psi(v,0,0,0), \qquad \forall v \in E. \tag{20}$$

Proof. Consider the set

$$\Upsilon = \{p : E \to F_\rho\}.$$

Let $\overline{\rho}$ be a function on Υ, defined by

$$\overline{\rho}(p) =: \inf\{\lambda > 0 : \rho(p(v)) \leq \lambda\psi(v,0,0,0), \quad \forall\, v \in E\}.$$

We have the same evidence as Theorem 5:

(a) The function $\overline{\rho}$ is a convex modular on Υ.
(b) $\Upsilon_{\overline{\rho}}$ is $\overline{\rho}$-complete.
(c) $\overline{\rho}$ holds the Fatou property.

Let us define a mapping $\chi : \Upsilon_{\overline{\rho} \to \Upsilon_{\overline{\rho}}}$ for all $v \in E$ and for $p \in \Upsilon_{\overline{\rho}}$ by

$$\chi p(v) = b^4 p\left(\frac{v}{b}\right).$$

Let $p, q \in \Upsilon_{\overline{\rho}}$ and $\lambda \in [0,1]$ with $\overline{\rho}(p - q) < \lambda$ (λ is an arbitrary constant). Consequently,

$$\rho(p(v) - q(v)) \leq \lambda\psi(v,0,0,0),$$

for all $v \in E$. We obtain by assumption and the above inequality that

$$\begin{aligned}
\rho\left(b^4 p\left(\frac{v}{b}\right) - b^4 q\left(\frac{v}{b}\right)\right) &\leq k^4\rho\left(p\left(\frac{v}{b}\right) - q\left(\frac{v}{b}\right)\right) \\
&\leq k^4\lambda\psi\left(\frac{v}{b},0,0,0\right) \\
&\leq \lambda L\psi(v,0,0,0),
\end{aligned}$$

for all $v \in E$. Hence,

$$\overline{\rho}(\chi p - \chi q) \leq L\overline{\rho}(p - q), \qquad p, q \in \Upsilon_{\overline{\rho}},$$

which proves that χ is a $\overline{\rho}$-contraction.

We will now show that χ has a bounded orbit at ϕ. Setting (v_1, v_2, v_3, v_4) by $(v, 0, 0, 0)$ in Equation (13), we obtain

$$\rho\left(b^4\phi(v) - \phi(bv)\right) \leq \psi(v,0,0,0), \qquad \forall v \in E. \tag{21}$$

It follows by replacing v with $\frac{v}{b}$ in Equation (21) that

$$\rho\left(b^4\phi\left(\frac{v}{b}\right) - \phi(v)\right) \leq \psi\left(\frac{v}{b},0,0,0\right), \qquad \forall v \in E. \tag{22}$$

Again, replacing v by $\frac{v}{b}$ in Equation (22), we obtain

$$\rho\left(b^4\phi\left(\frac{v}{b^2}\right) - \phi\left(\frac{v}{b}\right)\right) \quad \leq \quad \psi\left(\frac{v}{b^2},0,0,0\right), \qquad \forall v \in E. \tag{23}$$

Considering Equations (21)–(23), for all $v \in E$, we obtain

$$\begin{aligned}
\rho\left(b^{4(2)}\phi\left(\frac{v}{b^2}\right) - \phi(v)\right) &\leq \rho\left(b^{4(2)}\phi\left(\frac{v}{b^2}\right) - b^4\phi\left(\frac{v}{b}\right)\right) + \rho\left(b^4\phi\left(\frac{v}{b}\right) - \phi(v)\right) \\
&\leq k^4\rho\left(b^4\phi\left(\frac{v}{b^2}\right) - \phi\left(\frac{v}{b}\right)\right) + \rho\left(b^4\phi\left(\frac{v}{b}\right) - \phi(v)\right)
\end{aligned}$$

$$\leq b^4\psi\left(\frac{v}{b^2},0,0,0\right)+\psi\left(\frac{v}{b},0,0,0\right),$$

