mathematics

MDPI

Article

Modified Mann-Type Subgradient Extragradient Rules for Variational Inequalities and Common Fixed Points Implicating Countably Many Nonexpansive Operators †

Yun-Ling Cui, Lu-Chuan Ceng *, Fang-Fei Zhang, Cong-Shan Wang, Jian-Ye Li, Hui-Ying Hu and Long He

Department of Mathematics, Shanghai Normal University, Shanghai 200234, China; cuiyunling@shnu.edu.cn (Y.-L.C.); fangfeizhang@shnu.edu.cn (F.-F.Z.); congshanwang@shnu.edu.cn (C.-S.W.); jianyeli@shnu.edu.cn (J.-Y.L.); huiying@shnu.edu.cn (H.-Y.H.); longhe@shnu.edu.cn (L.H.)

* Correspondence: zenglc@shnu.edu.cn

† This research was partially supported by the 2020 Shanghai Leading Talents Program of the Shanghai Municipal Human Resources and Social Security Bureau (20LJ2006100), the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), and the Program for Outstanding Academic Leaders in Shanghai City (15XD1503100).

Abstract: In a real Hilbert space, let the CFPP, VIP, and HFPP denote the common fixed-point problem of countable nonexpansive operators and asymptotically nonexpansive operator, variational inequality problem, and hierarchical fixed point problem, respectively. With the help of the Mann iteration method, a subgradient extragradient approach with a linear-search process, and a hybrid deepest-descent technique, we construct two modified Mann-type subgradient extragradient rules with a linear-search process for finding a common solution of the CFPP and VIP. Under suitable assumptions, we demonstrate the strong convergence of the suggested rules to a common solution of the CFPP and VIP, which is only a solution of a certain HFPP.

Keywords: modified Mann-type subgradient extragradient rule; linear-search process; variational inequality problem; countable nonexpansive operators; strong convergence

MSC: 47J25; 47J20; 47H10; 47H09

Citation: Cui, Y.-L.; Ceng, L.-C.; Zhang, F.-F.; Wang, C.-S.; Li, J.-Y.; Hu, H.-Y.; He, L. Modified Mann-Type Subgradient Extragradient Rules for Variational Inequalities and Common Fixed Points Implicating Countably Many Nonexpansive Operators. *Mathematics* **2022**, *10*, 1949. https://doi.org/10.3390/math10111949

Academic Editors: Salvador Romaguera and Pasquale Vetro

Received: 3 May 2022
Accepted: 27 May 2022
Published: 6 June 2022

1. Introduction

Throughout this paper, we assume that P_C is the metric projection of H onto C, with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denoting the inner product and induced norm of real Hilbert space H and C being a convex and closed set satisfying $\emptyset \neq C \subset H$. Given nonlinear mapping $S : C \to H$, let the $\mathrm{Fix}(S)$ and $\mathbf{R}$ indicate the fixed-point set of S and the real-number set, respectively. In the fixed point theory, we recall an important class of mappings. A self-mapping S on C is known as being asymptotically nonexpansive iff $\exists \{\theta_i\}_{i=1}^{\infty} \subset [0, +\infty)$ s.t. $\lim_{i \to \infty} \theta_i = 0$ and

$$\|S^i u - S^i v\| \leq \|u - v\| + \theta_i \|u - v\| \quad \forall i \geq 1,\ u, v \in C. \tag{1}$$

In particular, whenever $\theta_i = 0\ \forall i \geq 1$, S is said to be nonexpansive. In the past several decades, the fixed point theory has played a key role in solving real-world problems such as the time-fractional biological population model [1], fractional multi-dimensional system of boundary value problems on the methylpropane graph [2], traumatic avoidance learning model [3], and so forth.

Given a self-mapping A on H, we consider the classical variational inequality problem (VIP) of finding $u \in C$ s.t. $\langle Au, v - u \rangle \geq 0 \ \ \forall v \in C$. Its solution set is written as $\mathrm{VI}(C, A)$. To the best of our awareness, one of the most effective techniques for treating the VIP is the extragradient one put forward by Korpelevich [4] in 1976, i.e., for any starting point $u_0 \in C$, $\{u_i\}$ is fabricated below

$$\begin{cases} v_i = P_C(u_i - \ell A u_i), \\ u_{i+1} = P_C(u_i - \ell A v_i) \quad \forall i \geq 0, \end{cases} \tag{2}$$

where $\ell \in (0, \frac{1}{L})$ and L is Lipschitz constant of A. Whenever $\mathrm{VI}(C, A) \neq \emptyset$, the sequence $\{u_i\}$ converges weakly to a point in $\mathrm{VI}(C, A)$. At present, the vast literature on Korpelevich's extragradient technique shows that many authors have paid great attention to it and enhanced this technique in different manners; for details, refer to [5–28] and references therein, to name but a few.

Very recently, Xie et al. [9] suggested the amended inertial extragradient approach with a line-search process for solving the pseudomonotone VIP in H. Let $f : H \to H$ be a contraction with constant $\delta \in [0, 1)$ and assume that $\Omega := \mathrm{VI}(C, A) \neq \emptyset$. Given the sequences $\{\alpha_i\}, \{\beta_i\} \subset (0, 1]$ such that $\lim_{i\to\infty} \beta_i = 0$ and $\sum_{i=1}^{\infty} \beta_i = \infty$. Their approach is formulated by Algorithm 1 below:

Algorithm 1 Modified inertial extragradient approach (see [9])

Initial Step: Let $\varsigma \in (0, 1)$, $\ell \in (0, 1)$, $\mu \in (0, 1)$ $\gamma \in (0, \infty)$, given any starting points x_1, x_0 in H.
Iterations: Given the iterates x_{i-1}, x_i $(i \geq 1)$, compute x_{i+1} below:
Step 1. Set $w_i = x_i + \alpha_i(x_i - x_{i-1})$.
Step 2. Calculate $v_i = P_C(w_i - \tau_i A w_i)$ and $z_i = P_C(w_i - \varsigma\tau_i A v_i)$, where $\tau_i := \gamma\ell^{m_i}$ and m_i is the smallest nonnegative integer m such that

$$\gamma\ell^m \langle Aw_i - Av_i, z_i - v_i \rangle \leq \mu \|w_i - v_i\| \|z_i - v_i\|.$$

If $w_i = v_i$ or $Av_i = 0$, then stop and v_i is an element of Ω. Otherwise, go to Step 3.
Step 3. Calculate $x_{i+1} = \beta_i f(z_i) + (1 - \beta_i) z_i$. If $w_i = z_i = x_{i+1}$, then $w_i \in \Omega$.
Again, set $i := i + 1$ and go to Step 1.

Under appropriate assumptions, they showed the strong convergence of $\{x_i\}$ to the solution $p = P_\Omega \circ f(p)$ provided $\lim_{i\to\infty} \frac{\alpha_i}{\beta_i} \|x_i - x_{i-1}\| = 0$. In the extragradient technique, two projections onto C have to be calculated per one iteration. In 2018, Thong and Hieu [22] first proposed the inertial subgradient extragradient method, and then proved the weak convergence of this method to an element of $\mathrm{VI}(C, A)$ under mild assumptions. In 2019, Thong and Hieu [17] proposed the inertial-type subgradient extragradient method with a linear-search process for settling the VIP with monotone and Lipschitzian operator A and the fixed-point problem (FPP) of a quasi-nonexpansive operator S with the demiclosedness in H. Assume that $\Omega := \mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Given the sequences $\{\alpha_i\} \subset [0, 1]$ and $\{\beta_i\} \subset (0, 1)$. Their method is formulated by Algorithm 2 below:

Algorithm 2 Inertial-type subgradient extragradient method (see [17])

Initial Step: Let $\nu \in (0, 1)$, $l \in (0, 1)$, $\gamma \in (0, \infty)$, given any starting points x_1, x_0 in H.
Iterations: Compute x_{i+1} below:
Step 1. Put $w_i = x_i + \alpha_i(x_i - x_{i-1})$ and calculate $v_i = P_C(w_i - \varsigma_i A w_i)$, where ς_i is picked to be the largest $\varsigma \in \{\gamma, \gamma l, \gamma l^2, \ldots\}$ s.t.

$$\varsigma \|Aw_i - Av_i\| \leq \nu \|w_i - v_i\|.$$

Step 2. Calculate $z_i = P_{C_i}(w_i - \varsigma_i A v_i)$ with $C_i := \{v \in H : \langle w_i - \varsigma_i A w_i - v_i, v - v_i \rangle \leq 0\}$.
Step 3. Calculate $x_{i+1} = (1 - \beta_i) w_i + \beta_i S z_i$. If $w_i = z_i = x_{i+1}$, then $w_i \in \Omega$.
Again, set $i := i + 1$ and go to Step 1.

Under suitable assumptions, it was proven in [17] that $\{x_i\}$ converges weakly to a point in Ω. Subsequently, Ceng and Shang [25] proposed the hybrid inertial subgradient extragradient rule with a linear-search process for settling the VIP with Lipschitzian pseudomonotonicity operator A and the common fixed-point problem (CFPP) of finite nonexpansive operators $\{S_k\}_{k=1}^N$ and asymptotically nonexpansive operator S on H. Assume that $\Omega := \mathrm{VI}(C,A) \cap (\bigcap_{k=0}^N \mathrm{Fix}(S_k)) \neq \emptyset$ with $S_0 := S$. Given a δ-contractive map $g : H \to H$ with $\delta \in [0,1)$, and an operator $F : H \to H$ of both η-strong monotonicity and κ-Lipschitz continuity, fulfilling $\delta < \zeta := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$ with $0 < \mu < \frac{2\eta}{\kappa^2}$. Let $\{\beta_i\}, \{\alpha_i\} \subset (0,1)$ and $\{\epsilon_i\} \subset [0,1]$ s.t. $\alpha_i + \beta_i < 1\ \forall i \geq 1$. In addition, one writes $S_i := S_{i\mathrm{mod}N}$ for each $i \geq 1$, where the mod function takes values in $\{1,\ldots,N\}$, that is, whenever $i = qN + j$ for some integers $q \geq 0$ and $0 \leq j < N$, one has that $S_i = S_N$ in the case of $j = 0$ and $S_i = S_j$ in the case of $0 < j < N$. Their rule is formulated by Algorithm 3 below:

Algorithm 3 Hybrid inertial subgradient extragradient rule (see [25])

Initial Step: Let $\nu \in (0,1)$, $l \in (0,1)$, $\gamma \in (0,\infty)$, given any starting points x_1, x_0 in H.
Iterations: Compute x_{i+1} below:
Step 1. Put $w_i = S_i x_i + \epsilon_i(S_i x_i - S_i x_{i-1})$ and calculate $y_i = P_C(w_i - \varsigma_i A w_i)$, with ς_i being picked to be the largest $\varsigma \in \{\gamma, \gamma l, \gamma l^2, \ldots\}$ s.t.

$$\varsigma\|Aw_i - Ay_i\| \leq \nu\|w_i - y_i\|.$$

Step 2. Calculate $q_i = P_{C_i}(w_i - \varsigma_i A y_i)$ with $C_i := \{y \in H : \langle w_i - \varsigma_i A w_i - y_i, y_i - y\rangle \geq 0\}$.
Step 3. Calculate $x_{i+1} = \alpha_i g(x_i) + \beta_i x_i + ((1-\beta_i)I - \alpha_i \mu F)S^i q_i$.
Again, set $i := i+1$ and go to Step 1.

Under appropriate assumptions, it was proven in [25] that, if $S^i q_i - S^{i+1} q_i \to 0$, then $\{x_i\}$ converges strongly to $q^* \in \Omega$ if and only if $x_i - x_{i+1} \to 0$ and $x_i - y_i \to 0$ as $i \to \infty$, with $q^* \in \Omega$ being only a solution to the hierarchical fixed point problem (HFPP): $q^* = P_\Omega(I - \mu F + g)q^*$.

In the rest of this paper, we always assume that the CFPP and HFPP denote the common fixed-point problem of countable nonexpansivity operators $\{S_i\}_{i=1}^\infty$ and asymptotical nonexpansivity operator $S_0 := S$ and hierarchical fixed-point problem, respectively. With the help of the Mann iteration method, a subgradient extragradient approach with a linear-search process, and hybrid deepest-descent technique, we construct two amended Mann-type subgradient extragradient rules with a linear-search process for finding a common solution of the CFPP of $\{S_i\}_{i=0}^\infty$ and the VIP for pseudomonotone operator A. Via suitable conditions, we show the strong convergence of the proposed rules to a point in $\Omega := \mathrm{VI}(C,A) \cap (\bigcap_{k=0}^\infty \mathrm{Fix}(S_k))$, which is only a solution of a certain HFPP. In the end, using the main results, we deal with the CFPP and VIP in an illustrated example.

The architecture of this paper is arranged as follows: In Section 2, we recollect certain concepts and basic tools for subsequent applications. In Section 3, we prove the strong convergence of the proposed rules. Finally, in Section 4, the main theorems are exploited to settle the CFPP and VIP in a demonstrated instance. Our rules are more general and more subtle than the above algorithms because they implicate settling the VIP for pseudomonotone operator and the CFPP for countable nonexpansive operators and an asymptotically nonexpansive operator. Our theorems ameliorate and develop the associated theorems pronounced in Xie et al. [9], Ceng and Shang [25], and Thong and Hieu [17].

2. Preliminaries

Given a sequence $\{v_i\} \subset H$, let $v_i \rightharpoonup v$ (resp., $v_i \to v$) represent the weak (resp., strong) convergence of $\{v_i\}$ to v. A mapping $S : C \to H$ is referred to as being

(a) of L-Lipschitz continuity (or of L-Lipschitzian property) iff $\exists L > 0$ s.t. $L\|p - q\| \geq \|Sp - Sq\|\ \forall p,q \in C$;
(b) of monotonicity iff $0 \leq \langle Sp - Sq, p - q\rangle\ \forall p,q \in C$;
(c) of pseudomonotonicity iff $\langle Sp, q - p\rangle \geq 0 \Rightarrow \langle Sq, q - p\rangle \geq 0\ \forall p,q \in C$;
(d) of η-strong monotonicity iff $\exists \eta > 0$ s.t. $\eta\|p - q\|^2 \leq \langle Sp - Sq, p - q\rangle\ \forall p,q \in C$;
(e) of sequential weak continuity iff $\forall \{q_i\} \subset C$, the relation holds: $q_i \rightharpoonup q \Rightarrow Sq_i \rightharpoonup Sq$.

Obviously, each monotonicity mapping is of pseudomonotonicity. However, the inverse is false. It is known that $\forall q \in H$, $\exists |$ (nearest point) $P_C q \in C$ s.t. $\|q - p\| \geq \|q - P_C q\|$ $\forall p \in C$. P_C is refereed to as a nearest point (or metric) projection from H onto C. The statements below are valid (see [29]):

(a) $\|P_C q - P_C p\|^2 \leq \langle q - p, P_C q - P_C p\rangle \ \forall q, p \in H$;
(b) $p = P_C q \Leftrightarrow \langle q - p, t - p\rangle \leq 0 \ \forall q \in H, t \in C$;
(c) $\|q - P_C q\|^2 + \|P_C q - t\|^2 \geq \|q - t\|^2 \ \forall q \in H, t \in C$;
(d) $\|q - p\|^2 = \|q\|^2 - \|p\|^2 - 2\langle q - p, p\rangle \ \forall q, p \in H$;
(e) $\|sq + (1-s)p\|^2 = s\|q\|^2 + (1-s)\|p\|^2 - s(1-s)\|q - p\|^2 \ \forall q, p \in H, s \in [0,1]$.

The following concept and two propositions can be found in [30].

Definition 1. *Let $\{\xi_i\}_{i=1}^\infty \subset [0,1]$ and suppose that $\{S_i\}_{i=1}^\infty$ is a sequence of nonexpansive operators from C into itself. For any $k \geq 1$, the self-mapping W_k on C is constructed as follows:*

$$\begin{cases} U_{k,k+1} = I, \\ U_{k,k} = \xi_k S_k U_{k,k+1} + (1-\xi_k) I, \\ U_{k,k-1} = \xi_{k-1} S_{k-1} U_{k,k} + (1-\xi_{k-1}) I, \\ \ldots \\ U_{k,i} = \xi_i S_i U_{k,i+1} + (1-\xi_i) I, \\ \ldots \\ U_{k,2} = \xi_2 S_2 U_{k,3} + (1-\xi_2) I, \\ W_k = U_{k,1} = \xi_1 S_1 U_{k,2} + (1-\xi_1) I. \end{cases} \tag{3}$$

Then, W_k is refereed to as a W-operator fabricated by $S_k, \ldots, S_2, S_1$ and $\xi_k, \ldots, \xi_2, \xi_1$.

Proposition 1. *Let $\{\xi_i\}_{i=1}^\infty \subset (0,1]$ and suppose that $\{S_i\}_{i=1}^\infty$ is a sequence of nonexpansive operators from C into itself, such that $\bigcap_{i=1}^\infty \mathrm{Fix}(S_i) \neq \emptyset$. Then,*

(a) W_k is of nonexpansivity and $\bigcap_{i=1}^k \mathrm{Fix}(S_i) = \mathrm{Fix}(W_k)$ $\forall k \geq 1$;
(b) $\forall q \in C, i \geq 1$, $\lim_{k\to\infty} U_{k,i} q$ exists;
(c) the operator W, formulated as $Wq := \lim_{k\to\infty} W_k q = \lim_{k\to\infty} U_{k,1} q$ $\forall q \in C$, is a nonexpansive operator s.t. $\bigcap_{i=1}^\infty \mathrm{Fix}(S_i) = \mathrm{Fix}(W)$, and it is refereed to as the W-operator fabricated by $S_1, S_2, \ldots$ and $\xi_1, \xi_2, \ldots$.

