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Article

Implementation of Horizontal Connections in the Course of Mathematics by Combining Pedagogical and Digital Technologies

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Abstract: The study examines three experiments of implementing the "learning through teaching" approach combined with the development of digital technologies and explores the influence of these methods on the quality of education for undergraduate engineering students at an urban technical university in Russia. In the first experiment, small independent groups of students within the same cohort developed individual Java modules with the goal of creating an intelligent system to support solving problems using graphs. In the second experiment, each student peer-taught the topic of their choice to three other students and then administered an oral exam to these students to assess their understanding. In the third experiment, each participant selected a problem to develop a solution and recorded a video that explained this solution to other students. All recorded videos were made available to all students, and the combined collection consisted of 100 videos. During the final exam, students were randomly assigned one of these problems and had to present their solutions to the instructor. Analysis of the experiments demonstrated that integrating "learning through teaching" led to an increase in student interest in the discipline and an improvement in conceptual understanding, more so for students in the role of teacher than in the role of learner. Overall, combining pedagogical and digital technologies improved the quality of education for engineering students.

Keywords: digital technologies; mathematics education; pedagogical technologies

MSC: 97C70

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1. Introduction

At the beginning of the article, we will clarify the concepts of pedagogical and digital technologies as they are interpreted in this study. Pedagogical technology is defined as a systematic method of creating, applying, and defining the entire process of teaching and learning, taking into account technical and human resources and their interaction, the aims of which are to optimize the forms of education. Digital technologies in education are considered a way of organizing a modern educational environment based on digital technologies.

The proposed work is based on the following approaches and pedagogical concepts, which will be discussed in detail in this part of the article:

(1) Using interdisciplinary connections as horizontal connections;
(2) Using informal connections that arise between students or students and a teacher in the course of project work as horizontal connections;
(3) Using a "learning by teaching" approach;
(4) Using the concept of boundary objects, which underlies the construction of a common information workspace.

1.1. Horizontal Connections

Horizontal connections may be considered as connections between different subjects studied in parallel. In the article, this concept is expanded to social aspects, that is, the connections between various objects of activity and people representing this activity. Horizontal connections include informal connections between students, between students and teachers, and between students and professionals that can be organized in order to better master the subject but are not regulated by the educational process of the university. The importance of such connections has increased significantly during distance learning in the COVID-19 pandemic years. Remote lectures and practices are boring, and new activities are needed that do not require a lot of time from the teacher. The difference between horizontal connections and informal learning is horizontal connections are built by teachers as a means of managing the learning process.

Let us consider various aspects of the use of horizontal connections in the educational process.

1.1.1. Interdisciplinary Connections

Interdisciplinary connections are given much attention in the current methodological literature [1–3]. In modern pedagogical science, interdisciplinarity is considered an integration and system-forming component [4,5] and a toolkit for the development of universal educational activities [6]. Many authors indicate that for many educators, interdisciplinary interaction means a selection of student tasks that demonstrate interdisciplinary connections. However, research has shown that the analysis of such tasks in unfamiliar subjects often has a formal character since subject teachers generally "do not feel" an unfamiliar subject and insert such tasks into lessons without enthusiasm.

Schoenfeld [7] suggests that interdisciplinary connections are horizontal (as opposed to vertical) connections that characterize the continuity of education in each subject through school and post-secondary education.

In his view of interdisciplinary connections, Schoenfeld considers the transfer of style from one subject to another rather than the transfer of the subject matter. As an example, he defines the most common teaching style in mathematics as one that involves solving traditional problems of a technical nature, calling it the "problem-solving" style. The interdisciplinary connection occurs when a "research" teaching style, which, according to the author, is characteristic of natural science subjects (for example, physics), is transferred to the teaching of mathematics [7].

The experience of teaching unfamiliar subjects, which has long been used in the universities of the former Soviet Union, is interesting to consider from this point of view. For example, at a technical university, a course in mathematical analysis could be offered to an algebraist, and vice versa. The view of an interdisciplinary connection as a transfer of style between different subjects enables us to identify not only the external connections of individual topics but also the possibility of a new vision of the subject itself. The most famous example of a transfer of the style of a natural science subject (in this case, physics) to the teaching of mathematics is the methodological work of Zel'dovich, who wrote a book [8] on how to teach mathematics to physicists and engineers. Of interest are also the works of Ryzhik, who proposed that we recognize the experimental results as valid methods of proof of mathematical facts [9,10].

1.1.2. Informal Connections

One of the ways to organize horizontal interaction is a project activity. In a project, the student creates a physical or virtual artifact that combines knowledge from different scientific areas or academic disciplines (subjects). The traditional organization of the educational process in mathematics in the form of discipline-specific lessons (in secondary schools) and lectures and seminars (in post-secondary education) does not align with the goals of the mutual influence of interdisciplinary activities. Thus, horizontal connections need to be studied in other forms of organization of the educational process. An example of such an informal education community is an online network of scientists and educators to support communication between scientists and students participating in scientific research [11,12].

