mathematics

MDPI

Article

A Reverse Hardy–Hilbert's Inequality Containing Multiple Parameters and One Partial Sum

Bicheng Yang [1], Shanhe Wu [2,*] and Xingshou Huang [3]

1 Institute of Applied Mathematics, Longyan University, Longyan 364012, China; bcyang@gdei.edu.cn
2 Department of Mathematics, Longyan University, Longyan 364012, China
3 School of Mathematics and Statistics, Hechi University, Yizhou 546300, China; hxs509@hcnu.edu.cn
* Correspondence: shanhewu@163.com or shanhewu@lyun.edu.cn

Abstract: In this work, by introducing multiple parameters and utilizing the Euler–Maclaurin summation formula and Abel's partial summation formula, we first establish a reverse Hardy–Hilbert's inequality containing one partial sum as the terms of double series. Then, based on the newly proposed inequality, we characterize the equivalent conditions of the best possible constant factor associated with several parameters. At the end of the paper, we illustrate that more new inequalities can be generated from the special cases of the reverse Hardy–Hilbert's inequality.

Keywords: reverse Hardy–Hilbert's inequality; partial sum; multiple parameters; best possible constant factor

MSC: 26D15; 26D20

check for updates

Citation: Yang, B.; Wu, S.; Huang, X. A Reverse Hardy–Hilbert's Inequality Containing Multiple Parameters and One Partial Sum. *Mathematics* **2022**, *10*, 2362. https://doi.org/10.3390/math10132362

Academic Editor: Ana-Maria Acu

Received: 1 June 2022
Accepted: 4 July 2022
Published: 5 July 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m \geq 0$, $b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. Then,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where $\frac{\pi}{\sin(\pi/p)}$ is the best possible constant factor. Inequality (1) is known in the literature as Hardy-Hilbert's inequality (see [1]).

By introducing parameters $\lambda_i \in (0,2]$ $(i = 1,2), \lambda_1 + \lambda_2 = \lambda \in (0,4]$, Krnić and Pečarić [2] provided a generalization of Hardy–Hilbert's inequality (1) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}, \tag{2}$$

where the constant factor $B(\lambda_1,\lambda_2)$ given by the beta function is the best possible one.

By introducing more parameters, Yang, Wu and Chen [3] established a further generalization of Hardy–Hilbert's inequality (1) as follows:

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\lambda}} &< \left(\frac{1}{\beta}k_{\lambda}(\lambda_2)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_{\lambda}(\lambda_1)\right)^{\frac{1}{q}} \\ &\times\left\{\sum_{m=1}^{\infty} m^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}},\end{aligned} \tag{3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, $\alpha,\beta \in (0,1]$, $\lambda \in (0,6]$, $\lambda_1 \in (0,\frac{2}{\alpha}] \cap (0,\lambda)$, $\lambda_2 \in (0,\frac{2}{\beta}] \cap (0,\lambda)$, $k_{\lambda}(\lambda_i) := B(\lambda_i,\lambda-\lambda_i)$ $(i = 1,2)$.

By constructing partial sums $A_m := \sum_{i=1}^{m} a_i$ and $B_n := \sum_{k=1}^{n} b_k$, Adiyasuren, Batbold and Azar [4] presented the following analogous version of Hardy–Hilbert's inequality with the best possible constant factor $\lambda_1\lambda_2 B(\lambda_1, \lambda_2)$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < \lambda_1\lambda_2 B(\lambda_1,\lambda_2)\Big(\sum_{m=1}^{\infty} m^{-p\lambda_1-1}A_m^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty} n^{-q\lambda_2-1}B_n^q\Big)^{\frac{1}{q}}, \tag{4}$$

where $\lambda_i \in (0,1] \cap (0,\lambda)$, $\lambda_1 + \lambda_2 = \lambda$ $(\lambda \in (0,2]; i = 1,2)$. Inequality (4) is the other kind of (2) involving two partial sums inside the two terms of series.

Recently, Liao, Wu and Yang [5] considered a variation of inequality (3); one partial sum $B_n = \sum_{k=1}^{n} b_k$ was embedded inside the terms of series, i.e.,

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\lambda}} &< \lambda\Big(\frac{1}{\beta}k_{\lambda+1}(\lambda_2+1)\Big)^{\frac{1}{p}}\Big(\frac{1}{\alpha}k_{\lambda+1}(\lambda_1)\Big)^{\frac{1}{q}} \\ &\times\Big\{\sum_{m=1}^{\infty} m^{p(1-\alpha\hat{\lambda}_1)-1}a_m^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty} n^{q[1-\beta(1+\hat{\lambda}_2)]-1}B_n^q\Big\}^{\frac{1}{q}},\end{aligned} \tag{5}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $\alpha,\beta \in (0,1]$, $\lambda \in (0,5]$, $\lambda_1 \in (0,\frac{2}{\alpha}] \cap (0,\lambda+1)$, $\lambda_2 \in (0,\frac{2}{\beta}-1] \cap (0,\lambda+1)$, $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, $k_{\lambda+1}(\lambda_i) := B(\lambda_i, \lambda+1-\lambda_i) (i=1,2)$.

Yang, Wu and Huang [6] established a reverse Hardy–Hilbert's inequality with one partial sum $B_n = \sum_{k=1}^{n} b_k$ as the term of the double series, as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m B_n}{(m^{\alpha}+n^{\beta})^{\lambda+1}} > \frac{1}{\lambda}\Big(\frac{1}{\beta}k_{\lambda}(\lambda_2)\Big)^{\frac{1}{p}}\Big(\frac{1}{\alpha}k_{\lambda}(\lambda_1)\Big)^{\frac{1}{q}}. \tag{6}$$

As a further study of the development methods of Hardy–Hilbert-type inequalities, some unconventional methods are adopted. For example, a half-discrete Hilbert-type inequality with the multiple upper limit function and the partial sums was provided by [7]. A reverse Hardy–Hilbert-type integral inequality involving one derivative function was published by [8]. Inequalities (4)–(6) and the work of [7,8] are meaningful extensions of (2) based on the Euler–Maclaurin summation formula, Abel's partial summation formula and the techniques of real analysis. Some applications of Hardy–Hilbert-type inequalities in the real analysis and operator theory can be found in the monograph [9]. In [10], Hong gave an equivalent condition between the best possible constant factor and the parameters in the extension of (4). Some other similar results are provided by [11–13].

Inspired by the work of [4–10], in this paper, we construct a reverse Hardy–Hilbert's inequality which contains one partial sum and some extra parameters inside the weight coefficients, the reverse Hardy–Hilbert's inequality has different structural forms by comparing with existing results mentioned above. Our method is mainly based on some skillful applications of the Euler–Maclaurin summation formula and Abel's partial summation formula. By means of the newly proposed inequality, we then discuss the equivalent conditions of the best possible constant factor associated with several parameters. As applications, we deal with some equivalent forms of the obtained inequality and illustrate how to derive more reverse inequalities of Hardy–Hilbert type from the current results.

2. Preliminaries

For convenience, let us first state the following conditions (C1) that would be used repeatedly in subsequent section:

(C1) $p<0$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda \in (0,3]$, $\eta_i \in [0,\frac{1}{4}]$, $\lambda_i \in (0,\frac{3}{2}] \cap (0,\lambda)$ $(i=1,2)$, $\eta_1+\eta_2=\eta$, $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$. $a_m, b_n \geq 0$ $(m,n \in \mathrm{N} = \{1,2,\cdots\})$, $0 < \sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p < \infty$, $0 < \sum_{n=1}^{\infty}(n-\eta_2)^{q(1-\hat{\lambda}_2)-1}b_n^q < \infty$, $A_m = \sum_{j=1}^{m} a_j$ with $A_m = o(e^{t(m-\eta_1)})$ $(t>0)$.