Proposition 2. *Let $\{\xi_i\}_{i=1}^\infty \subset (0, \varsigma]$ for certain $\varsigma \in (0,1)$ and suppose that $\{S_i\}_{i=1}^\infty$ is a sequence of nonexpansive operators from C into itself, such that $\bigcap_{i=1}^\infty \mathrm{Fix}(S_i) \neq \emptyset$. Then, $\lim_{k\to\infty} \sup_{p\in D} \|W_k p - Wp\| = 0$ for each bounded set $D \subset C$.*

Throughout this paper, we always assume that $\{\xi_i\}_{i=1}^\infty \subset (0, \varsigma]$ for some $\varsigma \in (0,1)$. Later on, we will make use of the following lemmas to demonstrate our main results.

Lemma 1 ([28])**.** *Let H_1 and H_2 be two real Hilbert spaces. Suppose that $F : H_1 \to H_2$ is of uniform continuity on each boundedness subset of H_1 and D is of boundedness in H_1. Then, $F(D)$ is of boundedness.*

It is clear that the relation below holds for the inner product in H:

$$2\langle p, q + p\rangle + \|q\|^2 \geq \|q + p\|^2 \quad \forall q, p \in H. \tag{4}$$

Lemma 2 ([31])**.** *Each Hilbert space fulfills Opial's condition, that is, $\forall \{q_n\} \subset H$ with $q_n \rightharpoonup q$, the relation $\liminf_{n\to\infty} \|q_n - p\| > \liminf_{n\to\infty} \|q_n - q\|$ $\forall p \in H, p \neq q$ is true.*

Lemma 3 ([9])**.** *Suppose that $F : C \to H$ of both pseudomonotonicity and continuity, given $u \in C$. Then, the relation holds: $\langle Fu, v - u\rangle \geq 0 \ \forall v \in C \ \Leftrightarrow \ \langle Fv, v - u\rangle \geq 0 \ \forall v \in C$.*

Lemma 4 ([32]). *Suppose that the sequence $\{a_\iota\} \subset [0,\infty)$ is such that $\lambda_\iota\gamma_\iota + (1-\lambda_\iota)a_\iota \geq a_{\iota+1}$ $\forall \iota \geq 1$, where the real sequences $\{\lambda_\iota\}$ and $\{\gamma_\iota\}$ satisfy the conditions: (a) $\{\lambda_\iota\} \subset [0,1]$ and $\sum_{\iota=1}^{\infty} \lambda_\iota = \infty$, and (b) $\limsup_{\iota\to\infty} \gamma_\iota \leq 0$ or $\sum_{\iota=1}^{\infty} |\lambda_\iota\gamma_\iota| < \infty$. Then, $\lim_{\iota\to\infty} a_\iota = 0$.*

Lemma 5 ([33]). *Let E be a Banach space and admit a duality mapping of weak continuity. Suppose that C is convex and closed such that $\emptyset \neq C \subset E$, and that S is asymptotically nonexpansive self-mapping on C such that $\mathrm{Fix}(S) \neq \emptyset$. Then, $I - S$ is demiclosed at zero, i.e., for $\{q_\iota\} \subset C$ satisfying both $q_\iota \rightharpoonup q \in C$ and $(I-S)q_\iota \to 0$, one has $(I-S)q = 0$, with I being the identity operator of E.*

The following lemmas are very crucial to the convergence analysis of our designed rules.

Lemma 6 ([34]). *Suppose that $\{\Phi_m\}$ is a sequence in $\mathbf{R}$, which does not decrease at infinity, that is, $\exists\{\Phi_{m_\iota}\} \subset \{\Phi_m\}$ s.t. $\Phi_{m_\iota} < \Phi_{m_\iota+1}$ $\forall \iota \geq 1$. The sequence $\{\varphi(m)\}_{m\geq m_0}$ of integers is formulated below:*

$$\varphi(m) = \max\{\iota \leq m : \Phi_\iota < \Phi_{\iota+1}\},$$

where $m_0 \geq 1$ s.t. $\{\iota \leq m_0 : \Phi_\iota < \Phi_{\iota+1}\} \neq \emptyset$. Then, the statements hold below:
(a) $\varphi(m_0) \leq \varphi(m_0+1) \leq \cdots$ and $\varphi(m) \to \infty$;
(b) $\Phi_{\varphi(m)} \leq \Phi_{\varphi(m)+1}$ and $\Phi_m \leq \Phi_{\varphi(m)+1}$ $\forall m \geq m_0$.

Lemma 7 ([32]). *Given a number λ in $(0,1]$, suppose that S is a nonexpansive self-mapping on C, and $S^\lambda : C \to H$ is the operator formulated as $S^\lambda p := Sp - \lambda\rho F(Sp)$ $\forall p \in C$, with $F : C \to H$ being of both κ-Lipschitz continuity and η-strong monotonicity. Then, S^λ is a contractive map for $\rho \in (0, \frac{2\eta}{\kappa^2})$, that is, $\|S^\lambda q - S^\lambda p\| \leq (1-\lambda\zeta)\|q-p\|$ $\forall q,p \in C$, with $\zeta = 1 - \sqrt{1-\rho(2\eta-\rho\kappa^2)} \in (0,1]$.*

3. Criteria of Strong Convergence

In what follows, let us suppose that the conditions are valid below.

$\{S_i\}_{i=1}^{\infty}$ is a sequence of nonexpansive operators on H and S is asymptotically nonexpansive operator on H with $\{\theta_i\}$.

W_n is the W-operator constructed by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$, with $\{\xi_i\}_{i=1}^{\infty} \subset (0,\varsigma]$ for certain $\varsigma \in (0,1)$.

A is of both pseudomonotonicity and L-Lipschitz continuity on H, s.t. $\|Au\| \leq \liminf_{n\to\infty} \|Av_n\|$ for each $\{v_n\} \subset C$ with $v_n \rightharpoonup u$.

g is a δ-contractive map on H with $\delta \in [0,1)$, and F is of η-strong monotonicity and κ-Lipschitz continuity on H s.t. $\delta < \zeta := 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)}$ with $0 < \mu < \frac{2\eta}{\kappa^2})$.

$\Omega = \mathrm{VI}(C,A) \cap (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)) \neq \emptyset$ where $S_0 := S$.

$\{\epsilon_n\}, \{\sigma_n\} \subset [0,1]$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ with $\alpha_n + \beta_n < 1$ $\forall n \geq 1$, s.t.

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sup_{n\geq 1}(\epsilon_n/\alpha_n) < \infty$ and $\lim_{n\to\infty}(\theta_n/\alpha_n) = 0$;
(iii) $1 > \limsup_{n\to\infty} \sigma_n \geq \liminf_{n\to\infty} \sigma_n > 0$;
(iv) $1 > \limsup_{n\to\infty} \beta_n \geq \liminf_{n\to\infty} \beta_n > 0$.

Lemma 8. *The linear-search process (6) in the following Algorithm 4 is well formulated, and the relation holds: $\gamma \geq \varsigma_n \geq \min\{\frac{\nu l}{L}, \gamma\}$.*

Proof. Note that $\|Au_n - AP_C(u_n - \gamma l^m Au_n)\| \leq L\|u_n - P_C(u_n - \gamma l^m Au_n)\|$. Then, (6) is valid for each $\gamma l^m \leq \frac{\nu}{L}$ and ς_n is well defined. Clearly, $\varsigma_n \leq \gamma$. When $\varsigma_n = \gamma$, the conclusion is true. When $\varsigma_n < \gamma$, from (6), one obtains $\|Au_n - AP_C(u_n - \frac{\varsigma_n}{l}Au_n)\| > \frac{\nu}{(\varsigma_n/l)}\|u_n - P_C(u_n - \frac{\varsigma_n}{l}Au_n)\|$, which immediately yields $\varsigma_n > \frac{\nu l}{L}$. Thus, the conclusion is true. $\square$

Lemma 9. *Suppose that the sequences $\{w_n\}, \{u_n\}, \{y_n\}, \{q_n\}$ are constructed in Algorithm 4. Then,*

$$\begin{aligned} \|q_n - p\|^2 &\le \|w_n - p\|^2 - (1-\nu)\|y_n - q_n\|^2 - (1-\nu)\|y_n - u_n\|^2 \\ &\quad - \sigma_n(1-\sigma_n)\|w_n - W_n w_n\|^2 \quad \forall p \in \Omega. \end{aligned} \tag{5}$$

Algorithm 4 The 1st modified Mann-type subgradient extragradient rule

Initial Steps: Let $l \in (0,1)$, $\nu \in (0,1)$, $\gamma \in (0,\infty)$, given any starting points x_1, x_0 in H.
Iterations: Calculate x_{n+1} $(n \ge 1)$ below:
Step 1. Set $w_n = x_n + \epsilon_n(x_n - x_{n-1})$ and $u_n = (1-\sigma_n)w_n + \sigma_n W_n w_n$, and calculate $y_n = P_C(u_n - \varsigma_n A u_n)$, with ς_n being picked to be the largest $\varsigma \in \{\gamma, \gamma l, \gamma l^2, \ldots\}$ s.t.

$$\varsigma\|Au_n - Ay_n\| \le \nu\|u_n - y_n\|. \tag{6}$$

Step 2. Calculate $q_n = P_{C_n}(u_n - \varsigma_n A y_n)$ with $C_n := \{y \in H : \langle u_n - \varsigma_n A u_n - y_n, y_n - y\rangle \ge 0\}$.
Step 3. Calculate

$$x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F)S^n q_n. \tag{7}$$

Put $n := n+1$ and return to Step 1.

Proof. It is clear that $C_n \supset C \supset \Omega$. Observe that $\forall p \in \Omega$,

$$\begin{aligned} &\|q_n - p\|^2 = \|P_{C_n}(u_n - \varsigma_n A y_n) - P_{C_n} p\|^2 \\ &\le \langle q_n - p, u_n - \varsigma_n A y_n - p\rangle \\ &= \tfrac{1}{2}(\|q_n - p\|^2 + \|u_n - p\|^2 - \|q_n - u_n\|^2) - \varsigma_n\langle q_n - p, A y_n\rangle. \end{aligned}$$

Thus, one has

$$\|q_n - p\|^2 \le \|u_n - p\|^2 - \|q_n - u_n\|^2 - 2\varsigma_n\langle q_n - p, A y_n\rangle. \tag{8}$$

Thanks to $q_n = P_{C_n}(u_n - \varsigma_n A y_n)$ where $C_n := \{y \in H : \langle u_n - \varsigma_n A u_n - y_n, y_n - y\rangle \ge 0\}$, one obtains $\langle u_n - \varsigma_n A u_n - y_n, y_n - q_n\rangle \ge 0$. Using the pseudomonotonicity of A, from (8) and (6), we deduce that

$$\begin{aligned} \|q_n - p\|^2 &\le \|u_n - p\|^2 - \|q_n - u_n\|^2 - 2\varsigma_n\langle A y_n, y_n - p + q_n - y_n\rangle \\ &\le \|u_n - p\|^2 - \|q_n - u_n\|^2 - 2\varsigma_n\langle A y_n, q_n - y_n\rangle \\ &= \|u_n - p\|^2 - \|q_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\langle u_n - \varsigma_n A y_n - y_n, q_n - y_n\rangle \\ &= \|u_n - p\|^2 - \|q_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\langle u_n - \varsigma_n A u_n - y_n, q_n - y_n\rangle \\ &\quad + 2\varsigma_n\langle A u_n - A y_n, q_n - y_n\rangle \\ &\le \|u_n - p\|^2 - \|q_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\nu\|u_n - y_n\|\|q_n - y_n\| \\ &\le \|u_n - p\|^2 - \|q_n - y_n\|^2 - \|y_n - u_n\|^2 + \nu(\|u_n - y_n\|^2 + \|q_n - y_n\|^2) \\ &= \|u_n - p\|^2 - (1-\nu)\|y_n - q_n\|^2 - (1-\nu)\|y_n - u_n\|^2. \end{aligned} \tag{9}$$

Owing to $u_n = (1-\sigma_n)w_n + \sigma_n W_n w_n$, one has

$$\begin{aligned} \|u_n - p\|^2 &= (1-\sigma_n)\|w_n - p\|^2 + \sigma_n\|W_n w_n - p\|^2 - \sigma_n(1-\sigma_n)\|w_n - W_n w_n\|^2 \\ &\le (1-\sigma_n)\|w_n - p\|^2 + \sigma_n\|w_n - p\|^2 - \sigma_n(1-\sigma_n)\|w_n - W_n w_n\|^2 \\ &= \|w_n - p\|^2 - \sigma_n(1-\sigma_n)\|w_n - W_n w_n\|^2. \end{aligned}$$

Consequently, this, together with (9), ensures that inequality (5) is true. □

Lemma 10. *Suppose that $\{w_n\}, \{u_n\}, \{x_n\}, \{q_n\}$ are boundedness sequences constructed in Algorithm 4. Assume that $S^n x_n - S^{n+1} x_n \to 0$, $u_n - x_n \to 0$, $w_n - q_n \to 0$ and $x_n - x_{n+1} \to 0$. Then, $\omega_w(\{x_n\}) \subset \Omega$, where $\omega_w(\{x_n\}) = \{q \in H : x_{n_l} \rightharpoonup q$ for some $\{x_{n_l}\} \subset \{x_n\}\}$.*

Proof. Take a fixed $q \in \omega_w(\{x_n\})$ arbitrarily. Then, $\exists\{x_{n_l}\} \subset \{x_n\}$ s.t. $x_{n_l} \rightharpoonup q \in H$. Thanks to $u_n - x_n \to 0$, we know that $\exists\{u_{n_l}\} \subset \{u_n\}$ s.t. $u_{n_l} \rightharpoonup q \in H$. In what follows, we claim $q \in \Omega$. In fact, by Lemma 9, we obtain that, for each $p \in \Omega$,

$$\begin{aligned}&(1-\nu)\|y_n - q_n\|^2 + (1-\nu)\|y_n - u_n\|^2 + \sigma_n(1-\sigma_n)\|w_n - W_n w_n\|^2 \\ &\leq \|w_n - p\|^2 - \|q_n - p\|^2 \leq \|w_n - q_n\|(\|w_n - p\| + \|q_n - p\|).\end{aligned}$$

Since $w_n - q_n \to 0$, $\nu \in (0,1)$ and $1 > \limsup_{n\to\infty} \sigma_n \geq \liminf_{n\to\infty} \sigma_n > 0$, from boundedness of $\{w_n\}, \{q_n\}$, we deduce that

$$\lim_{n\to\infty} \|y_n - q_n\| = \lim_{n\to\infty} \|y_n - u_n\| = \lim_{n\to\infty} \|w_n - W_n w_n\| = 0.$$

This immediately yields

$$\|u_n - q_n\| \leq \|u_n - y_n\| + \|y_n - q_n\| \to 0 \quad (n \to \infty).$$

Clearly, one has $\|w_n - x_n\| = \epsilon_n \|x_n - x_{n-1}\| \to 0$ (due to $\sup_{n\geq 1}(\epsilon_n/\alpha_n) < \infty$). Hence, we have

$$\begin{aligned}\|W_n x_n - x_n\| &\leq \|W_n x_n - W_n w_n\| + \|W_n w_n - w_n\| + \|w_n - x_n\| \\ &\leq 2\|w_n - x_n\| + \|W_n w_n - w_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{10}$$

Noticing $x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F)S^n q_n$, we obtain $x_{n+1} - S^n q_n = \alpha_n g(x_n) + \beta_n(x_n - S^n q_n) - \alpha_n \mu F S^n q_n$, which immediately yields

$$\begin{aligned}\|x_n - S^n q_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S^n q_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n \|g(x_n)\| + \beta_n \|x_n - S^n q_n\| + \alpha_n \|\mu F S^n q_n\|.\end{aligned}$$

Thus, it follows that

$$(1-\beta_n)\|x_n - S^n q_n\| \leq \|x_n - x_{n+1}\| + \alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|).$$

Since $x_{n+1} - x_n \to 0$, $\alpha_n \to 0$, $\liminf_{n\to\infty}(1-\beta_n) > 0$ and $\{x_n\}, \{q_n\}$ are of boundedness, one obtains

$$\lim_{n\to\infty} \|x_n - S^n q_n\| = 0.$$

We claim $\|x_n - Sx_n\| \to 0$ $(n \to \infty)$. Indeed, using the asymptotical nonexpansivity of S, one deduces that

$$\begin{aligned}\|x_n - Sx_n\| &\leq \|x_n - S^n q_n\| + \|S^n q_n - S^n x_n\| + \|S^n x_n - S^{n+1} x_n\| \\ &\quad + \|S^{n+1} x_n - S^{n+1} q_n\| + \|S^{n+1} q_n - Sx_n\| \\ &\leq \|x_n - S^n q_n\| + (1+\theta_n)\|q_n - x_n\| + \|S^n x_n - S^{n+1} x_n\| \\ &\quad + (1+\theta_{n+1})\|x_n - q_n\| + (1+\theta_1)\|S^n q_n - x_n\| \\ &= (2+\theta_1)\|x_n - S^n q_n\| + (2+\theta_n+\theta_{n+1})\|q_n - x_n\| + \|S^n x_n - S^{n+1} x_n\| \\ &\leq (2+\theta_1)\|x_n - S^n q_n\| + (2+\theta_n+\theta_{n+1})(\|q_n - u_n\| + \|u_n - x_n\|) \\ &\quad + \|S^n x_n - S^{n+1} x_n\|.\end{aligned}$$

Since $u_n - x_n \to 0$, $u_n - q_n \to 0$ and $x_n - S^n q_n \to 0$, we obtain

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{11}$$

In addition, let us show that $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$. In fact, note that

$$\|Wx_n - x_n\| \leq \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\| \leq \sup_{u\in D} \|Wu - W_n u\| + \|W_n x_n - x_n\|,$$

where $D = \{x_n : n \geq 1\}$. Using Proposition 2, from (10), we obtain

$$\lim_{n\to\infty} \|Wx_n - x_n\| = 0. \tag{12}$$

□

In what follows, we claim $q \in \mathrm{VI}(C, A)$. Indeed, noticing $u_n - y_n \to 0$ and $u_{n_l} \rightharpoonup q$, we have $y_{n_l} \rightharpoonup q$. In addition, noticing $\{y_n\} \subset C$ and $y_{n_l} \rightharpoonup q$, by the convexity and closedness of C, one obtains $q \in C$. Next, we discuss two situations. When $Aq = 0$, it is readily known that $q \in \mathrm{VI}(C, A)$ (due to $\langle Aq, y - q\rangle \geq 0\ \forall y \in C$).