In Soviet and Russian pedagogical practice, the organizational aspects of the connection between academic science and school have always been strong, especially in the secondary boarding schools for gifted children, established in the large scientific and academic centers of the country (e.g., Moscow, St. Petersburg, Novosibirsk). Another example is the school that operates in St. Petersburg, "Laboratory of Continuous Mathematical Education" (LCME) [13]. The work of the LCME is based on the concept of the integration of general and extra-curricular education and academic science. The purpose of the LCME is to attract secondary school students (16–18 years old) to scientific issues in various areas within the framework of mathematical, engineering, biological, and humanities educational tracks. For example, student participants in the LCME study mathematical problems at the same level as college students with a mathematics major. University faculty and graduate students take an active part in this work, and they involve schoolchildren in their own scientific work. Completed projects are presented at country and international level conferences, receiving well-deserved recognition [14,15].

Thompson [16] provides an overview of studies confirming the influence of horizontal connections on various aspects of learning:

> The importance and influence of informal student-faculty interaction on the differential patterns of student learning and growth has been documented for decades (Feldman, Newcomb, Lamport, Pascarella, Terenzini, Theophilides). The interaction has been identified as a primary agent of college culture, and as a significant influence on the attitudes, interests, and values of college students (Chickering, Feldman, Newcomb, Newman, Pascarella, Terenzini, Wallace). Other studies have revealed evidence that informal student-faculty interaction plays an important role in the learning environment (Churukian, Cooper, Stewart, Gudykunst, Davis, Young, Feldman, Rogers, Theophilides, Terenzini) (p. 35).

Thompson also notes the importance of informal interaction in the design of the learning environment: "the instructional quality and value of the learning environment is related to the quality of the interpersonal relationship between the faculty member and student" (p. 3).

1.2. Learning through Teaching

Using the "learning through teaching" approach to organize horizontal connections makes it possible to provide equal opportunities to all participants in the learning process by changing their roles. Studies of this approach focus on different aspects of learning.

When information technologies were just beginning to enter the practice of teaching, the studies on the "learning through teaching" approach mainly focused on various forms of tutoring at the school level [7,17–24]. At the same time, a number of results obtained in these studies showed the effectiveness of this approach for studying complex mathematical issues and conveying the meanings of mathematical concepts. When students teach, they develop a deeper and more enduring understanding of the material than from just studying the material [22]. Explaining to others potentially offers more learning opportunities than explaining to oneself because those who receive the explanation may also identify gaps and inconsistencies and may demand clarification or challenge the explanation [23]. On the other hand, students "benefit from the experience, knowledge and enthusiasm of their peers. While working on their projects, students have the opportunity to develop a range of transferable skills such as teamwork, presentation and communication skills, and creativity" [21] (p. 161).

It should be noted that many researchers [25,26] of the "learning by teaching" approach refer in their works to the theory of the zone of proximal development (ZPD), introduced by Vygotsky [27] and still under development [28]. The theory was further developed in the system of developmental education by Elkonin-Davydov [29,30]. Roscoe explains the relationship of "learning through teaching" with the ZPD as follows: "The ZPD represents what the learner could achieve if they put forth the effort and received the right help. In peer tutoring, somewhat more advanced students may be well-suited to help novice peers develop because the advanced students are actually operating within the novices' ZPD" [26] (p. 21).

The emergence of computer technologies in teaching has contributed to the formation of new directions in the study of the "learning through teaching" approach. An example of that is the study on the capabilities of pedagogical agents that imitate the behavior of a teacher [31]. As a rule, those studies examine the emotional impact of such an interaction on the student; however, the pedagogical agents themselves are not carriers of knowledge, and their functions are limited to "humanizing" the student's interaction with the material. A detailed review of pedagogical agents is given in [32]. The most significant step in the development of the idea of pedagogical agents was the introduction of a "pedagogical friend", which communicates the subject content to the student and is an analog of an expert system. According to the author [32], a learning companion is a kind of educational agent who plays a non-authoritative role in the social environment of learning.

A team of scientists led by Gautam Biswas from Vanderbilt University has been studying pedagogical agents for more than 15 years in another direction. In those studies, the pedagogical agent "Betty"—a special computer program—is trained using cognitive maps [33]. Students teach Betty by creating concept maps. Cognitive maps provide an expressive graphical language for creating domain knowledge structures, and this gives students the means to create complex structures without getting involved in complex programming tasks. Together with other pedagogical agents, Betty creates the basis for the implementation of the "learning by teaching" approach, when the object of teaching is the training of a computer system [34]. In our work, we used different interpretations of a "pedagogical friend" and a "pedagogical agent". The "pedagogical friend" in our study is the "Graphs" system described later in the article; it enables the students to develop tasks by asking them to "build a graph (or subgraph) with specified properties" on their own. The "Graphs" system also corrects the student's actions without giving the answer to the task. Thus, such systems support the research activities of students in the subject area, allowing them to set tasks and subtasks for the problem under study and check partial solutions [35].

1.3. Boundary Objects

The functioning of the information learning environment [36] presupposes the existence of communities that interact through certain artifacts or objects, called boundary objects. We use the notion of boundary objects as means of overcoming the boundaries between communities of practice, which allow us to interpret the transmitted data in different contexts of interacting communities. Boundary objects include both tangible and intangible objects of the environment. Star [37] suggests several methodological considerations in the study of boundary objects, emphasizing the need for an analytical framework in order to study the system of boundary objects and infrastructure. Nicolini, Mengis, and Swan [38] explore the possibility of building a common information space for cross-disciplinary collaboration; their study shows that the timely introduction of convenient boundary objects increases the efficiency of cooperation and reduces the time for solving the problem. The authors note the importance of learning different representations of concepts and managing the interpretations attributed to objects by the collaborating partners.

