

Article

Sharp Bounds of Hankel Determinant on Logarithmic Coefficients for Functions of Bounded Turning Associated with Petal-Shaped Domain

Lei Shi ¹, Muhammad Arif ^{2,*} , Ayesha Rafiq ³, Muhammad Abbas ² and Javed Iqbal ²¹ School of Mathematics and Statistics, Anyang Normal University, Anyang 455002, China; shimath@aynu.edu.cn² Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan; muhammad_abbas@awikum.edu.pk (M.A.); javedmath@awikum.edu.pk (J.I.)³ Institute of Space Technology, University of Islamabad, Islamabad 44000, Pakistan; ayesha.rafiq@mail.ist.edu.pk

* Correspondence: marifmaths@awikum.edu.pk

Abstract: The purpose of this article is to obtain the sharp estimates of the first four initial logarithmic coefficients for the class $\mathcal{BT}_s$ of bounded turning functions associated with a petal-shaped domain. Further, we investigate the sharp estimate of Fekete-Szegő inequality, Zalcman inequality on the logarithmic coefficients and the Hankel determinant $H_{2,1}(F_f/2)$ and $H_{2,2}(F_f/2)$ for the class $\mathcal{BT}_s$ with the determinant entry of logarithmic coefficients.

Keywords: Hankel determinant; bounded turning functions; petal-shaped domain; logarithmic coefficient bounds

MSC: 30C45; 30C50



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1. Introduction and Definitions

For a good sense of the terminology used throughout our primary results, some basic pertinent information from Geometric Function Theory must always be given and explained. Let us start with the letter $\mathcal{A}$, which stands for the normalised analytic functions family and $\mathcal{S}$ for the normalised univalent functions family. These fundamental concepts are defined in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and are provided by the set builder in the form of

$$\mathcal{A} = \left\{ f \in \mathcal{H}(\mathbb{D}) : f(z) = z + \sum_{l=2}^{\infty} a_l z^l \right\}, \quad (1)$$

where $\mathcal{H}(\mathbb{D})$ represents the family of analytic functions, and

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{D}\}.$$

Recently, Aleman and Constantin [1] gave a beautiful interaction between univalent function theory and fluid dynamics. In fact, they demonstrated a simple method that shows how to use a univalent harmonic map to obtain explicit solutions of incompressible two-dimensional Euler equations. The logarithmic coefficients β_n of $f \in \mathcal{S}$ are given by the below formula

$$F_f(z) := \log \left(\frac{f(z)}{z} \right) = 2 \sum_{n=1}^{\infty} \beta_n z^n \text{ for } z \in \mathbb{D}.$$

These coefficients contribute significantly, in many estimations, to the theory of univalent functions. In 1985, de Branges [2] obtained that for $n \geq 1$,

$$\sum_{l=1}^n l(n-l+1) |\beta_n|^2 \leq \sum_{l=1}^n \frac{n-l+1}{l},$$

and the equality holds if and only if f takes the form $z/(1 - e^{i\theta}z)^2$ for some $\theta \in \mathbb{R}$. Clearly, this inequality gives the famous Bieberbach–Robertson–Milin conjectures about Taylor-coefficients of f belonging to $\mathcal{S}$ in its most general form. For more about the proof of de Brange’s result, we refer to [3–5]. In 2005, Kayumov [6] was able to solve Brennan’s conjecture for conformal mappings by considering the logarithmic coefficients. We list a few papers that have conducted significant work on the study of logarithmic coefficients [7–14].

For the given functions $g_1, g_2 \in \mathcal{A}$, the subordination between g_1 and g_2 (mathematically written as $g_1 \prec g_2$), if an analytic function v appears in $\mathbb{D}$ with the restriction $v(0) = 0$ and $|v(z)| < 1$ in such a manner that $f(z) = g(v(z))$ hold. Moreover, if g_2 in $\mathbb{D}$ is univalent, the following connection holds:

$$g_1(z) \prec g_2(z), \quad (z \in \mathbb{D})$$

if and only if

$$g_1(0) = g_2(0) \text{ \& } g_1(\mathbb{D}) \subset g_2(\mathbb{D}).$$

By employing the principle of subordination, Ma and Minda [15] considered a unified version of the class $\mathcal{S}^*(\phi)$ in 1992, which is stated below as

$$\mathcal{S}^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \text{ for } z \in \mathbb{D} \right\},$$

where ϕ is a univalent function with $\phi'(0) > 0$ and $\Re \phi > 0$. Moreover, the region $\phi(\mathbb{D})$ is star-shaped about the point $\phi(0) = 1$ and is symmetric along the real line axis. In the past few years, numerous sub-families of the collection $\mathcal{S}$ have been examined as particular choices of the class $\mathcal{S}^*(\phi)$. For example,

- (i) If we choose $\phi(z) = \frac{1+(1-2\xi)z}{1-z}$ with $0 \leq \xi < 1$, then we achieved the class $\mathcal{S}^*(\xi) := \mathcal{S}^*\left(\frac{1+(1-2\xi)z}{1-z}\right)$ of starlike function family of order ξ . Furthermore, $\mathcal{S}^* := \mathcal{S}^*\left(\frac{1+z}{1-z}\right)$ is the familiar starlike function family.
- (ii) The family $\mathcal{S}_{\mathcal{L}}^* := \mathcal{S}^*(\phi(z))$ with $\phi(z) = \sqrt{1+z}$ was developed in [16] by Sokół and Stankiewicz. The function $\phi(z) = \sqrt{1+z}$ maps the region $\mathbb{D}$ onto the image domain, which is bounded by $|w^2 - 1| < 1$.
- (iii) By selecting $\phi(z) = 1 + \sin z$, the class $\mathcal{S}^*(\phi(z))$ lead to the family $\mathcal{S}_{\sin}^*$, which was explored in [17], while $\mathcal{S}_{\mathcal{L}}^* \equiv \mathcal{S}^*(e^z)$ has been produced in the article [18].
- (iv) The family $\mathcal{S}_{\cos}^* := \mathcal{S}^*(\cos(z))$ and $\mathcal{S}_{\cosh}^* := \mathcal{S}^*(\cosh(z))$ were contributed, respectively, by Raza and Bano [19], and Alotaibi et al. [20]. In both the papers, the authors studied good properties of these families.

For given parameters $q, n \in \mathbb{N} = \{1, 2, \dots\}$, the Hankel determinant $H_{q,n}(f)$ was defined by Pommerenke [21,22] for a function $f \in \mathcal{S}$ of the form Equation (1), which is given by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}. \quad (2)$$

The growth of $H_{q,n}(f)$ has been investigated for different sub-collections of univalent functions. Specifically, the absolute sharp bounds of the functional $H_{2,2}(f) = a_2a_4 - a_3^2$ were found in [23,24] for each of the sets $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{R}$, where the family $\mathcal{R}$ contained functions of bounded turning. This determinant has also been recently studied for two new subfamilies of bi-univalent functions in [25,26]. However, the exact estimate of this determinant for the family of close-to-convex functions is still undetermined [27]. Later on, many authors published their work regarding the upper bounds of the Hankel determinant for different sub-collections of univalent functions, see [28–37].