Let $Aq \neq 0$. Since $y_{n_l} \rightharpoonup q$ as $l \to \infty$, using the hypothesis on A, one obtains $\liminf_{l\to\infty} \|Ay_{n_l}\| \geq \|Aq\| > 0$. Hence, one might assume $\|Ay_{n_l}\| \neq 0\ \forall l \geq 1$. Moreover, using $y_n = P_C(u_n - \varsigma_n Au_n)$, one has $\langle u_n - \varsigma_n Au_n - y_n, y - y_n\rangle \leq 0\ \forall y \in C$, and hence

$$\frac{1}{\varsigma_n}\langle u_n - y_n, y - y_n\rangle + \langle Au_n, y_n - u_n\rangle \leq \langle Au_n, y - u_n\rangle \quad \forall y \in C. \tag{13}$$

Since A is uniform continuous, $\{Au_n\}$ is of boundedness (by Lemma 1). Noticing the boundedness of $\{y_n\}$, by Lemma 8 and (13), one obtains $\liminf_{l\to\infty}\langle Au_{n_l}, y - u_{n_l}\rangle \geq 0\ \forall y \in C$. In addition, it is readily known that $\langle Ay_n, y - y_n\rangle = \langle Ay_n - Au_n, y - u_n\rangle + \langle Au_n, y - u_n\rangle + \langle Ay_n, u_n - y_n\rangle$. Note that $u_n - y_n \to 0$ and A is uniform continuous. Thus, one obtains $Ay_n - Au_n \to 0$. This hence arrives at $\liminf_{l\to\infty}\langle Ay_{n_l}, y - y_{n_l}\rangle \geq 0\ \forall y \in C$.

In order to demonstrate $q \in \mathrm{VI}(C, A)$, one chooses $\{\kappa_l\} \subset (0, 1)$ s.t. $\kappa_l \downarrow 0\ (l \to \infty)$. For each l, one denotes by m_l the smallest natural number satisfying

$$\langle Ay_{n_i}, y - y_{n_i}\rangle + \kappa_l \geq 0 \quad \forall i \geq m_l. \tag{14}$$

Note that $\{\kappa_l\}$ is of decreasement. Thus, it is readily known that $\{m_l\}$ is an increasing. Using $Ay_{m_l} \neq 0\ \forall l \geq 1$ (owing to $\{Ay_{m_l}\} \subset \{Ay_{n_l}\}$), we set $v_{m_l} = \frac{Ay_{m_l}}{\|Ay_{m_l}\|^2}$, and obtain $\langle Ay_{m_l}, v_{m_l}\rangle = 1\ \forall l \geq 1$. Thus, from (14), one obtains $\langle Ay_{m_l}, y + \kappa_l v_{m_l} - y_{m_l}\rangle \geq 0\ \forall l \geq 1$. In addition, by the pseudomonotonicity of A, one has $\langle A(y + \kappa_l v_{m_l}), y + \kappa_l v_{m_l} - y_{m_l}\rangle \geq 0\ \forall l \geq 1$. This immediately arrives at

$$\langle Ay, y - y_{m_l}\rangle \geq \langle Ay - A(y + \kappa_l v_{m_l}), y + \kappa_l v_{m_l} - y_{m_l}\rangle - \kappa_l\langle Ay, v_{m_l}\rangle \quad \forall l \geq 1. \tag{15}$$

We show that $\lim_{l\to\infty} \kappa_l v_{m_l} = 0$. In fact, from $u_{n_l} \rightharpoonup q \in C$ and $u_n - y_n \to 0$, we obtain $y_{n_l} \rightharpoonup q$. Note that $\{y_{m_l}\} \subset \{y_{n_l}\}$ and $\kappa_l \downarrow 0\ (l \to \infty)$. Thus, one deduces that $0 \leq \limsup_{l\to\infty} \|\kappa_l v_{m_l}\| = \limsup_{l\to\infty} \frac{\kappa_l}{\|Ay_{m_l}\|} \leq \frac{\limsup_{l\to\infty} \kappa_l}{\liminf_{l\to\infty} \|Ay_{n_l}\|} = 0$. Therefore, one obtains $\kappa_l v_{m_l} \to 0\ (l \to \infty)$. Note that A is uniformly continuous, the sequences $\{y_{m_l}\}, \{v_{m_l}\}$ are of boundedness, and $\lim_{l\to\infty} \kappa_l v_{m_l} = 0$. Consequently, letting $l \to \infty$, one concludes that $\langle Ay, y - q\rangle = \liminf_{l\to\infty}\langle Ay, y - y_{m_l}\rangle \geq 0\ \forall y \in C$. By Lemma 3, one has $q \in \mathrm{VI}(C, A)$.

Next, we show that $q \in \Omega$. In fact, since (11) guarantees $x_{n_l} - Sx_{n_l} \to 0$, by Lemma 5, we obtain the demiclosedness of $I - S$ at zero. Thus, from $x_{n_l} \rightharpoonup q$, one obtains $(I - S)q = 0$, that is, $q \in \mathrm{Fix}(S)$. In addition, we claim $q \in \mathrm{Fix}(W) = \bigcap_{i=1}^\infty \mathrm{Fix}(S_i)$. Conversely, we suppose that $q \notin \mathrm{Fix}(W)$, that is, $Wq \neq q$. Using Lemma 2 and Proposition 1 (c), we obtain

$$\liminf_{l\to\infty} \|x_{n_l} - q\| < \liminf_{l\to\infty} \|x_{n_l} - Wq\| \leq \liminf_{l\to\infty}(\|x_{n_l} - Wx_{n_l}\| + \|Wx_{n_l} - Wq\|),$$

which together with (12) yields $\liminf_{l\to\infty} \|x_{n_l} - q\| < \liminf_{l\to\infty} \|x_{n_l} - q\|$, which leads to a contradiction. Thus, one has $q \in \bigcap_{i=1}^\infty \mathrm{Fix}(S_i)$. Consequently, $q \in \bigcap_{i=0}^\infty \mathrm{Fix}(S_i) \cap \mathrm{VI}(C, A)$, that is, $q \in \Omega$.

Theorem 1. *Suppose that $\{x_n\}$ is the sequence constructed in Algorithm 4. Then,*

$$x_n \to q^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} S^n x_n - S^{n+1} x_n \to 0, \\ \sup_{n\geq 1} \|x_{n-1} - x_n\| < \infty, \end{cases}$$

with $q^* \in \Omega$ being only a solution of the HFPP: $q^* = P_\Omega(I - \mu F + g)q^*$.

Proof. Because $1 > \limsup_{n\to\infty} \sigma_n \geq \liminf_{n\to\infty} \sigma_n > 0$ and $\lim_{n\to\infty} \theta_n/\alpha_n = 0$, we might suppose that $\{\sigma_n\} \subset [\bar{a}, \bar{b}] \subset (0,1)$ and $\beta_n(\zeta - \delta)/2 \geq \theta_n$ for all n. Let us show that $P_\Omega(I - \mu F + g) : H \to H$ is the contractive map on H. Indeed, using Lemma 7, one has

$$\|P_\Omega(I - \mu F + g)u - P_\Omega(I - \mu F + g)v\| \leq [1 - (\zeta - \delta)]\|u - v\| \quad \forall u, v \in H.$$

This ensures that $P_\Omega(I - \mu F + g)$ is a contractive map. Thus, it is readily known that there exists $q^* \in H$, which is only a fixed point of $P_\Omega(I - \mu F + g)$, that is, $q^* = P_\Omega(I - \mu F + g)q^*$. That is, there exists $q^* \in \Omega$, which is only a solution to the following VIP:

$$\langle (\mu F - g)q^*, q - q^* \rangle \geq 0 \quad \forall q \in \Omega. \tag{16}$$

We first show the necessity of the theorem. In fact, when $x_n \to q^* \in \Omega$, we know that $q^* = Sq^*$ and

$$\begin{aligned} \|S^n x_n - S^{n+1}x_n\| &\leq \|S^n x_n - q^*\| + \|q^* - S^{n+1}x_n\| \\ &\leq (1 + \theta_n)\|x_n - q^*\| + (1 + \theta_{n+1})\|q^* - x_n\| \\ &= (2 + \theta_n + \theta_{n+1})\|x_n - q^*\| \to 0 \quad (n \to \infty). \end{aligned}$$

Since $\|x_{n+1} - q^*\| + \|q^* - x_n\| \geq \|x_{n+1} - x_n\|$, one has

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

This immediately yields $\sup_{n\geq 1} \|x_{n-1} - x_n\| < \infty$.

In what follows, we claim the sufficiency of the theorem. To the goal, under the assumption $S^n x_n - S^{n+1}x_n \to 0$ with $\sup_{n\geq 1} \|x_{n-1} - x_n\| < \infty$, we divide the remainder of the proof into several claims. □

Claim 1. One claims the boundedness of $\{x_n\}$. In fact, picking a $q \in \Omega$ arbitrarily, one has that $Sq = q$, $W_n q = q$, and (5) leads to

$$\begin{aligned} \|q_n - q\|^2 \leq\ & \|w_n - q\|^2 - (1 - \nu)\|y_n - q_n\|^2 - (1 - \nu)\|y_n - u_n\|^2 \\ & - \sigma_n(1 - \sigma_n)\|w_n - W_n w_n\|^2 \quad \forall q \in \Omega, \end{aligned} \tag{17}$$

which hence yields

$$\|q_n - q\| \leq \|w_n - q\|. \tag{18}$$

By the formulation of w_n, one obtains

$$\|w_n - q\| \leq \|x_n - q\| + \epsilon_n\|x_n - x_{n-1}\| = \|x_n - q\| + \alpha_n \cdot \frac{\epsilon_n}{\alpha_n}\|x_n - x_{n-1}\|. \tag{19}$$

Noticing $\sup_{n\geq 1}(\epsilon_n/\alpha_n) < \infty$ and $\sup_{n\geq 1} \|x_n - x_{n-1}\| < \infty$, one obtains $\sup_{n\geq 1}(\epsilon_n/\alpha_n)\|x_n - x_{n-1}\| < \infty$, which guarantees that $\exists M_1 > 0$ s.t.

$$M_1 \geq \frac{\epsilon_n}{\alpha_n}\|x_n - x_{n-1}\|. \tag{20}$$

From (18)–(20), one obtains

$$\|q_n - q\| \leq \|w_n - q\| \leq \|x_n - q\| + \alpha_n M_1. \tag{21}$$

In addition, observe that

$$\|u_n - q\| \leq (1 - \sigma_n)\|w_n - q\| + \sigma_n\|W_n w_n - q\| \leq \|w_n - q\|, \tag{22}$$

which, together with (9) and (21), yields

$$\|q_n - q\| \le \|u_n - q\| \le \|w_n - q\| \le \|x_n - q\| + \alpha_n M_1. \quad (23)$$

Thus, using (23) and $\alpha_n + \beta_n < 1 \; \forall n \ge 1$, from Lemma 7, we obtain

$$\begin{aligned}
&\|x_{n+1} - q\| = \|\alpha_n g(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F)S^n q_n - q\| \\
&= \|\alpha_n(g(x_n) - q) + \beta_n(x_n - q) + (1 - \alpha_n - \beta_n)\{\tfrac{1-\beta_n}{1-\alpha_n-\beta_n} \\
&\quad \times [(I - \tfrac{\alpha_n}{1-\beta_n}\mu F)S^n q_n - (I - \tfrac{\alpha_n}{1-\beta_n}\mu F)q] + \tfrac{\alpha_n}{1-\alpha_n-\beta_n}(I - \mu F)q\}\| \\
&= \|\alpha_n(g(x_n) - g(p)) + \beta_n(x_n - q) + (1-\beta_n) \\
&\quad \times [(I - \tfrac{\alpha_n}{1-\beta_n}\mu F)S^n q_n - (I - \tfrac{\alpha_n}{1-\beta_n}\mu F)q] + \alpha_n(f - \mu F)q\| \\
&\le \alpha_n\|g(x_n) - g(q)\| + \beta_n\|x_n - q\| + (1-\beta_n) \\
&\quad \times \|(I - \tfrac{\alpha_n}{1-\beta_n}\mu F)S^n q_n - (I - \tfrac{\alpha_n}{1-\beta_n}\mu F)q\| + \alpha_n\|(g - \mu F)q\| \\
&\le \alpha_n\delta\|x_n - q\| + \beta_n\|x_n - q\| + (1-\beta_n) \\
&\quad \times (1 - \tfrac{\alpha_n}{1-\beta_n}\zeta)(1+\theta_n)\|q_n - q\| + \alpha_n\|(g - \mu F)q\| \\
&\le \alpha_n\delta\|x_n - q\| + \beta_n(\|x_n - q\| + \alpha_n M_1) + (1 - \beta_n - \alpha_n\zeta) \\
&\quad \times (\|x_n - q\| + \alpha_n M_1) + \theta_n\|q_n - q\| + \alpha_n\|(g - \mu F)q\| \\
&\le [\alpha_n\delta + \beta_n + (1 - \beta_n - \alpha_n\zeta)]\|x_n - q\| + \alpha_n M_1 \\
&\quad + \tfrac{\alpha_n(\zeta-\delta)(\|x_n - q\| + \alpha_n M_1)}{2} + \alpha_n\|(g - \mu F)q\| \\
&\le [1 - \tfrac{\alpha_n(\zeta-\delta)}{2}]\|x_n - q\| + \alpha_n(2M_1 + \|(g - \mu F)q\|) \\
&= [1 - \tfrac{\alpha_n(\zeta-\delta)}{2}]\|x_n - q\| + \tfrac{\alpha_n(\zeta-\delta)}{2} \cdot \tfrac{2(2M_1 + \|(g-\mu F)q\|)}{\zeta-\delta} \\
&\le \max\{\|x_n - q\|, \tfrac{2(2M_1 + \|(g-\mu F)q\|)}{\zeta-\delta}\},
\end{aligned}$$

which immediately arrives at

$$\|x_n - q\| \le \max\{\|x_1 - q\|, \frac{2(2M_1 + \|(g - \mu F)q\|)}{\zeta - \delta}\} \quad \forall n \ge 1.$$

Therefore, one obtains the boundedness of $\{x_n\}$. This ensures that $\{w_n\}$, $\{u_n\}$, $\{y_n\}$, $\{q_n\}$, $\{g(x_n)\}$, $\{Ay_n\}$ and $\{W_n w_n\}$, $\{S^n q_n\}$ are bounded.