The application of the theory of boundary objects in education has received less development. A recent review of the state of teaching mathematics with digital technology [39] refers to several studies that have used the theory of boundary objects in education specifically to serve as mediators of interaction between teachers and researchers. For example, Sinclair [40] uses the technological environment TouchTimes, an app that provides mathematical experiences based on embodied cognition as a boundary object for the interaction of elementary school teachers with the researchers. The introduction of boundary objects as a tool for constructing a common information space between teachers and researchers has occurred relatively recently [41]. At the same time, the possibility of using boundary objects for the interaction of teachers and students in carrying out research and project activities related to several subject areas remains unexplored.

Therefore, in this work, we study the possibilities of training computers to support the productive mental activity of students. We refer to this part of the study as the development of digital technologies in education. At the same time, the very creation of such support could be the subject of the application of various pedagogical technologies, such as "learning through teaching". The interaction of these two processes is examined in this study.

An example of a boundary object is a large project created jointly by all students. Such an approach was presented by one of the authors in the article "Computer Algebra System as a Pedagogical Problem" [42], in which, within the framework of a course of discrete mathematics, first-year students created a complete computer algebra system from 40 interacting modules for working with long numbers of the following types: natural, integer, rational, and polynomials with rational coefficients. Thus, horizontal connections were realized through interdisciplinary connections between mathematics and computer science (programming).

On the other hand, the creation of the system was carried out by teams of an arbitrary number of students. In each team, one of the students had the role of an architect, who determined the choice of language, data structure, and interaction interfaces between modules. Another student played the role of a quality manager who acted as the organizer of the work. The size of the team was determined by the relationship between students (friendships), the desire to learn a new programming language, and the need to join a particular team. There have been rare cases of teams of three or two people and even one person, but usually, the teams consisted of 10–12 people. The more people in the team, the less work falls on a single member; however, it is more difficult to organize interaction. Students chose modern means of interaction, for example, a version control system (GitHub), project management tools, etc. Thus, students were given great freedom in organizing work, which motivated most students. Students noted the impact of this form of work on the formation of soft skills. In this case, horizontal connections were manifested in the form of informal connections and joint project activities.

2. Theoretical Framework

The paper explores various ways of constructing a common information space to support the interaction of students in the process of studying mathematics. For this, various boundary objects are used, such as: the joint creation of a software product in mathematics, the use of distance services for students to teach each other, and the creation of a common video solution to complex problems in mathematics. The information space is considered in the context of the implementation of various horizontal connections between students. These connections are of a different nature: some of them are determined by interdisciplinary connections, others are formed during the implementation of the "learning through teaching" approach, and others are informal in nature and are initiated in the process of teamwork.

The purpose of the study is to study the possibilities of supporting students' independent cognitive activity by combining various pedagogical approaches and various ways of creating a common information space.

This mixed-methods study was guided by the following research questions:

(1) Is it possible to simultaneously solve several diverse pedagogical tasks so that they act as catalysts for each other, that is, so that the total effect is higher than if each task were solved independently?
(2) Can an informal pedagogical activity that is not part of the program requirements positively affect students' interest in the learning and understanding of the material being studied?

(3) How do we combine digital technologies with traditional pedagogical technologies in order to foster horizontal connections for managing the educational process?

There were three different experiments that introduced the "learning through teaching" approach into mathematics courses. The second experiment was carried out in the second semester and the other two in the third. The same students participate in all three experiments. The study was conducted over three consecutive semesters in mathematics courses with first- and second-year undergraduate computer engineering students enrolled in an urban university in the northeastern region of Russia. A total of 100 students participated in the study. The participants' age ranged from 17 to 19 years old. They all studied in the same cohort. Within the framework of the traditional educational process (lectures and seminars), various types of horizontal connections were introduced into the organization of educational activities, aimed at supporting informal interactions between the students. The following horizontal connections were implemented during each experiment.

3. Experiment 1

3.1. Experiment Design and Procedure

In the first experiment, students were assigned to develop a large project to support the course on combinatorics and graph theory with tasks; the project was to combine the program modules developed by students that implement algorithms on graphs into a joint system. To implement the project, the students needed to use informal connections to learn Java, a new programming language for them. A system of self-checking tasks, Wise Tasks Graphs, was used as a digital technology in this project.

The "learning through teaching" approach also played a significant role in this project:

(1) Students "trained a computer", forcing it to do what they had previously done at school and university "with their own hands".
(2) Working in teams, some students had to explicitly or implicitly teach the other students the language chosen for the project (some of the students already knew it, others wanted to learn it). They also had to teach each other the discrete mathematics algorithms, explaining the errors found during the verification and testing of the written modules.

This project is similar in its ideas to the project described in [42]. The project also has several pedagogical goals that are achieved through the use of various forms of horizontal connections.

The goal of the project, set before its participants, was to create a support system for working with graphs. By design, the system should have allowed an easy creation of tasks for constructing a graph or its subgraphs with given properties. This was achieved by using software modules that check certain properties of the graph. After the creation of such modules, the formulation of the task is carried out by either choosing the necessary characteristics (in this case, there may not be a solution) or building a graph and choosing the properties that will be included in the condition (in this case, it is obvious that at least one solution exists). Figures 1–4, taken from [43], are devoted to the technical aspects of the project.

A fourth-year student had to create a system by combining the efforts of all 100 participants. To do this, he had to provide technical opportunities for student interaction and create a system shell that would be used as a boundary object. This shell provided an opportunity for each student to choose one of the algorithms on graphs (the list of algorithms was the second boundary object) and turn it into a module of the system being created. Thus, each student felt like a member of a large team creating a common product.