According to the definition, it is not hard to calculate that for $f \in \mathcal{S}$, its logarithmic coefficients are given by

$$\gamma_1 = \frac{1}{2}a_2 \quad (3)$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) \quad (4)$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right) \quad (5)$$

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right). \quad (6)$$

Recently, Kowalczyk and Lecko [38,39] proposed the study of the Hankel determinant $H_{q,n}(F_f/2)$, whose elements are logarithmic coefficients of f , that is

$$H_{q,n}(F_f/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2q-2} \end{vmatrix}. \quad (7)$$

It is observed that $H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2$ is just corresponding to the well-known functional $H_{2,1}(f) = a_3 - a_2^2$ over the class $\mathcal{S}$ or its subclasses. Some basic calculations gives the expressions of $H_{q,n}(F_f/2)$ in the following, which we will discuss in the present paper.

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2, \quad (8)$$

$$H_{2,2}(F_f/2) = \gamma_2\gamma_4 - \gamma_3^2. \quad (9)$$

In [40], Kumar and Arora introduce an interesting subclass of the starlike function, defined by

$$\mathcal{S}_\rho^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \sinh^{-1} z \quad (z \in \mathbb{D}) \right\}. \quad (10)$$

Let $\phi(z) = 1 + \sinh^{-1} z$. It can be noted that $\phi(z) = 1 + \ln(z + \sqrt{1+z^2})$ and is convex in $\mathbb{D}$. In geometry, it maps the unit disk onto a petal-shaped domain $\Omega_\rho = \{\omega \in \mathbb{C} : |\sinh(\omega - 1)| < 1\}$ symmetric about the line $\Re\omega = 1$. Using this function, Barukab and his coauthors [41] considered a subclass of the bounded turning function, given by

$$\mathcal{BT}_s = \left\{ f \in \mathcal{A} : f'(z) \prec 1 + \sinh^{-1} z \quad (z \in \mathbb{D}) \right\}. \quad (11)$$

In the current article, our main goal is to calculate the sharp logarithmic coefficient-related problems for the class $\mathcal{BT}_s$ of bounded turning functions linked with the petal-shaped domain. The sharp bounds of Fekete-Szegő inequality, Zalcman inequality of logarithmic coefficients, $H_{2,1}(F_f/2)$ and $H_{2,2}(F_f/2)$ are obtained for the class $\mathcal{BT}_s$.

2. A Set of Lemmas

Let $\mathcal{P}$ represent the class of all functions p that are holomorphic in $\mathbb{D}$ with $\Re(p(z)) > 0$ and has series representation given in the form of

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}). \quad (12)$$

To prove the main results, we need the following lemmas.

Lemma 1 (see [42]). Let $p \in \mathcal{P}$ and be the form of (12). Then for $x, \tau, \rho \in \overline{\mathbb{D}}$,

$$2c_2 = c_1^2 + x(4 - c_1^2), \quad (13)$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\tau, \quad (14)$$

$$8c_4 = c_1^4 + (4 - c_1^2)x[c_1^2(x^2 - 3x + 3) + 4x] - 4(4 - c_1^2)(1 - |x|^2) \\ [c(x - 1)\tau + \bar{x}\tau^2 - (1 - |\tau|^2)\rho]. \quad (15)$$

Lemma 2. If $p \in \mathcal{P}$ and be the form of (12), we obtain

$$|c_n| \leq 2 \quad (n \geq 1). \quad (16)$$

and

$$|c_{n+k} - \mu c_n c_k| \leq 2 \max\{1, |2\mu - 1|\} = \begin{cases} 2 & \text{for } 0 \leq \mu \leq 1; \\ 2|2\mu - 1| & \text{otherwise.} \end{cases} \quad (17)$$

Also, If $B \in [0, 1]$ and $B(2B - 1) \leq D \leq B$, we obtain

$$|c_3 - 2Bc_1c_2 + Dc_1^3| \leq 2. \quad (18)$$

The inequalities in (16)–(18) are taken from [43–45], respectively.

Lemma 3 (see [46]). Let α, β, r and a satisfy the inequalities $0 < \alpha < 1, 0 < a < 1$ and

$$8a(1 - a)((\alpha\beta - 2r)^2 + (\alpha(a + \alpha) - \beta)^2) + \alpha(1 - \alpha)(\beta - 2a\alpha)^2 \leq 4a\alpha^2(1 - \alpha)^2(1 - a). \quad (19)$$

If $p \in \mathcal{P}$ is of the form (12), then

$$\left| rc_1^4 + ac_2^2 + 2\alpha c_1c_3 - \frac{3}{2}\beta c_1^2c_2 - c_4 \right| \leq 2.$$

3. Coefficient Inequalities for the Class $\mathcal{BT}_s$

We begin this section by finding the absolute values of the first four initial logarithmic coefficients for the function of class $\mathcal{BT}_s$.

Theorem 1. If $f \in \mathcal{BT}_s$ and has the series representation (1), then

$$|\gamma_1| \leq \frac{1}{4}, \quad (20)$$

$$|\gamma_2| \leq \frac{1}{6}, \quad (21)$$

$$|\gamma_3| \leq \frac{1}{8}, \quad (22)$$

$$|\gamma_4| \leq \frac{1}{10}. \quad (23)$$

These bounds are the best possible.

Proof. Let $f \in \mathcal{BT}_s$. Then, (11) can be written in the form of a Schwarz function, as

$$f'(z) = 1 + \sinh^{-1}(z), \quad (z \in \mathbb{D}). \quad (24)$$

If $p \in \mathcal{P}$, and it may be written in terms of Schwarz function $w(z)$ as

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots,$$

equivalently,

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \dots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \dots}. \quad (25)$$

From (1), we obtain

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \dots. \quad (26)$$

By simplification and using the series expansion of (25), we obtain

$$\begin{aligned} 1 + \sinh^{-1}(w(z)) &= 1 + \left(\frac{1}{2}c_1\right)z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_1c_2 + \frac{5}{48}c_1^3\right)z^3 \\ &+ \left(-\frac{1}{32}c_1^4 - \frac{1}{4}c_2^2 - \frac{1}{2}c_1c_3 + \frac{5}{16}c_1^2c_2 + \frac{1}{2}c_4\right)z^4 + \dots. \end{aligned} \quad (27)$$