We can easily determine by induction that

$$\begin{aligned}\rho\left(b^{4n}\phi\left(\frac{v}{b^n}\right)-\phi(v)\right) &\leq \frac{1}{b^4}\sum_{i=1}^{n} b^{4i}\psi\left(\frac{v}{b^i},0,0,0\right)\\ &\leq \frac{1}{b^4}\psi(v,0,0,0)\sum_{i=1}^{n}L^i\\ &\leq \frac{L}{b^4(1-L)}\psi(v,0,0,0),\end{aligned} \tag{24}$$

for all $v\in E$. Equation (24) gives

$$\begin{aligned}\rho\left(b^{4n}\phi\left(\frac{v}{b^n}\right)-b^{4m}\phi\left(\frac{v}{b^m}\right)\right) &\leq \frac{1}{2}\rho\left(2(b^{4n})\phi\left(\frac{v}{b^n}\right)-2\phi(v)\right)+\frac{1}{2}\rho\left(2(b^{4m})\phi\left(\frac{v}{b^m}\right)-2\phi(v)\right)\\ &\leq \frac{kL}{b^4(1-L)}\psi(v,0,0,0),\end{aligned}$$

for all $v\in E$, and all $n,m\in\mathbb{N}$. We can conclude that by defining $\overline{\rho}$,

$$\overline{\rho}(\chi^n\phi-\chi^m\phi)\leq\frac{kL}{b^4(1-L)}.$$

This means that the χ orbit is limited to ϕ. The sequence $\{\chi^n\phi\}$ $\overline{\rho}$-converges to $Q_4\in Y_{\overline{\rho}}$ from Theorem 1.5 in [31].

We have from the $\overline{\rho}$-contractivity of χ that

$$\overline{\rho}(\chi^n\phi-\chi Q_4)\leq L\overline{\rho}\left(\chi^{n-1}\phi-Q_4\right).$$

Letting $n\to\infty$ together with Fatou property, we have

$$\begin{aligned}\overline{\rho}(\chi Q_4-Q_4) &\leq \lim\inf_{n\to\infty}\overline{\rho}(\chi Q_4-\chi^n\phi)\\ &\leq L\lim\inf_{n\to\infty}\overline{\rho}\left(Q_4-\chi^{n-1}\phi\right)=0.\end{aligned}$$

Therefore, the function Q_4 is a fixed point of χ. Replacing (v_1,v_2,v_3,v_4) with $\left(\frac{v_1}{b^l},\frac{v_2}{b^l},\frac{v_3}{b^l},\frac{v_4}{b^l}\right)$ in inequality (13), we obtain

$$\rho\left(\Phi\left(b^{-l}v_1,b^{-l}v_2,b^{-l}v_3,b^{-l}v_4\right)\right)\leq\psi\left(b^{-l}v_1,b^{-l}v_2,b^{-l}v_3,b^{-l}v_4\right),$$

for all $v_1,v_2,v_3,v_4\in E$. Therefore,

$$\rho\left(b^{4l}\Phi\left(\frac{v_1}{b^l},\frac{v_2}{b^l},\frac{v_3}{b^l},\frac{v_4}{b^l}\right)\right)\leq k^{4l}\psi\left(\frac{v_1}{b^l},\frac{v_2}{b^l},\frac{v_3}{b^l},\frac{v_4}{b^l}\right).$$

Passing to the limit $l\to\infty$, we obtain

$$Q_4(v_1,v_2,v_3,v_4)=0,\qquad \forall v_1,v_2,v_3,v_4\in E.$$

Therefore, Q_4 is quartic from Theorem 7. Using the inequality (24), we obtain the inequality (20).

It is only left to show the uniqueness of Q_4. For this, consider another QM $Q_4^{'}:E\to F_\rho$ which satisfies the inequality (14). Then, $Q_4^{'}$ is a fixed point of χ. So, we write

$$\overline{\rho}\left(Q_4-Q_4^{'}\right)=\overline{\rho}\left(\chi Q_4-\chi Q_4^{'}\right)\leq L\overline{\rho}\left(Q_4-Q_4^{'}\right),$$

which implies that $\overline{\rho}\left(Q_4 - Q_4^{'}\right) = 0$ or $Q_4 = Q_4^{'}$. $\square$

Corollary 8. *Let $b \geq 3$ be an integer and also let $\psi : E^4 \to [0, +\infty)$ be a mapping such that*

$$\lim_{l \to \infty} b^{4l} \psi\left(\frac{v_1}{b^l}, \frac{v_2}{b^l}, \frac{v_3}{b^l}, \frac{v_4}{b^l}\right) = 0,$$

and

$$\psi\left(\frac{v_1}{b}, \frac{v_2}{b}, \frac{v_3}{b}, \frac{v_4}{b}\right) \leq \frac{L}{b^4} \psi(v_1, v_2, v_3, v_4),$$

for all $v_i \in E$; $i = 1, 2, 3, 4$, with $0 < L < 1$. If a mapping $\phi : E \to F$ is even with $\phi(0) = 0$ satisfies the inequality (7), then there is an unique QM $Q_4 : E \to F$ satisfying

$$\|Q_4(v) - \phi(v)\| \leq \frac{L}{b^4(1 - L)} \psi(v, 0, 0, 0), \qquad \forall v \in E.$$

Remark 3. *If we replace $\psi(v_1, v_2, v_3, v_4)$ with $\alpha\left(\sum_{i=1}^{4} \|v_i\|^p\right)$ and taking $L = b^{4-p}$ in Corollary 8, we fairy have the stability results for the sum of norms as follows:*

$$\|Q_4(v) - \phi(v)\| \leq \frac{\alpha \|v\|^p}{(b^p - b^4)}, \qquad \forall v \in E,$$

where p ($p > 4$) and α are constants.