Lemma 1. *(cf. [9] (2.2.3)) (i) If* $(-1)^i \frac{d^i}{dt^i} g(t) > 0$, $t \in [m, \infty)$ $(m \in \mathrm{N})$ *with* $g^{(i)}(\infty) = 0$ $(i = 0, 1, 2, 3)$, $P_i(t), B_i (i \in \mathrm{N})$ *are Bernoulli functions and Bernoulli numbers of i-order, then*

$$\int_m^\infty P_{2q-1}(t) g(t) dt = -\varepsilon_q \frac{B_{2q}}{2q} g(m) \ (0 < \varepsilon_q < 1; q = 1, 2, \cdots).$$

In particular, for $q = 1$, $B_2 = \frac{1}{6}$, *we have:*

$$-\frac{1}{12} g(m) < \int_m^\infty P_1(t) g(t) dt < 0; \tag{7}$$

for $q = 2$, $B_4 = -\frac{1}{30}$, *it follows that:*

$$0 < \int_m^\infty P_3(t) g(t) dt < \frac{1}{120} g(m). \tag{8}$$

(ii) (cf. [9], (2.3.2)) If $f(t)(> 0) \in C^3[m, \infty)$, $\lim_{t \to \infty} f^{(i)}(t) = f^{(i)}(\infty) = 0$ $(i = 0, 1, 2, 3)$, *then we have the following Euler–Maclaurin summation formula:*

$$\sum_{k=m}^\infty f(k) = \int_m^\infty f(t) dt + \frac{1}{2} f(m) + \int_m^\infty P_1(t) f'(t) dt, \tag{9}$$

$$\int_m^\infty P_1(t) f'(t) dt = -\frac{1}{12} f'(m) + \frac{1}{6} \int_m^\infty P_3(t) f'''(t) dt. \tag{10}$$

Lemma 2. *Suppose that* $s \in (0, 3]$, $s_2 \in (0, \frac{3}{2}] \cap (0, s)$, $k_s(s_2) := B(s_2, s - s_2)$, *we define the following weight coefficient:*

$$\varpi_s(s_2, m) := (m - \eta_1)^{s - s_2} \sum_{n=1}^\infty \frac{(n - \eta_2)^{s_2 - 1}}{(m + n - \eta)^s} \ (m \in \mathrm{N}). \tag{11}$$

Then, we have the following inequalities:

$$0 < k_s(s_2)(1 - O_1(\frac{1}{(m - \eta_1)^{s_2}})) < \varpi_s(s_2, m) < k_s(s_2) \ (m \in \mathrm{N}) \tag{12}$$

where $O_1(\frac{1}{(m-\eta_1)^{s_2}}) := \frac{1}{k_s(s_2)} \int_0^{\frac{1-\eta_2}{m-\eta 1}} \frac{u^{s_2 - 1}}{(1+u)^s} du$, *which satisfies* $0 < O_1(\frac{1}{(m-\eta_1)^{s_2}}) \leq \frac{1}{s_2 k_s(s_2)} (\frac{1-\eta_2}{m-\eta 1})^{s_2}$.

Proof. For fixed $m \in \mathrm{N}$, we set the following real function: $g(m, t) := \frac{(t-\eta_2)^{s_2-1}}{(m-\eta+t)^s}$ $(t > \eta_2)$. In the following, we divide two cases of $s_2 \in (0, 1) \cap (0, s)$ and $s_2 \in [1, \frac{3}{2}] \cap (0, s)$ to prove (12).

(i) For $s_2 \in (0, 1) \cap (0, s)$, since $(-1)^i g^{(i)}(m, t) > 0$ $(t > \eta_2; i = 0, 1, 2)$, by using Hermite–Hadamard's inequality (cf. [10]) and setting $u = \frac{t-\eta_2}{m-\eta_1}$, we find:

$$\begin{aligned} \varpi_s(s_2, m) &= (m - \eta_1)^{s - s_2} \sum_{n=1}^\infty g(m, n) < (m - \eta_1)^{s - s_2} \int_{\frac{1}{2}}^\infty g(m, t) dt \\ &= (m - \eta_1)^{s - s_2} \int_{\frac{1}{2}}^\infty \frac{t^{s_2 - 1}}{(m - \eta_1 + t - \eta_2)^s} dt = \int_{\frac{\frac{1}{2} - \eta_2}{m - \eta_1}}^\infty \frac{u^{s_2 - 1}}{(1+u)^s} du \\ &\leq \int_0^\infty \frac{u^{s_2 - 1}}{(1+u)^s} du = B(s_2, s - s_2) = k_\lambda(s_2). \end{aligned}$$

On the other hand, in view of the decreasing property of the series, setting $u = \frac{t-\eta_2}{m-\eta_1}$, we obtain:

$$\begin{aligned}\varpi_s(s_2,m) &= (m-\eta_1)^{s-s_2}\sum_{n=1}^{\infty} g(m,n) > (m-\eta_1)^{s-s_2}\int_1^{\infty} g(m,t)dt \\ &= \int_{\frac{1-\eta_2}{m-\eta_1}}^{\infty} \frac{u^{s_2-1}}{(1+u)^s}du = B(s_2, s-s_2) - \int_0^{\frac{1-\eta_2}{m-\eta_1}} \frac{u^{s_2-1}}{(1+u)^s}du \\ &= k_s(s_2)(1 - O_1(\frac{1}{(m-\eta_1)^{s_2}})) > 0,\end{aligned}$$

where $O_1(\frac{1}{(m-\eta_1)^{s_2}}) = \frac{1}{k_s(s_2)}\int_0^{\frac{1-\eta 2}{m-\eta_1}} \frac{u^{s_2-1}}{(1+u)^s}du > 0$, which satisfies the following inequality: $0 < \int_0^{\frac{1-\eta 2}{m-\eta_1}} \frac{u^{s_2-1}}{(1+u)^s}du < \int_0^{\frac{1-\eta 2}{m-\eta_1}} u^{s_2-1}du = \frac{1}{s_2}(\frac{1-\eta 2}{m-\eta_1})^{s_2} (m \in \mathrm{N})$.

Hence, we obtain (12).

(ii) For $s_2 \in [1, \frac{3}{2}] \cap (0, s)$, by (9), we have:

$$\begin{aligned}\sum_{n=1}^{\infty} g(m,n) &= \int_1^{\infty} g(m,t)dt + \tfrac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)dt \\ &= \int_{\eta_2}^{\infty} g(m,t)dt - h(m),\end{aligned}$$

where $h(m)$ is indicated as

$$h(m) := \int_{\eta_2}^{1} g(m,t)dt - \frac{1}{2}g(m,1) - \int_1^{\infty} P_1(t)g'(m,t)dt.$$

It is easy to observe that $-\frac{1}{2}g(m,1) = \frac{-(1-\eta_2)^{s_2-1}}{2(m-\eta+1)^s}$. Furthermore, integrating by parts, it follows that

$$\begin{aligned}&\int_{\eta 2}^1 g(m,t)dt = \int_{\eta 2}^1 \frac{(t-\eta 2^{s_2-1}}{(m-\eta+t)^s}dt = \frac{1}{s_2}\int_{\eta 2}^1 \frac{d(t-\eta)2^{s_2}}{(m-\eta+t)^s} \\ &= \frac{1}{s_2}\frac{(t-\eta)2^{s_2}}{(m-\eta+t)^s}\Big|_{\eta 2}^1 + \frac{s}{s_2}\int_{\eta 2}^1 \frac{(t-\eta 2^{s_2}dt}{(m-\eta+t)^{s+1}} = \frac{1}{s_2}\frac{(1-\eta)2^{s_2}}{(m-\eta+1)^s} + \frac{s}{s_2(s_2+1)}\int_{\eta 2}^1 \frac{d(t-\eta)2^{s_2+1}}{(m-\eta+t)^{s+1}} \\ &> \frac{1}{s_2}\frac{(1-\eta_2)^{s_2}}{(m-\eta+1)^s} + \frac{s}{s_2(s_2+1)}\Big[\frac{(t-\eta_2)^{s_2+1}}{(m-\eta+t)^{s+1}}\Big]_{\eta_2}^1 + \frac{s(s+1)}{s_2(s_2+1)(m-\eta+1)^{s+2}}\int_{\eta 2}^1 (t-\eta_2)^{s_2+1}dt \\ &= \frac{1}{s_2}\frac{(1-\eta_2)^{s_2}}{(m-\eta+1)^s} + \frac{s}{s_2(s_2+1)}\frac{(1-\eta_2)^{s_2+1}}{(m-\eta+1)^{s+1}} + \frac{s(s+1)(1-\eta_2)^{s_2+2}}{s_2(s_2+1)(s_2+2)(m-\eta+1)^{s+2}}.\end{aligned}$$