Comparing (26) and (27), we obtain

$$\begin{aligned} a_2 &= \frac{1}{4}c_1, \\ a_3 &= \frac{1}{6}c_2 - \frac{1}{12}c_1^2, \\ a_4 &= \frac{1}{8}c_3 - \frac{1}{8}c_1c_2 + \frac{5}{192}c_1^3, \\ a_5 &= -\frac{1}{160}c_1^4 - \frac{1}{20}c_2^2 - \frac{1}{10}c_1c_3 + \frac{5}{80}c_1^2c_2 + \frac{1}{10}c_4. \end{aligned} \quad (28)$$

Plugging (28) in (3)–(6), we obtain

$$\gamma_1 = \frac{1}{8}c_1, \quad (29)$$

$$\gamma_2 = \frac{1}{12}c_2 - \frac{11}{192}c_1^2, \quad (30)$$

$$\gamma_3 = \frac{5}{192}c_1^3 - \frac{1}{12}c_1c_2 + \frac{1}{16}c_3, \quad (31)$$

$$\gamma_4 = \frac{1}{20}c_4 - \frac{23}{720}c_2^2 - \frac{1033}{92160}c_1^4 + \frac{17}{288}c_1^2c_2 - \frac{21}{320}c_1c_3. \quad (32)$$

For γ_1 , implementing (16), in (29), we obtain

$$|\gamma_1| \leq \frac{1}{4}.$$

For γ_2 , we can write (30), as

$$\gamma_2 = \frac{1}{12}\left(c_2 - \frac{11}{16}c_1^2\right).$$

Using (17) we have

$$|\gamma_2| \leq \frac{1}{6}.$$

For γ_3 , we can write (31) as

$$|\gamma_3| = \frac{1}{16}\left|c_3 - 2\left(\frac{2}{3}\right)c_1c_2 + \frac{5}{12}c_1^3\right|.$$

From (18), we have

$$0 \leq B = \frac{2}{3} \leq 1, B = \frac{2}{3} \geq D = \frac{5}{12},$$

and

$$B(2B - 1) = \frac{2}{9} \leq D = \frac{5}{12}.$$

Application of triangle inequality plus (18) lead us to

$$|\gamma_3| \leq \frac{1}{8}.$$

For γ_4 , we can rewrite (32) as

$$\begin{aligned} \gamma_4 &= -\frac{1}{20} \left(\frac{1033}{4608} c_1^4 + \left(\frac{23}{36} \right) c_2^2 + 2 \left(\frac{21}{32} \right) c_1 c_3 - \frac{3}{2} \left(\frac{85}{108} \right) c_1^2 c_2 - c_4 \right). \\ &= -\frac{1}{20} \left(r c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2} \beta c_1^2 c_2 - c_4 \right), \end{aligned} \quad (33)$$

where

$$r = \frac{1033}{4608}, \quad a = \frac{23}{36}, \quad \alpha = \frac{21}{32}, \quad \beta = \frac{85}{108},$$

are such that

$$8a(1-a) \left((\alpha\beta - 2r)^2 + (\alpha(a+\alpha) - \beta)^2 \right) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leq 4a\alpha^2(1-\alpha)^2(1-a),$$

$0 < \alpha < 1, 0 < a < 1$, therefore by (19) and (33), we have

$$|\gamma_4| \leq \frac{1}{10}.$$

These outcomes are best possible. For this, we consider a function

$$f'_n(z) = 1 + \sinh^{-1}(z^n),$$

where $n = 1, 2, 3, 4$. Thus, we have

$$\begin{aligned} f_1(z) &= \int_0^z \left(1 + \sinh^{-1}(t) \right) dt = z + \frac{1}{2} z^2 - \frac{1}{24} z^4 + \dots, \\ f_2(z) &= \int_0^z \left(1 + \sinh^{-1}(t^2) \right) dt = z + \frac{1}{3} z^3 - \frac{1}{42} z^7 + \dots, \\ f_3(z) &= \int_0^z \left(1 + \sinh^{-1}(t^3) \right) dt = z + \frac{1}{4} z^4 - \frac{1}{60} z^{10} + \dots, \\ f_4(z) &= \int_0^z \left(1 + \sinh^{-1}(t^4) \right) dt = z + \frac{1}{5} z^5 - \frac{1}{78} z^{13} + \dots. \end{aligned}$$

□

Theorem 2. If $f \in \mathcal{BT}_s$ is of the form Equation (1), then

$$\left| \gamma_2 - \lambda \gamma_1^2 \right| \leq \max \left\{ \frac{1}{6}, \left| \frac{3|\lambda| + 3}{48} \right| \right\}, \text{ for } \lambda \in \mathbb{C}.$$

This inequality is sharp.

Proof. Employing (29), and (30), we may write

$$\left| \gamma_2 - \lambda \gamma_1^2 \right| = \left| \frac{1}{12} c_2 - \frac{11}{192} c_1^2 - \frac{\lambda}{64} c_1^2 \right|.$$

Application of (17), leads us to

$$|\gamma_2 - \lambda\gamma_1^2| \leq \frac{2}{12} \max \left\{ 1, \left| \frac{3\lambda + 11}{8} - 1 \right| \right\}.$$

After the simplification, we obtain

$$|\gamma_2 - \lambda\gamma_1^2| \leq \max \left\{ \frac{1}{6}, \left| \frac{3|\lambda| + 3}{48} \right| \right\}.$$

The required result is sharp and is determined by using (3) and (4) and

$$f_2(z) = \int_0^z \left(1 + \sinh^{-1}(t^2) \right) dt = z + \frac{1}{3}z^3 - \frac{1}{42}z^7 + \dots.$$

□

Theorem 3. If $f \in \mathcal{BT}_s$ has the form of Equation (1), then

$$|\gamma_1\gamma_2 - \gamma_3| \leq \frac{1}{8}.$$

This inequality is sharp.

Proof. Using (29)–(31), we have

$$|\gamma_1\gamma_2 - \gamma_3| = \frac{1}{16} \left| c_3 - 2 \left(\frac{3}{4} \right) c_1 c_2 + \frac{17}{32} c_1^3 \right|.$$

From (18), we have

$$0 \leq B = \frac{3}{4} \leq 1, B = \frac{3}{4} \geq D = \frac{17}{32},$$

and

$$B(2B - 1) = \frac{3}{8} \leq D = \frac{17}{32}.$$

Using (18), we obtain

$$|\gamma_1\gamma_2 - \gamma_3| \leq \frac{1}{8}.$$

This result is the best possible and is obtained by using (3)–(5) and

$$f_3(z) = \int_0^z \left(1 + \sinh^{-1}(t^3) \right) dt = z + \frac{1}{4}z^4 - \frac{1}{60}z^{10} + \dots.$$

□

Theorem 4. Let $f \in \mathcal{BT}_s$ be of the form Equation (1). Then

$$|\gamma_4 - \gamma_2^2| \leq \frac{1}{10}.$$

This inequality is the best possible.