3.4. Illustrative Examples

Here, in this section, we investigate a suitable example to verify that the stability of quartic FE (4) fails for a singular case. Following by the example of Gajda (see [37]), we examine the following counter-example which proves the instability in a particular conditions $b = 3$ and $\alpha = 4$ in Corollaries 3 and 5 of Equation (4).

Remark 4. *If a mapping $\phi : \mathbb{R} \to E$ satisfies the functional Equation (4), then*
(C1) $\phi(m^{c/4} v) = m^c \phi(v)$, for all $v \in \mathbb{R}, m \in \mathbb{Q}$ and $c \in \mathbb{Z}$,
(C2) $\phi(v) = v^4 \phi(1)$, for all $v \in \mathbb{R}$ if ϕ is continuous,
hold.

Example 1. *Consider $\phi : \mathbb{R} \to \mathbb{R}$ defined as*

$$\phi(v) = \sum_{p=0}^{\infty} \frac{\psi(3^p v)}{3^{4p}}, \tag{25}$$

where

$$\psi(v) = \begin{cases} \lambda v^4, & -1 < v < 1 \\ \lambda, & \text{else.} \end{cases}$$

Suppose that the function ϕ defined in Equation (25) which satisfies

$$|\Phi(v_1, v_2, v_3, v_4)| \leq \frac{3^{12}(360)\lambda}{80} \left(\sum_{j=1}^{4} |v_j|^4\right), \tag{26}$$

for all $v_1, v_2, v_3, v_4 \in \mathbb{R}$. We here obtain that there does not exist a QM $Q : \mathbb{R} \to \mathbb{R}$ satisfying

$$|\phi(v) - Q(v)| \leq \delta |v|^4, \tag{27}$$

for all $v \in \mathbb{R}$,, where λ and δ are constants.

Clearly, ϕ is bounded by $\frac{3^4}{80}\lambda$ on $\mathbb{R}$. If $\sum_{j=1}^{4}|v_j|^4 \geq \frac{1}{3^4}$ or 0, then

$$|\Phi(v_1,v_2,v_3,v_4)| < (360)\frac{3^4}{80}\lambda.$$

Thus, the inequality (26) is valid. Next, suppose that

$$0 < \sum_{j=1}^{4}|v_j|^4 < \frac{1}{3^4}.$$

Then, there is an integer $l > 0$ that satisfies

$$\frac{1}{3^{4(l+2)}} \leq \sum_{j=1}^{4}|v_j|^4 < \frac{1}{3^{4(l+1)}}. \tag{28}$$

So, $3^{4l}|v_1| < \frac{1}{3^4}, 3^{4l}|v_2| < \frac{1}{3^4}, 3^{4l}|v_3| < \frac{1}{3^4}, 3^{4l}|v_4| < \frac{1}{3^4}$, and

$$\left.\begin{array}{r} 3^t v_1, 3^t v_2, 3^t v_3, 3^t v_4 \\ 3^t v_1 + 3^t v_2 + 3^t v_3 + 3^t v_4 \\ \sum_{i=1}^{4}\left(-3^t v_i + \sum_{j=1}^{4} 3^t v_j\right) \\ \sum_{1\leq i<j<k\leq 4}\left(3^t(v_i+v_j+v_k)\right) \\ \sum_{1\leq i<j\leq 4}\left(3^t(v_i+v_j)\right) \\ \sum_{1\leq i<j\leq 4}\left(3^t(v_i-v_j)\right) \end{array}\right\} \in]-1,1[, \quad t = 0,1,\cdots,l-1.$$

