mathematics

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Hölder Inequalities for a Generalized Subclass of Univalent Functions Involving Borel Distributions

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Abstract: In this article, by making use of the Borel distributions series, we introduce a new family of normalized holomorphic functions in the open unit disk and investigate necessary and sufficient conditions for functions f to be in this new class. Furthermore, results on the modified Hadamard product, Hölder inequalities, and closure properties under integral transforms and subordination results are discussed in detail.

Keywords: univalent; starlike function; convex function; coefficient bounds; convolution properties; Borel distribution; Hölder inequality; subordination

MSC: 30C55; 30C45

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Citation: Murugusundaramoorthy, G.; Cotîrlă, L.-I. Hölder Inequalities for a Generalized Subclass of Univalent Functions Involving Borel Distributions. *Mathematics* **2022**, *10*, 2430. https://doi.org/10.3390/math10142430

Academic Editors: Gabriel Eduard Vilcu and Janusz Brzdęk

Received: 7 May 2022
Accepted: 9 July 2022
Published: 12 July 2022

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, with the normalization $f(0) = 0 = f'(0) - 1$. The class $\mathcal{A}$ includes the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in \mathbb{U}. \tag{1}$$

Let $\mathcal{S}$ be the subclass of all univalent (i.e., one-to-one) functions in $\mathcal{A}$.

A function $f \in \mathcal{A}$ is said to be a starlike function with respect to $\omega_0 \in f(\mathbb{U})$ if f maps $\mathbb{U}$ univalently onto a starlike region with respect to ω_0. If $\omega_0 = 0$, we say that f is a starlike function, and the class of those functions is denoted by $\mathcal{S}^*$. The function $f \in \mathcal{A}$ is said to be convex if f maps $\mathbb{U}$ univalently onto a convex region. That is, $f(\mathbb{U})$ is a convex region.

For more details about analytic univalent functions, see [1,2].

We say that $\mathbf{Y} \subseteq \mathbb{C}$ is called a starlike domain with respect to $\omega_0 \in \mathbf{Y}$, if each line segment joining ω_0 to every other point $\omega \in \mathbf{Y}$ lies entirely with in $\mathbf{Y}$. Furthermore, the domain $\mathbf{Y}$ is said to be a convex domain if, for $\omega_1, \omega_2 \in \mathbf{Y}$, the line segment joining these two points lies inside $\mathbf{Y}$.

In 1936, Robertson [3] defined the following subclasses of $\mathcal{S}$.

We know that a function $f \in \mathcal{A}$ given by (1) is starlike of order α $(0 \le \alpha < 1)$, if

$$\mathcal{R}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathbb{U},$$

and we denote this function class by $\mathcal{S}^*(\alpha)$.

A function $f \in \mathcal{A}$ is said to be convex of order α $(0 \le \alpha < 1)$ if

$$\mathcal{R}\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha,\ z \in \mathbb{U},$$

and this class is denoted by $\mathcal{K}(\alpha)$.

We observe that $\mathcal{K}(\alpha) = \{f \in A : zf'(z) \in \mathcal{S}^*(\alpha)\}$.

Moreover, we note that

$$\mathcal{S}^*(0) =: \mathcal{S}^* = \{f \in \mathcal{A} : \mathcal{R}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{U}\}$$

and

$$\mathcal{K} := \mathcal{K}(0) = \{f \in \mathcal{A} : \mathcal{R}\left(\frac{(zf'(z))'}{f'(z)}\right) > 0,\ z \in \mathbb{U}\}$$

represents the class of starlike and convex functions, (see [1], p. 41) respectively.

In 1915, Alexander [4] (also see [1], p. 43; [2] Volume I, Chapter 8) gave the analytic connection between starlike and convex functions as $f \in \mathcal{K}$, if and only if $zf'(z) \in \mathcal{S}^*$.

In view of the one-to-one correspondence between $\mathcal{K}$ and $\mathcal{S}^*$, Alexander defined the transform as $J[f](z) = \int_0^z \frac{f(t)}{t}\, dt$. That is, $J[f]$ is convex if and only if f is starlike.