Proof. From (30) and (32), we obtain

$$|\gamma_4 - \gamma_2^2| = \left| \frac{2671}{184320} c_1^4 - \frac{7}{180} c_2^2 - \frac{21}{320} c_1 c_3 + \frac{79}{1152} c_1^2 c_2 + \frac{1}{20} c_4 \right|.$$

After simplifying we have

$$\left| \gamma_4 - \gamma_2^2 \right| = \frac{1}{20} \left| \frac{2671}{9216} c_1^4 + \frac{7}{9} c_2^2 + 2 \left(\frac{21}{32} \right) c_1 c_3 - \frac{3}{2} \left(\frac{395}{432} \right) c_1^2 c_2 - c_4 \right|. \quad (34)$$

Comparing the right side of (34) with

$$\left| r c_1^4 + a c_2^2 + 2 \alpha c_1 c_3 - \frac{3}{2} \beta c_1^2 c_2 - c_4 \right|, \quad (35)$$

where

$$r = \frac{2671}{9216}, \quad a = \frac{7}{9}, \quad \alpha = \frac{21}{32}, \quad \beta = \frac{395}{432},$$

are such that

$$8a(1-a) \left((\alpha\beta - 2r)^2 + (\alpha(a+\alpha) - \beta)^2 \right) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leq 4a\alpha^2(1-\alpha)^2(1-a),$$

$0 < \alpha < 1, 0 < a < 1$, therefore by Equations (19) and (35), we have

$$\left| \gamma_4 - \gamma_2^2 \right| \leq \frac{1}{10}.$$

This required inequality is sharp and is determined by using Equations (4) and (6) and

$$f_4(z) = \int_0^z \left(1 + \sinh^{-1}(t^4) \right) dt = z + \frac{1}{5} z^5 - \frac{1}{78} z^{13} + \dots$$

□

4. Hankel Determinant with Logarithmic Coefficients for the Class $\mathcal{BT}_s$

Theorem 5. If f belongs to $\mathcal{BT}_s$, then

$$\left| H_{2,1}(F_f/2) \right| = \left| \gamma_1 \gamma_3 - \gamma_2^2 \right| \leq \frac{1}{36}.$$

The inequality is sharp.

Proof. From (29)–(31), we have

$$H_{2,1}(F_f/2) = -\frac{1}{36864} c_1^4 + \frac{1}{128} c_1 c_3 - \frac{1}{1152} c_1^2 c_2 - \frac{1}{144} c_2^2.$$

Using (13) and (14) to express c_2 and c_3 in terms of c_1 and, noting that without loss in generality we can write $c_1 = c$, with $0 \leq c \leq 2$, we obtain

$$\begin{aligned} \left| H_{2,1}(F_f/2) \right| &= \left| -\frac{1}{4096} c^4 - \frac{1}{512} c^2 x^2 (4 - c^2) - \frac{1}{576} x^2 (4 - c^2)^2 \right. \\ &\quad \left. + \frac{1}{256} c (4 - c^2) (1 - |x|^2) \tau \right|, \end{aligned}$$

with the aid of the triangle inequality and replacing $|\tau| \leq 1, |x| = b$, where $b \leq 1$ and taking $c \in [0, 2]$. So

$$\begin{aligned} \left| H_{2,1}(F_f/2) \right| &\leq \frac{1}{4096} c^4 + \frac{1}{512} c^2 b^2 (4 - c^2) + \frac{1}{576} b^2 (4 - c^2)^2 \\ &\quad + \frac{1}{256} c (4 - c^2) (1 - b^2) := \phi(c, b). \end{aligned}$$

It is a simple exercise to show that $\frac{\partial \phi}{\partial b} \geq 0$ on $[0, 1]$, so that $\phi(c, b) \leq \phi(c, 1)$. Putting $b = 1$ gives

$$\left| H_{2,1}(F_f/2) \right| \leq \frac{1}{4096}c^4 + \frac{1}{512}c^2(4 - c^2) + \frac{1}{576}(4 - c^2)^2 := \phi(c, 1).$$

Since $\frac{\partial \phi(c, 1)}{\partial c} < 0$, so $\phi(c, 1)$ is a decreasing function, and obtains its maximum value at $c = 0$ is

$$\left| H_{2,1}(F_f/2) \right| \leq \frac{1}{36}.$$

The required Hankel determinant is sharp and is obtained by using (3)–(5) and

$$f_2(z) = \int_0^z \left(1 + \sinh^{-1}(t^2) \right) dt = z + \frac{1}{3}z^3 - \frac{1}{42}z^7 + \dots.$$

□

Theorem 6. If f belongs to $\mathcal{BT}_s$, and has the form Equation (1). Then

$$\left| H_{2,2}(F_f/2) \right| = \left| \gamma_2\gamma_4 - \gamma_3^2 \right| \leq \frac{1}{64}.$$

This result is the best possible.

Proof. The $H_{2,2}(F_f/2)$ can be written as

$$H_{2,2}(F_f/2) = \gamma_2\gamma_4 - \gamma_3^2.$$

Putting (30)–(32), with $c_1 = c$ we obtain

$$\begin{aligned} H_{2,2}(F_f/2) = \frac{1}{17694720} & \left(-637c^6 + 432c^4c_2 + 8928c^3c_3 - 3456c^2c_2^2 - 50688c^2c_4 \right. \\ & \left. + 87552cc_2c_3 - 47104c_2^3 + 73728c_2c_4 - 69120c_3^2 \right). \end{aligned} \quad (36)$$

