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On Some Model Theoretic Properties of Totally Bounded Ultrametric Spaces

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Abstract: Continuing investigations initiated by the first author, we associate relational structures for metric spaces and investigate their model theoretic properties. In this paper, we consider ultrametric spaces. Among others, we show that any elementary substructure of the relational structure associated with a totally bounded ultrametric space $\mathcal{X}$ is dense in X. Further, we provide an explicit upper bound for a splitting chain of atomic types in ultrametric spaces of a finite spectrum. For ultrametric spaces, these results improve previous ones of the present authors and may have further practical applications in designing similarity detecting algorithms.

Keywords: model theoretic stability; ultrametric spaces; totally bounded spaces

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1. Introduction

Let $\mathcal{X} = \langle X, \varrho \rangle$ be a metric space. First, we recall a well-known method that associates a relational structure $\mathcal{A}(\mathcal{X})$ to $\mathcal{X}$. If d is a distance of $\mathcal{X}$, that is, $d \in ran(\varrho)$, then the binary relation R_d is defined to be

$$R_d = \{\langle a, b \rangle \in X^2 : \varrho(a, b) \leq d\}.$$

Thus, the relational structure $\mathcal{A}(\mathcal{X}) := \langle X, R_d \rangle_{d \in ran(\varrho)}$ completely describes $\mathcal{X}$ and, at the same time, it can be treated as a first order relational structure.

We also recall that the metric space $\mathcal{X} = \langle X, \varrho \rangle$ is an *ultrametric space* if and only if

$$\varrho(a, b) \leq \max\{\varrho(a, c), \varrho(c, b)\}$$

holds for all $a, b, c \in X$. Furthermore, a metric space $\mathcal{X} = \langle X, \varrho \rangle$ is defined to be *totally bounded* iff, for all $\varepsilon \in \mathbb{R}^+$, there exists a finite set $C_\varepsilon \subseteq X$ such that, for all $a \in X$, there exists $c \in C_\varepsilon$ with $\varrho(a, c) < \varepsilon$, or in another words, the union of the finitely many open ε-balls centered in the elements of C_ε covers X.

Investigations related to the present work have been initiated and motivated in [1,2]. The present results may have practical applications e.g., in designing similarity detecting algorithms. More concretely, consider the following setting:

- X is a set of instances of an abstract data type and
- ϱ is a distance function on X measuring similarity of elements of X (that is, if $\varrho(a, b)$ is small for some $a, b \in X$, then a and b are "similar").

The similarity detecting problem is the following. We are given a fixed set $A \subseteq X$ and, for an input $x \in X$, we would like to find some (or all) elements $a \in A$ which are "similar enough to x", that is, for which $\varrho(x,a)$ is smaller than a previously fixed $\varepsilon \in \mathbb{R}^+$. Very often, X may be infinite and A is finite but huge. The challenge is how to represent A. In these problems, usually the metric space is compact or at least totally bounded and often, it has an ultrametric distance function. For further details, we refer to [2,3]. More applications of finite metric spaces can be found in [4–6].

Structural investigations of $\mathcal{A}(\mathcal{X})$ may help to find suitable representations for finite subspaces of $\mathcal{X}$. In particular, "small" substructures of $\mathcal{A}(\mathcal{X})$ similar to the whole $\mathcal{A}(\mathcal{X})$ play a central role. Hence, elementary substructures of $\mathcal{A}(\mathcal{X})$, reflection principles and model theoretic stability for formulas of low complexity (atomic formulas, universal formulas, etc.) are particularly interesting.

From the results of [1], one can easily obtain that, if $\mathcal{X}$ is a totally bounded metric space, then $\mathcal{A}(\mathcal{X})$ is Δ-stable in the model theoretic sense (where Δ is the set of all atomic formulas of the language of $\mathcal{A}(\mathcal{X})$). The proof of this was reconstructed in [7] (see Theorem 2.3 therein), where, similarly to [8], we find an "almost isometry" from our space into a finite dimensional Euclidean space. In addition, in Theorem 2.4 of [7], we also characterized dense in itself, totally bounded metric spaces in terms of stability properties of their associated relational structures. Moreover, by [9], stability of generalized Urysohn spaces can be charactized by ultrametrizability.

In the present work, we will investigate totally bounded ultrametric spaces. As we mentioned at the end of the previous paragraph, according to Theorem 2.4 of [7], for a dense in itself metric space $\mathcal{X}$, certain stability properties of $\mathcal{A}(\mathcal{X})$ are equivalent with total boundedness of $\mathcal{X}$. Hence, this assumption seems to be reasonable. Furthermore, as we mentioned above, from the point of view of applications, total boundedness of $\mathcal{X}$ seems to be a mild assumption as well.