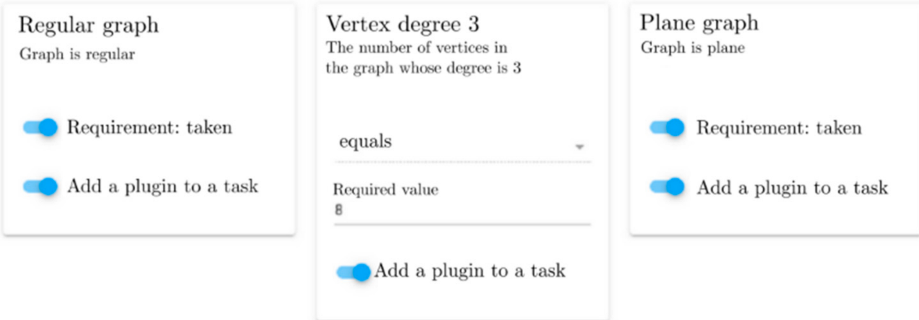

Figure 1. An example of choosing graph characteristics when creating a problem (the figure is taken from [43]).

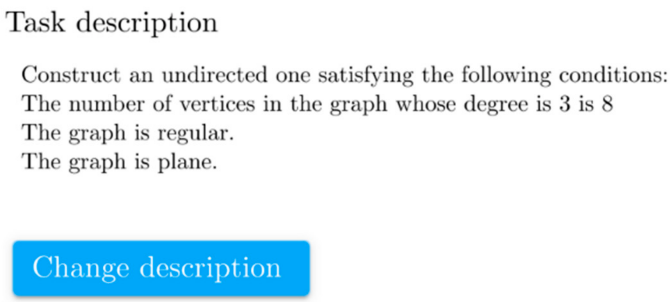

Figure 2. The text of the automatically generated problem for the selected properties of the solution (the figure is taken from [43]).

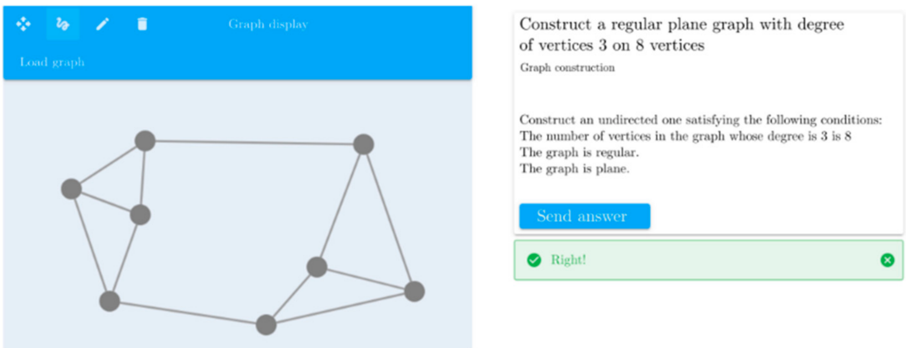

Figure 3. An example of a correct solution to the stated problem (the figure is taken from [43]).

Finally, the project also used a third form of horizontal connections that is associated with the "learning by teaching" approach. Most of the students at this point (second-year undergraduates) did not know the required programming language (Java). At the same time, there were about 10 people in the cohort who had experience with this language. These students were asked to play the role of advisers to 1–3 other students to help them write the required module. It is noteworthy that in the questionnaire conducted after the experiment, all students reported that they wrote the program themselves. More experienced students helped them, but not a single student took the opportunity to ask the student experts to write the program for them.

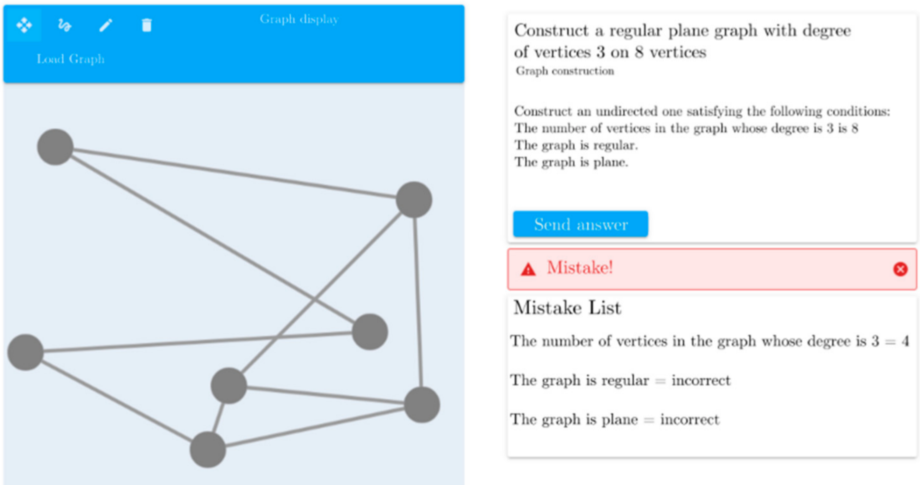

Figure 4. An example of an incorrect solution to the problem and the reaction of the system to errors (the figure is taken from [43]).