Geometric function theory properties fundamentally aim to categorize analytic functions that are defined in $\mathbb{U}$ and have certain analytic criteria, such as being univalent, convex, and starlike by connecting them to the geometric characterization. Moreover, the common geometric characterization of functions belonging to a class gives very clear limitations on the Taylor coefficients of the functions belonging to the class. Many results obtained regarding class S or certain subclasses of S are due to the attempt to prove the famous and easily stated Bieberbach conjecture, stating that $|a_n| \leq n, (n \geq 2)$ for every function $f \in \mathcal{S}$.

The Koebe function $k(z) = \frac{z}{(1-z)^2}, |z| < 1$ is the extremal function for class $\mathcal{S}$ (see [5]). It gives sharp growth and distortion bounds and the coefficient estimates bounds [6].

In 1975, Silverman [7] introduced another subclass $\mathcal{T}$, which denotes the subclass of $\mathcal{S}$ consisting of all functions whose nonzero coefficients, from the second on, are negative. That is, $\mathcal{T}$ the subclass of $\mathcal{S}$ comprises functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,\ \ z \in \mathbb{U}. \tag{2}$$

In this case, the classes of starlike and convex functions of order α are denoted by $\mathcal{ST}^*(\alpha)$ and $\mathcal{KT}(\alpha)$ with $\alpha \in [0, 1)$, respectively, see [7].

A function of the form (2) belongs to class $\mathcal{T}$ if and only if the inequality $\sum\limits_{n=2}^{\infty} na_n \leq 1$ is satisfied [7]. Similarly, the condition $\sum\limits_{n=2}^{\infty} n|a_n| \leq 1$ is sufficient for all functions $f \in \mathcal{S}$ of the form (1) . In fact, such functions are also starlike since $|\frac{zf'}{f} - 1| < 1, (z \in \mathbb{U})$ (see [7]). The class $\mathcal{T}$ can also be fruitful when constructing counterexamples for conjectures within larger families. Solutions in $\mathcal{T}$ sometimes provide insight into problems for $\mathcal{S}$ or other subclasses. The coefficient characterization makes $\mathcal{T}$ considerably more manageable than the class $\mathcal{S}$.

The elementary distributions, such as the Poisson, Pascal, binomial, beta negative binomial, and logarithmic, have been partially studied from a theoretical point of view (for more details, see [8–12]). Furthermore, there is increasing interest to study analytic functions associated with certain polynomials [13,14]. Lately, Wanas and Khuttar in [15] consider the power series whose coefficients are probabilities of Borel distributions, as below:

$$B(\ell, z) = z + \sum_{n=2}^{\infty} \frac{(\ell(n-1))^{n-2} e^{-\ell(n-1)}}{(n-1)!} z^n,\ (z \in \mathbb{U}; 0 < \ell \leq 1).$$

We conclude that the radius of convergence of the above power series is infinity by using the ratio test. Wanas and Khuttar in [15] also defined

$$\mathcal{B}(\ell,z) = 2z - B(\ell,z) = z - \sum_{n=2}^{\infty} \frac{(\ell(n-1))^{n-2}e^{-\ell(n-1)}}{(n-1)!} z^n, \ (z \in \mathbb{U}; 0 < \ell \leq 1).$$

Moreover, Wanas and Khuttar in [15] defined the linear operator $\mathcal{B}_\ell : \mathcal{A} \to \mathcal{A}$ as:

$$\begin{aligned} \mathcal{B}_\ell f(z) &= f(z) * \mathcal{B}(\ell,z) \\ &= z - \sum_{n=2}^{\infty} \frac{(\ell(n-1))^{n-2}e^{-\ell(n-1)}}{(n-1)!} a_n z^n, \\ &= z - \sum_{n=2}^{\infty} \mathcal{B}(n,\ell) a_n z^n, \end{aligned} \tag{3}$$

where

$$\mathcal{B}(n,\ell) = \frac{(\ell(n-1))^{n-2}e^{-\ell(n-1)}}{(n-1)!} \qquad (0 < \ell \leq 1). \tag{4}$$

and the symbol $*$ specifies the Hadamard product (convolution) of two series.