The main results of the present work are as follows. In Theorem 1, we prove that, if $\mathcal{X}$ is a totally bounded metric space and $\mathcal{Y}$ is an elementary substructure of $\mathcal{A}(\mathcal{X})$, then Y is dense in X. According to the remark right after the proof of Theorem 1, this theorem is not true in general (total boundedness cannot be deleted from the hypotheses). In Theorem 3, we give an explicit upper bound for the length of a splitting chain of atomic types; this may be used to design and analyze similarity detecting algorithms for ultrametic spaces with finite spectrum (we postpone related investigations later). In Corollary 3, we show that, if $\mathcal{X}$ is totally bounded, then one-point extensions of $\aleph_0$-saturated elementary substructures of $\mathcal{A}(\mathcal{X})$ remain ($\aleph_0$-saturated) elementary substructures. As a byproduct statement, in Corollary 2, we show that, for a totally bounded $\mathcal{X}$, the associated relational structure $\mathcal{A}(\mathcal{X})$ is not only Δ-stable, but, in fact, stable in the model theoretical sense. For ultrametric spaces, this improves the results presented in [7]. We do not know if Corollary 2 remains true without assuming that $\mathcal{X}$ is totally bounded.

The structure of the paper is rather simple: we close this section by fixing our notation. Section 2 contains general model theoretical investigations and Section 3 is devoted to investigate ultrametric spaces of the finite spectrum.

Notation

Our notation is mostly standard, but the following list may help. In general, for topological or model theoretic notions not recalled here, we refer to [10] (and respectively to [11] or [12]).

Throughout this, $\mathbb{N}$ denotes the set of natural numbers. In addition, $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of real numbers, and the set of positive real numbers, respectively.

Let $\mathcal{X} = \langle X, \varrho \rangle$ be a metric space, $a \in X$ and let γ be a non-negative real number. As usual, the open γ-ball $B(\gamma, a)$ at a is the set

$$B(\gamma, a) = \{x \in X : \varrho(a, x) < \gamma\}.$$

If $\mathcal{L}$ is a first order language, then $Form_{\mathcal{L}}$ denotes the set of formulas of $\mathcal{L}$.

2. Stability

Studying combinatorial and model theoretic properties of metric and ultrametric spaces has a great tradition (see, e.g., Section 6.4 of [13], [14,15], and the references therein). In this section, we investigate model theoretic stability of certain ultrametric spaces.

In this section, unless otherwise stated, Δ and Γ denote finite sets of formulas. As usual, we do not make a strict distinction between logically equivalent (but syntactically different) formulas.

We assume that the reader is familiar with the notion of structures, types, partial types, etc. For related definitions, we refer to [11,12].

Definition 1. *Let $\mathcal{A}$ be a structure, let $X \subseteq Y \subseteq A$ and let p be a partial type over Y. We recall from [12] that, by definition, p is (Δ, Γ)-splitting over X iff there are $\overline{b}, \overline{b}' \in Y$ and $\varphi(\overline{v}, \overline{w}) \in \Gamma$ such that $tp_\Delta(\overline{b}/X) = tp_\Delta(\overline{b}'/X)$ but $\varphi(\overline{v}, \overline{b}), \neg\varphi(\overline{v}, \overline{b}') \in p$.*

Furthermore, p splits over X iff it $(Form_{\mathcal{L}}, Form_{\mathcal{L}})$-splits over X where $Form_{\mathcal{L}}$ denotes the set of all formulas of the language $\mathcal{L}$ of $\mathcal{A}$.

Remark 1. *Keeping the notation of Definition 1, the following properties of splitting are immediate.*

(1) If p (Δ, Γ)-splits over X and $\Delta' \subseteq \Delta$, $\Gamma \subseteq \Gamma'$, then p (Δ', Γ')-splits over X.
(2) If p (Δ, Γ)-splits over X and $X' \subseteq X$, then p (Δ, Γ)-splits over X'.

Lemma 1. *Suppose $\mathcal{A}$ is a structure, $X \subseteq Y \subseteq A$ and $\bar{a} \in A - Y$. Suppose $\phi, \psi \in \Gamma$ are formulas.*

(1) If $tp_{\{\neg\phi\}}(\bar{a}/Y)$ $(\Delta, \{\neg\phi\})$-splits over X, then $tp_\Gamma(\bar{a}/Y)$ (Δ, Γ)-splits over X.
(2) If $tp_{\{\phi\wedge\psi\}}(\bar{a}/Y)$ $(\Delta, \{\phi \wedge \psi\})$-splits over X, then $tp_\Gamma(\bar{a}/Y)$ (Δ, Γ)-splits over X.

