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On Certain Generalizations of Rational and Irrational Equivariant Functions

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Abstract: In this paper, we address the case of a particular class of function referred to as the rational equivariant functions. We investigate which elliptic zeta functions arising from integrals of power of $\wp$, where $\wp$ is the Weierstrass $\wp$-function attached to a rank two lattice of $\mathbb{C}$, yield rational equivariant functions. Our concern in this survey is to provide certain examples of rational equivariant functions. In this sense, we establish a criterion in order to determine the rationality of equivariant functions derived from ratios of modular functions of low weight. Modular forms play an important role in number theory and many areas of mathematics and physics.

Keywords: rational equivariant functions; elliptic zeta functions; meromorphic function; Weierstrass $\wp$-function

MSC: 11F12; 35Q15; 32L10

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1. Introduction

Let us consider that the modular group $SL_2(\mathbb{Z})$ acts on on the upper half-plane $\mathbb{H}$ by Möbius transformation: for $\tau \in \mathbb{H}$ $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, we have $\gamma\tau = \frac{a\tau+b}{c\tau+b}$. The region delimited by $\Re\tau = -\frac{1}{2}$, $\Re\tau = \frac{1}{2}$ and $\mid \tau \mid= 1$ is the fundamental domain for this action. If f is a modular form for $SL_2(\mathbb{Z})$, then $f(\tau+1) = f(\tau)$. In other words, any modular function is periodic and thus has a Fourier expansion that can be written as a power series in the form $q = e^{2\pi i\tau}$. This representation is called the q-expansion of f. By this property, f is meromorphic at ∞ if its q-expansion has only a finite number of negative powers of q, and f is holomorphic at ∞ if the limit

$$f(\infty) := \lim_{\Im\tau\to\infty} f(\tau)$$

exists.

Equivalently, a modular function f is holomorphic at ∞ if its q-expansion has only non-negative powers of q. Finally, a cusp form is a holomorphic modular form that vanishes at ∞.

For a finite index subgroup Γ of $SL_2(\mathbb{Z})$, an equivariant function is a meromorphic function on the upper half-plane $\mathbb{H}$, which commutes with the action of Γ on $\mathbb{H}$. Namely,

$$f(\gamma\tau) = \gamma f(\tau)\ ,\ \gamma \in \Gamma\ ,\ \tau \in \mathbb{H}, \tag{1}$$

where γ acts by linear fractional transformations on both sides. These were extensively studied in connection with modular forms in [1–3] and have important applications to modular forms and vector-valued modular forms [4,5]. We could recall here several applications of equivariant functions. Thus, we describe here some of them as they found in [3]. In [6], we showed that the set of equivariant functions is parameterized by modular functions of weight 2. It turns out that the rationality property is connected to the analytic behaviour at the fixed points of the function. There is another construction of equivariant functions that uses logarithmic derivatives of modular functions f of any weight. The equivariant functions h_f that are constructed in this way are known as *rational equivariant functions*, because their associated function $\hat{h} := 1/(h(\tau) - \tau)$ in $\mathbb{H} \cup \{\infty\}$ has rational residues at all of its poles [2].

In this paper, we focus on the problem of producing examples of rational equivariant functions. To this aim, we prove a criterion which examines the rationality of equivariant functions constructed from ratios of modular functions of low weight. As consequence, it follows from this criterion that the equivariant functions

$$h_n(\tau) := \tau + \frac{2\pi i}{f_n(\tau) + \eta(1)}, \tag{2}$$

with $f_n = \Phi_n/\Psi_n$, are rational for all $n \leq 12$ and for $n = 14$.

In the next step, we turn our attention to the problem of establishing the non-rationality of given equivariant functions. In particular, we prove that h_{13} does not belong to the set of rational equivariant functions, and conjecture that for all $n \geq 15$, the functions h_n are non-rational. In support of the conjecture, we provide some analysis numerically.

An interesting elliptic aspect occurs along the modular dimension of equivariant functions structure, as we can see in [3]. The significant form of the equivariant function related to the weight 12 cusp form Δ given by

$$h_1(z) = z + \frac{6}{i\pi E_2(z)} = z + 12\frac{\Delta}{\Delta'},$$

is associated to the Weierstrass ζ-function. It is here denoted by E_2, the weight 2 Eisenstein series and $\zeta' = \wp$ with $\wp$ as the classical Weierstrass elliptic function.

From the standpoint of differential algebra, each equivariant form satisfies a differential equation of a degree at most of 6; this is something one would expect from a function that satisfies a large number of functional equations. To explain this phenomenon, consider the differential ring of modular forms and their derivatives, commonly known as the ring of quasi-modular forms, which has a transcendence degree of 3 and is simply $\mathbb{C}[E_2, E_4, E_6]$. We uncover essential differential features of the reciprocal of E_2, E_4, and E_6 once again when we specify concrete examples of equivariant forms from the Eisenstein series. It was demonstrated that $\frac{1}{E_2}$, $\frac{1}{E_4}$ and $\frac{1}{E_6}$ fulfill algebraic differential equations over $\mathbb{Q}$ using a Maillet theorem. Since the equivariant functions are differentially algebraic, this allows us to control gaps or growth coefficients in the expansion of these functions in q-series using well-known transcendence theory theorems like those of Maillet and Popken. In [3], the following elements of equivariant functions are emphasized: the connection between the Schwarz derivative and cross-ratio, then the the Schwarz derivative and equivariance, and finally, the cross-ratio and equivariance. This primary construct is the result of the following aspects: the infinitesimal counterpart of the cross-ratio represents the Schwarz derivative; the Schwarz differential equation is related to the Riccati equation; the cross-ratio of four solutions to the Riccati equation is a constant in the field $\mathbb{C}$; and finally the cross-ratio of four solutions to the Riccati equation is a constant in the field $\mathbb{C}$.

As a consequences of the above aspects, the equivariant functions are very fascinating objects for study. We can recall here another form of the equivariance concept, namely the platonic form, which occurs in the physics field.

We investigate in the paper two various techniques to reveal that an equivariant function h is not rational. One approach, which uses the classification of rational equivariant functions, is to prove that $\hat{h}$ has irrational residue at some pole. For instance, explicitly computing the residues at the poles of $\hat{h}_{13}$ reveals that they are quadratic irrationals. The disadvantage of this approach is that to find the poles and the residues of given equivariant functions such as h_n, we often need to find the roots of polynomials. This is difficult if the polynomials have a large degree.

Then, we propose another criterion for the non-rationality of equivariant functions, which is based on the notion of irreducible polynomials. There are criteria to test if a polynomial is irreducible, such as Eisenstein's criterion. The advantage of this approach in proving the irrationality of h_n is most evident for large values of n, because this criterion requires only the analysis of coefficients of large polynomials, not of their roots.

2. Basic Definitions and Facts

Let $\Lambda \subset \mathbb{C}$ be a lattice in $\mathbb{C}$, that is $\Lambda = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}$ with $\Im(\omega_2/\omega_1) > 0$. Such a lattice can be expressed with a different basis (ω_1', ω_2') if $\omega_1' = a\,\omega_1 + b\,\omega_2$ and $\omega_2' = c\,\omega_1 + d\,\omega_2$ with $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$; that is, $(\omega_1', \omega_2') = (\omega_1, \omega_2)\gamma^t$, where γ^t denotes the transpose of the matrix γ. The main reference in this section is [7].