Many differential and integral operators can be written with regards to the convolution of certain analytic functions. It is perceived that this formalism makes mathematical exploration easier and also helps to improve understanding around the symmetric and geometric properties of such operators. Silverman in [7] was the first to pave the way for the study of functions with negative coefficients of the form (2), after which various forms of such functions have been investigated by many researchers in the field of geometric function theory. The study of operators plays a significant role in geometric function theory (GFT). For more details about the importance of convolution in the geometric function theory (GFT), we refer to [16–21] and references cited therein.

Inspired by the earlier works on analytic functions with negative coefficients, see [22–28], and recent studies on analytic functions convoluting with Borel distributions conducted by Wanas and Khuttar [15], Ahmad et al. [29], El-Deeb et al. [30,31], and Srivastava et al. [32], we define the unified subclass of analytic functions with negative coefficients $\mathcal{WS}_{\mathcal{B}}^{\ell}(L,M,\alpha,\varrho,\vartheta,\lambda)$ concerning convolution structure $f(z) * \mathcal{B}(\ell,z)$ assumed in (3).

Definition 1. *For* $0 \leq \lambda \leq 1, 0 < \varrho \leq 1, -1 \leq M < L \leq 1, 0 \leq \vartheta \leq 1, 0 \leq \alpha < 1$, *we let* $\mathcal{WS}_{\mathcal{B}}^{\ell}(L,M,\alpha,\varrho,\vartheta,\lambda)$ *and we denote a new subclass of* $\mathcal{S}$ *concerning Borel distributions comprising functions* f *of the form (1) and satisfying the analytic condition*

$$\left| \frac{\frac{zQ_\lambda'(z)}{Q_\lambda(z)} - 1}{(M-L)\vartheta\left[\frac{zQ_\lambda'(z)}{Q_\lambda(z)} - \alpha\right] - M\left[\frac{zQ_\lambda'(z)}{Q_\lambda(z)} - 1\right]} \right| < \varrho, \qquad z \in \mathbb{U}, \tag{5}$$

where

$$Q_\lambda(z) = (1-\lambda)\mathcal{B}_\ell f(z) + \lambda z(\mathcal{B}_\ell f(z))'.$$

Thus, we obtain

$$\frac{zQ_\lambda'(z)}{Q_\lambda(z)} = \frac{z(\mathcal{B}_\ell f(z))' + \lambda z^2(\mathcal{B}_\ell f(z))''}{(1-\lambda)\mathcal{B}_\ell f(z) + \lambda z(\mathcal{B}_\ell f(z))'}, \qquad (0 < \ell \leq 1), \tag{6}$$

and $\mathcal{B}_\ell(f(z))$ *is given by (3).*

We also define

$$\mathcal{W}_{\mathcal{B}}^{\ell}(L,M,\alpha,\varrho,\vartheta,\lambda) \equiv \mathcal{WS}_{\mathcal{B}}^{\ell}(L,M,\alpha,\varrho,\vartheta,\lambda) \cap \mathcal{T}.$$

Example 1. *For $0 \leq \lambda \leq 1, 0 < \varrho \leq 1, 0 \leq \vartheta \leq 1$ and fixing $L = 1 - 2\zeta, (0 \leq \zeta < 1); M = -1$, let $\mathcal{WS}_{\mathcal{B}}^{\ell}(\zeta, \alpha, \varrho, \vartheta, \lambda)$. We denote a new subclass of $\mathcal{S}$ concerning Borel distributions comprising functions f of the form (1) and sustaining the analytic condition*

$$\left| \frac{\frac{zQ'_{\lambda}(z)}{Q_{\lambda}(z)} - 1}{2\vartheta(\zeta - 1)\left[\frac{zQ'_{\lambda}(z)}{Q_{\lambda}(z)} - \alpha\right] + \left[\frac{zQ'_{\lambda}(z)}{Q_{\lambda}(z)} - 1\right]} \right| < \varrho, \qquad z \in \mathbb{U}, \tag{7}$$

where $Q_{\lambda}(z) = (1 - \lambda)\mathcal{B}_{\ell} f(z) + \lambda z(\mathcal{B}_{\ell} f(z))'$.