3.2. Data Collection and Analysis

A survey was conducted at the end of the course. In the questionnaire, students had to evaluate the impact of developing modules for the "Graphs" system on mastering a new programming language, on understanding the main ideas of the graph theory course, and on the motivation to study graph theory. Additionally, survey participants could choose an answer that corresponded to a neutral assessment or choose their own answer; 39 people out of 100 participated in the survey. An analysis of the attitude of students to the work of creating modules for the "Graphs" system is shown in Figure 5.

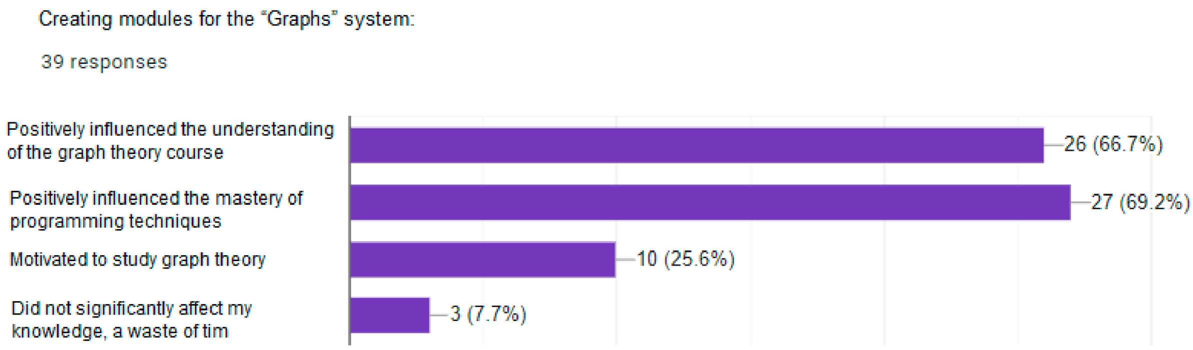

Figure 5. Analysis of the attitude of students to the work of creating modules for the Graphs system.

Thus, survey results showed the significant influence of the first experiment on student perceptions about their own mastery of mathematical ideas and the mastery of a new programming language. At the same time, such activities did not have a significant impact on student motivation to study mathematics itself, although there is a small effect.

In the described project for creating the "Graphs" system, the boundary objects were:

(1) A software shell that made it possible to assemble modules and ensure their interaction with a common interface for building a graph;
(2) A list of algorithms (modules) that was carefully selected to match the capabilities of the system and the abilities of students. This list was coordinated with the textbook, according to which students could master material that sometimes went beyond the course curriculum. We attribute this to the advantages of horizontal connections, which naturally lead to new topics and actually fulfilled the purpose of the project activity;

(3) The Java language, which was a boundary object in the field of programming, created a practicing community of users of this language within the framework of the student flow.

3.3. Results

The experiment showed the possibility of simultaneously solving several pedagogical problems through the creation of a common information space associated with the development of digital technology. Data analysis shows a high appreciation of the "learning by teaching" approach used. In this experiment, this approach was interpreted as "the teaching of a computer by a human". About 70% of the students surveyed (28% of all students) noted the positive impact of the experiment on both understanding the main ideas of the course and learning programming. The created system "Graphs" will be further used in research on the role of constructive tasks in teaching mathematics continuing research [44,45].

4. Experiment 2

4.1. Experiment Design and Procedure

In the second experiment, students were assigned to teach each other in the course of discrete mathematics. Each student had to choose one of the topics in the course curriculum and explain it to three other students, acting as an instructor. The students themselves could use any means convenient for them for explanations: presentations, virtual boards, etc. Following this activity, the student-instructor administered an oral exam to assess the understanding of the material by these three students. This activity was used as a preparation for the exam and was not a substitute for the course lectures and the regular exams. The explanations and oral exams were video-recorded.

This study was carried out in 2021, when, due to the COVID-19 pandemic, all teaching was conducted remotely and there was an evident lack of live interaction between agents of the educational process, which was largely replaced by direct instruction by university instructors.

Students were offered a list of topics for independent learning. The topics were selected by the course instructor and represented the key topics of the course. Each student had to select one topic and prepare it for teaching to three other students. Depending on the students' preparedness, they were allowed to go beyond the scope of the course in their explanations. The number of topics was more than 30, so each student in the group worked on his own topic; in total, the topics covered the entire course of discrete mathematics.

- Topic examples.
- Diophantine equations.
- Continuous fractions.
- Fast exponentiation. Additive chains.
- Chinese remainder theorem and its applications. Chinese code and its usage.
- Prime numbers: primality test.
- Factorization of polynomials with integer coefficients.
- Combinations and poker game.
- One-to-one correspondences and Catalan numbers.
- Enumerative combinatorics.
- Error correction coding.
- Bayes formula and its application. Hypothesis testing.

Each group leader distributed the topics among the students of the group. Student-learners (this means students when they play the role of students) were randomly but evenly distributed into small groups so that each student-teacher (this means students when they play the role of instructor) had three student-learners, and each student had a chance to learn about three different topics. This group worked during the semester. At the end of the semester, each of the student-teachers arranged an oral mini-exam, in which they asked student-learners questions about the material they had taught. The lectures to the students and the exams were video-recorded. These videos were collected by the course instructor as evidence of completed projects. At the same time, in accordance with the principles of non-invasive monitoring [35,36], students did not receive grades for these projects, reflecting the fact that these projects were designed to help all students for the final exam.

Students reacted differently to this project, which was one of several events through which students were motivated to actively work on the course.