The Weierstrass $\wp$-function is the elliptic function with respect to Λ given by:

$$\wp(\Lambda, z) \;=\; \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{(z-w)^2} - \frac{1}{\omega^2} \right).$$

The function is absolutely and uniformly convergent on certain compact sets of $\mathbb{C} \setminus \Lambda$ and provides a meromorphic function on $\mathbb{C}$ with poles of order 2 at the points of Λ and no other poles. The Weierstrass ζ-function is defined by the series

$$\zeta(\Lambda, z) \;=\; \frac{1}{z} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{z-w} + \frac{1}{\omega} + \frac{z}{\omega^2} \right). \tag{3}$$

We can also affirm that the function is absolutely and uniformly convergent on certain compact sets of $\mathbb{C} \setminus \Lambda$. In addition, it provides a meromorphic function on $\mathbb{C}$ with simple poles at the points of Λ and no other poles. Differentiating the above series we get for all $z \in \mathbb{C}$:

$$\frac{d}{dz}\zeta(\Lambda, z) \;=\; -\wp(\Lambda, z)\;.$$

Since $\wp$ is periodic relative to Λ, ζ is quasi-periodic in the sense that for all $\omega \in \Lambda$ and for all $z \in \mathbb{C}$, we have

$$\zeta(\Lambda, z+\omega) \;=\; \zeta(\Lambda, z) \;+\; \eta_\Lambda(\omega), \tag{4}$$

where $\eta_\Lambda(\omega)$ is independent of z. We call $\eta_\Lambda : \Lambda \longrightarrow \mathbb{C}$ the quasi-period map associated with ζ. It is clear that η_Λ is $\mathbb{Z}$-linear, and thus it is completely determined by the values of $\eta_\Lambda(\omega_1)$ and $\eta_\Lambda(\omega_2)$. The periods and the quasi-periods are related by the Legendre relation:

$$\omega_1\eta_\Lambda(\omega_2) - \omega_2\eta_\Lambda(\omega_1) = 2\pi i. \tag{5}$$

Let ω_1 and ω_2 be such that $\Im(\omega_2/\omega_1) > 0$ and set

$$M_{(\omega_1,\omega_2)} = \begin{bmatrix} \omega_2 & \eta(\omega_2) \\ \omega_1 & \eta(\omega_1) \end{bmatrix},$$

where η is the quasi-period map of the Weierstrass zeta function $\zeta(\omega_1\mathbb{Z}+\omega_2\mathbb{Z},z)$. Using the Legendre relation (5), we have

$$\det M_{(\omega_1,\omega_2)} = -2\pi i.$$

Definition 1. *Let $\mathcal{L}$ be the set of lattices $\Lambda_{(\omega_1,\omega_2)} = \omega_1\mathbb{Z}+\omega_2\mathbb{Z}$ with $\Im(\omega_2/\omega_1) > 0$. The elliptic zeta function of weight $k \in \mathbb{Z}$ is defined as a map*

$$\mathcal{Z} : \mathcal{L} \times \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$$

satisfying the following properties:

I. *For each $\Lambda = \omega_1\mathbb{Z}+\omega_2\mathbb{Z}$, the map*

$$\mathcal{Z}(\Lambda,\cdot) : \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$$

is quasi-periodic, that is

$$\mathcal{Z}(\Lambda,z+\omega) \;=\; \mathcal{Z}(\Lambda,z) + H_\Lambda(\omega)\ ,\ z \in \mathbb{C}\ ,\ \omega \in \Lambda\ ,$$

where the quasi-period function $H_\Lambda(\omega)$ does not depend on z;

II. *$\mathcal{Z}$ is homogeneous of weight k in the sense that*

$$\mathcal{Z}(\alpha\Lambda,\alpha z) \;=\; \alpha^k\, \mathcal{Z}(\Lambda,z)\ ,\ \alpha \in \mathbb{C}^*\ ,\ z \in \mathbb{C};$$

III. *If $\Lambda_\tau = \mathbb{Z}+\tau\mathbb{Z}$, $\tau \in \mathbb{H}$, then the quasi-periods $H_{\Lambda_\tau}(\tau)$ and $H_{\Lambda_\tau}(1)$ as functions of τ are meromorphic on $\mathbb{H}$.*

It follows from (I) that for each Λ, the quasi-period function H_Λ is $\mathbb{Z}$-linear, and therefore, it is completely determined by $H_\Lambda(\omega_1)$ and $H_\Lambda(\omega_2)$.

Let $\mathcal{Z}$ be an elliptic zeta function of weight k with the two quasi-periods $H(\omega_1)$ and $H(\omega_2)$. Set

$$\begin{bmatrix}\Phi\\ \Psi\end{bmatrix} = M^{-1}_{(\omega_1,\omega_2)}\begin{bmatrix}H(\omega_2)\\ H(\omega_1)\end{bmatrix}.$$

In other words,

$$2\pi i\Phi = \eta(\omega_2)H(\omega_1) - \eta(\omega_1)H(\omega_2) \tag{6}$$
$$2\pi i\Psi = \omega_1 H(\omega_2) - \omega_2 H(\omega_1). \tag{7}$$

Recall the Eisenstein series $G_2(\tau)$ defined by

$$G_2(\tau) = \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2} + \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2}$$

and the normalized weight-two Eisenstein series

$$E_2(\tau) = \frac{6}{\pi^2}G_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\ ,\ \ q = e^{2\pi i\tau},$$

where $\sigma_1(n)$ is the sum of positive divisors of n. The following properties one can easily deduce from the definition of the Weierstrass ζ-function (3), as they were described in [7], namely

$$\eta(1) = G_2(\tau),$$
$$\eta(\tau) = \tau G_2(\tau) - 2\pi i,$$

Further, if Δ denotes the weight 12 cusp form (the discriminant)

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \ \ q = e^{2\pi i \tau},$$

then

$$E_2(\tau) = \frac{1}{2\pi i} \frac{\Delta'(\tau)}{\Delta(\tau)}.$$

Let $\Lambda = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}$, $\Im(\omega_2/\omega_1) > 0$ be a lattice in $\mathbb{C}$. The Eisenstein series g_2 and g_3 are defined by

$$g_2(\Lambda) \ = \ 60 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^4}\,,\ g_3(\Lambda) \ = \ 140 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^4}.$$

When $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{H}$, g_2 and g_3, as functions of τ, they are modular forms of weight four and six, respectively.

We now study the effects of differentiation on modular forms. The following functions were studied by Ramanujan [8], who proved that they satisfy the following differential equations:

Proposition 1. *If f is a modular function of weight k, then*

$$Df := (2\pi i) f' + k\eta_1 f; \tag{8}$$

is a modular function of weight $k + 2$.

Ramanujan considered derivatives of the Eisenstein series [8] and showed the following formulas:

Example 1.

$$(2\pi i)G_2' = 6(\eta_1)^2 - 4g_2; \tag{9}$$
$$(2\pi i)g_2' = 6g_3 - 4\eta_1 g_2; \tag{10}$$
$$(2\pi i)g_3' = \frac{1}{3}g_2^2 - 6g_3\eta_1; \tag{11}$$
$$(2\pi i)\Delta' = -12\eta_1\Delta. \tag{12}$$

These allow us to study a special modular function, the j invariant, in terms of which any other modular function of weight 0 can be expressed explicitly.

Definition 2. *The j invariant function is given by*

$$j(\tau) = 1728 \frac{g_2^3(\tau)}{\Delta(\tau)}, \quad \tau \in \mathbb{H}.$$

We remark that the function j is a modular one of weight 0 since it is the ratio of two modular forms of weight 12. The function j has a simple pole at infinity and is holomorphic on $\mathbb{H}$ since Δ has a simple zero at infinity but vanishes nowhere else and $g_2(\infty) \neq 0$.