Claim 2. One claims that

$$\begin{aligned}
&(1 - \beta_n - \alpha_n\zeta)(1+\theta_n)\{(1-\nu)(\|y_n - q_n\|^2 + \|y_n - u_n\|^2) + \sigma_n(1-\sigma_n)\|w_n - W_n w_n\|^2\} \\
&\le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + (\alpha_n + \theta_n)M_4,
\end{aligned}$$

for certain $M_4 > 0$. In fact, one has

$$\begin{aligned}
x_{n+1} - q &= \alpha_n(g(x_n) - q) + \beta_n(x_n - q) + (1 - \alpha_n - \beta_n)\{\tfrac{1-\beta_n}{1-\alpha_n-\beta_n} \\
&\quad \times [(I - \tfrac{\alpha_n}{1-\beta_n}\mu F)S^n q_n - (I - \tfrac{\alpha_n}{1-\beta_n}\mu F)q] + \tfrac{\alpha_n}{1-\alpha_n-\beta_n}(I - \mu F)q\} \\
&= \alpha_n(g(x_n) - g(q)) + \beta_n(x_n - q) + (1-\beta_n) \\
&\quad \times [(I - \tfrac{\alpha_n}{1-\beta_n}\mu F)S^n q_n - (I - \tfrac{\alpha_n}{1-\beta_n}\mu F)q] + \alpha_n(g - \mu F)q.
\end{aligned}$$

Using the convex property of $\phi(s) = s^2 \; \forall s \in \mathbf{R}$, one obtains

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \|\alpha_n(g(x_n) - g(q)) + \beta_n(x_n - q) + (1-\beta_n) \\
&\quad \times [(I - \tfrac{\alpha_n}{1-\beta_n}\mu F)S^n q_n - (I - \tfrac{\alpha_n}{1-\beta_n}\mu F)q]\|^2 + 2\alpha_n\langle(g - \mu F)q, x_{n+1} - q\rangle \\
&\le [\alpha_n\delta\|x_n - q\| + \beta_n\|x_n - q\| + (1-\beta_n) \\
&\quad \times (1 - \tfrac{\alpha_n}{1-\beta_n}\zeta)(1+\theta_n)\|q_n - q\|]^2 + 2\alpha_n\langle(g - \mu F)q, x_{n+1} - q\rangle \\
&= [\alpha_n\delta\|x_n - q\| + \beta_n\|x_n - q\| + (1 - \beta_n - \alpha_n\zeta)(1+\theta_n)\|q_n - q\|]^2 \\
&\quad + 2\alpha_n\langle(g - \mu F)q, x_{n+1} - q\rangle \\
&\le \alpha_n\delta\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 + (1 - \beta_n - \alpha_n\zeta)(1+\theta_n)\|q_n - q\|^2 + \alpha_n M_2
\end{aligned} \quad (24)$$

(because of $\alpha_n\delta + \beta_n + (1-\beta_n-\alpha_n\zeta)(1+\theta_n) \leq 1+\alpha_n(\delta-\zeta)+\theta_n \leq 1-\frac{\alpha_n(\zeta-\delta)}{2}$), with $\sup_{n\geq 1} 2\|(g-\mu F)q\|\|x_n-q\| \leq M_2$ for certain $M_2 > 0$. Combining (17) and (24), one obtains

$$\begin{aligned}\|x_{n+1}-q\|^2 \leq{}& \alpha_n\delta\|x_n-q\|^2+\beta_n\|x_n-q\|^2+(1-\beta_n-\alpha_n\zeta)(1+\theta_n)[\|w_n-q\|^2\\&-(1-\nu)\|y_n-q_n\|^2-(1-\nu)\|y_n-u_n\|^2-\sigma_n(1-\sigma_n)\|w_n-W_nw_n\|^2]+\alpha_nM_2.\end{aligned} \tag{25}$$

In addition, from (23), we have

$$\begin{aligned}\|w_n-q\|^2 &\leq (\|x_n-q\|+\alpha_nM_1)^2\\&= \|x_n-q\|^2+\alpha_n(2M_1\|x_n-q\|+\alpha_nM_1^2)\\&\leq \|x_n-q\|^2+\alpha_nM_3,\end{aligned} \tag{26}$$

where $\sup_{n\geq 1}(2M_1\|x_n-q\|+\alpha_nM_1^2) \leq M_3$ for certain $M_3 > 0$. From (25) and (26), one obtains

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq \alpha_n\delta\|x_n-q\|^2+\beta_n[\|x_n-q\|^2+\alpha_nM_3]+(1-\beta_n-\alpha_n\zeta)(1+\theta_n)[\|x_n-q\|^2\\&\quad+\alpha_nM_3]-(1-\beta_n-\alpha_n\zeta)(1+\theta_n)[(1-\nu)\|y_n-q_n\|^2+(1-\nu)\|y_n-u_n\|^2\\&\quad+\sigma_n(1-\sigma_n)\|w_n-W_nw_n\|^2]+\alpha_nM_2\\&\leq [1-\alpha_n(\zeta-\delta)]\|x_n-q\|^2-(1-\beta_n-\alpha_n\zeta)(1+\theta_n)[(1-\nu)(\|y_n-q_n\|^2\\&\quad+\|y_n-u_n\|^2)+\sigma_n(1-\sigma_n)\|w_n-W_nw_n\|^2]+(\alpha_n+\theta_n)M_4\\&\leq \|x_n-q\|^2-(1-\beta_n-\alpha_n\zeta)(1+\theta_n)[(1-\nu)(\|y_n-q_n\|^2+\|y_n-u_n\|^2)\\&\quad+\sigma_n(1-\sigma_n)\|w_n-W_nw_n\|^2]+(\alpha_n+\theta_n)M_4,\end{aligned}$$

where $\sup_{n\geq 1}(\|x_n-q\|^2+M_3+M_2) \leq M_4$ for certain $M_4 > 0$. Consequently,

$$\begin{aligned}&(1-\beta_n-\alpha_n\zeta)(1+\theta_n)\{(1-\nu)(\|y_n-q_n\|^2+\|y_n-u_n\|^2)+\sigma_n(1-\sigma_n)\|w_n-W_nw_n\|^2\}\\&\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2+(\alpha_n+\theta_n)M_4.\end{aligned} \tag{27}$$

Claim 3. One claims that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq [1-\alpha_n(\zeta-\delta)]\|x_n-q\|^2+\alpha_n(\zeta-\delta)\{\tfrac{2}{\zeta-\delta}\langle(g-\mu F)q, x_{n+1}-q\rangle\\&\quad+\tfrac{M}{\zeta-\delta}(\tfrac{\epsilon_n}{\alpha_n}3\|x_n-x_{n-1}\|+\tfrac{\theta_n}{\alpha_n})\}\end{aligned}$$

for some $M > 0$. In fact, one has

$$\begin{aligned}\|w_n-q\|^2 &\leq (\|x_n-q\|+\epsilon_n\|x_n-x_{n-1}\|)^2\\&= \|x_n-q\|^2+\epsilon_n\|x_n-x_{n-1}\|(2\|x_n-q\|+\epsilon_n\|x_n-x_{n-1}\|).\end{aligned} \tag{28}$$

Using (23), (24) and (28), one obtains

$$\begin{aligned}&\|x_{n+1}-q\|^2\\&\leq \alpha_n\delta\|x_n-q\|^2+\beta_n\|x_n-q\|^2+(1-\beta_n-\alpha_n\zeta)(1+\theta_n)\|q_n-q\|^2\\&\quad+2\alpha_n\langle(g-\mu F)q, x_{n+1}-q\rangle\\&\leq \alpha_n\delta\|x_n-q\|^2+\beta_n\|x_n-q\|^2+(1-\beta_n-\alpha_n\zeta)\|w_n-q\|^2\\&\quad+\theta_n\|q_n-q\|^2+2\alpha_n\langle(g-\mu F)q, x_{n+1}-q\rangle\\&\leq \alpha_n\delta\|x_n-q\|^2+(1-\alpha_n\zeta)[\|x_n-q\|^2+\epsilon_n\|x_n-x_{n-1}\|(2\|x_n-q\|\\&\quad+\epsilon_n\|x_n-x_{n-1}\|)]+\theta_n\|q_n-q\|^2+2\alpha_n\langle(g-\mu F)q, x_{n+1}-q\rangle\\&\leq [1-\alpha_n(\zeta-\delta)]\|x_n-q\|^2+\epsilon_n\|x_n-x_{n-1}\|(2\|x_n-q\|\\&\quad+\epsilon_n\|x_n-x_{n-1}\|)+\theta_n\|q_n-q\|^2+2\alpha_n\langle(g-\mu F)q, x_{n+1}-q\rangle\\&\leq [1-\alpha_n(\zeta-\delta)]\|x_n-q\|^2+(\epsilon_n\|x_n-x_{n-1}\|3+\theta_n)M+2\epsilon_n\langle(g-\mu F)q, x_{n+1}-q\rangle\\&= [1-\alpha_n(\zeta-\delta)]\|x_n-q\|^2+\alpha_n(\zeta-\delta)[\tfrac{2\langle(g-\mu F)q, x_{n+1}-q\rangle}{\zeta-\delta}+\tfrac{M}{\zeta-\delta}(\tfrac{\epsilon_n}{\alpha_n}3\|x_n-x_{n-1}\|+\tfrac{\theta_n}{\alpha_n})]\end{aligned} \tag{29}$$

with $\sup_{n\geq 1}\{\|x_n-q\|, \epsilon_n\|x_n-x_{n-1}\|, \|q_n-q\|^2\} \leq M$ for certain $M > 0$.

Claim 4. One claims that $x_n \to q^* \in \Omega$, which is only a solution to the HFPP: $q^* = P_\Omega(I - \mu F + g)q^*$. In fact, using (29) with $q = q^*$, one obtains

$$\begin{aligned} \|x_{n+1} - q^*\|^2 &\le [1 - \alpha_n(\zeta - \delta)]\|x_n - q^*\|^2 + \alpha_n(\zeta - \delta)[\tfrac{2\langle (g-\mu F)q^*, x_{n+1}-q^*\rangle}{\zeta-\delta} \\ &\quad + \tfrac{M}{\zeta-\delta}(\tfrac{\epsilon_n}{\alpha_n}3\|x_n - x_{n-1}\| + \tfrac{\theta_n}{\alpha_n})]. \end{aligned} \tag{30}$$

Putting $\Phi_n = \|x_n - q^*\|^2$, one demonstrates $\Phi_n \to 0$ $(n \to \infty)$ in both aspects below.

Aspect 1. Suppose that $\exists$ (integer) $n_0 \ge 1$ s.t. $\{\Phi_n\}$ is non-increasing. It is clear that the limit $\lim_{n\to\infty}\Phi_n = d < \infty$ and $\lim_{n\to\infty}(\Phi_n - \Phi_{n+1}) = 0$. Setting $q = q^*$, by (27) and $\{\sigma_n\} \subset [\bar{a}, \bar{b}] \subset (0,1)$ one obtains

$$\begin{aligned} &(1 - \beta_n - \alpha_n\zeta)(1 + \theta_n)\{(1 - \nu)(\|y_n - q_n\|^2 + \|y_n - u_n\|^2) + \bar{a}(1 - \bar{b})\|w_n - W_n w_n\|^2\} \\ &\le (1 - \beta_n - \alpha_n\zeta)(1 + \theta_n)\{(1 - \nu)(\|y_n - q_n\|^2 + \|y_n - u_n\|^2) + \sigma_n(1 - \sigma_n)\|w_n - W_n w_n\|^2\} \\ &\le \|x_n - q^*\|^2 - \|x_{n+1} - q^*\|^2 + (\alpha_n + \theta_n)M_4 \le \Phi_n - \Phi_{n+1} + (\alpha_n + \theta_n)M_4. \end{aligned}$$

Noticing $\liminf_{n\to\infty}(1 - \beta_n) > 0$, $\alpha_n \to 0$, $\theta_n \to 0$ and $\Phi_n - \Phi_{n+1} \to 0$, one has

$$\lim_{n\to\infty}\|w_n - W_n w_n\| = \lim_{n\to\infty}\|y_n - u_n\| = \lim_{n\to\infty}\|y_n - q_n\| = 0. \tag{31}$$

Thus, it follows that

$$\|u_n - q_n\| \le \|u_n - y_n\| + \|y_n - q_n\| \to 0 \quad (n \to \infty). \tag{32}$$

Noticing $w_n - x_n = \epsilon_n(x_n - x_{n-1})$ and $u_n - w_n = \sigma_n(W_n w_n - w_n)$, we obtain

$$\begin{aligned} \|u_n - x_n\| &\le \|u_n - w_n\| + \|w_n - x_n\| \\ &= \sigma_n\|W_n w_n - w_n\| + \epsilon_n\|x_n - x_{n-1}\| \\ &\le \|W_n w_n - w_n\| + \alpha_n \cdot \tfrac{\epsilon_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\le \|W_n w_n - w_n\| + \alpha_n \cdot \sup_{n\ge1}\tfrac{\epsilon_n}{\alpha_n} \cdot \sup_{n\ge1}\|x_n - x_{n-1}\|. \end{aligned}$$

Since $\sup_{n\ge1}\frac{\epsilon_n}{\alpha_n} < \infty$, $\sup_{n\ge1}\|x_n - x_{n-1}\| < \infty$ and $\alpha_n \to 0$, using (31), one has

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{33}$$

Moreover, noticing $x_{n+1} - q^* = \beta_n(x_n - q^*) + (1 - \beta_n)(S^n q_n - q^*) + \alpha_n(g(x_n) - \mu F S^n q_n)$, we obtain from (23) that

$$\begin{aligned} &\|x_{n+1} - q^*\|^2 = \|\beta_n(x_n - q^*) + (1 - \beta_n)(S^n q_n - q^*) + \alpha_n(g(x_n) - \mu F S^n q_n)\|^2 \\ &\le \|\beta_n(x_n - q^*) + (1 - \beta_n)(S^n q_n - q^*)\|^2 + 2\langle \alpha_n(g(x_n) - \mu F S^n q_n), x_{n+1} - q^*\rangle \\ &\le \beta_n\|x_n - q^*\|^2 + (1 - \beta_n)\|S^n q_n - q^*\|^2 - \beta_n(1 - \beta_n)\|x_n - S^n q_n\|^2 \\ &\quad + 2\|\alpha_n(g(x_n) - \mu F S^n q_n)\|\|x_{n+1} - q^*\| \\ &\le \beta_n\|x_n - q^*\|^2 + (1 - \beta_n)(1 + \theta_n)^2\|q_n - q^*\|^2 - \beta_n(1 - \beta_n)\|x_n - S^n q_n\|^2 \\ &\quad + 2\alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|)\|x_{n+1} - q^*\| \\ &\le \beta_n(1 + \theta_n)^2(\|x_n - q^*\| + \alpha_n M_1)^2 + (1 - \beta_n)(1 + \theta_n)^2(\|x_n - q^*\| + \alpha_n M_1)^2 \\ &\quad - \beta_n(1 - \beta_n)\|x_n - S^n q_n\|^2 + 2\alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|)\|x_{n+1} - q^*\| \\ &= (1 + \theta_n)^2(\|x_n - q^*\| + \alpha_n M_1)^2 - \beta_n(1 - \beta_n)\|x_n - S^n q_n\|^2 \\ &\quad + 2\alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|)\|x_{n+1} - q^*\| \\ &= (1 + \theta_n)^2\|x_n - q^*\|^2 + (1 + \theta_n)^2\alpha_n M_1[2\|x_n - q^*\| + \alpha_n M_1] \\ &\quad - \beta_n(1 - \beta_n)\|x_n - S^n q_n\|^2 + 2\alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|)\|x_{n+1} - q^*\|. \end{aligned}$$

This hence arrives at

$$\begin{aligned} &\beta_n(1 - \beta_n)\|x_n - S^n q_n\|^2 \le (1 + \theta_n)^2\|x_n - q^*\|^2 - \|x_{n+1} - q^*\|^2 \\ &\quad + (1 + \theta_n)^2\alpha_n M_1[2\|x_n - q^*\| + \alpha_n M_1] + 2\alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|)\|x_{n+1} - q^*\| \\ &\le (1 + \theta_n)^2\Phi_n - \Phi_{n+1} + (1 + \theta_n)^2\alpha_n M_1[2\Phi_n^{\frac{1}{2}} + \alpha_n M_1] + 2\alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|)\Phi_{n+1}^{\frac{1}{2}}. \end{aligned}$$

Since $1 > \limsup_{n\to\infty} \beta_n \geq \liminf_{n\to\infty} \beta_n > 0$, $\theta_n \to 0$, $\alpha_n \to 0$, $\Phi_n - \Phi_{n+1} \to 0$ and $\lim_{n\to\infty} \Phi_n = d < +\infty$, from the boundedness of $\{g(x_n)\}, \{S^n q_n\}$, we infer that

$$\lim_{n\to\infty} \|x_n - S^n q_n\| = 0.$$

Thus, it follows that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n g(x_n) + (1-\beta_n)(S^n q_n - x_n) - \alpha_n \mu F S^n q_n\| \\ &\leq (1-\beta_n)\|S^n q_n - x_n\| + \alpha_n \|g(x_n) - \mu F S^n q_n\| \\ &\leq \|S^n q_n - x_n\| + \alpha_n (\|g(x_n)\| + \|\mu F S^n q_n\|) \to 0 \quad (n \to \infty). \end{aligned} \tag{34}$$

Since $\{x_n\}$ is bounded, we know that $\exists \{x_{n_\iota}\} \subset \{x_n\}$ s.t.

$$\limsup_{n\to\infty} \langle (g - \mu F) q^*, x_n - q^* \rangle = \lim_{\iota\to\infty} \langle (g - \mu F) q^*, x_{n_\iota} - q^* \rangle. \tag{35}$$

Noticing the reflexivity of H and boundedness of $\{x_n\}$, one might suppose that $x_{n_\iota} \rightharpoonup \tilde{q}$. Hence, using (35), we obtain

$$\begin{aligned} \limsup_{n\to\infty} \langle (g - \mu F) q^*, x_n - q^* \rangle &= \lim_{\iota\to\infty} \langle (g - \mu F) q^*, x_{n_\iota} - q^* \rangle \\ &= \langle (g - \mu F) q^*, \tilde{q} - q^* \rangle. \end{aligned} \tag{36}$$

Note that $\|w_n - x_n\| = \epsilon_n \|x_n - x_{n-1}\| \to 0$ (due to $\sup_{n\geq 1}(\epsilon_n/\alpha_n) < \infty$). Thus, we obtain

$$\|q_n - w_n\| \leq \|q_n - u_n\| + \|u_n - x_n\| + \|x_n - w_n\| \to 0 \quad (n \to \infty).$$

Noticing $x_{n+1} - x_n \to 0$, $u_n - x_n \to 0$, $w_n - q_n \to 0$ and $S^n x_n - S^{n+1} x_n \to 0$, from Lemma 10, one obtains $\tilde{q} \in\in \omega_w(\{x_n\}) \subset \Omega$. Thus, using (36) and (16), one has

$$\limsup_{n\to\infty} \langle (g - \mu F) q^*, x_n - q^* \rangle = \langle (g - \mu F) q^*, \tilde{q} - q^* \rangle \leq 0, \tag{37}$$

which, together with (34), yields

$$\begin{aligned} &\limsup_{n\to\infty} \langle (g - \mu F) q^*, x_{n+1} - q^* \rangle \\ &= \limsup_{n\to\infty} [\langle (g - \mu F) q^*, x_{n+1} - x_n \rangle + \langle (g - \mu F) q^*, x_n - q^* \rangle] \\ &\leq \limsup_{n\to\infty} [\|(g - \mu F) q^*\| \|x_{n+1} - x_n\| + \langle (g - \mu F) q^*, x_n - q^* \rangle] \leq 0. \end{aligned} \tag{38}$$

Since $\{\alpha_n(\zeta - \delta)\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n(\zeta - \delta) = \infty$, and

$$\limsup_{n\to\infty} [\frac{2\langle (g - \mu F) q^*, x_{n+1} - q^* \rangle}{\zeta - \delta} + \frac{M}{\zeta - \delta}(\frac{\epsilon_n}{\alpha_n} 3\|x_n - x_{n-1}\| + \frac{\theta_n}{\alpha_n})] \leq 0,$$

by the application of Lemma 4 to (30), one has $\lim_{n\to\infty} \|x_n - q^*\|^2 = 0$.