Let $w = 4 - c^2$ in (13)–(15). Now using the simplified form of these lemmas, we obtain

$$\begin{aligned} 432c^4c_2 &= 216c^6 + 216c^4wx, \\ 8928c^3c_3 &= -2232c^4wx^2 + 4464c^3w(1 - |x|^2)\tau + 4464c^4xw + 2232c^6, \\ 3456c^2c_2^2 &= 864c^6 + 1728c^4wx + 864c^2w^2x^2, \\ 50688c^2c_4 &= 6336c^6 + 6336c^4wx^3 - 19008c^4wx^2 + 19008c^4xw + 25344wc^2x^2 \\ &\quad - 25344c^3w(1 - |x|^2)\tau x - 25344c^2w(1 - |x|^2)\bar{x}\tau^2 + 25344c^2w \\ &\quad (1 - |x|^2)(1 - |\tau|^2)\rho + 25344c^3w(1 - |x|^2)\tau, \\ 87552cc_2c_3 &= -10944x^3w^2c^2 - 10944c^4wx^2 + 21888cxw^2(1 - |x|^2)\tau \\ &\quad + 21888c^2x^2w^2 + 21888c^3w(1 - |x|^2)\tau + 32832c^4xw + 10944c^6, \\ 47104c_2^3 &= 5888c^6 + 17664c^4wx + 17664c^2w^2x^2 + 5888w^3x^3, \end{aligned}$$

$$\begin{aligned}
73728c_2c_4 &= 4608c^6 + 4608c^4wx^3 - 13824c^4wx^2 + 18432c^4xw + 18432wc^2x^2 \\
&\quad - 18432c^3w + (1 - |x|^2)\tau x - 18432c^2w(1 - |x|^2)\bar{x}\tau^2 + 18432c^2w \\
&\quad \quad (1 - |x|^2)(1 - |\tau|^2)\rho + 18432c^3w(1 - |x|^2)\tau + 4608x^4w^2c^2 \\
&\quad - 13824x^3w^2c^2 + 13824c^2x^2w^2 + 18432x^3w^2 - 18432x^2w^2 \\
&\quad (1 - |x|^2)c\tau - 18432xw^2\bar{x}(1 - |x|^2)\bar{x}\tau^2 + 18432xw^2(1 - |x|^2) \\
&\quad (1 - |\tau|^2)\rho + 18432cxw^2(1 - |x|^2)\tau, \\
69120c_3^2 &= 4320x^4w^2c^2 - 17280x^2w^2(1 - |x|^2)c\tau - 17280x^3w^2c^2 - 8640c^4wx^2 \\
&\quad + 17280w^2(1 - |x|^2)^2\tau^2 + 34560cxw^2(1 - |x|^2)\tau + 17280c^2x^2w^2 \\
&\quad + 17280c^3w(1 - |x|^2)\tau + 17280c^4xw + 4320c^6.
\end{aligned}$$

Putting the above expressions in (36), we obtain,

$$\begin{aligned}
H_{2,3}(f) &= \frac{1}{17694720} \left\{ -5888x^3w^3 + 18432x^3w^2 - 17280w^2(1 - |x|^2)^2\tau^2 \right. \\
&\quad + 264c^4xw + 648c^4wx^2 - 96c^2x^2w^2 - 6912wc^2x^2 - 1728c^4wx^3 \\
&\quad - 7488x^3w^2c^2 + 288x^4w^2c^2 + 6912c^3w(1 - |x|^2)\tau x + 6912c^2w \\
&\quad (1 - |x|^2)\bar{x}\tau^2 - 6912c^2w(1 - |x|^2)(1 - |\tau|^2)\rho + 5760cxw^2 \\
&\quad (1 - |x|^2)\tau - 1152x^2w^2(1 - |x|^2)c\tau - 18432xw^2(1 - |x|^2)\bar{x}\tau^2 \\
&\quad \left. + 18432xw^2(1 - |x|^2)(1 - |\tau|^2)\rho - 45c^6 + 2160c^3w(1 - |x|^2)\tau \right\}.
\end{aligned}$$

Since $w = 4 - c^2$, it follows that

$$H_{2,2}(F_f/2) = \frac{1}{17694720} (r_1(c, x) + r_2(c, x)\tau + r_3(c, x)\tau^2 + r_4(c, x, \tau)\rho),$$

where $\rho, x, \tau \in \mathbb{D}$, and

$$\begin{aligned}
r_1(c, x) &= -45c^6 + (4 - c^2) \left[(4 - c^2) (-5120x^3 - 1600c^2x^3 - 96c^2x^2 + 288c^2x^4) \right. \\
&\quad \left. - 6912c^2x^2 + 264c^4x + 648c^4x^2 - 1728c^4x^3 \right], \\
r_2(c, x) &= (4 - c^2)(1 - |x|^2) \left[(4 - c^2) (-1152cx^2 + 5760cx) + 6912c^3x + 2160c^3 \right], \\
r_3(c, x) &= (4 - c^2)(1 - |x|^2) \left[(4 - c^2) (-1152|x|^2 - 17280) + 6912c^2\bar{x} \right], \\
r_4(c, x, \tau) &= (4 - c^2)(1 - |x|^2)(1 - |\tau|^2) \left[-6912c^2 + 18432x(4 - c^2) \right].
\end{aligned}$$

Now, by using $|x| = x, |\tau| = y$ and utilizing the fact $|\rho| \leq 1$, we obtain

$$\begin{aligned}
|H_{2,2}(F_f/2)| &\leq \frac{1}{17694720} (|r_1(c, x)| + |r_2(c, x)|y + |r_3(c, x)|y^2 + |r_4(c, x, \tau)|). \\
&\leq \frac{1}{17694720} (S(c, x, y)),
\end{aligned} \tag{37}$$

where

$$S(c, x, y) = t_1(c, x) + t_2(c, x)y + t_3(c, x)y^2 + t_4(c, x)(1 - y^2),$$

with

$$\begin{aligned}t_1(c, x) &= 45c^6 + (4 - c^2) \left[(4 - c^2) (5120x^3 + 1600c^2x^3 + 96c^2x^2 + 288c^2x^4) \right. \\&\quad \left. + 6912c^2x^2 + 264c^4x + 648c^4x^2 + 1728c^4x^3 \right], \\t_2(c, x) &= (4 - c^2) (1 - x^2) \left[(4 - c^2) (1152cx^2 + 5760cx) + 6912c^3x + 2160c^3 \right], \\t_3(c, x) &= (4 - c^2) (1 - x^2) \left[(4 - c^2) (1152x^2 + 17280) + 6912c^2x \right], \\t_4(c, x) &= (4 - c^2) (1 - x^2) \left[6912c^2 + 18432x(4 - c^2) \right].\end{aligned}$$

Now, we have to maximize $S(c, x, y)$ in the closed cuboid $\Theta : [0, 2] \times [0, 1] \times [0, 1]$.

For this, we have to discuss the maximum values of $S(c, x, y)$ in the interior of Θ , in the interior of its six faces and on its twelve edges.