Additionally, for $t = 0,1,\cdots,l-1$,

$$\begin{aligned} \Psi(v_1,v_2,v_3,v_4) &= 4\psi\left(\sum_{j=1}^{4} v_j\right) + \sum_{i=1;i\neq j}^{4}\psi\left(-v_i + \sum_{1\leq j\leq 4} v_j\right) \\ &\quad -4\sum_{1\leq i<j<k\leq 4}\psi(v_i+v_j+v_k) + \sum_{i=1}^{4}\psi(3v_i) \\ &\quad +2\left[\sum_{1\leq i<j\leq 4}\psi(v_i+v_j) - \psi(v_i-v_j)\right] - 77\sum_{i=1}^{4}\psi(v_i) \\ &= 0. \end{aligned}$$

Next, from inequality (28), we obtain that

$$\begin{aligned} |\Phi(v_1,v_2,v_3,v_4)| &\leq \sum_{t=0}^{\infty}\frac{1}{3^{4t}}|\Psi(3^t v_1,3^t v_2,3^t v_3,3^t v_4)| \\ &\leq \sum_{t=l}^{\infty}\frac{1}{3^{4t}}(360)\lambda. \end{aligned}$$

It follows from the inequality (28) that

$$|\Phi(v_1,v_2,v_3,v_4)| \leq \frac{3^{12}(360)\lambda}{80}\left(\sum_{j=1}^{4}|v_j|^4\right).$$

So, ϕ satisfies (26). On the contrary, consider there is a quartic solution $Q : \mathbb{R} \to \mathbb{R}$ which satisfies the inequality (27). Since ϕ is continuous and bounded, for all $v \in \mathbb{R}$, Q is bounded in an open interval and continuous.

Considering Remark 4, we observe that Q should be $Q(v) = cv^4$, $v \in \mathbb{R}$. So, we obtain

$$|\chi(v)| \leq (\delta + |c|)|v|^4,$$

for $v \in \mathbb{R}$. Choosing $l > 0$ such that $l\lambda > \delta + |c|$. If $v \in \left(0, \frac{1}{3^{l-1}}\right)$, then $3^t v \in (0,1)$ for all $t = 0, 1, \cdots, l-1$, one obtains

$$\phi(v) = \sum_{t=0}^{\infty} \frac{\psi(3^t v)}{3^{4t}} \geq \sum_{t=0}^{l-1} \frac{\lambda (3^t v)^4}{3^{4t}} = l\lambda v^4 > (\delta + |c|)\, |v^4|,$$

which is contradictary. Thus, Equation (4) is not stable which proves the instability in a particular conditions $b = 3$ and $\alpha = 4$ in Corollaries 3 and 5 of Equation (4)

The following counter-example is similar to above example.

Example 2. *Consider $\phi : \mathbb{R} \to \mathbb{R}$ defined as*

$$\phi(v) = \sum_{p=0}^{\infty} \frac{\psi(3^p v)}{3^{4p}}, \tag{29}$$

where

$$\psi(v) = \begin{cases} \lambda v^4, & -1 < v < 1 \\ \lambda, & else. \end{cases}$$

Suppose that the function ϕ defined in Equation (29) which satisfying

$$|\Phi(v_1, v_2, v_3, v_4)| \leq \frac{3^{12}(360)\lambda}{80} \left(\sum_{j=1}^{4} |v_j|^4 \right),$$

for all $v_1, v_2, v_3, v_4 \in \mathbb{R}$. We show that a QM $Q : \mathbb{R} \to \mathbb{R}$ does not exist that satisfies

$$|\phi(v) - Q(v)| \leq \delta |v|^4,$$

for all $v \in \mathbb{R}$, where λ and δ are constants. Following the lines of last example, one proves the instability in a particular conditions $b = 3$ and $\alpha = 1$ in Corollaries 4 and 6 of Equation (4).

4. Conclusions and Discussion

Many mathematicians obtain the stability results of various kinds of additive, quadratic, and cubic functional equations in various spaces. In our investigations, we first defined a new kind of quartic FN in the first section of this paper and obtained the general solution of our newly defined quartic FN. Additionally, we explored the stability results of this quartic FN in the setting of modular space using Hyers' technique by taking into our account three cases, that are: without utilising the Fatou property, without using the Δ_b-condition, and without using the Δ_b-condition and Fatou property. Moreover, by taking into our account the Fatou property and fixed-point approach, we established some stability results of our quartic FN in the framework of modular spaces. In addition, an appropriate counter-example is provided to demonstrate the non-stability of the singular case.

It is worth mentioning that one can further determine the stability results of this quartic FN in various frameworks, namely, quasi-β-normed spaces, fuzzy normed space, non-Archimedean spaces, random normed spaces, probabilistic normed spaces, intuitionistic fuzzy normed space and so on. The findings and techniques used in this study might be valuable to other researchers who want to conduct further work in this area.

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