Its derivative can be found by substituting terms using the values from Example 1:

$$(2\pi i)j' = \frac{1}{8}\frac{g_3}{g_2} j.$$

For a non-negative integer n, the power $\wp^n(z)$ can be written as a linear combination of 1, $\wp$ and successive derivatives of $\wp$:

$$\wp^n(\Lambda, z) \;=\; \Phi_n(\Lambda) - \Psi_n(\Lambda)\wp(\Lambda, z) + \sum_{k=1}^{n-1} \alpha_k \wp^{(2k)}, \tag{13}$$

where the coefficients α_k are polynomials in g_2 and g_3 with rational coefficients; see ([9], p. 108). In particular, $\Phi_0 = 1$, $\Psi_0 = 0$, $\Phi_1 = 0$ and $\Psi_1 = -1$.

For each lattice Λ and $z \in \mathbb{C}$, a primitive $\displaystyle\int \wp^n(u)du$ of $\wp^n$ has the form

$$\Phi_n(\Lambda)\, z + \Psi_n(\Lambda)\zeta(\Lambda, z) + E_n(\Lambda, z), \tag{14}$$

where for each Λ, $E_n(\Lambda, z)$ is a $\Lambda-$elliptic function. We define

$$\mathcal{Z}_n(\Lambda, z) \;:=\; \Phi_n(\Lambda)\, z + \Psi_n(\Lambda)\zeta(\Lambda, z).$$

It is clear that for each Λ, $\mathcal{Z}_n(\Lambda, z)$ is quasi-periodic with the quasi-period map given by

$$H_n(\omega) = \Phi_n(\Lambda)\, \omega + \Psi_n(\Lambda)\eta(\omega),$$

where η is the quasi-period map for the Weierstrass $\zeta-$function.

3. Rational Equivariant Functions

The key lemma for the definition of rational equivariant functions are the following:

Lemma 1 ([10])**.** *Let $f : \mathbb{H} \to \hat{\mathbb{C}}$ be a modular function of weight k for some $k \in \mathbb{Z}$. Then $h_f(\tau) = \tau + (kf(\tau)/f'(\tau))$ defines an equivariant function with $h = f'/(kf)$.*

The equivariant functions of this form are called rational:

Definition 3 (Rational equivariant functions)**.** *An equivariant function $h_f(\tau)$ is rational if there exists a modular function f of weight k such that*

$$h_f(\tau) = \tau + k\frac{f(\tau)}{f'(\tau)}.$$

Example 2. *The function*

$$h(\tau) = \frac{\eta(\tau)}{\eta(1)}$$

is a rational equivariant function, because by (9) *(Ramanujan formulas) and* (5) *(Legendre's equation) we have*

$$h_1(\tau) = \frac{\eta_2(\tau)}{\eta_1(\tau)} = \tau - \frac{2\pi i}{\eta_1(\tau)} = \tau + 12\frac{\Delta(\tau)}{\Delta'(\tau)}.$$

Example 3.

$$h_f(\tau) = \tau + 108\frac{f}{f'}$$

is a rational equivariant function with $f = \Delta^5 g_2^{12}$.

Proof. From Example 1 (Ramanujan's formula), we compute

$$f' = 5\Delta^4(-12\eta_1\Delta)g_2^{12} + 12\Delta^5 g_2^{11}(6g_3 - 4\eta_1 g_2)$$

Thus, $h_f(\tau)$ is a rational equivariant function. $\square$

In the case that h is an equivariant function different from the identity, it follows from [11] that

$$\widehat{h} = \frac{\eta_1}{2\pi i} + g$$

for some modular function g of weight 2. In particular, $\widehat{h}$ is meromorphic on $\mathbb{H} \cup \infty$, and $\widehat{h}(\infty) := \lim_{Im(\tau)\to\infty} \widehat{h}(\tau)$ is well-defined as an element of $\widehat{\mathbb{C}} := \mathbb{C} \cup \infty$. The rational equivariant functions are called in this way because of the following classification.

Proposition 2 ([2], Theorem 5.3)**.** *An equivariant function $h \neq id_{\mathbb{H}}$ is rational if and only if $\widehat{h}$ has simple poles with residues $\in \mathbb{Q}$, and $\widehat{h}(\infty)/(2\pi i) \in \mathbb{Q}$.*

4. Examples of Rational Equivariant Functions

We show in the next proposition that quotients of modular functions of low degree can be used to produce functions with rational residues yielding various examples of rational equivariant functions. Throughout this section, M_{2n} is the space of the modular function of weight $2n$.

Proposition 3. *Let $N, D \in \mathbb{Q}[g_2, g_3]$ with $N \in M_{2n}$ and $D \in M_{2n-2}$ where $n \leq 14$ and $n \neq 13$. Suppose also that D is not a cusp form, $ord_i D \leq 1$ and $ord_\rho D \leq 2$, where $\rho = e^{2\pi i/3}$. Then the meromorphic function $\phi = N/(2\pi i D)$ has only simple poles in $\mathbb{H}$ with rational residues, and $\phi(\infty)/(2\pi i)$ is rational.*

Proof. First, we know [12] Table 1.1 that $g_2(\tau)$ has a simple zero at $\tau = \rho$ and no other zero in the fundamental domain, while $g_3(\tau)$ has a simple zero at $\tau = i$ and no other zero in the fundamental domain. Since $ord_\rho D \leq 2$, the function D cannot be divisible by g_2^3 when considered as an element of $\mathbb{Q}[g_2, g_3]$. Analogously, D is not divisible by g_3^2, because $ord_i D \leq 1$. Now, we notice that the function $N/(2\pi i D)$ can be written as

$$\frac{N}{2\pi i D} = \frac{a g_3^4 + b g_2^3 g_3^2 + c g_2^6}{(2\pi i) g_2 g_3 (d g_2^3 + e g_3^2)} \tag{15}$$

for some suitable $a, b, c, d, e \in \mathbb{C}$. This can be seen from Table 1 using the fact that $N \in M_{2n}$ and $D \in M_{2n-2}$ and then multiplying or dividing by suitable factors to match the denominator in (15). Since $N, D \in \mathbb{Q}[g_2, g_3]$, we can choose for a, b, c, d, e to be in $\mathbb{Q}$. Further, as D is not divisible by g_2^3 or g_3^2, we can choose $d, e \neq 0$. Moreover, we can choose d, e such that $d g_2(\infty) + e g_3(\infty) \neq 0$ because D is not a cusp.

Table 1. Table of M_{2n} for $n \leq 14$.