Proof. To show (1) assume $tp_{\{\neg\phi\}}(\bar{a}/Y)$ $(\Delta, \neg\phi)$-splits over X. Then, there exist $\bar{b}, \bar{b}' \in Y$ such that

$$tp_\Delta(\bar{b}/X) = tp_\Delta(\bar{b}'/X) \text{ and } \neg\phi(\bar{a}, \bar{b}), \neg\neg\phi(\bar{a}, \bar{b}') \in tp_{\{\neg\phi\}}(\bar{a}/Y).$$

Thus, $\neg\phi(\bar{a}, \bar{b}), \phi(\bar{a}, \bar{b}') \in tp_\Gamma(\bar{a}/Y)$ showing that $tp_\Gamma(\bar{a}/Y)$ (Δ, Γ)-splits over X.

The proof of (2) is similar: assume $tp_{\{\phi\wedge\psi\}}(\bar{a}/Y)$ $(\Delta, \{\phi \wedge \psi\})$-splits over X. Then, there exist $\bar{b}, \bar{b}' \in Y$ such that

$$tp_\Delta(\bar{b}/X) = tp_\Delta(\bar{b}'/X) \text{ and } (\phi \wedge \psi)(\bar{a}, \bar{b}), \neg(\phi \wedge \psi)(\bar{a}, \bar{b}') \in tp_{\{\phi\wedge\psi\}}(\bar{a}/Y).$$

In particular, $\mathcal{A} \models \neg(\phi(\bar{a}, \bar{b}')) \vee \neg(\psi(\bar{a}, \bar{b}'))$. By symmetry, we may assume

$$\mathcal{A} \models \neg\phi(\bar{a}, \bar{b}'),$$

that is, $\neg\phi(\bar{a}, \bar{b}') \in tp_\Gamma(\bar{a}/Y)$. However, then $\phi, \bar{b}$ and $\bar{b}'$ show that $tp_\Gamma(\bar{a}/Y)$ (Δ, Γ)-splits over X. □

We will use the following notation. If Ω is a set of formulas, then

$$\exists\Omega := \Omega \cup \{\exists v\varphi : \varphi \in \Omega \text{ and } v \text{ is a variable occurring in } \varphi\}.$$

Lemma 2. *Suppose $\mathcal{A}$ is a structure, $X \subseteq Y \subseteq A$, X is finite and $\bar{a} \in A - Y$. Suppose*

(a) if $\bar{b} \in Y$, $\phi \in \Gamma$ and $\mathcal{A} \models \exists v\phi(v, \bar{a}, \bar{b})$, then there exists $c \in Y$ such that $\mathcal{A} \models \phi(c, \bar{a}, \bar{b})$ and
(b) for all finite $Y_0 \subseteq Y$, each Δ-type over Y_0 can be realized in Y.

If $tp_{\exists\Gamma}(\bar{a}/Y)$ $(\exists\Delta, \exists\Gamma)$-splits over X, then $tp_\Gamma(\bar{a}/Y)$ (Δ, Γ)-splits over X.

Proof. Assume $tp_{\exists\Gamma}(\bar{a}/Y)$ $(\exists\Delta, \exists\Gamma)$-splits over X. Then, there exist $\bar{b}, \bar{b}' \in Y$ and $\phi \in \Gamma$ such that

$$tp_{\exists\Delta}(\bar{b}/X) = tp_{\exists\Delta}(\bar{b}'/X) \text{ but } \mathcal{A} \models \exists v\phi(v, \bar{a}, \bar{b}) \wedge \neg\exists v\phi(v, \bar{a}, \bar{b}'). \quad (1)$$

By (a), there exists $c \in Y$ with $\mathcal{A} \models \phi(c, \bar{a}, \bar{b})$. Let $p(v, \bar{w}) = tp_{\Delta}(c\bar{b}/X)$ (where the variable v corresponds to c and $\bar{w}$ corresponds to $\bar{b}$). Then, c witnesses that $\exists v \wedge p(v, \bar{w}) \in tp_{\exists\Delta}(\bar{b}/X)$. Furthermore, as $tp_{\exists\Delta}(\bar{b}/X) = tp_{\exists\Delta}(\bar{b}'/X)$, we obtain $\exists v \wedge p(v, \bar{w}) \in tp_{\exists\Delta}(\bar{b}'/X)$. Then, $p(v, \bar{b}')$ is a Δ-type over $X \cup \bar{b}'$.