Aspect 2. Suppose that $\exists \{\Phi_{n_\iota}\} \subset \{\Phi_n\}$ s.t. $\Phi_{n_\iota} < \Phi_{n_\iota + 1}$ $\forall \iota \in \mathcal{N}$, with $\mathcal{N}$ being the set of all natural numbers. The self-mapping φ on $\mathcal{N}$ is formulated as

$$\varphi(n) := \max\{\iota \leq n : \Phi_\iota < \Phi_{\iota+1}\}.$$

Using Lemma 6, one obtains

$$\Phi_{\varphi(n)} \leq \Phi_{\varphi(n)+1} \quad \text{and} \quad \Phi_n \leq \Phi_{\varphi(n)+1}.$$

Putting $q = q^*$, from (27), we have

$$\begin{array}{l}(1-\beta_{\varphi(n)}-\alpha_{\varphi(n)}\zeta)(1+\theta_{\varphi(n)})\{(1-\nu)(\|y_{\varphi(n)}-q_{\varphi(n)}\|^2 \\ \quad +\|y_{\varphi(n)}-u_{\varphi(n)}\|^2)+\bar{a}(1-\bar{b})\|w_{\varphi(n)}-W_{\varphi(n)}w_{\varphi(n)}\|^2\} \\ \leq (1-\beta_{\varphi(n)}-\alpha_{\varphi(n)}\zeta)(1+\theta_{\varphi(n)})\{(1-\nu)(\|y_{\varphi(n)}-q_{\varphi(n)}\|^2 \\ \quad +\|y_{\varphi(n)}-u_{\varphi(n)}\|^2)+\sigma_{\varphi(n)}(1-\sigma_{\varphi(n)})\|w_{\varphi(n)}-W_{\varphi(n)}w_{\varphi(n)}\|^2\} \\ \leq \|x_{\varphi(n)}-q^*\|^2-\|x_{\varphi(n)+1}-q^*\|^2+(\alpha_{\varphi(n)}+\theta_{\varphi(n)})M_4 \\ = \Phi_{\varphi(n)}-\Phi_{\varphi(n)+1}+(\alpha_{\varphi(n)}+\theta_{\varphi(n)})M_4,\end{array} \tag{39}$$

which immediately yields

$$\lim_{n\to\infty}\|w_{\varphi(n)}-W_{\varphi(n)}w_{\varphi(n)}\| = \lim_{n\to\infty}\|y_{\varphi(n)}-u_{\varphi(n)}\| = \lim_{n\to\infty}\|y_{\varphi(n)}-q_{\varphi(n)}\| = 0.$$

Using the similar arguments to those of Aspect 1, one obtains

$$\lim_{n\to\infty}\|u_{\varphi(n)}-q_{\varphi(n)}\| = \lim_{n\to\infty}\|u_{\varphi(n)}-x_{\varphi(n)}\| = \lim_{n\to\infty}\|x_{\varphi(n)+1}-x_{\varphi(n)}\| = 0,$$

and

$$\limsup_{n\to\infty}\langle (g-\mu F)q^*, x_{\varphi(n)+1}-q^*\rangle \leq 0. \tag{40}$$

On the other hand, by (30), one has

$$\begin{aligned}\alpha_{\varphi(n)}(\zeta-\delta)\Phi_{\varphi(n)} &\leq \Phi_{\varphi(n)}-\Phi_{\varphi(n)+1}+\alpha_{\varphi(n)}(\zeta-\delta)[\tfrac{2\langle (g-\mu F)q^*, x_{\varphi(n)+1}-q^*\rangle}{\zeta-\delta} \\ &\quad +\tfrac{M}{\zeta-\delta}(\tfrac{\epsilon_{\varphi(n)}}{\alpha_{\varphi(n)}}3\|x_{\varphi(n)}-x_{\varphi(n)-1}\|+\tfrac{\theta_{\varphi(n)}}{\alpha_{\varphi(n)}})] \\ &\leq \alpha_{\varphi(n)}(\zeta-\delta)[\tfrac{2\langle (g-\mu F)q^*, x_{\varphi(n)+1}-q^*\rangle}{\zeta-\delta} \\ &\quad +\tfrac{M}{\zeta-\delta}(\tfrac{\epsilon_{\varphi(n)}}{\alpha_{\varphi(n)}}3\|x_{\varphi(n)}-x_{\varphi(n)-1}\|+\tfrac{\theta_{\varphi(n)}}{\alpha_{\varphi(n)}})],\end{aligned}$$

$$\limsup_{n\to\infty}\Phi_{\varphi(n)} \leq \limsup_{n\to\infty}[\frac{2\langle (g-\mu F)q^*, x_{\varphi(n)+1}-q^*\rangle}{\zeta-\delta}+\frac{M}{\zeta-\delta}(\frac{\epsilon_{\varphi(n)}}{\alpha_{\varphi(n)}}3\|x_{\varphi(n)}-x_{\varphi(n)-1}\|+\frac{\theta_{\varphi(n)}}{\alpha_{\varphi(n)}})] \leq 0.$$

Thus, $\lim_{n\to\infty}\|x_{\varphi(n)}-q^*\|^2 = 0$. In addition, note that

$$\begin{array}{l}\|x_{\varphi(n)+1}-q^*\|^2-\|x_{\varphi(n)}-q^*\|^2 \\ = 2\langle x_{\varphi(n)+1}-x_{\varphi(n)}, x_{\varphi(n)}-q^*\rangle+\|x_{\varphi(n)+1}-x_{\varphi(n)}\|^2 \\ \leq 2\|x_{\varphi(n)+1}-x_{\varphi(n)}\|\|x_{\varphi(n)}-q^*\|+\|x_{\varphi(n)+1}-x_{\varphi(n)}\|^2.\end{array} \tag{41}$$

Owing to $\Phi_n \leq \Phi_{\varphi(n)+1}$, one obtains

$$\begin{array}{l}\|x_n-q^*\|^2 \leq \|x_{\varphi(n)+1}-q^*\|^2 \\ \leq \|x_{\varphi(n)}-q^*\|^2+2\|x_{\varphi(n)+1}-x_{\varphi(n)}\|\|x_{\varphi(n)}-q^*\|+\|x_{\varphi(n)+1}-x_{\varphi(n)}\|^2 \to 0 \quad (n\to\infty).\end{array}$$

This means that $x_n \to q^*$ as $n \to \infty$.

In particular, when S is a nonexpansive operator, it is also asymptotically nonexpansive. In this case, the power S^n in Algorithm 4 can be simplified into S. In this way, we can obtain the following Theorem 2.

Theorem 2. *Suppose that S is of nonexpansivity on H and $\{x_n\}$ is constructed in the modification of Algorithm 4, i.e., for any starting points x_1, x_0 in H,*

$$\begin{cases} w_n = x_n+\epsilon_n(x_n-x_{n-1}), \\ u_n = (1-\sigma_n)w_n+\sigma_n W_n w_n, \\ y_n = P_C(u_n-\varsigma_n A u_n), \\ q_n = P_{C_n}(u_n-\varsigma_n A y_n), \\ x_{n+1} = \alpha_n g(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n\mu F)Sq_n \quad \forall n\geq 1,\end{cases} \tag{42}$$

with C_n and ς_n being picked as in Algorithm 4. Then, $x_n \to q^* \in \Omega \Leftrightarrow \sup_{n\geq 1}\|x_{n-1} - x_n\| < \infty$, with $q^* \in \Omega$ being only a solution of the HFPP: $q^* = P_\Omega(I - \mu F + g)q^*$.

Proof. We first pick a $q \in \Omega$ arbitrarily. Obviously, the necessity holds. Next, it is sufficient to demonstrate the sufficiency. To this goal, under the condition $\sup_{n\geq 1}\|x_{n-1} - x_n\| < \infty$, one divides the remainder of the proof into several claims. □

Claim 1. One claims the boundedness of $\{x_n\}$. In fact, using the similar inferences to those of Claim 1 in the proof of the above theorem, one obtains the claim.
Claim 2. One claims that

$$\begin{aligned}&(1-\beta_n-\alpha_n\zeta)\{(1-\nu)(\|y_n-q_n\|^2+\|y_n-u_n\|^2)+\sigma_n(1-\sigma_n)\|w_n-W_nw_n\|^2\}\\&\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2+\alpha_nM_4,\end{aligned}\tag{43}$$

for some $M_4 > 0$. In fact, using the similar inferences to those of Step 2 in the proof of the above theorem, one obtains the claim.

Claim 3. One claims that

$$\begin{aligned}\|x_{n+1}-q\|^2 \quad &\leq [1-\alpha_n(\zeta-\delta)]\|x_n-q\|^2+\alpha_n(\zeta-\delta)\{\tfrac{2}{\zeta-\delta}\langle(g-\mu F)q, x_{n+1}-q\rangle\\&\quad+\tfrac{3M}{\zeta-\delta}\cdot\tfrac{\epsilon_n}{\alpha_n}\|x_n-x_{n-1}\|\}\end{aligned}$$

for some $M > 0$. In fact, using the similar inferences to those of Claim 3 in the proof of the above theorem, one obtains the claim.

Claim 4. One claims that $x_n \to q^* \in \Omega$, which is only a solution to the HFPP: $q^* = P_\Omega(I - \mu F + g)q^*$. In fact, setting $q = q^*$, by Claim 3, one obtains

$$\begin{aligned}\|x_{n+1}-q^*\|^2 \quad &\leq [1-\alpha_n(\zeta-\delta)]\|x_n-q^*\|^2+\alpha_n(\zeta-\delta)\\&\quad\times\{\tfrac{2}{\zeta-\delta}\langle(g-\mu F)q^*, x_{n+1}-q^*\rangle+\tfrac{3M}{\zeta-\delta}\cdot\tfrac{\epsilon_n}{\alpha_n}\|x_n-x_{n-1}\|\}.\end{aligned}\tag{44}$$

Setting $\Phi_n = \|x_n - q^*\|^2$, one demonstrates $\Phi_n \to 0$ $(n \to \infty)$ in both aspects below.

Aspect 1. Suppose that $\exists$ (integer) $n_0 \geq 1$ s.t. $\{\Phi_n\}$ is non-increasing. Then, the limit $\lim_{n\to\infty}\Phi_n = d < \infty$ and $\lim_{n\to\infty}(\Phi_n - \Phi_{n+1}) = 0$. Using the similar inferences to those of Aspect 1 of Claim 4 in the proof of the above theorem, one obtains

$$\lim_{n\to\infty}\|w_n-W_nw_n\| = \lim_{n\to\infty}\|u_n-y_n\| = \lim_{n\to\infty}\|u_n-q_n\| = \lim_{n\to\infty}\|u_n-x_n\| = 0.\tag{45}$$

From (4) and (23), one has

$$\begin{aligned}&\|x_{n+1}-q^*\|^2 = \|\beta_n(x_n-q^*)+(1-\beta_n)(Sq_n-q^*)+\alpha_n(g(x_n)-\mu FSq_n)\|^2\\&\leq \|\beta_n(x_n-q^*)+(1-\beta_n)(Sq_n-q^*)\|^2+2\alpha_n\langle g(x_n)-\mu FSq_n, x_{n+1}-q^*\rangle\\&\leq \beta_n\|x_n-q^*\|^2+(1-\beta_n)\|Sq_n-q^*\|^2-\beta_n(1-\beta_n)\|x_n-Sq_n\|^2\\&\quad+2\alpha_n\langle g(x_n)-\mu FSq_n, x_{n+1}-q^*\rangle\\&\leq (\|x_n-q^*\|\alpha_nM_1)^2-\beta_n(1-\beta_n)\|x_n-Sq_n\|^2\\&\quad+2\alpha_n\|g(x_n)-\mu FSq_n\|\|x_{n+1}-q^*\|.\end{aligned}$$

This hence arrives at

$$\beta_n(1-\beta_n)\|x_n-Sq_n\|^2 \leq (\|x_n-q^*\|+\alpha_nM_1)^2-\|x_{n+1}-q^*\|^2+2\alpha_n\|g(x_n)-\mu FSq_n\|\|x_{n+1}-q^*\|.$$

Since $1 > \limsup_{n\to\infty}\beta_n \geq \liminf_{n\to\infty}\beta_n > 0$, $\alpha_n \to 0$ and $\lim_{n\to\infty}\Phi_n = d < +\infty$, from the boundedness of $\{g(x_n)\}, \{Sq_n\}$, we infer that

$$\lim_{n\to\infty}\|x_n-Sq_n\| = 0.$$

Therefore,

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n g(x_n) + (1-\beta_n)(Sq_n - x_n) - \alpha_n \mu F S q_n\| \\ &\le (1-\beta_n)\|Sq_n - x_n\| + \alpha_n \|g(x_n) - \mu F S q_n\| \\ &\le \|Sq_n - x_n\| + \alpha_n(\|g(x_n)\| + \|\mu F S q_n\|) \to 0 \quad (n \to \infty). \end{aligned} \tag{46}$$

Again utilizing the similar inferences to those of Aspect 1 of Claim 4 in the proof of the above theorem, one obtains $\lim_{n\to\infty} \|x_n - q^*\|^2 = 0$.

Aspect 2. Suppose that $\exists\{\Phi_{n_\iota}\} \subset \{\Phi_n\}$ s.t. $\Phi_{n_\iota} < \Phi_{n_\iota+1}$ $\forall \iota \in \mathcal{N}$, with $\mathcal{N}$ being the set of all natural numbers. The self-mapping φ on $\mathcal{N}$ is formulated as

$$\varphi(n) := \max\{\iota \le n : \Phi_\iota < \Phi_{\iota+1}\}.$$

From Lemma 6, one obtains

$$\Phi_{\varphi(n)} \le \Phi_{\varphi(n)+1} \quad \text{and} \quad \Phi_n \le \Phi_{\varphi(n)+1}.$$

Finally, by the similar inferences to those of Aspect 2 of Claim 4 in the proof of the above theorem, one can obtain the claim.

On the other hand, we put forward another modification of a Mann-type subgradient extragradient rule.

It is worth mentioning that (9) and Lemmas 8–10 remain true for Algorithm 5:

Algorithm 5 The 2nd modified Mann-type subgradient extragradient rule

Initial Step: Let $l \in (0,1)$, $\nu \in (0,1)$, $\gamma \in (0,\infty)$, given any starting points x_1, x_0 in H.
Iterations: Compute x_{n+1} $(n \ge 1)$ below:
Step 1. Set $w_n = x_n + \epsilon_n(x_n - x_{n-1})$ and $u_n = (1-\sigma_n)x_n + \sigma_n W_n w_n$, and calculate $y_n = P_C(u_n - \varsigma_n A u_n)$, with ς_n being picked to be the largest $\varsigma \in \{\gamma, \gamma l, \gamma l^2, \ldots\}$ s.t.

$$\varsigma\|Au_n - Ay_n\| \le \nu\|u_n - y_n\|. \tag{47}$$

Step 2. Calculate $q_n = P_{C_n}(u_n - \varsigma_n A y_n)$, where $C_n := \{y \in H : \langle u_n - \varsigma_n A u_n - y_n, y_n - y\rangle \ge 0\}$.
Step 3. Calculate

$$x_{n+1} = \alpha_n g(x_n) + \beta_n u_n + ((1-\beta_n)I - \alpha_n \mu F)S^n q_n. \tag{48}$$

Put $n := n+1$ and return to Step 1.

Theorem 3. *Suppose that $\{x_n\}$ is the sequence constructed in Algorithm 5. Then,*

$$x_n \to q^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} S^n x_n - S^{n+1}x_n \to 0, \\ \sup_{n\ge1} \|x_{n-1} - x_n\| < \infty, \end{cases}$$

with $q^* \in \Omega$ being only a solution of the HFPP: $q^* = P_\Omega(I - \mu F + g)q^*$.

Proof. By the similar inferences to those in the proof of the first theorem, one obtains that $\exists q^* \in \Omega$, which is only a solution of the HFPP: $q^* = P_\Omega(I - \mu F + g)q^*$. Obviously, the necessity holds.