By fixing the parameters $\lambda = 0$ and $\lambda = 1$, we can state various new subclasses of $\mathcal{S}$ which have not yet been investigated by association with Borel distributions. Furthermore, by specializing the parameter $\alpha, \varrho, \vartheta$, we can define certain new subclasses analogues to the subclasses discussed in [7,22,27,33,34] (also see references cited there in) of analytic functions associated with Borel distributions which are also new and have not yet been studied. Thus, our new subclass includes many subclasses studied in the literature.

In the following sections, for $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$, we obtain the modified Hadamard product, the Hölder inequality results, the closure properties under integral transforms, and the subordination results.

2. Characterization Property for $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$

Throughout our study, we let

$$0 \leq \lambda \leq 1, 0 < \varrho \leq 1, -1 \leq M < L \leq 1, 0 \leq \vartheta \leq 1; 0 \leq \alpha < 1, \quad \text{and} \quad 0 < \ell \leq 1.$$

In this section, we provide the necessary and sufficient conditions for $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$.

Theorem 1. *A function f of the form (1) is in the class $\mathcal{WS}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$ if*

$$\sum_{n=2}^{\infty} \mathcal{M}_n a_n \leq (1 - \alpha)(M - L)\varrho\vartheta,$$

where

$$\mathcal{M}_n = (1 + n\lambda - \lambda)[(n - 1)(1 - \varrho M) + \varrho\vartheta(M - L)(n - \alpha)]\mathcal{B}(n, \ell)$$

and $\mathcal{B}(n, \ell)$ is defined by (4).

Proof. Since the function $f(z)$ is of the form (1), we have

$$\begin{aligned} Q_{\lambda}(z) &= z + \sum_{n=2}^{\infty} (1 + n\lambda - \lambda)\mathcal{B}(n, \ell) a_n z^n, \\ zQ'_{\lambda}(z) &= z + \sum_{n=2}^{\infty} n(1 + n\lambda - \lambda)\mathcal{B}(n, \ell) a_n z^n. \end{aligned}$$

For $|z| = 1$, we have

$$\left| \frac{\frac{zQ'_{\lambda}(z)}{Q_{\lambda}(z)} - 1}{(M - L)\vartheta\left[\frac{zQ'_{\lambda}(z)}{Q_{\lambda}(z)} - \alpha\right] - B\left[\frac{zQ'_{\lambda}(z)}{Q_{\lambda}(z)} - 1\right]} \right| < \varrho, \qquad z \in \mathbb{U}.$$

It is suffices to show that

$$\begin{aligned}
&\left|zQ_\lambda'(z) - Q_\lambda(z)\right| - \varrho\left|(M-L)\vartheta\left[zQ_\lambda'(z) - \alpha Q_\lambda(z)\right] - B\left[zQ_\lambda'(z) - Q_\lambda(z)\right]\right| \quad (8)\\
= &\left|\sum_{n=2}^{\infty}(1+n\lambda-\lambda)(n-1)\mathcal{B}(n,\ell)a_n z^{n-1}\right| \\
- &\varrho\left|(M-L)(1-\alpha)\vartheta + \sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(M-L)(n-\alpha)\vartheta - B(n-1)]\mathcal{B}(n,\ell)a_n z^{n-1}]\right| \\
\leq &\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(n-1)(1-\varrho M) + \varrho\vartheta(M-L)(n-\alpha)]\mathcal{B}(n,\ell)a_n - \varrho\vartheta(M-L)(1-\alpha) \\
\leq &\ 0, \text{ by hypothesis.}
\end{aligned}$$

Thus, by maximum modulus theorem, $f \in \mathcal{WS}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$. □

Theorem 2. *Let f be a function of the form (2) and $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$ if and only if*

$$\sum_{n=2}^{\infty}\mathcal{M}_n a_n \leq (1-\alpha)(M-L)\varrho\vartheta, \tag{9}$$

where

$$\mathcal{M}_n = (1+n\lambda-\lambda)[(n-1)(1-\varrho M) + \varrho\vartheta(M-L)(n-\alpha)]\mathcal{B}(n,\ell) \tag{10}$$

and $\mathcal{B}(n,\ell)$ is given by (4).