4.2. Data Collection and Analysis

At the end of the course, a survey was conducted in which students evaluated the effectiveness of various activities carried out during the training. Among these questions were questions about the activities mentioned in the description of the experiment. These are issues related to the role of teacher and the role of learner.

Fifty-four students out of one hundred students in the cohort answered the questions of the questionnaire.

Figure 6 presents the results of the analysis of a questionnaire in which students evaluated the effect of various pedagogical technologies on them (it was possible to choose any number of significant factors of influence).

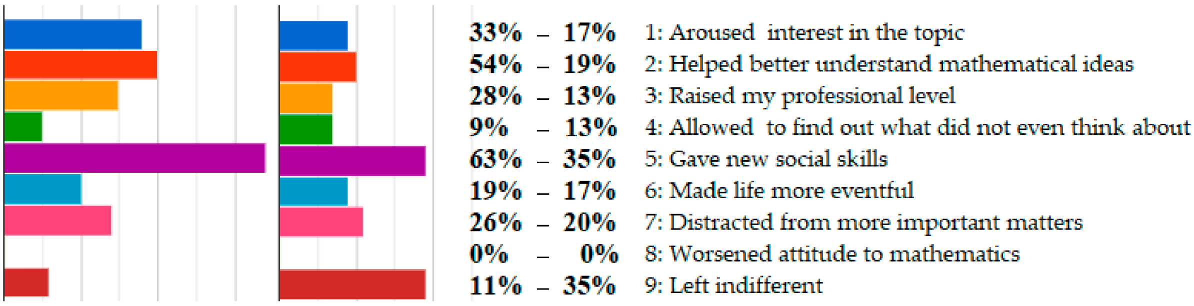

Figure 6. Student assessment of factors affected by the "learning through teaching" approach. On the left—assessment of factors in the role of an instructor; on the right—assessment of factors in the role of a student (students could choose any number of significant factors of influence).

The first thing that catches the eye when analyzing the results of the questionnaire (about half of all students participated in the survey) is the difference between attitudes towards the role of an instructor and the role of a student. Learning activity as a teacher affected the interest in the topic, the understanding of mathematical ideas, the improvement in professional skills, and the acquisition of new social skills significantly more (about twice as much). At the same time, the role of student left three times more students indifferent compared to the role of teacher.

The majority of students (63%) considered teaching other students to be important for acquiring new social skills. This factor was the dominating one among other factors. At the same time, 54% of students noted the importance of teaching for helping their understanding of mathematical ideas.

4.3. Results

This experiment showed that using the "learning by teaching" approach in organizing horizontal connections played an important role for students who acted as instructors.

These results are consistent with the findings reported in [46]. In that study, university students assumed the role of teachers in a project on the "horizontal learning" of schoolchildren as part of career guidance. As a result of the project, students developed their professional identities in relation to younger age groups. The study also showed that the project led to "an increase in subjective significance in the structure of personality identity, and also contributed to the formation of professionally significant competencies" [46] (p. 32).

5. Experiment 3

5.1. Experiment Design and Procedure

In the third experiment, students jointly created a solution manual for a textbook on graph theory. This book presented solutions to all problems included in the textbook as a resource for exam preparation in the course of combinatorics and graph theory. The Google Docs environment was used for collaboration on a document, and YouTube was used for posting videos with explanations of problem solutions.

In this experiment, horizontal connections were mediated by means of ICT more than in the previous experiment. In the previous experiment, video recordings were used only as a way to control educational activities in the part of the course the instructor based on the principles of non-invasive monitoring; in this experiment, the video recordings served as a boundary object for students' communication in the course on combinatorics and graph theory.

In this experiment, the students were offered a number of fairly complex problems from a graph theory textbook [47]. The tasks were presented in a Google Docs document, which all students in the cohort were able to edit. Each student had to choose one problem that was not already selected by someone else., Students had to video-record the analysis of the problem, and the link to the video was added to the document. After all solutions were completed, students were asked to review solutions developed by other students and pose questions and comments to the solutions, thereby developing an understanding of each problem. In these settings, however, the students' engagement was low, as only about 30% of students completed the work. Then, the settings were changed. Postgraduate students were assigned as reviewers; they had to listen to the videos explaining the solutions, evaluate their correctness and intelligibility, and make comments. It was also explained to the students that the exam problems would be randomly selected from these 100 problems, along with the theoretical questions. After that, the attitude of the students changed, and they became more engaged in the task by making comments on mistakes or the poor presentation of solutions in certain posted videos. It should be noted that in this experiment, the instructors also took the position of non-invasive monitoring, and students did not receive credit for solving problems. The whole project was presented as a study of the material, which, if desired, can be considered a preparation for the final exam.

5.2. Data Collection and Analysis

As mentioned earlier, the third experiment was carried out in parallel with the first, so the conditions for questioning were the same as in the first experiment.

The results are presented in Figure 7. The survey was completed by 39 participants (a return rate of about 45%).