Hence, (b) implies that there exists a realization $c' \in Y$ of $p(v, \bar{b}')$, in particular,

$$p(v, \bar{w}) = tp_{\Delta}(c\bar{b}/X) = tp_{\Delta}(c'\bar{b}'/X).$$

However, then $c\bar{b}, c'\bar{b}'$ and ϕ show that $tp_{\Gamma}(\bar{a}/Y)$ (Δ, Γ)-splits over X (we note that $\mathcal{A} \models \neg\phi(c', \bar{a}, \bar{b}')$ because of the last part of (1)). □

Theorem 1. *Suppose $\mathcal{X} = \langle X, \varrho \rangle$ is a totally bounded metric space. If $\mathcal{Y}$ is an elementary substructure of $\mathcal{A}(\mathcal{X})$, then Y is dense in X.*

Proof. Since $\mathcal{X}$ is assumed to be totally bounded, it follows that, for each $\varepsilon \in ran(\varrho)$, there exists $n \in \mathbb{N}$ and

$$\text{there exist } a_0, ..., a_{n-1} \in X \text{ such that } \{B(\varepsilon, a_i) : i < n\} \text{ is an } \varepsilon\text{-net}. \quad (2)$$

However, (2) can be formalized in the language of $\mathcal{A}(\mathcal{X})$. Hence,

$$\text{there are } a'_0, ..., a'_{n-1} \in Y \text{ such that } \{B(\varepsilon, a'_i) : i < n\} \text{ is an } \varepsilon\text{-net in } \mathcal{Y}.$$

Finally, as $\mathcal{Y}$ is an elementary substructure of $\mathcal{A}(\mathcal{X})$, it follows that $\{B(\varepsilon, a'_i) : i < n\}$ is an ε-net in $\mathcal{X}$ as well. □

It is natural to ask if Theorem 1 remains true without assuming that $\mathcal{X}$ is totally bounded. The following example shows that this condition cannot be completely eliminated, that is, some extra conditions for $\mathcal{X}$ must be assumed in order to reach the desired conclusion.

Let A be any uncountably infinite set and let X be the set of all functions from $\mathbb{N}$ into A. For any $f, g \in X$, define $\varrho(f, g)$ to be

$$\varrho(f, g) = \begin{cases} 0 & \text{if } f = g, \\ \frac{1}{n+1} & \text{if } f \neq g \text{ and } n \text{ is the smallest number for which } f(n) \neq g(n). \end{cases}$$

It is straightforward to check that $\langle X, \varrho \rangle$ is an ultrametric space. As $ran(\varrho) \subseteq \mathbb{Q}$, the language of $\mathcal{A}(\mathcal{X})$ is countable. Hence, there exists a countable elementary substructure $\mathcal{Y}$ of $\mathcal{A}(\mathcal{X})$. Finally, we show that Y is not dense in X. To do so, let $R = \{f(0) : f \in Y\}$. Since Y is countable, so is R and hence there exists $a \in A - R$. Let $g \in X$ be such that $g(0) = a$. However, then, clearly, Y and $B(\frac{1}{2}, g)$ are disjoint (recall that $B(\frac{1}{2}, g)$ denoting the open ball centered at g with radius $\frac{1}{2}$).

Lemma 3. *Suppose $\mathcal{X} = \langle X, \varrho \rangle$ is a metric space and let Δ be the set of atomic formulas of the language of $\mathcal{A}(\mathcal{X})$. Let $A_0 \subseteq A_1 \subseteq B \subseteq X$ and $a, b \in X - B$. Let $f : B \cup \{a, b\} \to B \cup \{a, b\}$ be the function interchanging a and b and leaving all elements of B fixed. Suppose in addition that*

(1) $tp_{\Delta}(a/A_1) = tp_{\Delta}(b/A_1)$;
(2) the types $tp_{\Delta}(a/B)$ and $tp_{\Delta}(b/B)$ do not split over A_0;
(3) if a Δ-type over A_0 can be realized in B, then it also can be realized in A_1.

Then, f is a partial isomorphism of $\mathcal{A}(\mathcal{X})$ (that is, f is an isomorphism between the substructures of $\mathcal{A}(\mathcal{X})$ generated by its domain and range).

Proof. Let $x \in ran(\varrho)$, $c, d \in B \cup \{a, b\}$ be arbitrary. If $c, d \in B$ or $\{c, d\} = \{a, b\}$, then, clearly,

$$\mathcal{A}(\mathcal{X}) \models \varrho(c,d) \leq x \text{ iff } \mathcal{A}(\mathcal{X}) \models \varrho(f(c), f(d)) \leq x.$$

Hence, without loss of generality, we may assume that $c = a$ and $d \in B$.

By (3), there is an element $d' \in A_1$ such that $tp_\Delta(d/A_0) = tp_\Delta(d'/A_0)$. In addition, we have

$$tp_\Delta(a/d) \overset{by\ (2)}{=} tp_\Delta(a/d') \overset{by\ (1)}{=} tp_\Delta(b/d') \overset{by\ (2)}{=} tp_\Delta(b/d)$$

hence $\mathcal{A}(\mathcal{X}) \models \varrho(a,d) \leq x$ iff $\mathcal{A}(\mathcal{X}) \models \varrho(f(a), f(d)) \leq x$, as desired. □

Lemma 4. *Suppose $\mathcal{X} = \langle X, \varrho \rangle$ is an ultrametric space and let Δ be the set of atomic formulas of the language of $\mathcal{A}(\mathcal{X})$. Suppose $A \subseteq X$ and $a, b \in X$ are such that*

$$\varrho(a,b) < inf\{\varrho(a,c) : c \in A\}.$$

Then, $tp_\Delta(a/A) = tp_\Delta(b/A)$.