We find that:

$$\begin{aligned}-g'(m,t) &= -\frac{(s_2-1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(t-\eta_2)^{s_2-1}}{(m-\eta+t)^{s+1}} = \frac{(1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(t-\eta_2)^{s_2-2}[(m-\eta+t)-(m-\eta_1)]}{(m-\eta+t)^{s+1}} \\ &= \frac{(1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} - \frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}} \\ &= \frac{(s+1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} - \frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}},\end{aligned}$$

and for $s_2 \in [1, \frac{3}{2}] \cap (0, s)$, we deduce that

$$(-1)^i \frac{d^i}{dt^i}\Big[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}\Big] > 0, (-1)^i \frac{d^i}{dt^i}\Big[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}\Big] > 0 \ (t > \eta_2; i = 0,1,2,3).$$

By utilizing (8)–(10), for $a := 1 - \eta_2 \in [\frac{3}{4}, 1]$, we obtain:

$$\begin{aligned}
&(s+1-s_2)\int_1^\infty P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}dt > -\frac{s+1-s_2}{12(m-\eta+1)^s}a^{s_2-2},\\
&-(m-\eta_1)s\int_1^\infty P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}dt > \frac{(m-\eta_1)sa^{s_2-2}}{12(m-\eta+1)^{s+1}} - \frac{(m-\eta_1)s}{720}\Big[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}\Big]''_{t=1}\\
&> \frac{(m-\eta+1)s-as}{12(m-\eta+1)^{s+1}}a^{s_2-2} - \frac{(m-\eta+1)s}{720}\Big[\frac{(s+1)(s+2)a^{s_2-2}}{(m-\eta+1)^{s+3}} + \frac{2(s+1)(2-s_2)a^{s_2-3}}{(m-\eta+1)^{s+2}} + \frac{(2-s_2)(3-s_2)a^{s_2-4}}{(m-\eta+1)^{s+1}}\Big]\\
&= \frac{sa^{s_2-2}}{12(m-\eta+1)^s} - \frac{sa^{s_2-1}}{12(m-\eta+1)^{s+1}} - \frac{s}{720}\Big[\frac{(s+1)(s+2)a^{s_2-2}}{(m-\eta+1)^{s+2}} + \frac{2(s+1)(2-s_2)a^{s_2-3}}{(m-\eta+1)^{s+1}} + \frac{(2-s_2)(3-s_2)a^{s_2-4}}{(m-\eta+1)^s}\Big],
\end{aligned}$$

and then we have:

$$h(m) > \frac{a^{s_2-4}}{(m-\eta+1)^s}h_1 + \frac{sa^{s_2-3}}{(m-\eta+1)^{s+1}}h_2 + \frac{s(s+1)a^{s_2-2}}{(m-\eta+1)^{s+2}}h_3,$$

where h_i $(i = 1,2,3)$ are formulated as $h_1 := \frac{a^4}{s_2} - \frac{a^3}{2} - \frac{(1-s_2)a^2}{12} - \frac{s(2-s_2)(3-s_2)}{720}$, $h_2 := \frac{a^4}{s_2(s_2+1)} - \frac{a^2}{12} - \frac{(s+1)(2-s_2)}{360}$, $h_3 := \frac{a^4}{s_2(s_2+1)(s_2+2)} - \frac{s+2}{720}$.

Moreover, for $s \in (0,3]$, $s_2 \in [1, \frac{3}{2}] \cap (0,s)$, $a \in [\frac{3}{4}, 1]$, we find $h_1 > \frac{a^2}{12s_2}[s_2^2 - (6a+1)s_2 + 12a^2] - \frac{1}{120}$.

In view of $\frac{\partial}{\partial a}[s_2^2 - (6a+1)s_2 + 12a^2] = 6(4a - s_2) \geq 6(4 \cdot \frac{3}{4} - \frac{3}{2}) > 0$, and $\frac{\partial}{\partial s_2}[s_2^2 - (6a+1)s_2 + 12a^2] = 2s_2 - (6a+1) \leq 2 \cdot \frac{3}{2} - (6 \cdot \frac{3}{4} + 1) < 0$, we obtain:

$$\begin{aligned}
&h_1 \geq \frac{(3/4)^2}{12(3/2)}[(\tfrac{3}{2})^2 - (6 \cdot \tfrac{3}{4} + 1)\tfrac{3}{2} + 12(\tfrac{3}{4})^2] - \tfrac{1}{120} = \tfrac{3}{128} - \tfrac{1}{120} > 0,\\
&h_2 > a^2(\tfrac{4a^2}{15} - \tfrac{1}{12}) - \tfrac{1}{90} \geq (\tfrac{3}{4})^2[\tfrac{4}{15}(\tfrac{3}{4})^2 - \tfrac{1}{12}] - \tfrac{1}{90} = \tfrac{3}{80} - \tfrac{1}{90} > 0,\\
&h_3 \geq \tfrac{8a^4}{105} - \tfrac{5}{720} \geq \tfrac{8}{105}(\tfrac{3}{4})^4 - \tfrac{1}{144} = \tfrac{27}{1120} - \tfrac{1}{144} > 0,
\end{aligned}$$

and hence we have $h(m) > 0$.

On the other hand, we have:

$$\begin{aligned}
\sum_{n=1}^\infty g(m,n) &= \int_1^\infty g(m,t)dt + \tfrac{1}{2}g(m,1) + \int_1^\infty P_1(t)g'(m,t)dt\\
&= \int_1^\infty g(m,t)dt + H(m),
\end{aligned}$$

where $H(m)$ is indicated as $H(m) := \frac{1}{2}g(m,1) + \int_1^\infty P_1(t)g'(m,t)dt$.

We have already obtained that $\frac{1}{2}g(m,1) = \frac{a^{s_2-1}}{2(m-\eta+1)^s}$ and $g'(m,t) = -\frac{(s+1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}$.