1. Interior points of cuboid Θ :

Let $(c, x, y) \in (0, 2) \times (0, 1) \times (0, 1)$, and differentiating partially $S(c, x, y)$ with respect to y , we obtain

$$\begin{aligned}\frac{\partial S}{\partial y} &= 144(4 - c^2)(1 - x^2) \left[16y(x - 1) \left((4 - c^2)(x - 15) + 6c^2 \right) \right. \\&\quad \left. + 8c \left(x(4 - c^2)(x + 5) + 6c^2 \left(x + \frac{5}{16} \right) \right) \right].\end{aligned}$$

Putting $\frac{\partial S}{\partial y} = 0$, yields

$$y = \frac{8c(x(4 - c^2)(x + 5) + 6c^2(x + \frac{5}{16}))}{16(x - 1)((4 - c^2)(15 - x) - 6c^2)} = y_0.$$

If y_0 is a critical point inside Θ , then $y_0 \in (0, 1)$, which is possible only if

$$c^3(48x + 15) + 8cx(4 - c^2)(x + 5) + 16(1 - x)(4 - c^2)(15 - x) < 96c^2(1 - x). \quad (38)$$

and

$$c^2 > \frac{4(15 - x)}{21 - x}. \quad (39)$$

For the existence of the critical points, we have to obtain the solutions which satisfy both inequalities in Equations (38) and (39).

Let $g(x) = \frac{4(15-x)}{21-x}$. As $g'(x) < 0$ in $(0, 1)$, it can be observed that $g(x)$ is decreasing over $(0, 1)$. Hence $c^2 > \frac{14}{5}$. It is not difficult to be verified that the inequality Equation (38) can not hold true in this situation for $x \in [\frac{2}{5}, 1)$. Thus, there is no critical point of $S(c, x, y)$ exist in $(0, 2) \times [\frac{2}{5}, 1) \times (0, 1)$.

Suppose that there is a critical point $(\tilde{c}, \tilde{x}, \tilde{y})$ of S existing in the interior of cuboid Θ , clearly, it must satisfy that $\tilde{x} < \frac{2}{5}$. From the above discussion, it can be also known that

$\tilde{c}^2 \geq \frac{292}{103}$ and $\tilde{y} \in (0, 1)$. Presently, we will prove that $S(\tilde{c}, \tilde{x}, \tilde{y}) < 276480$. For $(c, x, y) \in \left(\sqrt{\frac{292}{103}}, 2\right) \times (0, \frac{2}{5}) \times (0, 1)$, by invoking $x < \frac{2}{5}$ and $1 - x^2 < 1$; it is not hard to observe that

$$\begin{aligned} t_1(c, x) &\leq 45c^6 + (4 - c^2) \left[(4 - c^2) \left(5120 \left(\frac{2}{5} \right)^3 + 1600c^2 \left(\frac{2}{5} \right)^3 + 96c^2 \left(\frac{2}{5} \right)^2 \right. \right. \\ &\quad \left. \left. + 288c^2 \left(\frac{2}{5} \right)^4 \right) + 6912c^2 \left(\frac{2}{5} \right)^2 + 264c^4 \left(\frac{2}{5} \right) + 648c^4 \left(\frac{2}{5} \right)^2 \right. \\ &\quad \left. + 1728c^4 \left(\frac{2}{5} \right)^3 \right], \\ &= 45c^6 + \frac{1}{625} (4 - c^2) (121712c^4 + 799232c^2 + 819200) := \varsigma_1(c), \\ t_2(c, x) &\leq (4 - c^2) \left[(4 - c^2) \left(1152c \left(\frac{2}{5} \right)^2 + 5760c \left(\frac{2}{5} \right) \right) + 6912c^3 \left(\frac{2}{5} \right) + 2160c^3 \right], \\ &= \frac{1}{25} (4 - c^2) (60912c^3 + 248832c) := \varsigma_2(c), \\ t_3(c, x) &\leq (4 - c^2) \left[(4 - c^2) \left(1152 \left(\frac{2}{5} \right)^2 + 17280 \right) + 6912c^2 \left(\frac{2}{5} \right) \right], \\ &= \frac{1}{25} (4 - c^2) (-367488c^2 + 1746432) := \varsigma_3(c), \\ t_4(c, x) &\leq (4 - c^2) \left[6912c^2 + 18432 \left(\frac{2}{5} \right) (4 - c^2) \right], \\ &= \frac{1}{5} (4 - c^2) (-2304c^2 + 147456) := \varsigma_4(c). \end{aligned}$$

Therefore, we have

$$S(c, x, y) \leq \varsigma_1(c) + \varsigma_4(c) + \varsigma_2(c)y + [\varsigma_3(c) - \varsigma_4(c)]y^2 := \Gamma_2(c, y).$$

Obviously, it can be observed that

$$\frac{\partial \Gamma_2}{\partial y} = \varsigma_2(c) + 2y[\varsigma_3(c) - \varsigma_4(c)],$$

and

$$\frac{\partial^2 \Gamma_2}{\partial y^2} = 2[\varsigma_3(c) - \varsigma_4(c)] = \frac{2}{25} (4 - c^2) (-355968c^2 + 1009152).$$

Since $\varsigma_3(c) - \varsigma_4(c) \leq 0$ for $c \in \left(\sqrt{\frac{292}{103}}, 2\right)$, we obtain that $\frac{\partial^2 \Gamma_2}{\partial y^2} \leq 0$ for $y \in (0, 1)$ and thus it follows that

$$\begin{aligned} \frac{\partial \Gamma_2}{\partial y} &\geq \frac{\partial \Gamma_2}{\partial y} \Big|_{y=1} = \frac{(4 - c^2)}{25} (60912c^3 - 711936c^2 + 248832c + 2018304) \geq 0, \\ &\text{for } c \in \left(\sqrt{\frac{292}{103}}, 2\right). \end{aligned}$$

Therefore, we have

$$\Gamma_2(c, y) \leq \Gamma_2(c, 1) = \varsigma_1(c) + \varsigma_2(c) + \varsigma_3(c) := Y_2(c).$$

It is easy to be calculated that $Y_2(c)$ attains its maximum value 74510.302 at $c \approx 1.683731$. Thus, we have

$$S(c, x, y) < 276480, \quad (c, x, y) \in \left(\sqrt{\frac{292}{103}}, 2 \right) \times \left(0, \frac{2}{5} \right) \times (0, 1).$$

Hence $S(\tilde{c}, \tilde{x}, \tilde{y}) < 276480$. This implies that S is less than 276480 at all the critical points in the interior of Θ . Therefore, S has no optimal solution in the interior of Θ .

2. Interior of all the six faces of cuboid Θ :

(i) On the face $c = 0$, $S(c, x, y)$ takes the form

$$L_1(x, y) = S(0, x, y) = 2048 \left(40x^3 + 9(1 - x^2) \left(y^2(x - 1)(x - 15) + 16x \right) \right), \quad x, y \in (0, 1).$$

Then,

$$\frac{\partial L_1}{\partial y} = 36864y(1 - x^2)(x - 1)(x - 15), \quad x, y \in (0, 1).$$

Thus $L_1(x, y)$ has no critical point in the interval $(0, 1) \times (0, 1)$.