Proof. Let $c \in A$ be arbitrary. Then,

$$\varrho(b,c) \leq max\{\varrho(b,a), \varrho(a,c)\} = \varrho(a,c).$$

Furthermore, suppose seeking a contradiction that $\varrho(b,c) < \varrho(a,b)$. Then, we would have

$$\varrho(a,c) \leq max\{\varrho(a,b), \varrho(b,c)\} = \varrho(a,b),$$

which is impossible. It follows that $\varrho(a,c) = \varrho(b,c)$ holds for any $c \in A$, as desired. □

Notational Convention. In order to make our notation more reader friendly, till the end of the present section, we will apply the following notational conventions. If $\mathcal{X}$ is a metric space, then, for simplicity, we will denote $\mathcal{A}(\mathcal{X})$ by $\mathcal{A}$ and the underlying set of $\mathcal{A}(\mathcal{X})$ by A. Furthermore, Δ will denote the set of atomic formulas of the language of $\mathcal{A}(\mathcal{X})$.

Lemma 5. *Let $\mathcal{X}$ be a totally bounded ultrametric space. With the above notational convention, suppose, for each $Y \subseteq A$ and $a \in A - Y$, there exists a finite $X \subseteq Y$ such that $tp_\Delta(a/Y)$ does not $(Form_\mathcal{L}, \Delta)$-split over X.*

If Y is the universe of an $\aleph_0$-saturated elementary substructure of $\mathcal{A}$ and $a \in A$, then $Y \cup \{a\}$ generates an $\aleph_0$-saturated elementary substructure of $\mathcal{A}$.

Proof. If $a \in Y$, then the statement is obvious. Hence, in the rest of the proof, we assume $a \in A - Y$. Throughout this proof, if $U \subseteq A$, then the substructure of $\mathcal{A}$ generated by U will be denoted by $\mathcal{A}|_U$.

First, we shall show that

$$\mathcal{A}|_Y \text{ is an elementary substructure of } \mathcal{A}|_{Y \cup \{a\}}. \tag{3}$$

To do so, assume $\underline{d} \in Y$, $u \in Y \cup \{a\}$ and φ is a formula such that $\mathcal{A}|_{Y \cup \{a\}} \models \varphi(u, \underline{d})$. It is enough to check that there exists $v \in Y$ such that $\mathcal{A}|_{Y \cup \{a\}} \models \varphi(v, \underline{d})$. If $u \in Y$, then we are done, so we may assume $u = a$.

Observe that, by assumption, there exists a finite subset $A_0 \subseteq Y$ such that $tp_\Delta(a/Y)$ does not $(Form_\mathcal{L}, \Delta)$-split over A_0. We claim that, in fact,

$$tp_\Delta(a/A - \{a\}) \text{ does not } (Form_\mathcal{L}, \Delta)\text{-split over } A_0. \tag{4}$$

To check this, assume, seeking a contradiction, that there are $x, y \in A - \{a\}$ such that $tp(x/A_0) = tp(y/A_0)$ but $\varrho(a,x) \neq \varrho(a,y)$. Let

$$\varepsilon := inf\Big(\{\varrho(a,e) : e \in A_0 \cup \{x,y\}\}\Big).$$

Since $A_0 \subseteq Y$ and $a \in A - Y$, we have $\varepsilon > 0$. Furthermore, by Lemma 1, there exists $a' \in Y$ such that $\varrho(a,a') < \frac{\varepsilon}{2}$. By Lemma 4, $\varrho(a,x) = \varrho(a',x)$ and $\varrho(a,y) = \varrho(a',y)$. As $\mathcal{A}|_Y$ is an elementary substructure of $\mathcal{A}$, there exist $x', y' \in Y$ such that

$$tp(x/A_0 \cup \{a'\}) = tp(x'/A_0 \cup \{a'\})$$

and

$$tp(y/A_0 \cup \{a'\}) = tp(y'/A_0 \cup \{a'\}).$$

In particular, $tp_\Delta(a'/A_0 \cup \{x',y'\})$ $(Form_\mathcal{L}, \Delta)$-splits over A_0. However, then, by Lemma 4, $tp_\Delta(a/A_0 \cup \{x',y'\})$ would $(Form_\mathcal{L}, \Delta)$-split over A_0, which is impossible by the choice of A_0 (and because $x', y' \in Y$). Hence, (4) is true.