$M_0 =$	$\mathbb{C}$
$M_2 =$	0
$M_4 =$	$\mathbb{C}g_2$
$M_6 =$	$\mathbb{C}g_3$
$M_8 =$	$\mathbb{C}g_2^2$
$M_{10} =$	$\mathbb{C}g_2g_3$
$M_{12} =$	$\mathbb{C}g_2^3 + \mathbb{C}g_3^2$
$M_{14} =$	$\mathbb{C}g_2^2g_3$
$M_{16} =$	$\mathbb{C}g_2^4 + \mathbb{C}g_2g_3^2$
$M_{18} =$	$\mathbb{C}g_2^3g_3 + \mathbb{C}g_3^3$
$M_{20} =$	$\mathbb{C}g_2^5 + \mathbb{C}g_2^2g_3^2$
$M_{22} =$	$\mathbb{C}g_2^4g_3 + \mathbb{C}g_2g_3^3$
$M_{24} =$	$\mathbb{C}g_3^4 + \mathbb{C}g_2^3g_3^2 + \mathbb{C}g_2^6$
$M_{26} =$	$\mathbb{C}g_2^5g_3 + \mathbb{C}g_2^2g_3^3$
$M_{28} =$	$\mathbb{C}g_2g_3^4 + \mathbb{C}g_2^4g_3^2 + \mathbb{C}g_2^7$

Clearly, $\psi = dg_2^3 + eg_3^2$ is a modular function of weight 12. By [12] Corollary 3.8, it follows that

$$\frac{1}{2}ord_i(\psi) + \frac{1}{3}ord_\rho(\psi) + ord_\infty(\psi) + \sum_{\substack{\tau\in\Gamma(1)\backslash\mathbb{H}^* \\ \tau\neq i,\rho,\infty}} ord_\tau(\psi) = 1. \tag{16}$$

We notice that $ord_i(\psi) = ord_\rho(\psi) = ord_\infty(\psi) = 0$, because ψ is not a cusp, $\psi(i) = dg_2(i) \neq 0$ and $\psi(\rho) = eg_3(\rho) \neq 0$. Moreover, both g_2 and g_3 are holomorphic on all $\mathbb{H}^*$, so $ord_\tau(\psi)$ is a non-negative integer for all $\tau \in \Gamma(1)\backslash\mathbb{H}^*$. We deduce that ψ has exactly one simple zero at some $\tau_0 \in \Gamma(1)\backslash\mathbb{H}^*$ with $\tau_0 \notin \{i, \rho, \infty\}$.

Coming back to the function ϕ, we see from (15) that it can have poles only in the $SL_2(\mathbb{Z})$-orbits of i, ρ and τ_0. On these points, ϕ is either holomorphic or it has a simple pole. Since the denominator in (15) has simple zeros, we can compute the residues of ϕ via

$$res_\tau\phi = \frac{(ag_3^4 + bg_2^3g_3^2 + cg_2^6)(\tau)}{(2\pi i)(g_2g_3\psi)'(\tau)}$$

for any $\tau \in \mathbb{H}$. Thus, in the next computations, it requires the derivatives of this denominator:

$$(g_2g_3\psi)' = g_2'g_3(dg_2^3 + eg_3^2) + g_2g_3'(dg_2^3 + eg_3^2) + g_2g_3(3dg_2^2g_2' + 2eg_3g_3')$$

Now we find the residue of ϕ at i using Ramanujan's formula (11):

$$res_i\phi = \frac{cg_2^6(i)}{(2\pi i)dg_2^4(i)g_3'(i)} = \frac{cg_2^6(i)}{\frac{1}{3}dg_2^6(i)} = \frac{3c}{d}.$$

Similarly, we can also use Ramanujan's formula (10) to compute the residue of ϕ at ρ:

$$res_\rho\phi = \frac{ag_3^4(\rho)}{(2\pi i)eg_2'g_3^3(\rho)} = \frac{ag_3^4(\rho)}{6eg_3^4(\rho)} = \frac{a}{6e}.$$

Now, we compute the residue of ϕ at τ_0. Since ψ is zero at τ_0, this gives $g_2^3(\tau_0) = -\frac{e}{d}g_3^4(\tau_0)$, which is useful to simplify the numerator in the following computation. In order to keep the formulas simple, here we write only g_2, g_3 in place of $g_2(\tau_0), g_3(\tau_0)$:

$$\begin{aligned}
res_{\tau_0}\phi &= \frac{ag_3^4 + bg_2^3g_3^2 + cg_2^6}{g_2g_3(3dg_2^2g_2' + 2eg_3g_3')} \\
&= \frac{-a\frac{d}{e}g_2^3g_3^2 + bg_2^3g_3^2 - c\frac{e}{d}g_2^3g_3^2}{g_2g_3[3dg_2^2(6g_3 - 4\eta_1g_2) + 2eg_3(\frac{1}{3}g_2^2 - 6g_3\eta_1)]} \\
&= \frac{g_2^3g_3^2\frac{-ad^2+bed-ce^2}{ed}}{-12\eta_1g_2g_3[dg_2^3 + eg_3^2] + g_2^3g_3^2[18d + \frac{2}{3}e]} \\
&= \frac{g_2^3g_3^2\frac{-ad^2+bed-ce^2}{ed}}{0 + \frac{2}{3}g_2^3g_3^2[27d + e]} \\
&= \frac{3(-ad^2 + bed - ce^2)}{2de(27d + e)}.
\end{aligned}$$

Finally, we compute the value of ϕ at infinity

$$\phi(\infty) = \frac{ag_3^4(\infty) + bg_2^3(\infty)g_3^2(\infty) + cg_2^6(\infty)}{(2\pi i)g_2(\infty)g_3(\infty)(dg_2^3(\infty) + eg_3^2(\infty))}.$$

Using that $g_2(\infty) = (2\pi i)^4/12$ and $g_3(\infty) = -(2\pi i)^6/216$, it follows

$$\frac{1}{2\pi i}\phi(\infty) = \frac{\frac{a}{(216)^6} + \frac{b}{12^3(216)^2} + \frac{c}{12^6}}{\frac{1}{12}\frac{-1}{216}\left(\frac{d}{12^3} + \frac{e}{216^2}\right)}.$$

This concludes the proposition. □

This Proposition 3 has the following immediate consequence.

Corollary 1. *For all $n = 1, \ldots, 12$ and $n = 14$ the function h_n is rational.*

Proof. By [6] Propostion 7.1, we have

$$h_n = \tau + \frac{2\pi i}{f_n(\tau) + \eta_1} \quad \text{with} \quad f_n = \frac{\Phi_n}{\Psi_n},$$

and so

$$\widehat{h}_n = \frac{1}{h_n - \tau} = \frac{f_n + \eta_1}{2\pi i}.$$

Here we recall that $\Phi_n \in M_{2n}$, $\Psi_n \in M_{2n-2}$ and $\Phi_n, \Psi_n \in \mathbb{Q}[g_2, g_3]$ by [6] Proposition 3.2. We now prove that $ord_i\Psi_n \leq 1$ and $ord_\rho\Psi_n \leq 2$. We list in Table 2 the values of Φ_n and Ψ_n for $n \leq 14$, which are computed recursively from the definitions. From this table, we see that Ψ_n is not divisible by g_3^2 and so

$$ord_i\Psi_n \leq 1.$$

We also check that Ψ_n is not divisible by g_2^3, and so it has no triple zeros at ρ:

$$ord_\rho\Psi_n \leq 2.$$

Now it is necessary to show that Ψ_n is not a cusp form, that is $\Phi_n(\infty) \neq 0$, which is equivalent to verify that Ψ_n is not divisible by $\Delta = g_2^3 + 27g_3^2$ when considered as an element of $\mathbb{C}[g_2, g_3]$. This requirement is fulfilled and one can verify this through Table 2. Proposition 3 implies $f_n/(2\pi i)$ has only simple poles in $\mathbb{H}$ with rational residues and $f_n(\infty)/(2\pi i)^2$ is rational. It is known that η_1 is holomorphic on $\mathbb{H}$ and

$$\frac{\eta_1(\infty)}{(2\pi i)^2} = \frac{\pi^2 E_2(i\infty)}{3(2\pi i)^2} = \frac{1}{12} \in \mathbb{Q}.$$

In conclusion, we get that $\widehat{h_n}$ has simple poles with rational residues and $h_n(\infty)/(2\pi i)$ is rational. That proves h_n is a rational equivariant function by the classification stated in Proposition 2. □