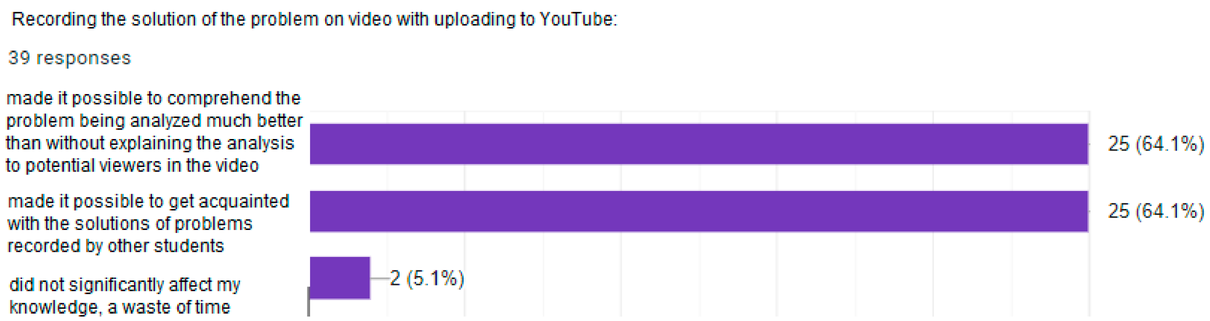

Figure 7. Influence of recording explanations of problem solutions for other students to master the problem solutions.

The survey included three statements for students to select from. The first statement suggested that recording videos of problem solutions helped students in the comprehension of the problem. The second statement suggested that watching videos created by other students helped them in problem comprehension. The last statement suggested that either activity did not affect student knowledge.

The statements about the influence of videos of these two activities on the understanding of the material were tested. In both cases, the null hypotheses were rejected, suggesting a positive impact of both activities on learning. Figure 7 shows that the impact of videotaping their own solutions on comprehension is the same as studying solutions recorded by other students. This suggests that, in the opinion of students, explaining the material to other students is just as important as the traditional way of learning through reading/watching educational materials.

Statements from the other part of the survey made it possible to evaluate various aspects of students' behavior when interacting with the videos. Thus, it can be seen that about half of the students viewed videos selectively or incompletely. A third of the students tried to choose easier tasks. At the same time, a quarter of the students read other people's analyses of problem solutions critically and found inaccuracies or errors in them. Almost no one considered the solutions presented by students as the ultimate truth, which often happens when the problem analysis is presented by a teacher.

Analysis of the problem required 40% of students to turn to additional sources, which indicates that the proposed pedagogical approach supports an active approach to learning the material (Figure 8).

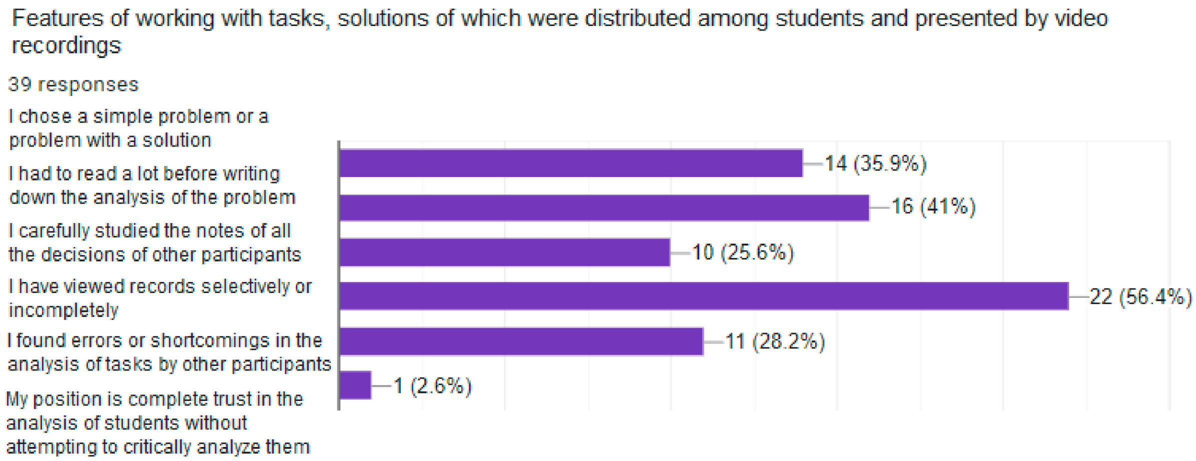

Figure 8. Distribution of student responses about working with videos of problem solutions.

5.3. Results

The third experiment showed that in order to create a common information space in which the "learning through teaching" approach will be effective, it is advisable to use mathematical tasks as boundary objects. Moreover, these should be complex tasks that require additional study of the literature. The solution to such problems will be accompanied by errors, which are the subject of discussion by the students themselves and the students with instructors. Such conclusions can be drawn from both observations of the work of students and their opinions expressed in the questionnaires.

6. Discussion

When planning work, the principle "you chase two hares, you won't catch one" is often used. In our opinion, pedagogical technologies cannot be developed based on only one theory and on setting only one goal. With each pedagogical action, the teacher solves many problems, some of which have already received psychological, didactic, or methodological understanding; some are solved intuitively and relate to the field of culture and art and are transmitted through information and social environment.

In solving the problems posed in the article, the authors do not seek to separate some pedagogical technologies from others and study them independently but, on the contrary, to combine them to solve various and outwardly unrelated pedagogical tasks.

As part of this section, we will discuss the answers to the following questions:

(1) What are "horizontal connections" in teaching mathematics?
(2) How can a "learning by teaching" approach be used in conjunction with digital technologies in today's education?

We see the following aspects of horizontal connections in the educational process:

(1) Horizontal connections can be considered relative to the subject being studied. These connections are interdisciplinary.
(2) Horizontal connections can be viewed from the point of view of the teacher's attitudes. These connections are defined by teaching traditions and the styles and goals of teaching.
(3) Horizontal connections can be viewed as connections between students in a collaborative learning process. These connections are informal and are linked to the "learning by teaching" approach.