Note that there are finitely many Δ-types over A_0 and each of them can be realized in $\mathcal{A}$, hence in $\mathcal{A}|_Y$ as well. Let $A_1 \subseteq Y$ be finite such that all Δ-types over A_0 can be realized in A_1. For each finite set, $s \subseteq Y$, let b_s be a realization of

$$tp^{\mathcal{A}}(a/A_1 \cup s)$$

in $\mathcal{A}|_Y$. Finally, let $f_s : Y \cup \{a\} \to Y \cup \{a\}$ be the function interchanging a and b_s and leaving all elements of $Y - \{b_s\}$ fixed. We claim that, for all s,

$$f_s \text{ is an automorphism of } \mathcal{A}|_{Y \cup \{a\}}. \tag{5}$$

To check this, observe that the conditions of Lemma 3 are satisfied by our construction.

Now, consider the case when $s = \emptyset$. By (5), $f_\emptyset$ is still an automorphism of $\mathcal{A}|_{Y \cup \{a\}}$. It follows that $\mathcal{A}|_{Y \cup \{a\}} \models \varphi(f_\emptyset(a), f_\emptyset(\underline{d}))$. However, by construction, $f_\emptyset(a) = b_\emptyset \in Y$ and $f_\emptyset(\underline{d}) = \underline{d}$. Thus, (3) has been established.

It remains to show that $\mathcal{A}|_{Y \cup \{a\}}$ is an elementary substructure of $\mathcal{A}$. To do so, assume $\underline{d} \in Y$ and φ is a formula such that $\mathcal{A} \models \varphi(a, \underline{d})$. We shall show

$$\mathcal{A}|_{Y \cup \{a\}} \models \varphi(a, \underline{d}). \tag{6}$$

Let s be any finite subset of Y containing the range of $\underline{d}$. Then, by construction, $b_s \in Y$ and $\mathcal{A} \models \varphi(b_s, \underline{d})$. Since Y generates an elementary substructure of $\mathcal{A}$, we also have $\mathcal{A}|_Y \models \varphi(b_s, \underline{d})$. Furthermore, by (3), $\mathcal{A}|_Y$ is an elementary substructure of $\mathcal{A}|_{Y \cup \{a\}}$ hence $\mathcal{A}|_{Y \cup \{a\}} \models \varphi(b_s, \underline{d})$. Finally, by (5), f_s is an automorphism of $\mathcal{A}|_{Y \cup \{a\}}$ mapping b_s onto a and leaving $\underline{d}$ fixed. Therefore, $\mathcal{A}|_{Y \cup \{a\}} \models \varphi(a, \underline{d})$, that is, (6) holds, as desired.

To show that $\mathcal{A}|_{Y \cup \{a\}}$ is $\aleph_0$-saturated, assume $s \subseteq Y$ is finite and p is a type over $s \cup \{a\}$. We shall show that p can be realized in $\mathcal{A}|_{Y \cup \{a\}}$. By the previous parts, $f_s(p)$ is a type over $s \cup b_s \subseteq Y$. Since $\mathcal{A}|_Y$ is assumed to be $\aleph_0$-saturated, there exists $c \in Y$ that realizes $f_s(p)$. However, then $f_s(c)$ realizes p and we are done. $\square$

Now, for certain ultrametric spaces, we can improve Lemma 2.

Lemma 6. *Assume the hypotheses of Lemma 5. Let Γ and Φ be any finite set of formulas. With the notational convention before Lemma 5, suppose $X \subseteq Y \subseteq A$, X is finite and $\bar{a} \in A - Y$. Suppose*

Y generates an $\aleph_0$-saturated substructure of $\mathcal{A}$.

If $tp_{\exists\Gamma}(\bar{a}/Y)$ $(\exists\Phi, \exists\Gamma)$-splits over X, then $tp_\Gamma(\bar{a}/Y)$ (Φ, Γ)-splits over X.

Proof. By Lemma 5, $Y \cup \{\bar{a}\}$ generates an $\aleph_0$-saturated elementary substructure of $\mathcal{A}$, hence Lemma 2(a) and (b) are satisfied. □

Theorem 2. *Assume the hypotheses of Lemma 5. With the notational convention before Lemma 5, let $X \subseteq Y \subseteq A$ and $\bar{a} \in A - Y$ such that Y is the universe of an $\aleph_0$-saturated elementary substructure of $\mathcal{A}$.*

If $tp_\Delta(\bar{a}/Y)$ does not $(Form_\mathcal{L}, \Delta)$-split over X, then $tp(\bar{a}/Y)$ does not split over X.