For $s_2 \in [1, \frac{3}{2}] \cap (0,s), 0 < s \leq 3$, by (7), we acquire:

$$\begin{aligned}
&-(s+1-s_2)\int_1^\infty P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}dt > 0,\\
&(m-\eta_1)s\int_1^\infty P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}dt > \frac{-(m-\eta_1)sa^{s_2-2}}{12(m-\eta+1)^{s+1}} = \frac{-(m-\eta+1)s+as}{12(m-\eta+1)^{s+1}}a^{s_2-2}\\
&= \frac{-s}{12(m-\eta+1)^s}a^{s_2-2} + \frac{s}{12(m-\eta+1)^{s+1}}a^{s_2-1} > \frac{-s}{12(m-\eta+1)^s}a^{s_2-2}.
\end{aligned}$$

Then, we have:

$$\begin{aligned}
H(m) &> \frac{a^{s_2-1}}{2(m-\eta+1)^s} - \frac{sa^{s_2-2}}{12(m-\eta+1)^s} = (\tfrac{a}{2} - \tfrac{s}{12})\frac{a^{s_2-2}}{(m-\eta+1)^s}\\
&\geq (\tfrac{1}{2} \cdot \tfrac{3}{4} - \tfrac{3}{12})\frac{a^{s_2-2}}{(m-\eta+1)^s} = (\tfrac{3}{8} - \tfrac{3}{12})\frac{a^{s_2-2}}{(m-\eta+1)^s} > 0.
\end{aligned}$$

Therefore, we derive the inequalities:

$$\int_1^\infty g(m,t)dt < \sum_{n=1}^\infty g(m,n) < \int_{\eta_2}^\infty g(m,t)dt.$$

By virtue of the results of the case (i), we obtain (12). The proof of Lemma 2 is complete. □

Lemma 3. *Under the assumption (C1), we have the following reverse Hardy–Hilbert's inequality:*

$$\begin{aligned} I_0 := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n-\eta)^{\lambda}} > (k_{\lambda}(\lambda_2))^{\frac{1}{p}}(k_{\lambda}(\lambda_1))^{\frac{1}{q}} \\ \times[\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}(1-O_2(\frac{1}{(n-\eta_2)^{\lambda_1}}))(n-\eta_2)^{q(1-\hat{\lambda}_2)-1}b_n^q]^{\frac{1}{q}}. \end{aligned} \tag{13}$$

Proof. By symmetry, for $s_1 \in (0, \frac{3}{2}] \cap (0, s)$, $k_s(s_1) = B(s_1, s-s_1)$, we can obtain the following inequalities for the next weight coefficient:

$$\begin{aligned} 0 < k_s(s_1)(1-O_2(\frac{1}{(n-\eta_2)^{s_1}})) \\ < \omega_s(s_1,n) := (n-\eta_2)^{s-s_1}\sum_{m=1}^{\infty}\frac{(m-\eta_1)^{s_1-1}}{(m+n-\eta)^s} < k_s(s_1)(n\in \mathrm{N}), \end{aligned} \tag{14}$$

where $O_2(\frac{1}{(n-\eta_2)^{s_1}}) := \frac{1}{k_s(s_1)}\int_0^{\frac{1-\eta 1}{n-\eta_2}}\frac{u^{s_1-1}}{(1+u)^s}du (> 0)$.

By applying the reverse Hölder's inequality (cf. [14]), we obtain:

$$\begin{aligned} I_0 = & \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n-\eta)^{\lambda}}[\frac{(m-\eta_1)^{(1-\lambda)1/q}}{(n-\eta_2)^{(1-\lambda_2)/p}}a_m][\frac{(n-\eta_2)^{(1-\lambda_2)/p}}{(m-\eta_1)^{(1-\lambda)1/q}}b_n] \\ \geq & [\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+n-\eta)^s}\frac{(m-\eta_1)^{(1-\lambda)1(p-1)}a_m^p}{(n-\eta_2)^{1-\lambda_2}}]^{\frac{1}{p}}[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n-\eta)^{\lambda}}\frac{(n-\eta_2)^{(1-\lambda_2)(q-1)}b_n^q}{(m-\eta_1)^{1-\lambda 1}}]^{\frac{1}{q}} \\ = & [\sum_{m=1}^{\infty}\varpi_{\lambda}(\lambda_2,m)(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}\omega_{\lambda}(\lambda,1n)(n-\eta_2)^{q(1-\hat{\lambda}_2)-1}b_n^q]^{\frac{1}{q}}. \end{aligned}$$

Now, by using (12) and (14) (for $s = \lambda,\ s_i = \lambda_i\ (i = 1,2)$), in view of $p < 0,\ 0 < q < 1$, we obtain (13). Lemma 3 is proved. □

Lemma 4. *If $t > 0$, then we have the following inequality:*

$$\sum_{m=1}^{\infty}e^{-t(m-\eta_1)}a_m \leq t\sum_{m=1}^{\infty}e^{-t(m-\eta_1)}A_m. \tag{15}$$

Proof. In view of $A_m e^{-t(m-\eta_1)} = o(1)(m\to\infty)$, using Abel's summation by parts formula, we find:

$$\begin{aligned} \sum_{m=1}^{\infty}e^{-t(m-\eta_1)}a_m &= \lim_{m\to\infty}A_m e^{-t(m-\eta_1)} + \sum_{m=1}^{\infty}A_m[e^{-t(m-\eta_1)} - e^{-t(m-\eta_1+1)}] \\ &= \sum_{m=1}^{\infty}A_m[e^{-t(m-\eta_1)} - e^{-t(m-\eta_1+1)}] = (1-e^{-t})\sum_{m=1}^{\infty}e^{-t(m-\eta_1)}A_m. \end{aligned}$$

Since $1 - e^{-t} < t\ (t > 0)$, we acquire inequality (15). This completes the proof of Lemma 4. □

3. Main Results

Theorem 1. *Under the assumption (C1), we have the following reverse Hardy–Hilbert's inequality:*

$$\begin{aligned} I &:= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n-\eta)^{\lambda+1}} > \frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}} \\ &\times\Big[\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}\Big(1-O_2\Big(\frac{1}{(n-\eta_2)^{\lambda_1}}\Big)\Big)(n-\eta_2)^{q(1-\hat{\lambda}_2)-1}b_n^q\Big]^{\frac{1}{q}}. \end{aligned} \tag{16}$$

where $O_2(\frac{1}{(n-\eta_2)^{\lambda_1}}) := \frac{1}{k_\lambda(\lambda_1)}\int_0^{\frac{1-\eta 1}{n-\eta_2}}\frac{u^{\lambda_1-1}}{(1+u)^\lambda}du$*. In particular, for* $\lambda_1+\lambda_2=\lambda$*, we have*

$$0<\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\lambda_1)-1}a_m^p<\infty, 0<\sum_{n=1}^{\infty}\Big(1-O_2\Big(\frac{1}{(n-\eta_2)^{\lambda_1}}\Big)\Big)(n-\eta_2)^{q(1-\lambda_2)-1}b_n^q<\infty,$$

and the following reverse inequality:

$$\begin{aligned} I &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n-\eta)^{\lambda+1}} > \frac{1}{\lambda}B(\lambda_1,\lambda_2) \\ &\times\Big[\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\lambda_1)-1}a_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}\Big(1-O_2\Big(\frac{1}{(n-\eta_2)^{\lambda_1}}\Big)\Big)(n-\eta_2)^{q(1-\lambda_2)-1}b_n^q\Big]^{\frac{1}{q}}. \end{aligned} \tag{17}$$

Proof. In view of the formula $\frac{1}{(m+n-\eta)^{\lambda+1}} = \frac{1}{\Gamma(\lambda+1)}\int_0^\infty t^{(\lambda+1)-1}e^{-(m+n-\eta)t}dt$, by using (15), it follows that:

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda+1)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_m b_n\int_0^\infty t^{(\lambda+1)-1}e^{-(m+n-\eta)t}dt \\ &= \frac{1}{\Gamma(\lambda+1)}\int_0^\infty t^{\lambda-1}\Big[t\sum_{m=1}^{\infty}e^{-(m-\eta_1)t}A_m\Big]\sum_{n=1}^{\infty}e^{-(n-\eta_2)t}b_n dt \\ &\geq \frac{1}{\Gamma(\lambda+1)}\int_0^\infty t^{\lambda-1}\sum_{m=1}^{\infty}e^{-(m-\eta_1)t}a_m\sum_{n=1}^{\infty}e^{-(n-\eta_2)t}b_n dt \\ &= \frac{1}{\Gamma(\lambda+1)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_m b_n\int_0^\infty t^{\lambda-1}e^{-(m+n-\eta)t}dt \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda+1)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n-\eta)^\lambda} = \frac{I_0}{\lambda}. \end{aligned}$$