Proof. In view of Theorem 1, we need only to prove the necessity. If $f \in \mathcal{T}$ is given by (2), then

$$\left|\frac{\frac{zQ_\lambda'(z)}{Q_\lambda(z)} - 1}{(M-L)\vartheta\left[\frac{zQ_\lambda'(z)}{Q_\lambda(z)} - \alpha\right] - M\left[\frac{zQ_\lambda'(z)}{Q_\lambda(z)} - 1\right]}\right|$$

$$= \left|\frac{\sum_{n=2}^{\infty}(1+n\lambda-\lambda)(n-1)a_n\mathcal{B}(n,\ell)z^n}{(M-L)(1-\alpha)\vartheta z + \sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(M-L)(n-\alpha)\vartheta - M(n-1)]a_n\mathcal{B}(n,\ell)z^n}\right| < \varrho. \tag{11}$$

Since $\mathcal{R}(z) < |z|$ for all z, we have

$$\mathcal{R}\left\{\frac{\sum_{n=2}^{\infty}(n-1)(1+n\lambda-\lambda)\mathcal{B}(n,\ell)a_n|z|^{n-1}}{(M-L)(1-\alpha)\vartheta - \sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(M-L)(n-\alpha)\vartheta - B(n-1)]\mathcal{B}(n,\ell)a_n|z|^{n-1}}\right\} < \varrho.$$

By choosing the value of z on the real axis so that $f'(z)$ is real and letting $z \to 1^-$ through real values, we obtain

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(n-1)(1-\varrho M) + \varrho\vartheta(M-L)(n-\alpha)]\mathcal{B}(n,\ell)a_n \leq \varrho\vartheta(M-L)(1-\alpha)$$

and hence the proof is complete. □

Example 2. *For the function*

$$f(z) = z + \sum_{k=2}^{\infty}\frac{(1-\alpha)(M-L)\varrho}{\mathcal{M}_n}v_k z^k \quad (z \in \mathbb{U}),$$

where $\mathcal{M}_n$ is given by (10), and such that $\sum_{k=2}^{\infty} v_k = 1$, we have

$$\begin{aligned}&\sum_{k=2}^{\infty}\mathcal{M}_n|a_k|\\&=\sum_{k=2}^{\infty}\mathcal{M}_n\left(\frac{(1-\alpha)(M-L)\varrho}{\mathcal{M}_n}v_k\right)\\&=(1-\alpha)(M-L)\varrho\sum_{k=2}^{\infty}v_k=(1-\alpha)(M-L)\varrho.\end{aligned}$$

Corollary 1. *If the function $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$, then*

$$|a_n| \leq \frac{\varrho\vartheta(1-\alpha)(M-L)}{\mathcal{M}_n}.$$

For functions

$$f(z) := z - \frac{\varrho\vartheta(1-\alpha)(M-L)}{\mathcal{M}_2}z^2$$

where

$$\mathcal{M}_2 = (1+\lambda)[(1-\varrho M) + \varrho\vartheta(M-L)(2-\alpha)]\mathcal{B}(2,\ell),$$

the result is sharp.

Proof. The proof is quite straightforward, left for reader. □

In the following section, employing the techniques of Schild and Silverman [35], we determine some convolution properties for $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$.