Table 2. Table of Φ_n and Ψ_n for $n \leq 14$.

n	Φ_n	Ψ_n
3	$\frac{g_3}{10}$	$\frac{3g_2}{20}$
4	$\frac{5g_2^2}{336}$	$\frac{g_3}{7}$
5	$\frac{g_2g_3}{30}$	$\frac{7g_2^2}{240}$
6	$\frac{15g_2^3}{4928}+\frac{g_3^2}{55}$	$\frac{87g_2g_3}{1540}$
7	$\frac{433g_2^2g_3}{43680}$	$\frac{77g_2^3}{12480}+\frac{5g_3^2}{182}$
8	$\frac{13g_2^4}{19712}+\frac{7g_2g_3^2}{660}$	$\frac{167g_2^2g_3}{9240}$
9	$\frac{383g_2^3g_3}{136136}+\frac{7g_3^3}{1870}$	$\frac{77g_2^4}{56576}+\frac{6021g_2g_3^2}{340340}$
10	$\frac{2873g_2^5+86848g_2^2g_3^2}{19475456}$	$\frac{3251g_2^3g_3+3520g_3^3}{608608}$
11	$\frac{20327g_2^4g_3}{26138112}+\frac{7g_2g_3^3}{2244}$	$\frac{209g_2^5}{678912}+\frac{134g_2^2g_3^2}{17017}$
12	$\frac{663g_2^6}{19689472}+\frac{775529g_2^3g_3^2}{475931456}+\frac{7g_3^4}{8602}$	$\frac{2884469g_2^4g_3+9834816g_2g_3^3}{1903725824}$
13	$\frac{2623663g_2^5g_3+21088240g_2^2g_3^3}{12415603200}$	$\frac{4807g_2^6}{67891200}+\frac{44139g_2^3g_3^2}{14780480}+\frac{11g_3^4}{8645}$
14	$\frac{1221025g_2^7+86159616g_2^4g_3^2+138098688g_2g_3^4}{156649439232}$	$\frac{1367889g_2^5g_3+9613504g_2^2g_3^3}{3263529984}$

5. A Non-Rational Equivariant Function

In this section, we prove the non-rationality of h_{13}. The strategy here is to compute explicitly the residue at some pole of $\widehat{h_{13}}$ and verify that it is an irrational number.

Theorem 1. *h_{13} is not rational.*

Proof. By Proposition 2, it suffices to show that there exists a pole $z_0 \in \mathbb{H}$ of $\widehat{h}_{13}$ such that $res_{z_0}(\widehat{h_{13}}) \notin \mathbb{Q}$ to prove that h_{13} is not a rational equivariant function. We begin by recalling that h_{13} satisfies

$$h_{13} = \frac{\Phi_{13}z - \Psi_{13}\eta_2}{\Phi_{13}z - \Psi_{13}\eta_1}.$$

Therefore, we have

$$\widehat{h_{13}} = \frac{\Phi_{13}}{2\pi i\Psi_{13}} - \frac{\eta_1}{2\pi i}.$$

The values of Φ_{13} and Ψ_{13} are listed in Table 2:

$$\Psi_{13} = \alpha g_2^6 + \beta g_2^3 g_3^2 + \gamma g_3^4,$$
$$\Phi_{13} = \delta g_2^5 g_3 + \epsilon g_2^2 g_3^3,$$

where

$$\alpha = \frac{4807}{67891200}, \quad \beta = \frac{44139}{14780480}, \quad \gamma = \frac{11}{8645}$$
$$\delta = \frac{2623663}{12415603200}, \quad \text{and} \quad \epsilon = \frac{21088240}{12415603200}$$

These modular functions are written as polynomials in g_2 and g_3. In order to deal with polynomials in one variable only, it is useful to introduce the modular function $x : \mathbb{H} \to \hat{\mathbb{C}}$ of weight 0 given by the formula $x = \frac{g_2^3}{g_3^2}$. Then we have

$$\Psi_{13} = g_3^4(\alpha x^2 + \beta x + \gamma), \tag{17}$$
$$\Phi_{13} = g_2^2 g_3^3(\delta x + \epsilon). \tag{18}$$

For the computations of the residue of $\widehat{h_{13}}$, we need the derivative of Ψ_{13}.

$$(2\pi i)\Psi'_{13} = 6\alpha g_2^5(2\pi i)g'_2 + 3(2\pi i)\beta g_2^2 g_3^2 g'_2 + 2(2\pi i)\beta g_2^3 g_3 g'_3 + 4\gamma(2\pi i)g_3^3 g'_3. \tag{19}$$

By the Ramanujan identities (1) and after some simplification, Equation (19) becomes:

$$(2\pi i)\Psi'_{13} = -\eta_1 g_3^4(24\alpha x^2 + 24\beta x + 24\gamma) + (36\alpha g_2^5 g_3 + 18\beta g_2^2 g_3^3 + 2/3\beta g_2^5 g_3 + 4/3\gamma g_2^2 g_3^3)$$

and so

$$(2\pi i)\Psi'_{13} = 24\eta_1\Psi_{13} + g_2^2 g_3^3\left[\left(36\alpha + \frac{2}{3}\beta\right)x + \left(18\beta + \frac{4}{3}\gamma\right)\right]. \tag{20}$$

By (17), we have that $\tau \in \mathbb{H}$ is a zero of Ψ_{13} if and only if $g_3(\tau) = 0$ or

$$\alpha x(\tau)^2 + \beta x(\tau) + \gamma = 0. \tag{21}$$

Notice that $g_3(\tau) = 0$ if and only if $\tau = i$. Let us now apply the following lemma.

Lemma 2. *The modular function $x : \mathbb{H}^* \to \hat{\mathbb{C}}$ given by $x = \frac{g_2^3}{g_3^2}$ is surjective.*

Proof. In fact, any nonconstant modular function of weight zero is surjective. Alternatively, we can give a direct proof as follows. First it is well-known that the j-function induces a bijection between $\Gamma(1)\backslash\mathbb{H}^*$ and $\hat{\mathbb{C}}$ [12] Theorem 4.1. Since the projection $\mathbb{H}^* \to \Gamma(1)\backslash\mathbb{H}^*$ is surjective, we have that $j : \mathbb{H}^* \to \hat{\mathbb{C}}$ is surjective. Now, we have that $x = r \circ j$, where $r : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is given by $r(t) = 27^{-1}(1 - 1728t^{-1})$. However, r is bijective with inverse $r^{-1}(t) = 1728(1 - 27t)^{-1}$. Therefore, x is surjective. □

By Lemma 2, there exists $\tau_0 \in \mathbb{H}^*$ such that

$$x(\tau_0) = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{22}$$

Thus, (21) is satisfied by $\tau = \tau_0$. Moreover, we can see from an explicit calculation that $x(\tau_0) \neq i$ and $x(\tau_0) \neq \rho$. In particular, this means that $g_2(\tau_0) \neq 0$, $g_3(\tau_0) \neq 0$ and $\Psi_{13}(\tau_0) = 0$. Thus, τ_0 is a simple pole of $\widehat{h_{13}}$.