Furthermore, by means of (13), we obtain (16). The proof of Theorem 1 is complete. □

Remark 1. *For* $s=\lambda+1\in(1,3]$, $s_2=\widetilde{\lambda}_2\in(0,\frac{3}{2}]\cap(0,\lambda+1)$ *from (11) and (12), we have* $\lambda\in(0,2]$*, and the following inequality:*

$$\varpi_{\lambda+1}(\widetilde{\lambda}_2,m)=(m-\eta_1)^{\lambda+1-\widetilde{\lambda}_2}\sum_{n=1}^{\infty}\frac{(n-\eta_2)^{\widetilde{\lambda}_2-1}}{(m+n-\eta)^{\lambda+1}}<k_{\lambda+1}(\widetilde{\lambda}_2)(m\in\mathrm{N}). \tag{18}$$

Theorem 2. *If* $\lambda_1+\lambda_2=\lambda\in(0,2]$, $\lambda_1\in(0,1]\cap(0,\lambda)$, $\lambda_2\in(0,\frac{3}{2}]\cap(0,\lambda)$*, then the constant factor* $\frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ *in (16) is the best possible.*

Proof. (i) For the case of $\lambda_1\in(0,1)\cap(0,\lambda)$, we prove that the constant factor $\frac{1}{\lambda}B(\lambda_1,\lambda_2)$ in (17) is the best possible.

For any $0<\varepsilon<\min\{|p|(1-\lambda_1),q\lambda_2\}$, we set $\widetilde{a}_m := m^{\lambda_1-\frac{\varepsilon}{p}-1}, \widetilde{b}_n := n^{\lambda_2-\frac{\varepsilon}{q}-1}(m,n\in\mathrm{N})$. Since $0<\varepsilon<|p|(1-\lambda_1)$, we have $0<\lambda_1-\frac{\varepsilon}{p}<1(p<0)$, and $f(t):=t^{\lambda_1-\frac{\varepsilon}{p}-1}$ is strictly decreasing with respect to $t > 0$. Thus, by the decreasing property of the series, we have $\widetilde{A}_m := \sum_{i=1}^{m} \widetilde{a}_i = \sum_{i=1}^{m} i^{\lambda_1 - \frac{\varepsilon}{p} - 1} < \int_0^m t^{\lambda_1 - \frac{\varepsilon}{p} - 1} dt = \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} m^{\lambda_1 - \frac{\varepsilon}{p}}$.

If there exists a constant $M \geq \frac{1}{\lambda} B(\lambda_1, \lambda_2)$ such that (17) is valid when we replace $\frac{1}{\lambda} B(\lambda_1, \lambda_2)$ by M, then, in particular, for $\eta_i = \eta = 0$ $(i = 1, 2)$, using a substitution of $a_m = \widetilde{a}_m$, $b_n = \widetilde{b}_n$ and $A_m = \widetilde{A}_m$ in (17), we have:

$$\widetilde{I} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\widetilde{A}_m \widetilde{b}_n}{(m+n)^{\lambda+1}} > M\Big[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} \widetilde{a}_m^p\Big]^{\frac{1}{p}} \Big[\sum_{n=1}^{\infty} (1 - O_2(\frac{1}{n^{\lambda_1}})) n^{q(1-\lambda_2)-1} \widetilde{b}_n^q\Big]^{\frac{1}{q}}. \tag{19}$$

By (19) and the decreasing property of the series, we obtain:

$$\begin{aligned}
\widetilde{I} &> M\Big[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} m^{p\lambda_1 - \varepsilon - p}\Big]^{\frac{1}{p}} \Big[\sum_{n=1}^{\infty} (1 - O_2(\tfrac{1}{n^{\lambda_1}})) n^{q(1-\lambda_2)-1} n^{q\lambda_2 - \varepsilon - q}\Big]^{\frac{1}{q}} \\
&= M\Big(1 + \sum_{m=2}^{\infty} m^{-1-\varepsilon}\Big)^{\frac{1}{p}} \Big(\sum_{n=1}^{\infty} n^{-1-\varepsilon} - \sum_{m=1}^{\infty} O_2(\tfrac{1}{n^{\lambda_1 + 1 + \varepsilon}})\Big)^{\frac{1}{q}} \\
&> M\Big(1 + \int_1^{\infty} x^{-1-\varepsilon} dx\Big)^{\frac{1}{p}} \Big(\int_1^{\infty} y^{-1-\varepsilon} dy - O(1)\Big)^{\frac{1}{q}} \\
&> \frac{M}{\varepsilon} (\varepsilon + 1)^{\frac{1}{p}} (1 - \varepsilon O(1))^{\frac{1}{q}}.
\end{aligned}$$

By (18), for $\eta_i = \eta = 0$ $(i = 1, 2)$, $\widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q} \in (0, \frac{3}{2}) \cap (0, \lambda)$, we have:

$$\begin{aligned}
\widetilde{I} &< \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} \sum_{m=1}^{\infty} \Big[m^{\lambda - (\lambda_2 - \frac{\varepsilon}{q}) + 1} \sum_{n=1}^{\infty} \frac{1}{(m+n)^{\lambda+1}} n^{\lambda_2 - \frac{\varepsilon}{q} - 1}\Big] m^{-\varepsilon - 1} \\
&= \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} \sum_{m=1}^{\infty} \varpi_{\lambda+1}(\widetilde{\lambda}_2, n) m^{-\varepsilon - 1} < \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} k_{\lambda+1}(\widetilde{\lambda}_2)\Big(1 + \sum_{m=2}^{\infty} m^{-\varepsilon - 1}\Big) \\
&< \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} k_{\lambda+1}(\widetilde{\lambda}_2)\Big(1 + \int_1^{\infty} x^{-\varepsilon - 1} dx\Big) = \frac{1}{\varepsilon} \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} k_{\lambda+1}(\widetilde{\lambda}_2)(\varepsilon + 1).
\end{aligned}$$

This yields:

$$\frac{1}{\lambda_1 - \frac{\varepsilon}{p}} k_{\lambda+1}(\widetilde{\lambda}_2)(\varepsilon + 1) > \varepsilon \widetilde{I} > M(\varepsilon + 1)^{\frac{1}{p}} (1 - \varepsilon O(1))^{\frac{1}{q}}.$$

Putting $\varepsilon \to 0^+$ into the above inequality, by virtue of the continuity of the beta function, we obtain $\frac{1}{\lambda} B(\lambda_1, \lambda_2) = \frac{1}{\lambda_1} B(\lambda_1 + 1, \lambda_2) = \frac{1}{\lambda_1} k_{\lambda+1}(\lambda_2) \geq M$.

Hence, $M = \frac{1}{\lambda} B(\lambda_1, \lambda_2)$ is the best possible constant factor in (17).

(ii) For the case of $\lambda_1 = 1 (1 < \lambda \leq 2)$, for any $0 < \varepsilon < 1$, replacing λ by $\lambda - \varepsilon$ in (17), setting $\lambda_1 = 1 - \varepsilon$, $\lambda_2 = \lambda - 1$, by case (i), we have the following inequality with the best possible constant factor $\frac{1}{\lambda - \varepsilon} B(1 - \varepsilon, \lambda - 1)$:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_m b_n}{(m + n - \eta)^{\lambda - \varepsilon + 1}} > \frac{1}{\lambda - \varepsilon} B(1 - \varepsilon, \lambda - 1)\Big[\sum_{m=1}^{\infty} (m - \eta_1)^{p\varepsilon - 1} a_m^p\Big]^{\frac{1}{p}} K_{q,\varepsilon}, \tag{20}$$

where $K_{q,\varepsilon} := [\sum_{n=1}^{\infty} (1 - O_2(\frac{1}{(n-\eta_2)^{1-\varepsilon}}))(n - \eta_2)^{q(2-\lambda)-1} b_n^q]^{\frac{1}{q}}$.