3. Convolution Properties

Let the functions $f_j(z)$ $(j = 1, 2)$ be defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}z^n \quad (a_{n,j} \geq 0; j = 1, 2). \tag{12}$$

The modified Hadamard product of functions $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}z^n. \tag{13}$$

Theorem 3. *Let $f_1(z)$ given by the relation (12) be in the class $\mathcal{W}_{\mathcal{B}}^{\ell}(A, B, \xi_1, \varrho, \vartheta, \lambda)$ and the function $f_2(z)$ defined by (12) be in the class $\mathcal{W}_{\mathcal{B}}^{\ell}(A, B, \xi_2, \varrho, \vartheta, \lambda)$. If the sequence $\{\mathcal{M}_n\}$ is non-decreasing, then $(f_1 * f_2)(z) \in \mathcal{W}_{\mathcal{B}}^{\ell}(A, B, \aleph, \varrho, \vartheta, \lambda)$, where*

$$\aleph = 1 - \frac{(1-\xi_1)(1-\xi_2)[1-\varrho M+\varrho\vartheta(M-L)]\varrho\vartheta(M-L)}{(1+\lambda)\Lambda(\varrho,\vartheta,\xi_1,2)\Lambda(\varrho,\vartheta,\xi_2,2)\mathcal{B}(2,\ell) - [\varrho\vartheta(M-L)]^2(1-\xi_1)(1-\xi_2)}, \tag{14}$$

where

$$\Lambda(\varrho, \vartheta, \xi_1, 2) = [(1-\varrho M) + \varrho\vartheta(M-L)(2-\xi_1)]$$

and

$$\Lambda(\varrho, \vartheta, \xi_2, 2) = [(1-\varrho M) + \varrho\vartheta(M-L)(2-\xi_2)].$$

Proof. Suppose $(f_1 * f_2)(z) \in \mathcal{W}_{\mathcal{B}}^{\ell}(A, B, \aleph, \varrho, \vartheta, \lambda)$. Then, by Theorem 2, it is enough to show that

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\aleph)]}{\varrho\vartheta(M-L)(1-\aleph)}\mathcal{B}(n,\ell)a_{n,1}a_{n,2} \leq 1, \tag{15}$$

where $\aleph$ is given by (14).

Since $f_1 \in \mathcal{W}_{\mathcal{B}}^{\ell}(A, B, \xi_1, \varrho, \vartheta, \lambda)$, in view of Theorem 2, we have

$$\sum_{n=2}^{\infty} \frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_1)]}{\varrho\vartheta(M-L)(1-\xi_1)}\mathcal{B}(n,\ell)a_{n,1} \leq 1 \tag{16}$$

and for $f_2 \in \mathcal{W}_{\mathcal{B}}^{\ell}(A, B, \xi_2, \varrho, \vartheta, \lambda)$, in view of Theorem 2 we have

$$\sum_{n=2}^{\infty} \frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_2)]}{\varrho\vartheta(M-L)(1-\xi_2)}\mathcal{B}(n,\ell)a_{n,2} \leq 1. \tag{17}$$

For brevity , we let

$$\Lambda(\varrho,\vartheta,\aleph,n) = (1-B\varrho)(n-1)+\varrho\vartheta(M-L)(n-\aleph),$$

$$\Lambda(\varrho,\vartheta,\xi_1,n) = [(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_1)]$$

and

$$\Lambda(\varrho,\vartheta,\xi_2,n) = [(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_2)].$$

On the other hand, by the Cauchy–Schwarz inequality , we have

$$\sum_{n=2}^{\infty} \frac{(1+n\lambda-\lambda)[\Lambda(\varrho,\vartheta,\xi_1,n)]^{1/2}[\Lambda(\varrho,\vartheta,\xi_1,n)]^{1/2}}{\sqrt{(1-\xi_1)(1-\xi_2)}}\mathcal{B}(n,\ell)\sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{18}$$

From (16) and (17), it follows that

$$\sum_{n=2}^{\infty} \frac{(1+n\lambda-\lambda)^2\Lambda(\varrho,\vartheta,\xi_1,n)\mathcal{B}(n,\ell)\Lambda(\varrho,\vartheta,\xi_2,n)\mathcal{B}(n,\ell)}{[\varrho\vartheta(M-L)]^2(1-\xi_1)(1-\xi_2)}a_{n,1}a_{n,2} \leq 1. \tag{19}$$