(ii) On the face $c = 2$, $S(c, x, y)$ becomes

$$S(2, x, y) = 2880 < 276480.$$

(iii) On the face $x = 0$, $S(c, x, y)$ reduces to

$$L_2(c, y) = S(c, 0, y) = 45c^6 + (4 - c^2) \left(2160c^3y + (-24192c^2 + 69120)y^2 + 6912c^2 \right).$$

Differentiating $L_2(c, y)$ partially with respect to y

$$\frac{\partial L_2}{\partial y} = (4 - c^2) \left(2160c^3 + (-48384c^2 + 138240)y \right).$$

Putting $\frac{\partial L_2}{\partial y} = 0$, we obtain

$$y = \frac{5c^3}{16(7c^2 - 20)} = y_1.$$

For the given range of y , y_1 should belong to $(0, 1)$, which is possible only if $c > c_0$, $c_0 \approx 1.76094199$. Moreover, the derivative of $L_2(c, y)$, partially with respect to c , is

$$\begin{aligned} \frac{\partial L_2}{\partial c} &= 270c^5 + (4 - c^2) \left(6480c^2y - 48384cy^2 + 13824c \right) - 13824c^3 \\ &\quad - 4320c^4y + (48384c^2 - 138240)cy^2. \end{aligned} \quad (40)$$

By substituting the value of y in (40), plugging $\frac{\partial L_2}{\partial c} = 0$ and simplifying, we obtain

$$\frac{\partial L_2}{\partial c} = -27c \left(35c^8 + 49576c^6 - 385072c^4 + 983040c^2 - 819200 \right) = 0. \quad (41)$$

A calculation gives the solution of (41) in the interval $(0, 1)$ that is $c \approx 1.3851278$. Thus, $L_2(c, y)$ has no optimal point in the interval $(0, 2) \times (0, 1)$.

(iv) On the face $x = 1$, $S(c, x, y)$ yields

$$L_3(c, y) = S(c, 1, y) = 45c^6 + (4 - c^2) \left((4 - c^2)(1984c^2 + 5120) + 6912c^2 + 2640c^4 \right).$$

Then

$$\frac{\partial L_3}{\partial c} = -3666c^5 - 28416c^3 + 36864c.$$

Putting $\frac{\partial L_3}{\partial c} = 0$ and solving, we obtain $c \approx 1.0639470$. Thus, we have

$$S(c, 1, y) \leq \max L_3(c, y) = 92795.48842 < 276480.$$

(v) On the face $y = 0$, $S(c, x, y)$ becomes

$$\begin{aligned} L_4(c, x) = S(c, x, 0) = & 288c^6x^4 - 128c^6x^3 - 552c^6x^2 - 2304c^4x^4 - 264c^6x \\ & - 19200c^4x^3 + 45c^6 + 1824c^4x^2 + 4608c^2x^4 + 19488c^4x \\ & + 132096c^2x^3 - 6912c^4 + 1536c^2x^2 - 147456c^2x \\ & - 212992x^3 + 27648c^2 + 294912x. \end{aligned}$$

Presently, differentiating partially with respect to c , then, with respect to x and simplifying, we have

$$\begin{aligned} \frac{\partial L_4}{\partial c} = & 1728c^5x^4 - 768c^5x^3 - 3312c^5x^2 - 9216c^3x^4 - 1584c^5x \\ & - 76800c^3x^3 + 270c^5 + 7296c^3x^2 + 9216cx^4 + 77952c^3x \\ & + 264192cx^3 - 27648c^3 + 3072cx^2 - 294912cx + 55296c. \end{aligned} \quad (42)$$

and

$$\begin{aligned} \frac{\partial L_4}{\partial x} = & 1152c^6x^3 - 384c^6x^2 - 1104c^6x - 9216c^4x^3 - 264c^6 - 57600c^4x^2 \\ & + 3648c^4x + 18432c^2x^3 + 19488c^4 + 396288c^2x^2 + 3072c^2x \\ & - 147456c^2 - 638976x^2 + 294912. \end{aligned} \quad (43)$$

A numerical computation demonstrates that the solution does not exist for the system of Equations (42) and (43) in $(0, 2) \times (0, 1)$. Hence $L_4(c, x)$ has no optimal solution in the interval $(0, 2) \times (0, 1)$.

(vi) On the face $y = 1$, $S(c, x, y)$ yields

$$\begin{aligned} L_5(c, x) = S(c, x, 1) = & 288c^6x^4 - 128c^6x^3 - 1152c^5x^4 - 552c^6x^2 + 1152c^5x^3 \\ & - 3456c^4x^4 - 264c^6x + 3312c^5x^2 + 6144c^4x^3 + 9216c^3x^4 + 45c^6 \\ & - 1152c^5x - 21216c^4x^2 + 18432c^3x^3 + 13824c^2x^4 - 2160c^5 \\ & - 5856c^4x - 17856c^3x^2 - 43008c^2x^3 - 18432cx^4 + 17280c^4 \\ & - 18432c^3x + 158208c^2x^2 - 92160cx^3 - 18432x^4 + 8640c^3 \\ & + 27648c^2x + 18432cx^2 + 81920x^3 - 138240c^2 + 92160cx \\ & - 258048x^2 + 276480. \end{aligned}$$

Partial derivative of $L_5(c, x)$ with respect to c and then with respect to x , we have

$$\begin{aligned} \frac{\partial L_5}{\partial c} = & 1728c^5x^4 - 768c^5x^3 - 5760c^4x^4 - 3312c^5x^2 + 5760c^4x^3 - 13824c^3x^4 \\ & - 1584c^5x + 16560c^4x^2 + 24576c^3x^3 + 27648c^2x^4 + 270c^5 - 5760c^4x \\ & - 84864c^3x^2 + 55296c^2x^3 + 27648cx^4 - 10800c^4 - 23424c^3x - 53568 \\ & c^2x^2 - 86016cx^3 - 18432x^4 + 69120c^3 - 55296c^2x + 316416cx^2 \\ & - 92160x^3 + 25920c^2 + 55296cx + 18432x^2 - 276480c + 92160x. \end{aligned} \quad (44)$$

and

$$\begin{aligned} \frac{\partial L_5}{\partial x} = & 1152c^6x^3 - 384c^6x^2 - 4608c^5x^3 - 1104c^6x + 3456c^5x^2 - 13824c^4x^3 \\ & - 264c^6 + 6624c^5x + 18432c^4x^2 + 36864c^3x^3 - 1152c^5 - 42432c^4x \\ & + 55296c^3x^2 + 55296c^2x^3 - 5856c^4 - 35712c^3x - 129024c^2x^2 \\ & - 73728cx^3 - 18432c^3 + 316416c^2x - 276480cx^2 - 73728x^3 \\ & + 27648c^2 + 36864cx + 245760x^2 + 92160c - 516096x. \end{aligned} \quad (45)$$

As in the above case, we conclude the same result for the face $y = 0$, that is the system of Equations (44) and (45) has no solution in $(0, 2) \times (0, 1)$.