Proof. Let $\Delta_0 = \Delta$, and, if Δ_n has already been defined for some $n \in \mathbb{N}$, then let

$$\Delta_{n+1} = \Delta \cup \{\neg\phi, \phi \wedge \psi : \phi, \psi \in \Delta_n\} \cup \exists\Delta_n.$$

To complete the proof, we shall show that, for all $n \in \mathbb{N}$,

$$tp_{\Delta_n}(\bar{a}/Y) \text{ does not } (Form_\mathcal{L}, \Delta_n)\text{-split over } X. \tag{7}$$

We apply induction on n. For $n = 0$, (7) holds by assumption. Next, assume (7) is true for some $n \in \mathbb{N}$. Let $\phi, \psi \in \Delta_n$ be arbitrary. Then, (the contrapositive forms of) Lemmas 1 and 6 imply respectively that

- $tp_{\neg\phi}(\bar{a}/Y)$ does not $(Form_\mathcal{L}, \{\phi\})$-split over X;
- $tp_{\phi\wedge\psi}(\bar{a}/Y)$ does not $(Form_\mathcal{L}, \{\psi \wedge \psi\})$-split over X and
- $tp_{\exists\Delta_n}(\bar{a}/Y)$ does not $(Form_\mathcal{L}, \exists\Delta_n)$-split over X.

Consequently, (7) remains true for $n + 1$, and the induction is complete.

Finally observe that, if $tp(\bar{a}, Y)$ would split over X, then, for some $n \in \mathbb{N}$, $tp_{\Delta_n}(\bar{a}/Y)$ would $(Form_\mathcal{L}, \Delta_n)$-split over X. Hence, the proof is complete. □

The next corollary, in our context, is a kind of converse of Lemma 2.7 of Chapter I.2 of [12]. The idea of the proof is similar to the methods applied in [16].

Corollary 1. *Assume the hypotheses of Lemma 5. Then, $\mathcal{A}$ is stable.*

In more detail, suppose $\mathcal{X}$ is a totally bounded ultrametric space and with the notational convention before Lemma 5, suppose, for each $Y \subseteq A$ and $a \in A - Y$, there exists a finite $X \subseteq Y$ such that $tp_\Delta(a/Y)$ does not $(Form_\mathcal{L}, \Delta)$-split over X. Then, $\mathcal{A}$ is stable.

Proof. It is enough to show that, if $\mathcal{A}'$ is an elementary extension of $\mathcal{A}$ and $Y \subseteq \mathcal{A}'$ is the universe of an $\aleph_0$-saturated elementary substructure of $\mathcal{A}'$ with $|Y| = 2^{2^{\aleph_0}}$, then the number of types over Y is at most (in fact, precisely) $|Y|$. Without loss of generality, we may assume that $\mathcal{A}'$ is $|Y|^+$-saturated.

For each finite subset $X \subseteq Y$, choose $Y_X \subseteq Y$ with $X \subseteq Y_X$ and which is the universe of an $\aleph_0$-saturated elementary substructure of $\mathcal{A}'$ with cardinality $2^{\aleph_0}$. By assumption, for all $a \in A' - Y$, there exists a finite $X(a) \subseteq Y$ such that

$$tp_\Delta(a/Y) \text{ does not } (Form_\mathcal{L}, \Delta)\text{-split over } X(a). \tag{8}$$

For each $a \in A - Y$, let $\gamma(a) = \langle X(a), tp(a/Y_{X(a)})\rangle$. Clearly,

$$|ran(\gamma)| \leq |[Y]^{2^{\aleph_0}}| \cdot 2^{2^{\aleph_0}} = |Y|.$$

Combining this with our saturation assumption on $\mathcal{A}'$, it is enough to show that

$$\text{for all } a, b \in A - Y \text{ if } \gamma(a) = \gamma(b), \text{ then } tp(a/Y) = tp(b/Y).$$

To do so, assume $a, b \in A - Y$ and $\gamma(a) = \gamma(b)$. By Theorem 2, $tp(a/Y_{X(a)})$ does not split over $X(a)$ and similarly (by $\gamma(a) = \gamma(b)$), $tp(b/Y_{X(a)})$ does not split over $X(a)$. Suppose $\phi(v, \bar{b}) \in tp(a/Y)$; we shall show $\phi(v, \bar{b}) \in tp(b/Y)$. Let $\bar{c}$ be a realization of $tp(\bar{b}/X(a))$ in $Y_{X(a)}$ (such a realization exists because $Y_{X(a)}$ is $\aleph_0$-saturated). Since $tp(a/Y)$ does not split over $X(a)$, it follows that $\phi(v, \bar{c}) \in tp(a/Y_{X(a)})$. Combining this with $\gamma(a) = \gamma(b)$, we obtain $\phi(v, \bar{c}) \in tp(b/Y_{X(a)})$. However, $tp(b/Y_{X(a)})$ does not split over $X(a)$, whence $\phi(v, \bar{b}) \in tp(b/Y)$, as desired. □

3. Ultrametric Spaces of Finite Spectrum

In somewhat different contexts, variants of the next theorem appear in [1].