It is clear that the residue of $\widehat{h_{13}}$ at τ_0 is the same as the residue of Φ_{13}/Ψ_{13}. By the usual formula, this residue is equal to

$$res_{\tau_0}(\widehat{h_{13}}) = \frac{\Phi_{13}(\tau_0)}{(2\pi i)\Psi'_{13}(\tau_0)}$$

as long as $\Phi_{13}(\tau_0) \neq 0$ and $\Psi'_{13}(\tau_0) \neq 0$. The values of $\Phi_{13}(\tau_0)$ and $\Psi_{13}(\tau_0)$ are calculated from (18) and (20). Performing all simplifications, we finally get:

$$res_{\tau_0}(\widehat{h_{13}}) = \frac{A + B\sqrt{55211205}}{C},$$

for some nonzero integers A, B, C. Since this is not a rational number, we get by Proposition 2 that h_{13} is not a rational equivariant function. □

6. Irreducibility of Denominators Implies Non-Rationality

It seems that the rationality of the equivariant functions h_n for $n \leq 12$ or $n = 14$ is more the exception than the rule. In this section, we replace the ad-hoc arguments of the previous section with considerations that are valid for every h_n.

First, in the previous proof, we rewrote Φ_{13} and Ψ_{13} up to factors of the form $g_2^a g_3^b$ as polynomials in the modular function of weight 0 and degree 1 $x = g_2^3/g_3^2$. There is another natural choice we can make, namely to rewrite everything in terms of the modular function j.

The functions h_n are constructed in terms of the modular functions Φ_n and Ψ_n of weight 2. The next proposition shows that each ratio Φ_n/Ψ_n belongs to a nice class of modular functions of weight 2, namely those that can be written as $R(j)j'$, where R is some rational function with rational coefficients. This expression is particularly useful to compute the residues.

Proposition 4. *For every $n \in \mathbb{N}$ with $n \geq 3$, we have $\frac{\Phi_n}{2\pi i \Psi_n} = R_n(j)j'$ where $R_n \in \mathbb{Q}(t)$.*

Proof. The key observation is that $g_2\Phi_n$ and $g_3\Psi_n$ are modular functions of the same weight, which can be written as polynomials in g_2 and g_3 with rational coefficients. This is clear from the recursion in the definition of Φ_n and Ψ_n. If $x = g_2^3/g_3^2$, then we can write

$$g_2\Phi_n = g_2^a g_3^b P(x)$$
$$g_3\Psi_n = g_2^a g_3^b Q(x)$$

for some $a \in \{0,1,2\}$, some $b \in \{0,1\}$ and some polynomials P, Q with rational coefficients. The proposition is then proved by noticing the formulas

$$x = \frac{1}{27}(1 - 1728j)$$

and

$$\frac{g_3}{(2\pi i)g_2} = \frac{1}{18j}j'.$$

□

Proposition 4 can also be proof by a different method, and the following lemma is required for this purpose.

Lemma 3. *For every $n \in \mathbb{N}$ with $n \geq 5$, we have $\frac{g_2}{g_3}\frac{\Psi_n}{\Psi_{n-1}} = \tilde{R}_n(j)$ and $\frac{\Psi_n}{g_2\Psi_{n-2}} = \tilde{\tilde{R}}_n(j)$ for some $\tilde{R}_n, \tilde{\tilde{R}}_n \in \mathbb{Q}(t)$.*

Proof. We are going to use mathematical induction. By direct computation, we get

$$\frac{g_2}{g_3}\frac{\Psi_4}{\Psi_3} = \frac{g_2}{g_3}\frac{\frac{-2}{14}g_3}{\frac{-3}{20}g_2} = \frac{20}{21} \tag{23}$$

$$\frac{g_2}{g_3}\frac{\Psi_5}{\Psi_4} = \frac{g_2}{g_3}\frac{\frac{7}{240}g_2^2}{\frac{1}{7}g_3} = \frac{49g_2^3}{240g_3^2} = \frac{49x}{240} \tag{24}$$

$$\frac{\Psi_5}{g_2\Psi_3} = \frac{\frac{-7}{240}g_2^2}{\frac{-3}{20}g_2^2} = \frac{7}{36}. \tag{25}$$

where $x = g_2^3/g_3^2$ as stated in Lemma 2. Now, suppose that both statements are true up to n, and we prove they are true for $n+1$. Using the recursion formula from [9], p. 109, (see also [3] §9) we have

$$\begin{aligned}\frac{g_2}{g_3}\frac{\Psi_{n+1}}{\Psi_n} &= g_2\frac{r(n)g_2\Psi_{n-1} + s(n)g_3\Psi_{n-1}}{g_3\Psi_n} \\ &= \frac{g_2}{g_3}\left[r(n)g_2\frac{\Psi_{n-1}}{\Psi_n} + s(n)g_3\frac{\Psi_{n-2}}{\Psi_n}\right] \\ &= r(n)\frac{x}{\tilde{R}_n(j)} + s(n)\frac{1}{\tilde{\tilde{R}}_n(j)}.\end{aligned}$$

Since $r(n), s(n) \in \mathbb{Q}$ and $x = g_2^3/g_3^2 = r(j) \in \mathbb{Q}(j)$, then

$$\frac{g_2}{g_3}\frac{\Psi_{n+1}}{\Psi_n} \in \mathbb{Q}(j).$$

Moreover,

$$\begin{aligned}\frac{\Psi_{n+1}}{g_2\Psi_{n-1}} &= \frac{r(n)g_2\Psi_{n-1} + s(n)g_3\Psi_{n-2}}{g_2\Psi_{n-1}} \\ &= r(n) + s(n)\frac{g_3\Psi_{n-2}}{g_2\Psi_{n-1}} \\ &= r(n) + s(n)\frac{1}{\tilde{R}_{n-1}(j)},\end{aligned}$$

and this implies $\Psi_{n+1}/(g_2\Psi_{n-1}) \in \mathbb{Q}(j)$. □

Another Proof of Proposition 4. This proof is also based on mathematical induction. The first part consists of direct computation for $n = 3,4,5,6$:

$$\frac{\Phi_3}{2\pi i\Psi_3} = \frac{2g_3}{(2\pi i)3g_2} = \frac{1}{27j}j', \tag{26}$$

$$\frac{\Phi_4}{2\pi i\Psi_4} = \frac{5g_2^2}{(2\pi i)48g_3} = \frac{5}{32(-1728+j)}j', \tag{27}$$

$$\frac{\Phi_5}{2\pi i\Psi_5} = \frac{8g_3}{(2\pi i)7g_2} = \frac{4}{63j}j', \tag{28}$$

$$\frac{\Phi_6}{2\pi i\Psi_6} = \frac{8g_2^2}{(2\pi i)7g_3} + \frac{28g_3^2}{(2\pi i)87g_2} = \frac{75}{(928(-1728+j))}j' + \frac{14}{783j}j'. \tag{29}$$