Since for $\lambda_1 = 1$ we have:

$$\begin{gathered}
(m - \eta_1)^{p\varepsilon - 1} \leq (m - \eta_1)^{-1} = (m - \eta_1)^{p(1-\lambda_1)-1}, \\
\sum_{m=1}^{\infty} (m - \eta_1)^{p\varepsilon - 1} a_m^p \leq \sum_{m=1}^{\infty} (m - \eta_1)^{p(1-\lambda_1)-1} a_m^p < \infty,
\end{gathered}$$

it follows that $\lim_{\varepsilon \to 0^+} \sum_{m=1}^{\infty} (m - \eta_1)^{p\varepsilon - 1} a_m^p = \sum_{m=1}^{\infty} (m - \eta_1)^{p(1-1)-1} a_m^p$, and in the same way, we conclude that $\lim_{\varepsilon \to 0^+} K_{q,\varepsilon} = K_{q,0}$ is valid.

If there exists a constant factor $M \geq \frac{1}{\lambda}B(1, \lambda - 1) = \frac{1}{\lambda(\lambda-1)}(1 < \lambda \leq 2)$, such that (17) (for $\lambda_1 = 1, \lambda_2 = \lambda - 1$) is valid when we replace $\frac{1}{\lambda(\lambda-1)}$ by M, namely

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n-\eta)^{\lambda+1}} > M[\sum_{m=1}^{\infty}(m-\eta_1)^{-1}a_m^p]^{\frac{1}{p}}K_{q,0}, \tag{21}$$

Then, by using Fatou lemma (cf. [15]) and (20), it follows that

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n-\eta)^{\lambda-\varepsilon+1}}/[\sum_{m=1}^{\infty}(m-\eta_1)^{p\varepsilon-1}a_m^p]^{\frac{1}{p}}K_{q,\varepsilon}\}\\ &\geq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n-\eta)^{\lambda+1}}/[\sum_{m=1}^{\infty}(m-\eta_1)^{-1}a_m^p]^{\frac{1}{p}}K_{q,0} > M.\end{aligned}$$

By the property of limitation, there exists a constant $\delta_0 \in (0,1)$, such that for any $\delta \in (0, \delta_0)$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n-\eta)^{\lambda-\delta+1}}/[\sum_{m=1}^{\infty}(m-\eta_1)^{p[1-(1-\delta)]-1}a_m^p]^{\frac{1}{p}}K_q > M,$$

namely,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n-\eta)^{\lambda-\delta+1}} > M[\sum_{m=1}^{\infty}(m-\eta_1)^{p[1-(1-\delta)]-1}a_m^p]^{\frac{1}{p}}K_{q,\delta}.$$

Since the constant factor $\frac{1}{\lambda-\delta}B(1-\delta, \lambda-1)$ in (20) (for $\varepsilon = \delta$) is the best possible, we have $\frac{1}{\lambda-\delta}B(1-\delta, \lambda-1) \geq M$. Letting $\delta \to 0^+$, we have $\frac{1}{\lambda(\lambda-1)} = \frac{1}{\lambda}B(1, \lambda-1) \geq M$, which implies that $M = \frac{1}{\lambda(\lambda-1)}$ is the best possible factor of (17) (for $\lambda_1 = 1$, $\lambda_2 = \lambda - 1$). This completes the proof of Theorem 2. □

Theorem 3. *Under the assumption (C1), if the constant factor* $\frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ *in (16) is the best possible, then for*

$$\lambda - \lambda_1 - \lambda_2 \in [(\lambda - \lambda_2 - \frac{3}{2})q, (\lambda - \lambda_2)q) \cap (-\lambda_2 q, (\frac{3}{2} - \lambda_2)q] (\supset \{0\}), \tag{22}$$

we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. Note that for $\hat{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \hat{\lambda}_2 = \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find $\hat{\lambda}_1 + \hat{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda$.

If $\lambda - \lambda_1 - \lambda_2 \in [(\lambda - \lambda_2 - \frac{3}{2})q, (\lambda - \lambda_2)q) (\supset \{0\})$, then we have $0 < \hat{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} \leq \frac{3}{2}$; if $\lambda - \lambda_1 - \lambda_2 \in (-\lambda_2 q, (\frac{3}{2} - \lambda_2)q] (\supset \{0\})$, then we have $0 < \hat{\lambda}_2 \leq \frac{3}{2}$. By using (22), we obtain $0 < \hat{\lambda}_i < \lambda$ $(i = 1, 2)$, and then we deduce that $\hat{\lambda}_i \in (0, \frac{3}{2}] \cap (0, \lambda)$ $(i = 1, 2)$.

By applying (17), we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n-\eta)^{\lambda+1}}\\ &> \frac{1}{\lambda}B(\hat{\lambda}_1, \hat{\lambda}_2)[\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}(1 - O_2(\frac{1}{(n-\eta_2)^{\hat{\lambda}_1}}))(n-\eta_2)^{q(1-\hat{\lambda}_2)-1}b_n^q]^{\frac{1}{q}}.\end{aligned} \tag{23}$$

If the constant factor $\frac{1}{\lambda}(k_{\lambda}(\lambda_2))^{\frac{1}{p}}(k_{\lambda}(\lambda_1))^{\frac{1}{q}}$ in (16) is the best possible, then by using (23), we have the following inequality: $\frac{1}{\lambda}(k_{\lambda}(\lambda_2))^{\frac{1}{p}}(k_{\lambda}(\lambda_1))^{\frac{1}{q}} \geq \frac{1}{\lambda}B(\hat{\lambda}_1,\hat{\lambda}_2) = \frac{1}{\lambda}k_{\lambda}(\hat{\lambda}_1)$ $(\in \mathrm{R}_+ = (0,\infty))$, namely, $(k_{\lambda}(\lambda_2))^{\frac{1}{p}}(k_{\lambda}(\lambda_1))^{\frac{1}{q}} \geq k_{\lambda}(\hat{\lambda}_1)$.

By employing the reverse Hölder's inequality (cf. [14]), we obtain:

$$\begin{aligned}
&k_{\lambda}(\hat{\lambda}_1) = k_{\lambda}(\tfrac{\lambda-\lambda_2}{p}+\tfrac{\lambda_1}{q}) \\
&= \int_0^{\infty}\tfrac{1}{(1+u)^{\lambda}}u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}du = \int_0^{\infty}\tfrac{1}{(1+u)^{\lambda}}(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du \\
&\geq [\int_0^{\infty}\tfrac{1}{(1+u)^{\lambda}}u^{\lambda-\lambda_2-1}du]^{\frac{1}{p}}[\int_0^{\infty}\tfrac{1}{(1+u)^{\lambda}}u^{\lambda_1-1}du]^{\frac{1}{q}} \\
&= [\int_0^{\infty}\tfrac{1}{(1+v)^{\lambda}}v^{\lambda_2-1}dv]^{\frac{1}{p}}[\int_0^{\infty}\tfrac{1}{(1+u)^{\lambda}}u^{\lambda_1-1}du]^{\frac{1}{q}} \\
&= (k_{\lambda}(\lambda_2))^{\frac{1}{p}}(k_{\lambda}(\lambda_1))^{\frac{1}{q}},
\end{aligned} \tag{24}$$

which implies that $k_{\lambda}(\hat{\lambda}_1) = (k_{\lambda}(\lambda_2))^{\frac{1}{p}}(k_{\lambda}(\lambda_1))^{\frac{1}{q}}$, namely, (24) keeps the form of equality.