Teaching mathematics to students with computer majors (note that information technology will soon take a significant place in the structure of education for all students at technical universities) is carried out in parallel with the study of computer science, programming languages, and technologies. Thus, the study of interdisciplinary relationships between mathematics and computer science is the most natural one. That is the reason we chose courses in discrete mathematics, combinatorics, and graph theory for the experiments. These courses are directly related to computer science through the study of computational processes, data representation, etc.

Within the framework of a mathematical course, it is important that the algorithms are written correctly and that the student can justify their correctness. From a pedagogical point of view, it is convenient for students in the course of mathematics to be perceived by the teacher as "programmers", and they, as professionals, can determine this side of their work themselves (similarly, in programming classes, students can be treated as specialists who have already studied certain mathematical algorithms, which will give them confidence in using their own mathematical knowledge) [46]. Such freedom creates motivation and the possibility of informal horizontal connections for the exchange of experience and "know-how" between students and leads to the development of soft skills.

In what ways should digital technologies be used in the organization of the educational process in order to maintain the effectiveness of traditional pedagogical approaches?

In our experiments, we used two different types of digital technologies:

- Specially designed technologies that support a research approach to solving mathematical problems;
- Open-source digital technologies to support the informal horizontal interactions of students.

The potential of designing digital technologies is examined by the authors in [48,49]. The study used the opportunity to develop such technologies by organizing the horizontal interactions of students. In the course of the study, it was shown that the "learning through teaching" approach opens up a variety of new opportunities in the context of the digitalization of education and the enrichment of the information learning environment:

- The development of technologies that support simple user programming and modeling makes it possible to consider the transfer of knowledge by a student to a computer in one way or another as "computer training";
- The joint development of digital projects, whether it is a software system or a collection of solutions to problems in mathematics, allows us to consider digital technologies as a convenient intermediary between students in transferring their knowledge to the community.

It should be noted that a certain resistance to this pedagogical technology was observed among some students at the beginning of the experiment (accepting the role of an instructor apparently contradicted the students' attitude to passive participation in the educational process, which was possibly formed due to excessive exposure to direct instruction in the previous 10–15 years of schooling). Some students even filed a complaint with the dean's office about the overload of various activities in the discrete mathematics course. However, this attitude was reversed by the end of the semester. Moreover, when the students were instructed to develop requirements for the organization of the colloquium during the following semester, unexpectedly, for the instructor, the students offered to include video recordings of explanations of their own parts of the work so that other participants in the colloquium could familiarize themselves with the material in advance. This result is supported by the following excerpt from the colloquium rules proposed by the students:

Each of the team members, except for the architect and those responsible for quality and mathematical justification, must implement one algorithm and record a video where the algorithm is analyzed in detail from the point of view of mathematics and the algorithm code itself, with an emphasis on language features. Then, the video must be uploaded to YouTube, a link placed in the table, and a test based on the video material created (questions should be related to the features of the algorithm and programming language), a link to which should also be placed in the table. If, at the end of the test, there are questions or misunderstandings, then the participant can ask the author for help.

This example suggests that pedagogical technologies that are in conflict with the current attitudes of students may be met with resistance; however, if these approaches are research-based and field-tested, students will begin to accept new ways of teaching and transfer these approaches to organizing their own social activities.

7. Conclusions

The use of horizontal connections within the framework of learning a specific subject allows the teacher to solve several different pedagogical tasks in parallel. In particular, this approach allows us to combine digital technologies with pedagogical approaches by separating the "vertical" influence of the teacher responsible for student learning of the subject and organizing the "horizontal" student interaction. As part of the horizontal interaction, students use digital technologies known to them to communicate with each other and also to interact with a computer, converting subject knowledge into computer models and algorithms.

An important role in the organization of horizontal interaction is played by boundary objects, the material carriers of the context that allow different communities of practice to interact. For example, a software shell that allowed students to combine modules that implement algorithms on graphs as tools for checking the solution of mathematical problems in graph theory allowed the teacher of a discrete mathematics course who does not know Java to actively interact with students enrolled in the course on graph theory. Moreover, some of the students, who did not know the Java language before the start of the project, mastered it in the course of the project since the "Graphs" system became a material object that provided the basis for programming in the new language.

Another example of the successful use of a boundary object was the creation of a video tutorial on difficult problems in graph theory. The restrictions on the video viewing time and the clarity of the explanation, together with the purpose of using this resource to prepare for the exam, turned the video of a problem solution into a boundary object for students to interact with each other and with teachers.

The "learning through teaching" approach has great potential for organizing horizontal connections. Thus, the creation of software modules based on the implementation of mathematical algorithms can be considered as "teaching a computer" to solve mathematical problems in graphs. Recording student solutions to problems on video for use by other students forces them to consider the clarity of the explanation and to structure the presentation in a way that is effective for teaching others. As experiments have shown, such indirect learning by students from each other is more acceptable for them than learning through direct interaction.

The experiments implemented in this study show that the "learning through teaching" approach has great prospects due to the development of digital technologies. These perspectives are related to the expansion of the functionality of computer programs so that they can act as learners while remaining "smart" enough to provide meaningful feedback (formative assessment). Students are much better at being teachers than they are at being learners.

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