Now by using the recursion formulas in [9], p. 109, we find

$$\begin{aligned}
\frac{\Phi_n}{2\pi i \Psi_n} &= \frac{r(n)g_2\Phi_{n-2} + s(n)g_3\Phi_{n-3}}{2\pi i(r(n)g_2\Psi_{n-2} + s(n)g_3\Psi_{n-3})} \\
&= \frac{r(n)g_2\Phi_{n-2}}{2\pi i(r(n)g_2\Psi_{n-2} + s(n)g_3\Psi_{n-3})} + \frac{s(n)g_3\Phi_{n-3}}{2\pi i(r(n)g_2\Psi_{n-2} + s(n)g_3\Psi_{n-3})} \\
&= \left(\frac{2\pi i(r(n)g_2\Psi_{n-2} + s(n)g_3\Psi_{n-3})}{r(n)g_2\Phi_{n-2}}\right)^{-1} \\
&\quad + \left(\frac{2\pi i(r(n)g_2\Psi_{n-2} + s(n)g_3\Psi_{n-3})}{s(n)g_3\Phi_{n-3}}\right)^{-1} \\
&= \left(\frac{2\pi i\Psi_{n-2}}{\Phi_{n-2}} + \frac{2\pi i(s(n)g_3\Psi_{n-2}\Psi_{n-3})}{r(n)g_2\Psi_{n-2}\Phi_{n-2}}\right)^{-1} \\
&\quad + \left(\frac{2\pi i r(n)g_2\Psi_{n-2}\Psi_{n-3}}{s(n)g_3\Psi_{n-3}\Phi_{n-3}} + \frac{2\pi i\Psi_{n-3}}{\Phi_{n-3}}\right)^{-1} \\
&= \left(\frac{1}{R_{n-2}(j)j'} + \frac{s(n)}{r(n)\tilde{R}_{n-2}(j)R_{n-2}(j)j'}\right)^{-1} \\
&\quad + \left(\frac{r(n)\tilde{R}_{n-2}(j)}{s(n)R_{n-3}(j)j'} + \frac{1}{R_{n-3}(j)j'}\right)^{-1} \\
&= \underbrace{\left(\frac{r(n)\tilde{R}_{n-2}(j)R_{n-2}(j)}{s(n) + r(n)\tilde{R}_{n-2}(j)} + \frac{s(n)R_{n-3}(j)}{s(n) + r(n)\tilde{R}_{n-2}(j)}\right)}_{R_n(j)} j'.
\end{aligned}$$

This concludes the proof. □

Table 3 lists the expressions of $R_n(j)$ for $n \leq 14$. For every $n \in \mathbb{N}$ with $n \geq 3$, we introduce the polynomials $p_n, q_n \in \mathbb{Z}[t]$ so that $R_n = p_n/q_n$ and the fraction p_n/q_n is written in reduced form.

Notice that the rational function R_n for $n \leq 12$ and for $n = 14$ decomposes as a sum of fractions that have linear denominators with rational coefficients. On the contrary, the denominator of R_{13} is an irreducible polynomial of the second degree. This is the motivation for the following criterion of non-rationality.

Theorem 2. *Let $n \in \mathbb{N}$ with $n \geq 3$ and suppose q_n is irreducible in $\mathbb{Q}[t]$ with a degree of at least 2 and $p_n \neq cq_n'$ for some $c \in \mathbb{Q}$. Then, h_n is an irrational equivariant function.*

Proof. We are going to use the following lemma.

Lemma 4. *Let $p, q \in \mathbb{Q}[t]$ such that $\deg p \leq \deg q$, p is not identically zero and q is irreducible. Then p and q do not have common roots.*

Suppose that h_n has rational residues at all its poles and let j_0 be a zero of q_n. Because j is surjective, there exist $\tau_0 \in \mathbb{H}$ such that $j_0 = j(\tau_0)$. Thus, $q_n(j(\tau_0)) = 0$, $p_n(j(\tau_0)) \neq 0$ and $q_n'(j(\tau_0)) \neq 0$ by applying 4. Then τ_0 is a simple pole of h_n.

Let

$$r = res_{\tau_0} h_n = \frac{p_n(j(\tau_0))}{q_n'(j(\tau_0))}$$

for some $r \in \mathbb{Q}$ by assumption, which gets us

$$p_n(j_0) = rq_n'(j_0).$$

Now let $p_n(j_0) - rq'_n(j_0) = m(j_0)$. Thus, we get $m(j_0) = 0$. However, $p_n - rq'_n = m$ with a degree of m is at most the maximum between the degrees of p_n, q'_n and $\deg m \le \deg q_n$, so by Lemma 4, we get $m(j_0) \neq 0$, which is a contradiction. □

Table 3. Table of $R_n(j)$.

n	$R_n(j)$
3	$\frac{1}{27j}$
4	$\frac{5}{32(-1728+j)}$
5	$\frac{4}{63j}$
6	$\frac{75}{(928(-1728+j))} + \frac{14}{783j}$
7	$\frac{433}{(-1382400+5651j)}$
8	$\frac{585}{(10688(-1728+j))} + \frac{49}{1503j}$
9	$\frac{637}{54189j} + \frac{165599575}{(162567(-8220672+14639j))}$
10	$\frac{8619}{208064(-1728+j)} + \frac{12760776}{3251(-6082560+91297j)}$
11	$\frac{637}{28944j} + \frac{55120391}{3216(-237109248+282053j)}$
12	$\frac{439569}{(13186144(-1728+j))} + \frac{12103}{1383021j} + \frac{76910381850969359}{(63321923823(-5664854016+29238493j))}$
13	$\frac{4(-36440478720+91927141j)}{(125791622922240-4758534328320j+4420585843j^2)}$
14	$\frac{1221025}{43772448(-1728+j)} + \frac{22477}{1351899j} + \frac{371958496352913151}{205471974579(-16612134912+46546507j)}$

7. Examples of Irrational Equivariant Functions via Irreducibility

There is a well-known criterion to test if a polynomial is irreducible.

Lemma 5 (Eisenstein Criterion)**.** *Let p be a prime number and $Q(x)$ be a polynomial with integer coeffecients such that*

$$Q(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0.$$

If we find p such that p divides every a_i, and $i \neq n$, then p does not divide a_n and p^2 does not divide a_0. Then, we get that Q is irreducible over $\mathbb{Q}$.

This criterion of irreducibility, in conjunction with our criterion of irrationality Theorem 2, can be used to prove more examples of irrational equivariant functions. As an easy example, let us prove again that h_{13} is irrational.

Corollary 2. *h_{13} is irrational.*

Proof. First, let us prove that q_{13} is irreducible. From Table 3, it follows that

$$q_{13} = 125791622922240 - 4758534328320t + 4420585843t^2.$$

Now apply Eisenstein's criterion with the prime $p = 5$, which implies q_{13} is irreducible. Therefore, h_{13} is irrational by Theorem 2. □

More technical examples can be provided by using Theorem 2. Now we provide another example that shows h_n is irreducible for some $n > 14$ without computing the residues at its poles.

Lemma 6. *h_n for $n \cong 1, 3, 4, 5$ modulo 6, we have $p_n \neq q'_n$*

Proof. From the mathematical computation file, we can see that $\deg p_n \neq \deg q'_n$. □

Example 4.

$$h_{20} = \frac{32149306475g_2^9 + 5179242972288g_2^6g_3^2 + 45510257049600g_2^3g_3^4 + 20823072571392g_3^6}{48g_2g_3\left(53632004899g_2^6 + 1382526006400g_2^3g_3^2 + 2156880961536g_3^4\right)}.$$

8. Conclusions

Throughout this study we derive the criterion for rationality and non-rationality of the equivariant function h_n. The validity of the derived condition is done theoretically for $1 \leq n \leq 14$. However, for $n \geq 15$, we are unable to verify the criterion thematically.

Based on our extensive numerical experiment as presented in the above table, we state the following conjecture.

Conjecture 1. *For every $n \geq 15$, the equivariant function h_n is not a rational equivariant function.*

One possible method to prove the conjecture would be to suppose that α is the zero of Ψ_n and the pole of h_n, where $n > 14$, and then we get a P-series that changes its sign every n. This would require an advanced technique that can consider many special cases.