Note that (24) keeps the form of equality if and only if there exist constants A and B such that they are not both zero satisfying (cf. [15]) $Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1}$ *a.e.* in R_+. Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ *a.e.* in R_+, and $\lambda-\lambda_2-\lambda_1 = 0$. Hence, we have $\lambda_1+\lambda_2 = \lambda$. Theorem 3 is proved. □

4. Equivalent Forms and Some Particular Inequalities

Theorem 4. *Under the assumption (C1), we have the following reverse inequality equivalent to (16):*

$$\begin{aligned}
J &:= \{\sum_{n=1}^{\infty}\frac{(n-\eta_2)^{p\hat{\lambda}_2-1}}{(1-O(\frac{1}{(n-\eta_2)^{\lambda_1}}))^{p-1}}[\sum_{m=1}^{\infty}\frac{A_m}{(m+n-\eta)^{\lambda+1}}]^p\}^{\frac{1}{p}} \\
&> \frac{1}{\lambda}(k_{\lambda}(\lambda_2))^{\frac{1}{p}}(k_{\lambda}(\lambda_1))^{\frac{1}{q}}[\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}.
\end{aligned} \tag{25}$$

In particular, for $\lambda_1+\lambda_2 = \lambda$, we have $0 < \sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\lambda_1)-1}a_m^p < \infty$, and the following reverse inequality equivalent to (19):

$$\{\sum_{n=1}^{\infty}\frac{(n-\eta_2)^{p\lambda_2-1}}{(1-O(\frac{1}{(n-\eta_2)^{\lambda_1}}))^{p-1}}[\sum_{m=1}^{\infty}\frac{A_m}{(m+n-\eta)^{\lambda+1}}]^p\}^{\frac{1}{p}} > \frac{1}{\lambda}B(\lambda_1,\lambda_2)[\sum_{m=1}^{\infty}(m-\eta_1)^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}. \tag{26}$$

Proof. Suppose that (25) is valid. By using the reverse Höder's inequality (cf. [14]), we have

$$\begin{aligned}
I &= \sum_{n=1}^{\infty}[\frac{(n-\eta_2)^{\frac{-1}{p}+\hat{\lambda}_2}}{(1-O_2(\frac{1}{(n-\eta_2)^{\lambda_1}}))^{\frac{1}{q}}}\sum_{m=1}^{\infty}\frac{A_m}{(m+n-\eta)^{\lambda+1}}][(1-O_2(\frac{1}{(n-\eta_2)^{\lambda_1}}))^{\frac{1}{q}}(n-\eta_2)^{\frac{1}{p}-\hat{\lambda}_2}b_n] \\
&\geq J[\sum_{n=1}^{\infty}(1-O_2(\frac{1}{(n-\eta_1)^{\lambda_1}}))(n-\eta_1)^{q(1-\hat{\lambda}_2)-1}b_n^q]^{\frac{1}{q}}.
\end{aligned} \tag{27}$$

Then, from (25) and (27), we obtain (16).

On the other hand, assuming that (16) is valid, we set

$b_n := \frac{(n-\eta_2)^{p\hat{\lambda}_2-1}}{(1-O(\frac{1}{(n-\eta_2)^{\lambda_1}}))^{p-1}}[\sum_{m=1}^{\infty}\frac{A_m}{(m+n-\eta)^{\lambda+1}}]^{p-1}$, $n \in \mathrm{N}$. Then, it follows that

$J = [\sum_{n=1}^{\infty}(1-O(\frac{1}{(n-\eta_2)^{\lambda_1}}))(n-\eta_2)^{q(1-\hat{\lambda}_2)-1}b_n^q]^{\frac{1}{p}}$.

If $J = \infty$, then (25) is naturally valid; if $J = 0$, then it is impossible that it makes (25) valid, namely, $J > 0$. Suppose that $0 < J < \infty$. By virtue of (16), we have

$$\sum_{n=1}^{\infty}(1 - O_2(\frac{1}{(n-\eta_2)^{\lambda_1}}))(n - \eta_2)^{q(1-\hat{\lambda}_2)-1}b_n^q = J^p = I$$
$$> \frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}[\sum_{m=1}^{\infty}(m - \eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p]^{\frac{1}{p}} J^{p-1},$$
$$J > \frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}[\sum_{m=1}^{\infty}(m - \eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}.$$

Thus, we obtain (25), which implies that (25) is equivalent to (16). The Theorem 4 is proved. □

Remark 2. *By the same way as above, in view of assumption (C1), if $0 < p < 1, q < 0, \frac{1}{p} + \frac{1}{q} = 1$, then we can obtain the following reverse equivalent inequalities containing one partial sums:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n-\eta)^{\lambda+1}}$$
$$> \frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}[\sum_{m=1}^{\infty}(1 - O_1(\frac{1}{(m-\eta_1)^{\lambda_2}}))(m - \eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}(n - \eta_2)^{q(1-\hat{\lambda}_2)-1}b_n^q]^{\frac{1}{q}},$$
$$\{\sum_{n=1}^{\infty}(n - \eta_2)^{p\hat{\lambda}_2-1}[\sum_{m=1}^{\infty}\frac{A_m}{(m+n-\eta)^{\lambda+1}}]^p\}^{\frac{1}{p}}$$
$$> \frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}[\sum_{m=1}^{\infty}(1 - O_1(\frac{1}{(m-\eta_1)^{\lambda_2}}))(m - \eta_1)^{p(1-\hat{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}.$$

Theorem 5. *If $\lambda_1 + \lambda_2 = \lambda(\in (0, 2])$ satisfying $\lambda_1 \in (0, 1] \cap (0, \lambda)$ and $\lambda_2 \in (0, \frac{3}{2}] \cap (0, \lambda)$, then the constant factor $\frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ in (25) is the best possible. On the other hand, by virtue of the assumption (C1), if the constant factor $\frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ in (25) is the best possible, then for*

$$\lambda - \lambda_1 - \lambda_2 \in [(\lambda - \lambda_2 - \frac{3}{2})q, (\lambda - \lambda_2)q) \cap (-\lambda_2 q, (\frac{3}{2} - \lambda_2)q](\supset \{0\}),$$

we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. If $\lambda_1 + \lambda_2 = \lambda(\in (0, 2])$ satisfying $\lambda_1 \in (0, 1] \cap (0, \lambda)$ and $\lambda_2 \in (0, \frac{3}{2}] \cap (0, \lambda)$, then by using Theorem 2, we conclude that the constant factor $\frac{1}{\lambda}(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$ in (16) is the best possible. By employing (27), we can prove that the constant factor in (25) is still the best possible.

On the other hand, if the same constant factor in (25) is the best possible, then by the equivalency of (25) and (16), in view of $J^q = I$ (in the proof of Theorem 4), it follows that the same constant factor in (16) is still the best possible. By applying Theorem 2, in view of the assumption, we have $\lambda_1 + \lambda_2 = \lambda$. The proof of Theorem 5 is complete. □

Remark 3. *(i) Taking $\eta = \eta_1 = \eta_2 = 0$ in (17) and (26), we obtain the following reverse equivalent inequalities:*

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n)^{\lambda+1}}\\&> \frac{1}{\lambda}B(\lambda_1,\lambda_2)\Big[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}(1-O_2(\frac{1}{n^{\lambda_1}}))n^{q(1-\lambda_2)-1}b_n^q\Big]^{\frac{1}{q}},\\&\Big\{\sum_{n=1}^{\infty}\frac{n^{p\lambda_2-1}}{(1-O(\frac{1}{n^{\lambda_1}}))^{p-1}}\Big[\sum_{m=1}^{\infty}\frac{A_m}{(m+n)^{\lambda+1}}\Big]^p\Big\}^{\frac{1}{p}} > \frac{1}{\lambda}B(\lambda_1,\lambda_2)\Big[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\Big]^{\frac{1}{p}}.\end{aligned} \tag{28}$$