Thus, we need to find largest $\aleph$, such that

$$\begin{aligned}&\sum_{n=2}^{\infty} \frac{(1+n\lambda-\lambda)[\Lambda(\varrho,\vartheta,\aleph,n)]}{\varrho\vartheta(M-L)(1-\aleph)}\mathcal{B}(n,\ell)a_{n,1}a_{n,2}\\ \leq\ &\sum_{n=2}^{\infty} \frac{(1+n\lambda-\lambda)[\Lambda(\varrho,\vartheta,\xi_1,n)]^{1/2}[\Lambda(\varrho,\vartheta,\xi_2,n)]^{1/2}\mathcal{B}(n,\ell)}{[\varrho\vartheta(M-L)]\sqrt{(1-\xi_1)(1-\xi_2)}}\sqrt{a_{n,1}a_{n,2}}\end{aligned}$$

or, equivalently, that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{1-\aleph}{\sqrt{(1-\xi_1)(1-\xi_2)}}\frac{[\Lambda(\varrho,\vartheta,\xi_1,n)]^{1/2}[\Lambda(\varrho,\vartheta,\xi_2,n)]^{1/2}}{[\Lambda(\varrho,\vartheta,\aleph,n)]}, \quad n \geq 2.$$

In view of (18), it is sufficient to find the largest $\aleph$, such that

$$\begin{aligned}&\frac{[\varrho\vartheta(M-L)]\sqrt{(1-\xi_1)(1-\xi_2)}}{(1+n\lambda-\lambda)[\Lambda(\varrho,\vartheta,\xi_1,n)]^{1/2}[\Lambda(\varrho,\vartheta,\xi_2,n)]^{1/2}\mathcal{B}(n,\ell)}\\ &\leq \frac{1-\aleph}{\sqrt{(1-\xi_1)(1-\xi_2)}}\frac{[\Lambda(\varrho,\vartheta,\xi_1,n)]^{1/2}[\Lambda(\varrho,\vartheta,\xi_2,n)]^{1/2}}{[\Lambda(\varrho,\vartheta,\aleph,n)]},\end{aligned}$$

which yields

$$\begin{aligned}&\aleph[(1+n\lambda-\lambda)\Lambda(\varrho,\vartheta,\xi_1,n)\Lambda(\varrho,\vartheta,\xi_2,n)\mathcal{B}(n,\ell)-[\varrho\vartheta(M-L)]^2(1-\xi_1)(1-\xi_2)]\\ &\quad\leq (1+n\lambda-\lambda)\Lambda(\varrho,\vartheta,\xi_1,n)\Lambda(\varrho,\vartheta,\xi_2,n)\mathcal{B}(n,\ell)-n[\varrho\vartheta(M-L)]^2(1-\xi_1)(1-\xi_2)\\ &\quad -\varrho\vartheta(n-1)(1-\varrho M)(M-L)(1-\xi_1)(1-\xi_2).\end{aligned}$$

That is,

$$\aleph \leq 1 - \frac{[n[\varrho\vartheta(M-L)]^2 - \varrho\vartheta(1-\varrho M)(M-L) + [\varrho\vartheta(M-L)]^2](1-\xi_1)(1-\xi_2)}{(1+n\lambda-\lambda)\Lambda(\varrho,\vartheta,\xi_1,n)\Lambda(\varrho,\vartheta,\xi_2,n)\mathcal{B}(n,\ell) - [\varrho\vartheta(M-L)]^2(1-\xi_1)(1-\xi_2)}.$$

Let

$$\Phi(n) = \frac{[n[\varrho\vartheta(M-L)]^2 - (n-1)\varrho\vartheta(1-\varrho M)(M-L) + [\varrho\vartheta(M-L)]^2](1-\alpha_1)(1-\alpha_2)}{(1+n\lambda-\lambda)\Lambda(\varrho,\vartheta,\xi_1,n)\Lambda(\varrho,\vartheta,\xi_2,n)\mathcal{B}(n,\ell) - [\varrho\vartheta(M-L)]^2(1-\xi_1)(1-\xi_2)}.$$