3	$\frac{1}{7}$	$\frac{5}{28}$	$\frac{g_3}{10}$	$\frac{3g_2}{20}$	$\frac{2g_3}{3g_2}$
4	$\frac{1}{6}$	$\frac{7}{36}$	$\frac{5g_2^2}{336}$	$\frac{g_3}{7}$	$\frac{5g_2^2}{48g_3}$
5	$\frac{2}{11}$	$\frac{9}{44}$	$\frac{g_2g_3}{30}$	$\frac{7g_2^2}{240}$	$\frac{8g_3}{7g_2}$
6	$\frac{5}{26}$	$\frac{11}{52}$	$\frac{15g_2^3}{4928}+\frac{g_3^2}{55}$	$\frac{87g_2g_3}{1540}$	$\frac{75g_2^3+448g_3^2}{1392g_2g_3}$
7	$\frac{1}{5}$	$\frac{13}{60}$	$\frac{433g_2^2g_3}{43680}$	$\frac{77g_2^3}{12480}+\frac{5g_3^2}{182}$	$\frac{866g_2^2g_3}{539g_2^3+2400g_3^2}$
8	$\frac{7}{34}$	$\frac{15}{68}$	$\frac{13g_2^4}{19712}+\frac{7}{660}g_2g_3^2$	$\frac{167g_2^2g_3}{9240}$	$\frac{195g_2^3+3136g_3^2}{5344g_2g_3}$
9	$\frac{4}{19}$	$\frac{17}{76}$	$\frac{383g_2^3g_3}{136136}+\frac{7g_3^3}{1870}$	$\frac{77g_2^4}{56576}+\frac{6021g_2g_3^2}{340340}$	$\frac{32g_3\left(1915g_2^3+2548g_3^2\right)}{g_2\left(29645g_2^3+385344g_3^2\right)}$
10	$\frac{3}{14}$	$\frac{19}{84}$	$\frac{2873g_2^5+86848g_2^2g_3^2}{19475456}$	$\frac{3251g_2^3g_3+3520g_3^3}{608608}$	$\frac{g_2^2\left(2873g_2^3+86848g_3^2\right)}{32g_3\left(3251g_2^3+3520g_3^2\right)}$
11	$\frac{5}{23}$	$\frac{21}{92}$	$\frac{20327g_2^4g_3}{26138112}+\frac{7g_2g_3^3}{2244}$	$\frac{209g_2^5}{678912}+\frac{134g_2^2g_3^2}{17017}$	$\frac{2g_3\left(20327g_2^3+81536g_3^2\right)}{g_2\left(16093g_2^3+411648g_3^2\right)}$
12	$\frac{11}{50}$	$\frac{23}{100}$	$\frac{663g_2^6}{19689472}+\frac{775529g_2^3g_3^2}{475931456}+\frac{7g_3^4}{8602}$	$\frac{2884469g_2^4g_3+9834816g_2g_3^3}{1903725824}$	$\frac{1025661g_2^6+49633856g_2^3g_3^2+24786944g_3^4}{16g_2g_3\left(2884469g_2^3+9834816g_3^2\right)}$
13	$\frac{2}{9}$	$\frac{25}{108}$	$\frac{2623663g_2^5g_3+21088240g_2^2g_3^3}{12415603200}$	$\frac{4807g_2^6}{67891200}+\frac{44139g_2^3g_3^2}{14780480}+\frac{11g_3^4}{8645}$	$\frac{8g_2^2g_3\left(2623663g_2^3+21088240g_3^2\right)}{7032641g_2^6+296614080g_2^3g_3^2+126382080g_3^4}$
14	$\frac{13}{58}$	$\frac{27}{116}$	$\frac{1221025g_2^7+86159616g_2^4g_3^2+138098688g_2g_3^4}{156649439232}$	$\frac{1367889g_2^5g_3+9613504g_2^2g_3^3}{3263529984}$	$\frac{1221025g_2^6+86159616g_2^3g_3^2+138098688g_3^4}{48g_2g_3\left(1367889g_2^3+9613504g_3^2\right)}$
15	$\frac{7}{31}$	$\frac{29}{124}$	$\frac{357961827g_2^6g_3+4798866560g_2^3g_3^3+1150822400g_3^5}{6309491302400}$	$\frac{207965241g_2^7+13056798880g_2^4g_3^2+18349025280g_2g_3^4}{12618982604800}$	$\frac{2g_3\left(357961827g_2^6+4798866560g_2^3g_3^2+1150822400g_3^4\right)}{g_2\left(207965241g_2^6+13056798880g_2^3g_3^2+18349025280g_3^4\right)}$
16	$\frac{5}{22}$	$\frac{31}{132}$	$\frac{885243125g_2^8+85637913216g_2^5g_3^2+286372884480g_2^2g_3^4}{485613261619200}$	$\frac{288367567g_2^6g_3+3447978560g_2^3g_3^3+726696960g_3^5}{2529235737600}$	$\frac{g_2^2\left(885243125g_2^6+85637913216g_2^3g_3^2+286372884480g_3^4\right)}{192g_3\left(288367567g_2^6+3447978560g_2^3g_3^2+726696960g_3^4\right)}$
17	$\frac{8}{35}$	$\frac{33}{140}$	$\frac{53574809g_2^7g_3+1077589770g_2^4g_3^3+863116800g_2g_3^5}{3549088857600}$	$\frac{586083861g_2^8+51221444480g_2^5g_3^2+153089658880g_2^2g_3^4}{151427791257600}$	$\frac{128g_3\left(53574809g_2^6+1077589770g_2^3g_3^2+863116800g_3^4\right)}{3g_2\left(586083861g_2^6+51221444480g_2^3g_3^2+153089658880g_3^4\right)}$
18	$\frac{17}{74}$	$\frac{35}{148}$	$\frac{56478511375g_2^9+7168169898624g_2^6g_3^2+41120873041920g_2^3g_3^4+5479755939840g_3^6}{131439322811596800}$	$\frac{20976619763g_2^7g_3+381885338240g_2^4g_3^3+273891179520g_2g_3^5}{684579806310400}$	$\frac{56478511375g_2^9+7168169898624g_2^6g_3^2+41120873041920g_2^3g_3^4+5479755939840g_3^6}{192g_2g_3\left(20976619763g_2^6+381885338240g_2^3g_3^2+273891179520g_3^4\right)}$
19	$\frac{3}{13}$	$\frac{37}{156}$	$\frac{1385553199195g_2^8g_3+39015774948096g_2^5g_3^3+67038887546880g_2^2g_3^5}{347375353144934400}$	$\frac{211966996395g_2^9+24590804736256g_2^6g_3^2+127894506311680g_2^3g_3^4+15285828321280g_3^6}{231583568763289600}$	$\frac{2g_2^2g_3\left(1385553199195g_2^6+39015774948096g_2^3g_3^2+67038887546880g_3^4\right)}{3\left(211966996395g_2^9+24590804736256g_2^6g_3^2+127894506311680g_2^3g_3^4+15285828321280g_3^6\right)}$
20	$\frac{19}{82}$	$\frac{39}{164}$	$\frac{91205g_2^{10}}{894918721536}+\frac{7615817g_2^7g_3^2}{463859834880}+\frac{13227265g_2^4g_3^4}{91684795488}+\frac{247g_2g_3^6}{3741870}$	$\frac{53632004899g_2^8g_3+1382526006400g_2^5g_3^3+2156880961536g_2^2g_3^5}{6571966140579840}$	$\frac{32149306475g_2^9+5179242972288g_2^6g_3^2+45510257049600g_2^3g_3^4+20823072571392g_3^6}{48g_2g_3\left(53632004899g_2^6+1382526006400g_2^3g_3^2+2156880961536g_3^4\right)}$

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