Hence, (17) (resp. (16)) is an extension of inequality (28).
In particular, for $\lambda = 2, \lambda_1 = \lambda_2 = 1$, we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n)^3} > \frac{1}{2}\Big(\sum_{m=1}^{\infty} m^{-1}a_m^p\Big)^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}(1-O_2(\frac{1}{n}))n^{-1}b_n^q\Big]^{\frac{1}{q}},\\&\Big\{\sum_{n=1}^{\infty}\frac{n^{p-1}}{(1-O(\frac{1}{n}))^{p-1}}\Big[\sum_{m=1}^{\infty}\frac{A_m}{(m+n)^3}\Big]^p\Big\}^{\frac{1}{p}} > \frac{1}{2}\Big(\sum_{m=1}^{\infty} m^{-1}a_m^p\Big)^{\frac{1}{p}}.\end{aligned}$$

(ii) Putting $\lambda = 1, \lambda_1 = \lambda_2 = \frac{1}{2}$ in (17) and (26), we obtain the following reverse inequalities with the best possible constant factor π:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n-\eta)^2} > \pi\Big[\sum_{m=1}^{\infty}(m-\eta_1)^{\frac{p}{2}-1}a_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}(1-O_2(\frac{1}{(n-\eta_2)^{1/2}}))(n-\eta_2)^{\frac{q}{2}-1}b_n^q\Big]^{\frac{1}{q}}, \tag{29}$$

$$\Big\{\sum_{n=1}^{\infty}\frac{(n-\eta_1)^{\frac{p}{2}-1}}{(1-O(\frac{1}{(n-\eta_1)^{1/2}}))^{p-1}}\Big[\sum_{m=1}^{\infty}\frac{A_m}{(m+n-\eta)^2}\Big]^p\Big\}^{\frac{1}{p}} > \pi\Big[\sum_{m=1}^{\infty}(m-\eta_1)^{\frac{p}{2}-1}a_m^p\Big]^{\frac{1}{p}}. \tag{30}$$

Choosing $\eta_1 = \eta_2 = \eta = 0$ in (29) and (30), we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n)^2} > \pi\Big(\sum_{m=1}^{\infty} m^{\frac{p}{2}-1}a_m^p\Big)^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}(1-O_2(\frac{1}{n^{1/2}}))n^{\frac{q}{2}-1}b_n^q\Big]^{\frac{1}{q}},\\&\Big\{\sum_{n=1}^{\infty}\frac{n^{\frac{p}{2}-1}}{(1-O(\frac{1}{n^{1/2}}))^{p-1}}\Big[\sum_{m=1}^{\infty}\frac{A_m}{(m+n)^2}\Big]^p\Big\}^{\frac{1}{p}} > \pi\Big(\sum_{m=1}^{\infty} m^{\frac{p}{2}-1}a_m^p\Big)^{\frac{1}{p}}.\end{aligned}$$

Choosing $\eta_1 = \eta_2 = \frac{1}{4}, \eta = \frac{1}{2}$ in (29) and (30), we obtain:

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m b_n}{(m+n-\frac{1}{2})^2} > \pi\Big[\sum_{m=1}^{\infty}(m-\frac{1}{4})^{\frac{p}{2}-1}a_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}(1-O_2(\frac{1}{(n-\frac{1}{4})^{1/2}}))(n-\frac{1}{4})^{\frac{q}{2}-1}b_n^q\Big]^{\frac{1}{q}},\\&\Big\{\sum_{n=1}^{\infty}\frac{(n-\frac{1}{4})^{\frac{p}{2}-1}}{(1-O(\frac{1}{(n-\frac{1}{4})^{1/2}}))^{p-1}}\Big[\sum_{m=1}^{\infty}\frac{A_m}{(m+n-\frac{1}{2})^2}\Big]^p\Big\}^{\frac{1}{p}} > \pi\Big[\sum_{m=1}^{\infty}(m-\frac{1}{4})^{\frac{p}{2}-1}a_m^p\Big]^{\frac{1}{p}}.\end{aligned}$$

5. Conclusions

In this paper, inspired by the work of [4–10], we construct a reverse Hardy–Hilbert's inequality which contains one partial sum and some extra parameters inside the weight coefficients in Theorem 1. Our method is mainly based on some skillful applications of the Euler–Maclaurin summation formula and Abel's partial summation formula. By means of the newly proposed inequality, we then discuss the equivalent conditions of the best possible constant factor associated with several parameters in Theorems 2 and 3. As applications, we deal with some equivalent forms of the obtained inequality and illustrate how to derive more reverse inequalities of the Hardy–Hilbert type from the current results in Theorems 4 and 5. The lemmas and theorems reveal rich connotations and significance of this type of inequality.

Author Contributions: B.Y. carried out the mathematical studies and drafted the manuscript. S.W. and X.H. participated in the design of the study and performed the numerical analysis. All authors contributed equally in the preparation of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation (Nos.11961021, 11561019), the Hechi University Research Foundation for Advanced Talents under Grant (No.2021GCC024), the Characteristic Innovation Project of Guangdong Provincial Colleges and Universities (No.2020KTSCX088) and the Natural Science Foundation of Fujian Province of China (No.2020J01365).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no competing interest.

References

1. Hardy, G.H.; Littlewood, J.E.; Polya, G. *Inequalities*; Cambridge University Press: Cambridge, UK, 1934.
2. Krnić, M.; Pečarić, J. Extension of Hilbert's inequality. *J. Math. Anal. Appl.* **2006**, *324*, 150–160. [CrossRef]
3. Yang, B.; Wu, S.; Chen, Q. A new extension of Hardy-Hilbert's inequality containing kernel of double power functions. *Mathematics* **2020**, *8*, 894. [CrossRef]
4. Adiyasuren, V.; Batbold, T.; Azar, L.E. A new discrete Hilbert-type inequality involving partial sums. *J. Inequal. Appl.* **2019**, *2019*, 127. [CrossRef]
5. Liao, J.; Wu, S.; Yang, B. A multiparameter Hardy-Hilbert-type inequality containing partial sums as the terms of series. *J. Math.* **2021**, *2021*, 5264623. [CrossRef]
6. Yang, B.; Wu, S.; Huang, X. A reverse Hardy-Hilbert's inequality involving one partial sum as the terms of double series. *J. Funct. Spaces* **2022**, *2022*, 2175463. [CrossRef]
7. Wang, A.; Hong, Y.; Yang, B. On a new half-discrete Hilbert-type inequality with the multiple upper limit function and the partial sums. *J. Appl. Anal. Comput.* **2022**, *12*, 814–830. [CrossRef]
8. Chen, Q.; Yang, B. A reverse Hardy–Hilbert-type integral inequality involving one derivative function. *J. Inequal. Appl.* **2020**, *2020*, 259. [CrossRef]
9. Yang, B.C. *The Norm of Operator and Hilbert-Type Inequalities*; Science Press: Beijing, China, 2009.
10. Hong, Y.; Wen, Y. A necessary and sufficient condition of that Hilbert type series inequality with homogeneous kernel has the best constant factor. *Ann. Math.* **2016**, *37*, 329–336.
11. Hong, Y. On the structure character of Hilbert's type integral inequality with homogeneous kernel and application. *J. Jilin Univ.* **2017**, *55*, 189–194.
12. Liu, Q. On a mixed Kernel Hilbert-type integral inequality and its operator expressions with norm. *Math. Meth. Appl. Sci.* **2021**, *44*, 593–604. [CrossRef]
13. Yang, L.; Yang, R. Some new Hardy-Hilbert-type inequalities with multiparameters. *AIMS Math.* **2022**, *7*, 840–854. [CrossRef]
14. Kuang, J.C. *Applied Inequalities*; Shangdong Science and Technology Press: Jinan, China, 2004.
15. Kuang, J.C. *Real and Functional Analysis*; Higher Education Press: Beijing, China, 2015.