In what follows, one claims the sufficiency. To the goal, under the assumption $S^n x_n - S^{n+1}x_n \to 0$ with $\sup_{n\ge1} \|x_{n-1} - x_n\| < \infty$, one divides the claim of the sufficiency into several claims. □

Claim 1. One claims the boundedness of $\{x_n\}$. In fact, using the similar inferences to those of Claim 1 in the proof of the first theorem, one has that (19) and (20) hold. It is easy to see from (9) that

$$\|q_n - q\| \le \|u_n - q\| \le (1-\sigma_n)\|x_n - q\| + \sigma_n\|w_n - q\| \le \|x_n - q\| + \alpha_n M_1 \quad \forall n \ge 1. \tag{49}$$

Hence, using $\alpha_n + \beta_n < 1$, Lemma 7, and (49), we obtain

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n(g(x_n) - g(q)) + \beta_n(u_n - q) + (1-\beta_n) \\
&\quad \times [(I - \tfrac{\alpha_n}{1-\beta_n}\mu F)S^n q_n - (I - \tfrac{\alpha_n}{1-\beta_n}\mu F)q + \tfrac{\alpha_n}{1-\beta_n}(g - \mu F)q]\| \\
&\le \alpha_n\|g(x_n) - g(q)\| + \beta_n\|u_n - q\| + (1-\beta_n) \\
&\quad \times \|(I - \tfrac{\alpha_n}{1-\beta_n}\mu F)S^n q_n - (I - \tfrac{\alpha_n}{1-\beta_n}\mu F)q + \tfrac{\alpha_n}{1-\beta_n}(g - \mu F)q\| \\
&\le \alpha_n\delta\|x_n - q\| + \beta_n\|u_n - q\| + (1-\beta_n) \\
&\quad \times (1 - \tfrac{\alpha_n}{1-\beta_n}\zeta)(1+\theta_n)\|q_n - q\| + \alpha_n\|(g - \mu F)q\| \\
&\le \alpha_n\delta\|x_n - q\| + \beta_n(\|x_n - q\| + \alpha_n M_1) + (1 - \beta_n - \alpha_n\zeta) \\
&\quad \times (\|x_n - q\| + \alpha_n M_1) + \theta_n\|q_n - q\| + \alpha_n\|(g - \mu F)q\| \\
&\le [\alpha_n\delta + \beta_n + (1 - \beta_n - \alpha_n\zeta)]\|x_n - q\| + \alpha_n M_1 \\
&\quad + \tfrac{\alpha_n(\zeta-\delta)(\|x_n - q\| + \alpha_n M_1)}{2} + \alpha_n\|(g - \mu F)q\| \\
&\le [1 - \tfrac{\alpha_n(\zeta-\delta)}{2}]\|x_n - q\| + \alpha_n(2M_1 + \|(g - \mu F)q\|) \\
&= [1 - \tfrac{\alpha_n(\zeta-\delta)}{2}]\|x_n - q\| + \tfrac{\alpha_n(\zeta-\delta)}{2} \cdot \tfrac{2(2M_1 + \|(g-\mu F)q\|)}{\zeta-\delta} \\
&\le \max\{\|x_n - q\|, \tfrac{2(2M_1 + \|(g-\mu F)q\|)}{\zeta-\delta}\},
\end{aligned}$$

which immediately yields

$$\|x_n - q\| \le \max\{\|x_1 - q\|, \frac{2(2M_1 + \|(g - \mu F)q\|)}{\zeta - \delta}\} \quad \forall n \ge 1.$$

Therefore, we show the boundedness of $\{x_n\}$. This ensures that the sequences $\{w_n\}, \{u_n\}, \{y_n\}, \{q_n\}, \{g(x_n)\}, \{Ay_n\}, \{W_n w_n\}, \{S^n q_n\}$ are bounded.

Claim 2. One claims that

$$\begin{aligned}
&(1 - \beta_n - \alpha_n\zeta)(1+\theta_n)\{(1-\nu)(\|y_n - q_n\|^2 + \|y_n - u_n\|^2) + \sigma_n(1-\sigma_n)\|x_n - W_n w_n\|^2\} \\
&\le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + (\alpha_n + \theta_n)M_4,
\end{aligned}$$

for some $M_4 > 0$—in fact, since

$$\begin{aligned}
x_{n+1} - q &= \alpha_n(g(x_n) - g(q))\beta_n(u_n - q) + (1-\beta_n) \\
&\quad \times [(I - \tfrac{\alpha_n}{1-\beta_n}\mu F)S^n q_n - (I - \tfrac{\alpha_n}{1-\beta_n}\mu F)q] + \alpha_n(g - \mu F)q.
\end{aligned}$$

Using the same inferences as those of (24), one has

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \alpha_n\delta\|x_n - q\|^2 + \beta_n\|u_n - q\|^2 + (1 - \beta_n - \alpha_n\zeta)(1+\theta_n)\|q_n - q\|^2 \\
&\quad + 2\alpha_n\langle(g - \mu F)q, x_{n+1} - q\rangle \\
&\le \alpha_n\delta\|x_n - q\|^2 + \beta_n\|u_n - q\|^2 + (1 - \beta_n - \alpha_n\zeta)(1+\theta_n)\|q_n - q\|^2 + \alpha_n M_2,
\end{aligned} \tag{50}$$

with $M_2 \ge \sup_{n\ge1} 2\|(g - \mu F)q\|\|x_n - q\|$ for certain $M_2 > 0$. In addition, using (19) and (20), one obtains

$$\|w_n - q\|^2 \le (\|x_n - q\| + \alpha_n M_1)^2 \le \|x_n - q\|^2 + \alpha_n M_3,$$

with $M_3 \ge \sup_{n\ge1}(2M_1\|x_n - q\| + \alpha_n M_1^2)$ for certain $M_3 > 0$. Moreover, using (49), we deduce that

$$\|u_n - q\|^2 \le \|x_n - q\|^2 + \alpha_n M_3,$$

which, together with (9) and (50), leads to

$$
\begin{aligned}
&\|x_{n+1}-q\|^2 \le \alpha_n\delta\|x_n-q\|^2+\beta_n\|u_n-q\|^2+(1-\beta_n-\alpha_n\zeta)(1+\theta_n)[\|u_n-q\|^2\\
&\quad -(1-\nu)\|y_n-q_n\|^2-(1-\nu)\|y_n-u_n\|^2]+\alpha_nM_2\\
&\le \alpha_n\delta\|x_n-q\|^2+\beta_n\|u_n-q\|^2+(1-\beta_n-\alpha_n\zeta)(1+\theta_n)\{(1-\sigma_n)\|x_n-q\|^2\\
&\quad +\sigma_n\|W_nw_n-q\|^2-\sigma_n(1-\sigma_n)\|x_n-W_nw_n\|^2-(1-\nu)(\|y_n-q_n\|^2\\
&\quad +\|y_n-u_n\|^2)\}+\alpha_nM_2\\
&\le \alpha_n\delta(\|x_n-q\|^2+\alpha_nM_3)+\beta_n(\|x_n-q\|^2+\alpha_nM_3)+(1-\beta_n-\alpha_n\zeta)(1+\theta_n)\\
&\quad \times\{(1-\sigma_n)(\|x_n-q\|^2+\alpha_nM_3)+\sigma_n(\|x_n-q\|^2+\alpha_nM_3)-\sigma_n(1-\sigma_n)\|x_n-W_nw_n\|^2\\
&\quad -(1-\nu)(\|y_n-q_n\|^2+\|y_n-u_n\|^2)\}+\alpha_nM_2\\
&= [\alpha_n\delta+\beta_n+(1-\beta_n-\alpha_n\zeta)(1+\theta_n)](\|x_n-q\|^2+\alpha_nM_3)-(1-\beta_n-\alpha_n\zeta)(1+\theta_n)\\
&\quad \times\{(1-\nu)(\|y_n-q_n\|^2+\|y_n-u_n\|^2)+\sigma_n(1-\sigma_n)\|x_n-W_nw_n\|^2\}+\alpha_nM_2\\
&\le [\alpha_n\delta+\beta_n+(1-\beta_n-\alpha_n\zeta)+\theta_n](\|x_n-q\|^2+\alpha_nM_3)-(1-\beta_n-\alpha_n\zeta)(1+\theta_n)\\
&\quad \times\{(1-\nu)(\|y_n-q_n\|^2+\|y_n-u_n\|^2)+\sigma_n(1-\sigma_n)\|x_n-W_nw_n\|^2\}+\alpha_nM_2\\
&= [1-\alpha_n(\zeta-\delta)+\theta_n](\|x_n-q\|^2+\alpha_nM_3)-(1-\beta_n-\alpha_n\zeta)(1+\theta_n)\\
&\quad \times\{(1-\nu)(\|y_n-q_n\|^2+\|y_n-u_n\|^2)+\sigma_n(1-\sigma_n)\|x_n-W_nw_n\|^2\}+\alpha_nM_2\\
&\le [1-\alpha_n(\zeta-\delta)]\|x_n-q\|^2+\theta_n\|x_n-q\|^2+\alpha_nM_3-(1-\beta_n-\alpha_n\zeta)(1+\theta_n)\\
&\quad \times\{(1-\nu)(\|y_n-q_n\|^2+\|y_n-u_n\|^2)+\sigma_n(1-\sigma_n)\|x_n-W_nw_n\|^2\}+\alpha_nM_2\\
&\le \|x_n-q\|^2-(1-\beta_n-\alpha_n\zeta)(1+\theta_n)\{(1-\nu)(\|y_n-q_n\|^2+\|y_n-u_n\|^2)\\
&\quad +\sigma_n(1-\sigma_n)\|x_n-W_nw_n\|^2\}+(\alpha_n+\theta_n)M_4,
\end{aligned}
\tag{51}
$$

where $M_4 \ge \sup_{n\ge1}(\|x_n-q\|^2+M_3+M_2)$ for certain $M_4>0$. Thus, we obtain

$$
\begin{aligned}
&(1-\beta_n-\alpha_n\zeta)(1+\theta_n)\{(1-\nu)(\|y_n-q_n\|^2+\|y_n-u_n\|^2)+\sigma_n(1-\sigma_n)\|x_n-W_nw_n\|^2\}\\
&\le \|x_n-q\|^2-\|x_{n+1}-q\|^2+(\alpha_n+\theta_n)M_4.
\end{aligned}
\tag{52}
$$

Claim 3. One claims that

$$
\begin{aligned}
\|x_{n+1}-q\|^2 \quad &\le [1-\alpha_n(\zeta-\delta)]\|x_n-q\|^2+\alpha_n(\zeta-\delta)\{\tfrac{2}{\zeta-\delta}\langle(g-\mu F)q,x_{n+1}-q\rangle\\
&\quad +\tfrac{M}{\zeta-\delta}(\tfrac{\epsilon_n}{\alpha_n}3\|x_n-x_{n-1}\|+\tfrac{\theta_n}{\alpha_n})\}
\end{aligned}
$$

for certain $M>0$. In fact, one has

$$
\|w_n-q\|^2 \le \|x_n-q\|^2+\epsilon_n\|x_n-x_{n-1}\|(2\|x_n-q\|+\epsilon_n\|x_n-x_{n-1}\|),
$$

and hence

$$
\begin{aligned}
&\|u_n-q\|^2 \le (1-\sigma_n)\|x_n-q\|^2+\sigma_n\|W_nw_n-q\|^2\\
&\le (1-\sigma_n)\|x_n-q\|^2+\sigma_n\{\|x_n-q\|^2+\epsilon_n\|x_n-x_{n-1}\|(2\|x_n-q\|+\epsilon_n\|x_n-x_{n-1}\|)\}\\
&\le \|x_n-q\|^2+\epsilon_n\|x_n-x_{n-1}\|(2\|x_n-q\|+\epsilon_n\|x_n-x_{n-1}\|).
\end{aligned}
\tag{53}
$$

From (49), (50) and (53), one obtains

$$
\begin{aligned}
&\|x_{n+1}-q\|^2\\
&\le \alpha_n\delta\|x_n-q\|^2+\beta_n\|u_n-q\|^2+(1-\beta_n-\alpha_n\zeta)\|q_n-q\|^2\\
&\quad +\theta_n\|q_n-q\|^2+2\alpha_n\langle(g-\mu F)q,x_{n+1}-q\rangle\\
&\le \alpha_n\delta\|x_n-q\|^2+(1-\alpha_n\zeta)[\|x_n-q\|^2+\epsilon_n\|x_n-x_{n-1}\|(2\|x_n-q\|\\
&\quad +\epsilon_n\|x_n-x_{n-1}\|)]+\theta_n\|q_n-q\|^2+2\alpha_n\langle(g-\mu F)q,x_{n+1}-q\rangle\\
&\le [1-\alpha_n(\zeta-\delta)]\|x_n-q\|^2+(\epsilon_n\|x_n-x_{n-1}\|3+\theta_n)M+2\alpha_n\langle(g-\mu F)q,x_{n+1}-q\rangle\\
&= [1-\alpha_n(\zeta-\delta)]\|x_n-q\|^2+\alpha_n(\zeta-\delta)[\tfrac{2\langle(g-\mu F)q,x_{n+1}-q\rangle}{\zeta-\delta}+\tfrac{M}{\zeta-\delta}(\tfrac{\epsilon_n}{\alpha_n}3\|x_n-x_{n-1}\|+\tfrac{\theta_n}{\alpha_n})],
\end{aligned}
\tag{54}
$$

where $M \ge \sup_{n\ge1}\{\|x_n-q\|,\epsilon_n\|x_n-x_{n-1}\|,\|q_n-q\|^2\}$ for certain $M>0$.

Claim 4. One claims that $x_n \to q^* \in \Omega$, which is only a solution to the HFPP: $q^* = P_\Omega(I - \mu F + g)q^*$. In fact, using (54) with $q = q^*$, one obtains

$$\begin{aligned}\|x_{n+1} - q^*\|^2 &\le [1 - \alpha_n(\zeta - \delta)]\|x_n - q^*\|^2 + \alpha_n(\zeta - \delta)[\tfrac{2\langle (g-\mu F)q^*, x_{n+1}-q^*\rangle}{\zeta-\delta} \\ &\quad + \tfrac{M}{\zeta-\delta}(\tfrac{\epsilon_n}{\alpha_n}3\|x_n - x_{n-1}\| + \tfrac{\theta_n}{\alpha_n})].\end{aligned} \tag{55}$$

Setting $\Phi_n = \|x_n - q^*\|^2$, one demonstrates $\Phi_n \to 0$ $(n \to \infty)$ in both aspects below.

Aspect 1. Suppose that $\exists$ (integer) $n_0 \ge 1$ s.t. $\{\Phi_n\}$ is non-increasing. Then, the limit $\lim_{n\to\infty} \Phi_n = d < \infty$ and $\lim_{n\to\infty}(\Phi_n - \Phi_{n+1}) = 0$. Using (52) with $q = q^*$ and $\{\sigma_n\} \subset [\bar{a}, \bar{b}] \subset (0,1)$, one obtains

$$\begin{aligned}&(1 - \beta_n - \alpha_n\zeta)(1 + \theta_n)\{(1 - \nu)(\|y_n - q_n\|^2 + \|y_n - u_n\|^2) + \bar{a}(1 - \bar{b})\|x_n - W_n w_n\|^2\} \\ &\le (1 - \beta_n - \alpha_n\zeta)(1 + \theta_n)\{(1 - \nu)(\|y_n - q_n\|^2 + \|y_n - u_n\|^2) + \sigma_n(1 - \sigma_n)\|x_n - W_n w_n\|^2\} \\ &\le \Phi_n - \Phi_{n+1} + (\alpha_n + \theta_n)M_4.\end{aligned}$$

Noticing $\liminf_{n\to\infty}(1 - \beta_n) > 0$, $\alpha_n \to 0$, $\theta_n \to 0$ and $\Phi_n - \Phi_{n+1} \to 0$, one has

$$\lim_{n\to\infty} \|x_n - W_n w_n\| = \lim_{n\to\infty} \|y_n - u_n\| = \lim_{n\to\infty} \|y_n - q_n\| = 0. \tag{56}$$

Hence, one obtains

$$\|u_n - q_n\| \le \|u_n - y_n\| + \|y_n - q_n\| \to 0 \quad (n \to \infty). \tag{57}$$

Since $w_n - x_n = \epsilon_n(x_n - x_{n-1})$ and $u_n - x_n = \sigma_n(W_n w_n - x_n)$, we obtain

$$\lim_{n\to\infty} \|w_n - x_n\| = \lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{58}$$

Moreover, noticing $x_{n+1} - q^* = \beta_n(u_n - q^*) + (1 - \beta_n)(S^n q_n - q^*) + \alpha_n(g(x_n) - \mu F S^n q_n)$, we obtain from (49) that

$$\begin{aligned}\|x_{n+1} - q^*\|^2 &= \|\beta_n(u_n - q^*) + (1 - \beta_n)(S^n q_n - q^*) + \alpha_n(g(x_n) - \mu F S^n q_n)\|^2 \\ &\le \beta_n\|u_n - q^*\|^2 + (1 - \beta_n)(1 + \theta_n)^2\|q_n - q^*\|^2 - \gamma_n(1 - \gamma_n)\|u_n - S^n q_n\|^2 \\ &\quad + 2\alpha_n\langle g(x_n) - \mu F S^n q_n, x_{n+1} - q^*\rangle \\ &\le (1 + \theta_n)^2(\|x_n - q^*\| + \alpha_n M_1)^2 - \beta_n(1 - \beta_n)\|u_n - S^n q_n\|^2 \\ &\quad + 2\alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|)\|x_{n+1} - q^*\| \\ &= (1 + \theta_n)^2\|x_n - q^*\|^2 + (1 + \theta_n)^2\alpha_n M_1[2\|x_n - q^*\| + \alpha_n M_1] \\ &\quad - \beta_n(1 - \beta_n)\|u_n - S^n q_n\|^2 + 2\alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|)\|x_{n+1} - q^*\|.\end{aligned}$$

This hence arrives at

$$\begin{aligned}&\beta_n(1 - \beta_n)\|u_n - S^n q_n\|^2 \le (1 + \theta_n)^2\|x_n - q^*\|^2 - \|x_{n+1} - q^*\|^2 \\ &\quad + (1 + \theta_n)^2\alpha_n M_1[2\|x_n - q^*\| + \alpha_n M_1] + 2\alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|)\|x_{n+1} - q^*\| \\ &\le (1 + \theta_n)^2\Phi_n - \Phi_{n+1} + (1 + \theta_n)^2\alpha_n M_1[2\Phi_n^{\frac{1}{2}} + \alpha_n M_1] + 2\alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|)\Phi_{n+1}^{\frac{1}{2}}.\end{aligned}$$

Since $1 > \limsup_{n\to\infty} \beta_n \ge \liminf_{n\to\infty} \beta_n > 0$, $\theta_n \to 0$, $\alpha_n \to 0$, $\Phi_n - \Phi_{n+1} \to 0$ and $\lim_{n\to\infty} \Phi_n = d < +\infty$, from the boundedness of $\{g(x_n)\}$, $\{S^n q_n\}$, we infer that

$$\lim_{n\to\infty} \|u_n - S^n q_n\| = 0.$$

Thus, it follows from Algorithm 5 that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n g(x_n) + \beta_n(u_n - x_n) + (1 - \beta_n)(S^n q_n - x_n) - \alpha_n \mu F S^n q_n\| \\ &= \|\alpha_n g(x_n) + (u_n - x_n) + (1 - \beta_n)(S^n q_n - u_n) - \alpha_n \mu F S^n q_n\| \\ &\le \|u_n - x_n\| + \|S^n q_n - u_n\| + \alpha_n(\|g(x_n)\| + \|\mu F S^n q_n\|) \to 0 \quad (n \to \infty).\end{aligned} \tag{59}$$

Utilizing the same inferences as those of (38), one obtains

$$\limsup_{n\to\infty}\langle(g-\mu F)q^*, x_{n+1}-q^*\rangle \le 0. \tag{60}$$

Since $\{\alpha_n(\zeta-\delta)\}\subset[0,1]$, $\sum_{n=1}^{\infty}\alpha_n(\zeta-\delta)=\infty$, and

$$\limsup_{n\to\infty}[\frac{2\langle(g-\mu F)q^*, x_{n+1}-q^*\rangle}{\zeta-\delta}+\frac{M}{\zeta-\delta}(\frac{\epsilon_n}{\alpha_n}3\|x_n-x_{n-1}\|+\frac{\theta_n}{\alpha_n})]\le 0.$$

Therefore, by the application of Lemma 4 to (55), one has $\lim_{n\to\infty}\|x_n-q^*\|^2=0$.