3. On the Edges of Cuboid Θ :

(i) On the edge $x = 0$ and $y = 0$, $S(c, x, y)$ takes the form

$$S(c, 0, 0) = 45c^6 - 6912c^4 + 27648c^2 = L_6(c).$$

It is clear that

$$L'_6(c) = 270c^5 - 27648c^3 + 55296c.$$

Putting $L'_6(c) = 0$ and solving, we obtain $c_0 \approx 1.4285192$ at which $S(c, 0, 0) = L_6(c)$ receives its maximum. Thus

$$S(c, 0, 0) \leq \max L_6(c) = L_6(c_0) = 28018.979 < 276480.$$

(ii) On the edge $x = 0$ and $y = 1$, $S(c, x, y)$ becomes

$$S(c, 0, 1) = 45c^6 - 2160c^5 + 17280c^4 + 8640c^3 - 138240c^2 + 276480 = L_7(c).$$

It follows that

$$L'_7(c) = 270c^5 - 10800c^4 + 69120c^3 + 25920c^2 - 276480c.$$

Noting that $L'_7(c) < 0$ in $[0, 2]$, $L_7(c)$ is decreasing over $[0, 2]$. Thus $L_7(c)$ has its maxima at $c = 0$. Therefore, $\max L_7(c) = L_7(0) = 276480$. Hence

$$S(c, 0, 1) \leq 276480.$$

(iii) On the edge $c = 0$ and $x = 0$, $S(c, x, y)$ reduces to

$$S(0, 0, y) = 276480y^2 = L_8(y).$$

Since $L'_8(y) > 0$ in $[0, 1]$, it is clear that $L_8(y)$ is increasing over $[0, 1]$. Thus, $L_8(y)$ has its maxima at $y = 1$. Therefore, $\max L_8(y) = L_8(1) = 276480$. Hence

$$S(0, 0, y) \leq 276480.$$

(iv) On the edges of $S(c, 1, 0)$ and $S(c, 1, 1)$

Since $S(c, 1, y)$ is free of y , therefore

$$S(c, 1, 0) = S(c, 1, 1) = -611c^6 - 7104c^4 + 18432c^2 + 81920 = L_9(c).$$

and

$$L'_9(c) = -3666c^5 - 28416c^3 + 36864c.$$

Putting $L'_9(c) = 0$, gives the critical point $c_0 \approx 1.06394704$ at which $S(c, 1, 0) = S(c, 1, 1) = L_9(c)$ attains its maximum, therefore $\max L_9(c) = L_9(c_0) = 92795.4884$. Thus

$$S(c, 1, 0) = S(c, 1, 1) \leq 92795.4884 < 276480.$$

(v) On the edge $c = 0$ and $x = 1$, $S(c, x, y)$ becomes

$$S(0, 1, y) = 81920 < 276480.$$

(vi) On the edge $c = 2$, $S(c, x, y)$ reduces to

$$S(2, x, y) = 2880 < 276480.$$

$S(2, x, y)$ is independent of x and y , therefore

$$S(2, 0, y) = S(2, 1, y) = S(2, x, 0) = S(2, x, 1) = 2880 < 276480.$$

(vii) On the edge $c = 0$ and $y = 1$, $S(c, x, y)$ takes the form

$$S(0, x, 1) = -18432x^4 + 81920x^3 - 258048x^2 + 276480 = L_{10}(x).$$

Clearly

$$L'_{10}(x) = -73728x^3 + 245760x^2 - 516096x.$$

Note that $L'_{10}(x) < 0$ in $[0, 1]$, $L_{10}(x)$ is decreasing over $[0, 1]$. Thus, $L_{10}(x)$ has its maxima at $x = 0$. Therefore, $\max L_{10}(x) = L_{10}(0) = 276480$. Hence

$$S(0, x, 1) \leq 276480.$$

(viii) On the edge $c = 0$ and $y = 0$, $S(c, x, y)$ becomes

$$S(0, x, 0) = -212992x^3 + 294912x = L_{11}(x).$$

and

$$L'_{11}(x) = -638976x^2 + 294912.$$

Putting $L'_{11}(x) = 0$, gives the critical point $x_0 \approx 0.6793662$ at which $L_{11}(x)$ receives its maximum. Therefore, $\max L_{11}(x) = L_{11}(x_0) = \frac{196608}{13}\sqrt{6}\sqrt{13}$. Thus,

$$S(0, x, 0) \leq \frac{196608}{13}\sqrt{6}\sqrt{13} < 276480.$$

Thus, from the above cases we conclude that

$$S(c, x, y) \leq 276480 \quad \text{on } [0, 2] \times [0, 1] \times [0, 1].$$

From Equation (37) we have

$$\left| H_{2,2}(F_f/2) \right| \leq \frac{1}{17694720} (S(c, x, y)) \leq \frac{1}{64} \approx 0.0156.$$

If $f \in \mathcal{BT}_s$, then the sharp bound for this Hankel determinant is determined by using Equations (4)–(6) and

$$f_3(z) = \int_0^z \left(1 + \sinh^{-1}(t^3) \right) dt = z + \frac{1}{4}z^4 - \frac{1}{60}z^{10} + \dots$$

□

5. Conclusions

Due to the great importance of logarithmic coefficients, Kowalczyk and Lecko [38,39] proposed the topic of studying the Hankel determinant with the entry of logarithmic coefficients. In the current article, we considered a subclass of bounded turning functions denoted as $\mathcal{BT}_s$. This family of univalent functions was connected with a petal-shaped domain with $f'(z)$ subordinated to $1 + \sinh^{-1} z$. We gave an estimate for some initial

logarithmic coefficients and some related inequalities problems on logarithmic coefficients. The bounds of Hankel determinant with logarithmic coefficients as the entry for this class were determined. All the estimations were proven to be sharp.

In proving our main results, finding the upper bounds of the Hankel determinant for functions belonging to $\mathcal{BT}_s$ were transformed to a maximum value problem of a function with three variables in a domain of cuboid. Based on the analysis of all the possibilities that the maxima might occur, we were able to obtain the sharp upper bounds for this class. Since some of the calculations are very complicated, numerical analysis are used. Obviously, this method is useful sometimes to find bounds for functions of different subfamilies of univalent functions. However, in most cases, it is not so lucky to obtain the sharp results.

The use of the familiar quantum or basic (or q -) calculus, as shown in similar recent articles [47–49], could be a promising area for future study based on our present investigation. Many authors have investigated the third and fourth-order Hankel determinants in recent years, see [50–52]. The methodology provided in this article might potentially be used to study these higher-order Hankel determinants.

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