Theorem 3. *Let $\mathcal{X} = \langle X, \varrho \rangle$ be an ultrametric space of finite spectrum, let $Y \subseteq A$ and let $\Delta = \{R_\alpha : \alpha \in ran(\varrho)\}$ be the set of atomic formulas of $\mathcal{A}(\mathcal{X})$. Then, for each $a \in A - Y$, there exists a finite $B \subseteq Y$ such that $tp_\Delta(a/Y)$ does not (Δ, Δ)-split over B. In addition, $|B| \leq 2 \cdot \binom{|ran(\varrho)|-1}{2}$.*

If $\mathcal{X}$ is an ultrametric space with an arbitrary spectrum, then the same conclusion holds for all finite reducts of $\mathcal{A}(\mathcal{X})$.

Proof. Fix $a \in A - Y$ and assume, seeking a contradiction, that there is no B satisfying the conclusion of the theorem. For each $i \leq 1 + \binom{|ran(\varrho)|-1}{2}$, we define finite subsets $B_i \subseteq Y$ and $c_i, d_i \in Y$ by recursion such that the following stipulations are satisfied:

(a) $B_{i+1} = B_i \cup \{c_{i+1}, d_{i+1}\}$ is finite;
(b) $tp_\Delta(c_{i+1}/B_i) = tp_\Delta(d_{i+1}/B_i)$;
(c) $\varrho(a, c_{i+1}) > \varrho(a, d_{i+1}) > 0$.

Let $B_0 = \emptyset$; then, clearly, (a)–(c) are satisfied. Assume $i \leq \binom{|ran(\varrho)|-1}{2}$ and B_j, c_j, d_j have already been defined for all $j \leq i$, such that (a)–(c) are satisfied. According to the indirect assumption in the first sentence of the present proof,

$$tp_\Delta(a/Y) \qquad (\Delta, \Delta)\text{-splits over } B_i.$$

It follows that there exist $c_{i+1}, d_{i+1} \in Y$ such that $tp_\Delta(c_{i+1}/B_i) = tp_\Delta(d_{i+1}/B_i)$ but $\varrho(a, c_{i+1}) > \varrho(a, d_{i+1})$. Define B_{i+1} to be $B_{i+1} = B_i \cup \{c_{i+1}, d_{i+1}\}$. Clearly, (a)–(c) remain true.

In this way, B_i, c_i, d_i has been defined for all $i \leq 1 + \binom{|ran(\varrho)|-1}{2}$. Let

$$T = \{\langle \alpha, \beta \rangle : \alpha > \beta > 0, \alpha, \beta \in ran(\varrho)\}.$$

Clearly, $|T| \leq \binom{|ran(\varrho)|-1}{2}$ and for each $i \leq 1 + \binom{|ran(\varrho)|-1}{2}$, we have $\langle \varrho(a, c_i), \varrho(a, d_i) \rangle \in T$. However, then, by the pigeon hole principle, there exist $\alpha > \beta \in ran(\varrho) - \{0\}$ and $j < k \leq 1 + \binom{|ran(\varrho)|-1}{2}$ such that

$$\alpha = \varrho(a, c_j) = \varrho(a, c_k) \text{ and } \beta = \varrho(a, d_j) = \varrho(a, d_k).$$

Now, observe that

$$\varrho(d_j, d_k) \leq max\{\varrho(d_j, a), \varrho(a, d_k)\} = \beta$$

hence, by (b), we obtain $\varrho(c_k, d_j) \leq \beta$. In addition,

$$\varrho(c_k, d_k) \leq max\{\varrho(c_k, d_j), \varrho(d_j, d_k)\} \leq \beta.$$

Therefore,

$$\alpha = \varrho(c_k, a) \leq max\{\varrho(c_k, d_k), \varrho(d_k, a)\} \leq \beta,$$

which is impossible because, by stipulation (c) of our construction, $\alpha > \beta$. This contradiction completes the proof.

The last sentence of the theorem can be proved similarly. □

Corollary 2. *Totally bounded ultrametric spaces are stable.*

Proof. Since $\mathcal{A}(\mathcal{X})$ is stable if and only if all of its finite reducts are stable, one can combine Corollary 1 and Theorem 3. □

Corollary 3. *If $\mathcal{X}$ is a totally bounded metric space, $\mathcal{Y}$ is an $\aleph_0$-saturated elementary substructure of $\mathcal{A}(\mathcal{X})$ and $a \in X$ is arbitrary, then $Y \cup \{a\}$ generates an $\aleph_0$-saturated elementary substructure of $\mathcal{A}(\mathcal{X})$.*

Proof. Combine Lemma 5 and Theorem 3. □

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