Aspect 2. Suppose that $\exists\{\Phi_{n_\iota}\}\subset\{\Phi_n\}$ s.t. $\Phi_{n_\iota}<\Phi_{n_\iota+1}\ \forall\iota\in\mathcal{N}$, with $\mathcal{N}$ being the set of all natural numbers. The self-mapping φ on $\mathcal{N}$ is formulated as

$$\varphi(n):=\max\{\iota\le n:\Phi_\iota<\Phi_{\iota+1}\}.$$

From Lemma 6, one obtains

$$\Phi_{\varphi(n)}\le\Phi_{\varphi(n)+1}\quad\text{and}\quad\Phi_n\le\Phi_{\varphi(n)+1}.$$

Finally, by the similar inferences to those of Aspect 2 of Claim 4 in the proof of the first theorem, one can derive the claim.

In particular, when S is a nonexpansive operator, it is also asymptotically nonexpansive. In this case, the power S^n in Algorithm 5 can be simplified into S. In this way, we can obtain the following Theorem 3.

Theorem 4. *Suppose that S is of nonexpansivity on H and $\{x_n\}$ is constructed in the modification of Algorithm 5, i.e., for any starting points x_1, x_0 in H,*

$$\begin{cases} w_n=x_n+\epsilon_n(x_n-x_{n-1}),\\ u_n=(1-\sigma_n)x_n+\sigma_n W_n w_n,\\ y_n=P_C(u_n-\varsigma_n Au_n),\\ q_n=P_{C_n}(u_n-\varsigma_n Ay_n),\\ x_{n+1}=\alpha_n g(x_n)+\beta_n u_n+((1-\beta_n)I-\alpha_n\mu F)Sq_n \quad \forall n\ge 1,\end{cases} \tag{61}$$

with C_n and ς_n being picked as in Algorithm 5. Then, $x_n\to q^*\in\Omega \Leftrightarrow \sup_{n\ge1}\|x_{n-1}-x_n\|<\infty$, with $q^*\in\Omega$ being only a solution of the HFPP: $q^*=P_\Omega(I-\mu F+g)q^*$.

Proof. We first pick a $q\in\Omega$ arbitrarily. Obviously, the necessity holds. Next, it is sufficient to demonstrate the sufficiency. To the goal, under the condition $\sup_{n\ge1}\|x_{n-1}-x_n\|<\infty$, one divides the surplus of the proof into several claims. □

Claim 1. One claims the boundedness of $\{x_n\}$. In fact, using the similar inferences to those of Claim 1 in the proof of the third theorem, one obtains the claim.

Claim 2. One claims that

$$\begin{aligned}&(1-\beta_n-\alpha_n\zeta)\{(1-\nu)(\|y_n-q_n\|^2+\|y_n-u_n\|^2)+\sigma_n(1-\sigma_n)\|x_n-W_nw_n\|^2\}\\ &\le\|x_n-q\|^2-\|x_{n+1}-q\|^2+\alpha_nM_4,\end{aligned} \tag{62}$$

for certain $M_4>0$. In fact, using the similar inferences to those of Claim 2 in the proof of the third theorem, one obtains the claim.

Claim 3. One claims that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le[1-\alpha_n(\zeta-\delta)]\|x_n-q\|^2+\alpha_n(\zeta-\delta)\{\tfrac{2}{\zeta-\delta}\langle(g-\mu F)q, x_{n+1}-q\rangle\\ &\quad+\tfrac{3M}{\zeta-\delta}\cdot\tfrac{\epsilon_n}{\alpha_n}\|x_n-x_{n-1}\|\}\end{aligned}$$

for certain $M > 0$. In fact, using the similar inferences to those of Claim 3 in the proof of the third theorem, one obtains the claim.

Claim 4. One claims that $x_n \to q^* \in \Omega$, which is only a solution to the HFPP: $q^* = P_\Omega(I - \mu F + g)q^*$. In fact, setting $q = q^*$, by Claim 3, one obtains

$$\begin{array}{ll} \|x_{n+1} - q^*\|^2 & \leq [1 - \alpha_n(\zeta - \delta)]\|x_n - q^*\|^2 + \alpha_n(\zeta - \delta) \\ & \quad \times \{\frac{2}{\zeta - \delta}\langle(g - \mu F)q^*, x_{n+1} - q^*\rangle + \frac{3M}{\zeta - \delta} \cdot \frac{\epsilon_n}{\alpha_n}\|x_n - x_{n-1}\|\}. \end{array} \tag{63}$$

Putting $\Phi_n = \|x_n - q^*\|^2$, one shows $\Phi_n \to 0$ $(n \to \infty)$ in both aspects below.

Aspect 1. Suppose that $\exists$ (integer) $n_0 \geq 1$ s.t. $\{\Phi_n\}$ is non-increasing. Then, the limit $\lim_{n\to\infty} \Phi_n = d < +\infty$ and $\lim_{n\to\infty}(\Phi_n - \Phi_{n+1}) = 0$. Using the similar inferences to those of Aspect 1 of Claim 4 in the proof of the third theorem, one obtains

$$\lim_{n\to\infty} \|x_n - W_n w_n\| = \lim_{n\to\infty} \|u_n - y_n\| = \lim_{n\to\infty} \|u_n - q_n\| = \lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{64}$$

From (4) and (49), one has

$$\begin{array}{l} \|x_{n+1} - q^*\|^2 = \|\beta_n(u_n - q^*) + (1 - \beta_n)(Sq_n - q^*) + \alpha_n(g(x_n) - \mu FSq_n)\|^2 \\ \leq \|\beta_n(u_n - q^*) + (1 - \beta_n)(Sq_n - q^*)\|^2 + 2\alpha_n\langle g(x_n) - \mu FSq_n, x_{n+1} - q^*\rangle \\ \leq \beta_n\|u_n - q^*\|^2 + (1 - \beta_n)\|Sq_n - q^*\|^2 - \beta_n(1 - \beta_n)\|u_n - Sq_n\|^2 \\ \quad + 2\alpha_n\langle g(x_n) - \mu FSq_n, x_{n+1} - q^*\rangle \\ \leq (\|x_n - q^*\| + \alpha_n M_1)^2 - \beta_n(1 - \beta_n)\|u_n - Sq_n\|^2 \\ \quad + 2\alpha_n\|g(x_n) - \mu FSq_n\|\|x_{n+1} - q^*\|, \end{array}$$

which immediately yields

$$\beta_n(1 - \beta_n)\|u_n - Sq_n\|^2 \leq (\|x_n - q^*\| + \alpha_n M_1)^2 - \|x_{n+1} - q^*\|^2 + 2\alpha_n\|g(x_n) - \mu FSq_n\|\|x_{n+1} - q^*\|.$$

Since $1 > \limsup_{n\to\infty} \beta_n \geq \liminf_{n\to\infty} \beta_n > 0$, $\alpha_n \to 0$ and $\lim_{n\to\infty} \Phi_n = d < +\infty$, from the boundedness of $\{g(x_n)\}, \{Sq_n\}$, we infer that

$$\lim_{n\to\infty} \|u_n - Sq_n\| = 0.$$

Therefore,

$$\begin{array}{ll} \|x_{n+1} - x_n\| & = \|\alpha_n g(x_n) + \beta_n(u_n - x_n) + (1 - \beta_n)(Sq_n - x_n) - \alpha_n\mu FSq_n\| \\ & = \|\alpha_n g(x_n) + (u_n - x_n) + (1 - \beta_n)(Sq_n - u_n) - \alpha_n\mu FSq_n\| \\ & \leq \|u_n - x_n\| + \|Sq_n - u_n\| + \alpha_n\|g(x_n) - \mu FSq_n\| \\ & \leq \|u_n - x_n\| + \|Sq_n - u_n\| + \alpha_n(\|g(x_n)\| + \|\mu FSq_n\|) \to 0 \quad (n \to \infty). \end{array} \tag{65}$$

Again utilizing the similar inferences to those of Aspect 1 of Claim 4 in the proof of the third theorem, one obtains $\lim_{n\to\infty} \|x_n - q^*\|^2 = 0$.

Aspect 2. Suppose that $\exists\{\Phi_{n_\iota}\} \subset \{\Phi_n\}$ s.t. $\Phi_{n_\iota} < \Phi_{n_\iota+1}$ $\forall\iota \in \mathcal{N}$, with $\mathcal{N}$ being the set of all natural numbers. The self-mapping φ on $\mathcal{N}$ is formulated as

$$\varphi(n) := \max\{\iota \leq n : \Phi_\iota < \Phi_{\iota+1}\}.$$

Using Lemma 6, one obtains

$$\Phi_{\varphi(n)} \leq \Phi_{\varphi(n)+1} \quad \text{and} \quad \Phi_n \leq \Phi_{\varphi(n)+1}.$$

Finally, by the similar inferences to those of Aspect 2 of Claim 4 in the proof of the third theorem, one can obtain the claim.

It is remarkable that, in comparison with the associated theorems in Xie et al. [9], Ceng and Shang [25], and Thong and Hieu [17], our theorems ameliorate and develop them in the aspects below.

(i) The issue for one to find a point in $\mathrm{VI}(C, A)$ (see [9]) is developed into the issue for us to find a point in $\mathrm{VI}(C, A) \cap (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i))$ with both each S_i being of nonexpansivity and $S_0 = S$ being of asymptotical nonexpansivity. The modified inertial extragradient rule with a linear-search process for settling the VIP in [9] is developed into our modified Mann-type subgradient extragradient rule with a linear-search process for settling the CFPP and VIP, which is on the basis of the Mann iteration method, subgradient extragradient approach with a linear-search process, and the hybrid deepest-descent technique.

(ii) The issue for ones to find a point in $\mathrm{VI}(C, A) \cap \mathrm{Fix}(S)$ with a quasi-nonexpansive operator S in [17] is developed into the issue for us to find a point in $\mathrm{VI}(C, A) \cap (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i))$ with both S_i being of nonexpansivity and $S_0 = S$ being of asymptotical nonexpansivity. The inertial subgradient extragradient rule with a linear-search process for settling the VIP and FPP in [17] is developed into our modified Mann-type subgradient extragradient rule with a linear-search process for settling the CFPP and VIP, which is on the basis of the Mann iteration method, subgradient extragradient approach with a linear-search process, and the hybrid deepest-descent technique.

(iii) The issue for one to find a point in $\mathrm{VI}(C, A) \cap (\bigcap_{i=0}^{N} \mathrm{Fix}(S_i))$ with finite nonexpansive operators $\{S_i\}_{i=1}^{N}$ (see [25]) is developed into the issue for us to find a point in $\mathrm{VI}(C, A) \cap (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i))$ with countable nonexpansive operators $\{S_i\}_{i=1}^{\infty}$. The hybrid inertial subgradient extragradient rule with a linear-search process in [25] is developed into our modified Mann-type subgradient extragradient rule with a linear-search process, e.g., the original inertial step $w_n = S_n x_n + \epsilon_n (S_n x_n - S_n x_{n-1})$ is developed into the modified Mann iteration step: $w_n = x_n + \alpha_n (x_n - x_{n-1})$ and $u_n = (1 - \sigma_n) w_n + \sigma_n W_n w_n$. In addition, it was shown in [25] that, under the condition $S^n q_n - S^{n+1} q_n \to 0$, the relation holds:

$$x_n \to q^* \in \Omega \;\Leftrightarrow\; \|x_n - y_n\| + \|x_n - x_{n+1}\| \to 0 \quad \text{with } q^* = P_\Omega (I - \mu F + g) q^*.$$

In this paper, using Lemma 6, we show that, under the condition $S^n x_n - S^{n+1} x_n \to 0$, the relation holds:

$$x_n \to q^* \in \Omega \;\Leftrightarrow\; \sup_{n \geq 1} \|x_{n-1} - x_n\| < \infty \quad \text{with } q^* = P_\Omega (I - \mu F + g) q^*.$$

4. Implementability and Applicability of Rules

In what follows, we provide an illustrated instance to demonstrate the implementability and applicability of proposed rules. Put $\mu = 2$, $\gamma = 1$, $\nu = l = \frac{1}{2}$, $\sigma_n = \frac{1}{3}$, $\epsilon_n = \alpha_n = \frac{1}{3(n+1)}$ and $\beta_n = \frac{n}{3(n+1)}$. First, we construct an example of $\Omega = \mathrm{VI}(C, A) \cap (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)) \neq \emptyset$ with $S_0 := S$, where $A : H \to H$ is of both pseudomonotonicity and Lipschitz continuity, $S : H \to H$ is of asymptotical nonexpansivity and each $S_i : H \to H$ is of nonexpansivity. We put $H = \mathbf{R}$ and use the $\langle r, s \rangle = rs$ and $\|\cdot\| = |\cdot|$ to denote its inner product and induced norm, respectively. Moreover, we set $C = [-2, 5]$. The starting points x_1, x_0 are arbitrarily picked in $[-2, 5]$. Let $g(x) = F(x) = \frac{1}{2} x \ \forall x \in H$ with

$$\delta = \frac{1}{2} < \zeta = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} = 1 - \sqrt{1 - 2(2 \cdot \frac{1}{2} - 2(\frac{1}{2})^2)} = 1.$$

Let $A : H \to H$ and $S, S_i : H \to H$ be formulated by $Ax = 1/(1 + |\sin x|) - 1/(1 + |x|)$, $Sx = 3 \sin x / 5$ and $S_i x = Tx = \sin x \ \forall x \in H, i \geq 1$, respectively. We now claim that A is pseudomonotone and Lipschitz continuous. In fact, one has

$$\|Ax - Ay\| \leq \frac{\|y - x\|}{(1 + \|y\|)(1 + \|x\|)} + \frac{\|\sin y - \sin x\|}{(1 + \|\sin y\|)(1 + \|\sin x\|)} \leq 2\|x - y\| \quad \forall x, y \in H$$

This means that A is of Lipschitz continuity. In addition, one shows that A is of pseudomonotonicity. It can be readily seen that

$$\begin{aligned}&\langle Ax, y-x\rangle = (1/(1+|\sin x|) - 1/(1+|x|))(y-x) \geq 0\\ &\Rightarrow \ \langle Ay, y-x\rangle = (1/(1+|\sin y|) - 1/(1+|y|)(y-x) \geq 0 \quad \forall x, y \in H.\end{aligned}$$

Meanwhile, it is easily known that S is of asymptotical nonexpansivity with $\theta_n = (\frac{3}{5})^n \ \forall n \geq 1$, such that $\|S^{n+1}x_n - S^n x_n\| \to 0$ as $n \to \infty$. Indeed, we observe that

$$\|S^n x - S^n y\| \leq \frac{3}{5}\|S^{n-1}x - S^{n-1}y\| \leq \cdots \leq (\frac{3}{5})^n \|x-y\| \leq (1+\theta_n)\|x-y\|,$$

and

$$\|S^{n+1}x_n - S^n x_n\| \leq (\frac{3}{5})^{n-1}\|S^2 x_n - Sx_n\| = (\frac{3}{5})^{n-1}\|\frac{3}{5}\sin(Sx_n) - \frac{3}{5}\sin x_n\| \leq 2(\frac{3}{5})^n \to 0.$$

It is clear that $\mathrm{Fix}(S) = \{0\}$ and

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = \lim_{n\to\infty}\frac{(3/5)^n}{1/3(n+1)} = 0.$$

In addition, it is easy to see that $S_i = T$ is of nonexpansivity and $\mathrm{Fix}(S_i) = \{0\}$. Thus, $\Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(T) \cap \mathrm{Fix}(S) = \{0\}$.

Example 1. *Noticing $W_n = T$ and $(1-\beta_n)I - \alpha_n\mu F = (1 - \frac{n}{3(n+1)})I - \frac{1}{3(n+1)}2 \cdot \frac{1}{2}I = \frac{2}{3}I$, we rewrite Algorithm 4 as follows:*

$$\begin{cases} w_n = x_n + \frac{1}{3(n+1)}(x_n - x_{n-1}), \\ u_n = \frac{2}{3}w_n + \frac{1}{3}Tw_n, \\ y_n = P_C(u_n - \varsigma_n Au_n), \\ q_n = P_{C_n}(u_n - \varsigma_n Ay_n), \\ x_{n+1} = \frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + \frac{n}{3(n+1)}x_n + \frac{2}{3}S^n q_n, \end{cases} \tag{66}$$

with C_n and ς_n being picked as in Algorithm 4 for every n. Hence, using Theorem 1, one has that $x_n \to 0 \in \Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ if and only if $\sup_{n\geq 1}|x_n - x_{n-1}| < \infty$.

Example 2. *From the nonexpansivity of $Sx := \frac{3}{5}\sin x$, one obtains the following modification of Algorithm 4:*

$$\begin{cases} w_n = x_n + \frac{1}{3(n+1)}(x_n - x_{n-1}), \\ u_n = \frac{2}{3}w_n + \frac{1}{3}Tw_n, \\ y_n = P_C(u_n - \varsigma_n Au_n), \\ q_n = P_{C_n}(u_n - \varsigma_n Ay_n), \\ x_{n+1} = \frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + \frac{n}{3(n+1)}x_n + \frac{2}{3}Sq_n, \end{cases} \tag{67}$$

with C_n and ς_n being picked in the above way. Thus, using Theorem 2, one knows that $x_n \to 0 \in \Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ if and only if $\sup_{n\geq 1}|x_n - x_{n-1}| < \infty$.

Example 3. *Noticing $W_n = T$ and $(1-\beta_n)I - \alpha_n\mu F = (1 - \frac{n}{3(n+1)})I - \frac{1}{3(n+1)}2 \cdot \frac{1}{2}I = \frac{2}{3}I$, we rewrite Algorithm 5 as follows:*

$$\begin{cases} w_n = x_n + \frac{1}{3(n+1)}(x_n - x_{n-1}), \\ u_n = \frac{2}{3}x_n + \frac{1}{3}Tw_n, \\ y_n = P_C(u_n - \varsigma_n Au_n), \\ q_n = P_{C_n}(u_n - \varsigma_n Ay_n), \\ x_{n+1} = \frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + \frac{n}{3(n+1)}u_n + \frac{2}{3}S^n q_n, \end{cases} \tag{68}$$

with C_n and ς_n being picked as in Algorithm 5 for every n. Hence, using Theorem 3, one has that $x_n \to 0 \in \Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ if and only if $\sup_{n\geq 1} |x_n - x_{n-1}| < \infty$.

Example 4. *From the nonexpansivity of $Sx := \frac{3}{5}\sin x$, one obtains the following modification of Algorithm 5:*

$$\begin{cases} w_n = x_n + \frac{1}{3(n+1)}(x_n - x_{n-1}), \\ u_n = \frac{2}{3}x_n + \frac{1}{3}Tw_n, \\ y_n = P_C(u_n - \varsigma_n Au_n), \\ q_n = P_{C_n}(u_n - \varsigma_n Ay_n), \\ x_{n+1} = \frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + \frac{n}{3(n+1)}u_n + \frac{2}{3}Sq_n, \end{cases} \tag{69}$$

with C_n and ς_n being picked in the above way. Thus, using Theorem 4, one knows that $x_n \to 0 \in \Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ if and only if $\sup_{n\geq 1} |x_n - x_{n-1}| < \infty$.

It is noteworthy that the above two modified Mann-type subgradient extragradient algorithms with a linear-search process (i.e., Algorithms 4 and 5) are both applied for finding a point in the common solution set $\Omega = \mathrm{VI}(C, A) \cap (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i))$ with countable nonexpansive operators $\{S_i\}_{i=1}^{\infty}$ and asymptotically nonexpansive operator S. Under the same conditions imposed on the parameter sequences, we show the strong convergence of these two different algorithms to an element $q^* \in \Omega$, which is also a unique solution of the HFPP: $q^* = P_\Omega(I - \mu F + g)q^*$; see Theorems 1 and 3 for more details. Note that Algorithm 4 is very similar to Algorithm 5 because these two different algorithms belong to the same class of modified Mann-type subgradient extragradient rules with a linear-search process. It is not difficult to find that Algorithm 4 is extended to develop Algorithm 5, e.g., (i) the original Mann iterative step $u_n = (1 - \sigma_n)w_n + \sigma_n W_n w_n$ in Algorithm 4 is developed into the modified Mann iterative step $u_n = (1 - \sigma_n)x_n + \sigma_n W_n w_n$ in Algorithm 5, and (ii) the original viscosity hybrid deepest-descent step $x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)S^n q_n$ in Algorithm 4 is developed into the modified viscosity hybrid deepest-descent step $x_{n+1} = \alpha_n g(x_n) + \beta_n u_n + ((1 - \beta_n)I - \alpha_n \mu F)S^n q_n$ in Algorithm 5. In the above Examples 1 and 3, the iterative schemes (66) and (68) are numerical examples of Algorithms 4 and 5, respectively, and both are applied for finding $0 \in \Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$. Compared with scheme (66), the scheme (68) improves and develops it in the following aspects:

(i) The original Mann iterative step $u_n = \frac{2}{3}w_n + \frac{1}{3}Tw_n$ in (66) is developed into the modified Mann iterative step $u_n = \frac{2}{3}x_n + \frac{1}{3}Tw_n$ in (68);

(ii) The original viscosity hybrid deepest-descent step $x_{n+1} = \frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + \frac{n}{3(n+1)}x_n + \frac{2}{3}S^n q_n$ in (66) is developed into the modified viscosity hybrid deepest-descent step $x_{n+1} = \frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + \frac{n}{3(n+1)}u_n + \frac{2}{3}S^n q_n$ in (68).

Finally, applying Theorems 1 and 3 to schemes (66) and (68), respectively, we obtain that $x_n \to 0 \in \Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ if and only if $\sup_{n\geq 1} |x_n - x_{n-1}| < \infty$.

5. Conclusions

In real Hilbert spaces, we have designed two modified Mann-type subgradient extragradient rules with a linear-search process for settling the variational inequality problem (VIP) for a Lipschitz continuity and pseudomonotonicity operator A, and the common fixed-point problem (CFPP) for countable nonexpansivity operators $\{S_i\}_{i=1}^{\infty}$ and an asymptotical nonexpansivity operator $S_0 := S$. Under the lack of the sequential weak continuity and Lipschitz constant of the cost operator A, we have demonstrated the strong convergence of the constructed algorithms to a common element of the solution set of the VIP and the common fixed-point set of operators $\{S_i\}_{i=0}^{\infty}$, which is only a solution of a certain hierarchical fixed-point problem (HFPP). In addition, an illustrated example is provided to demonstrate the implementability and applicability of our proposed rules.

It is worth pointing out that there are our contributions to the research area of finding a common solution of the VIP and CFPP in three aspects below:

First, we extend the problem considered in [25], that is, the problem of finding a point in $\mathrm{VI}(C, A) \cap (\bigcap_{i=0}^{N} \mathrm{Fix}(S_i))$ with finite nonexpansive operators $\{S_i\}_{i=1}^{N}$ is developed into the problem of finding a point in $\mathrm{VI}(C, A) \cap (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i))$ with countable nonexpansive operators $\{S_i\}_{i=1}^{\infty}$.

Second, we improve the rules proposed in [25], that is, the hybrid inertial subgradient extragradient rule with a linear-search process in [25] is developed into our modified Mann-type subgradient extragradient rule with a linear-search process, e.g., the original inertial step $w_n = S_n x_n + \epsilon_n(S_n x_n - S_n x_{n-1})$ is developed into the modified Mann iteration step: $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and $u_n = (1 - \sigma_n)w_n + \sigma_n W_n w_n$.

Finally, we weaken the convergence criteria presented in [25]. Indeed, it was shown in [25] that, under the condition $S^n q_n - S^{n+1} q_n \to 0$, the relation holds:

$$x_n \to q^* \in \Omega \;\Leftrightarrow\; \|x_n - y_n\| + \|x_n - x_{n+1}\| \to 0 \quad \text{with } q^* = P_\Omega(I - \mu F + g)q^*.$$

In this article, using Lemma 6 (i.e., Maingé's lemma [34]), we show that, under the condition $S^n x_n - S^{n+1} x_n \to 0$, the relation holds:

$$x_n \to q^* \in \Omega \;\Leftrightarrow\; \sup_{n\geq 1} \|x_{n-1} - x_n\| < \infty \quad \text{with } q^* = P_\Omega(I - \mu F + g)q^*.$$

In addition, it is worth mentioning that part of our future research is aimed at acquiring the strong convergence results for the modifications of our proposed rules with a Nesterov inertial extrapolation step and adaptive stepsizes.

Author Contributions: Conceptualization, F.-F.Z. and H.-Y.H.; Data curation, F.-F.Z.; Formal analysis, Y.-L.C., C.-S.W., J.-Y.L. and L.H.; Funding acquisition, L.-C.C.; Investigation, Y.-L.C., L.-C.C., F.-F.Z., C.-S.W., J.-Y.L., H.-Y.H. and L.H.; Methodology, H.-Y.H.; Project administration, L.-C.C.; Resources, Y.-L.C.; Software, C.-S.W. and J.-Y.L.; Supervision, L.-C.C.; Validation, L.H.; Writing—original draft, Y.-L.C. and L.-C.C.; Writing—review & editing, L.-C.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the 2020 Shanghai Leading Talents Program of the Shanghai Municipal Human Resources and Social Security Bureau (20LJ2006100), the Innovation Program of Shanghai Municipal Education Commission (15ZZ068) and the Program for Outstanding Academic Leaders in Shanghai City (15XD1503100).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Rashid, S.; Ashraf, R.; Bonyah, E. On analytical solution of time-fractional biological population model by means of generalized integral transform with their uniqueness and convergence analysis. *J. Funct. Spaces* **2022**, *2022*, 7021288. [CrossRef]
2. Rezapour, S.; Deressa, C.T.; Hussain, A.; Etemad, S.; George, R.; Ahmad, B. A theoretical analysis of a fractional multi-dimensional system of boundary value problems on the methylpropane graph via fixed point technique. *Mathematics* **2022**, *10*, 568. [CrossRef]
3. Turab, A.; Sintunavarat, W. On the solution of the traumatic avoidance learning model approached by the Banach fixed point theorem. *J. Fixed Point Theory Appl.* **2020**, *22*, 12. [CrossRef]
4. Korpelevich, G.M. The extragradient method for finding saddle points and other problems. *Ekon. Mat. Metod.* **1976**, *12*, 747–756.
5. Yao, Y.; Liou, Y.C.; Kang, S.M. Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method. *Comput. Math. Appl.* **2010**, *59*, 3472–3480. [CrossRef]
6. Jolaoso, L.O.; Shehu, Y.; Yao, J.C. Inertial extragradient type method for mixed variational inequalities without monotonicity. *Math. Comput. Simul.* **2022**, *192*, 353–369. [CrossRef]
7. Ceng, L.C.; Petrusel, A.; Yao, J.C.; Yao, Y. Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces. *Fixed Point Theory* **2018**, *19*, 487–501. [CrossRef]

8. Ceng, L.C.; Petrusel, A.; Qin, X.; Yao, J.C. Two inertial subgradient extragradient algorithms for variational inequalities with fixed-point constraints. *Optimization* **2021**, *70*, 1337–1358. [CrossRef]
9. Xie, Z.; Cai, G.; Li, X.; Dong, Q.L. Strong convergence of the modified inertial extragradient method with line-search process for solving variational inequality problems in Hilbert spaces. *J. Sci. Comput.* **2021**, *88*, 19. [CrossRef]
10. Yao, Y.; Shahzad, N.; Yao, J.C. Convergence of Tseng-type self-adaptive algorithms for variational inequalities and fixed point problems. *Carpathian J. Math.* **2021**, *37*, 541–550. [CrossRef]
11. He, L.; Cui, Y.L.; Ceng, L.C.; Zhao, T.Y.; Wang, D.Q.; Hu, H.Y. Strong convergence for monotone bilevel equilibria with constraints of variational inequalities and fixed points using subgradient extragradient implicit rule. *J. Inequal. Appl.* **2021**, *2021*, 37. [CrossRef]
12. Zhao, T.Y.; Wang, D.Q.; Ceng, L.C.; He, L.; Wang, C.Y.; Fan, H.L. Quasi-inertial Tseng's extragradient algorithms for pseudomonotone variational inequalities and fixed point problems of quasi-nonexpansive operators. *Numer. Funct. Anal. Optim.* **2020**, *42*, 69–90. [CrossRef]
13. Denisov, S.V.; Semenov, V.V.; Chabak, L.M. Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators. *Cybern. Syst. Anal.* **2015**, *51*, 757–765. [CrossRef]
14. Cai, G.; Shehu, Y.; Iyiola, O.S. Strong convergence results for variational inequalities and fixed point problems using modified viscosity implicit rules. *Numer. Algorithms* **2018**, *77*, 535–558. [CrossRef]
15. Yang, J.; Liu, H.; Liu, Z. Modified subgradient extragradient algorithms for solving monotone variational inequalities. *Optimization* **2018**, *67*, 2247–2258. [CrossRef]
16. Vuong, P.T. On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities. *J. Optim. Theory Appl.* **2018**, *176*, 399–409. [CrossRef] [PubMed]
17. Thong, D.V.; Hieu, D.V. Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems. *Numer. Algorithms* **2019**, *80*, 1283–1307. [CrossRef]
18. Shehu, Y.; Iyiola, O.S. Strong convergence result for monotone variational inequalities. *Numer. Algorithms* **2019**, *76*, 259–282. [CrossRef]
19. Thong, D.V.; Dong, Q.L.; Liu, L.L.; Triet, N.A.; Lan, N.P. Two fast converging inertial subgradient extragradient algorithms with variable stepsizes for solving pseudo-monotone VIPs in Hilbert spaces. *J. Comput. Appl. Math.* **2022**, *410*, 19. [CrossRef]
20. Vuong, P.T.; Shehu, Y. Convergence of an extragradient-type method for variational inequality with applications to optimal control problems. *Numer. Algorithms* **2019**, *81*, 269–291. [CrossRef]
21. Shehu, Y.; Dong, Q.L.; Jiang, D. Single projection method for pseudo-monotone variational inequality in Hilbert spaces. *Optimization* **2019**, *68*, 385–409. [CrossRef]
22. Thong, D.V.; Hieu, D.V. Modified subgradient extragradient method for variational inequality problems. *Numer. Algorithms* **2018**, *79*, 597–610. [CrossRef]
23. Kraikaew, R.; Saejung, S. Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. *J. Optim. Theory Appl.* **2014**, *163*, 399–412. [CrossRef]
24. Ceng, L.C.; Wen, C.F. Systems of variational inequalities with hierarchical variational inequality constraints for asymptotically nonexpansive and pseudocontractive mappings. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. Racsam* **2019**, *113*, 2431–2447. [CrossRef]
25. Ceng, L.C.; Shang, M.J. Hybrid inertial subgradient extragradient methods for variational inequalities and fixed point problems involving asymptotically nonexpansive mappings. *Optimization* **2021**, *70*, 715–740. [CrossRef]
26. Ceng, L.C.; Petrusel, A.; Qin, X.; Yao, J.C. Pseudomonotone variational inequalities and fixed points. *Fixed Point Theory* **2021**, *22*, 543–558. [CrossRef]
27. Reich, S.; Thong, D.V.; Dong, Q.L.; Li, X.H.; Dung, V.T. New algorithms and convergence theorems for solving variational inequalities with non-Lipschitz mappings. *Numer. Algorithms* **2021**, *87*, 527–549. [CrossRef]
28. Iusem, A.N.; Nasri, M. Korpelevich's method for variational inequality problems in Banach spaces. *J. Global Optim.* **2011**, *50*, 59–76. [CrossRef]
29. Goebel, K.; Reich, S. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*; Marcel Dekker: New York, NY, USA, 1984.
30. Chang, S.S.; Lee, H.W.J.; Chan, C.K. A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization. *Nonlinear Anal.* **2009**, *70*, 3307–3319. [CrossRef]
31. Opial, Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Am. Math. Soc.* **1967**, *73*, 591–597. [CrossRef]
32. Xu, H.K.; Kim, T.H. Convergence of hybrid steepest-descent methods for variational inequalities. *J. Optim. Theory Appl.* **2003**, *119*, 185–201. [CrossRef]
33. Ceng, L.C.; Xu, H.K.; Yao, J.C. The viscosity approximation method for asymptotically nonexpansive mappings in Banach spaces. *Nonlinear Anal.* **2008**, *69*, 1402–1412. [CrossRef]
34. Maingé, P.E. Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. *Set-Valued Anal.* **2008**, *16*, 899–912. [CrossRef]