Since $\Phi(n)$ is non decreasing function of $n(n \geq 2)$, then we have $\aleph \leq 1 - \Phi(2)$. That is,

$$\aleph \leq 1 - \frac{(1-\xi_1)(1-\xi_2)[1-\varrho M + \varrho\vartheta(M-L)]\varrho\vartheta(M-L)}{(1+\lambda)\Lambda(\varrho,\vartheta,\xi_1,2)\Lambda(\varrho,\vartheta,\alpha_2,2)\mathcal{B}(2,\ell) - [\varrho\vartheta(M-L)]^2(1-\xi_1)(1-\xi_2)}$$

and hence the proof is complete. □

Remark 1. *Fixing $\xi_1 = \alpha = \xi_2$, we have*

$$\aleph \leq 1 - \frac{(1-\alpha)^2[1-\varrho M + \varrho\vartheta(M-L)]\varrho\vartheta(M-L)}{(1+\lambda)[1-\varrho M + \varrho\vartheta(M-L)(2-\alpha)]^2\mathcal{B}(2,\ell) - [\varrho\vartheta(M-L)]^2(1-\alpha)^2}. \tag{20}$$

Theorem 4. *Let the functions $f_j(z)(j = 1,2)$ defined by (12) be in the class $\mathcal{W}_{\mathcal{B}}^{\ell}(L,M,\alpha,\varrho,\vartheta,\lambda)$. Then, the function*

$$h(z) = z - \sum_{n=2}^{\infty}(a_{n,1}^2 + a_{n,2}^2)z^n \tag{21}$$

belongs to the class $\mathcal{W}_{\mathcal{B}}^{\ell}(A,B,\delta,\varrho,\vartheta,\lambda)$, where

$$\delta = 1 - \frac{2(1-\alpha)^2[1-\varrho M + \varrho\vartheta(M-L)]\varrho\vartheta(M-L)}{(1+\lambda)[1-\varrho M + \varrho\vartheta(M-L)(2-\alpha)]^2\mathcal{B}(2,\ell) - 2(1-\alpha)^2[\varrho\vartheta(M-L)]^2}.$$

Proof. By virtue of Theorem 2, it is sufficient to prove that

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M) + \varrho\vartheta(M-L)(n-\delta)]\mathcal{B}(n,\ell)}{\varrho\vartheta(M-L)(1-\delta)}(a_{n,1}^2 + a_{n.2}^2) \leq 1. \tag{22}$$

Since $f_j(z)(j = 1,2) \in \mathcal{W}_{\mathcal{B}}^{\ell}(L,M,\alpha,\varrho,\vartheta,\lambda)$ we have

$$\begin{aligned}\sum_{n=2}^{\infty}&\left\{\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\alpha)]\mathcal{B}(n,\ell)}{\varrho\vartheta(M-L)(1-\alpha)}\right\}^2 a_{n,j}^2 \\ &\leq \sum_{n=2}^{\infty}\left\{\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\alpha)]\mathcal{B}(n,\ell)a_{n,j}}{\varrho\vartheta(M-L)(1-\alpha)}\right\}^2 \leq 1\end{aligned} \tag{23}$$

which yields that

$$\sum_{n=2}^{\infty}\frac{1}{2}\left\{\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M) + \varrho\vartheta(M-L)(n-\alpha)]\mathcal{B}(n,\ell)}{\varrho\vartheta(M-L)(1-\alpha)}\right\}^2(a_{n,1}^2 + a_{n,2}) \leq 1. \tag{24}$$

Therefore, we need to find the largest δ, such that

$$\begin{aligned}&\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M) + \varrho\vartheta(M-L)(n-\delta)]\mathcal{B}(n,\ell)}{\varrho\vartheta(M-L)(1-\delta)} \\ &\qquad \leq \frac{1}{2}\left[\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M) + \varrho\vartheta(M-L)(n-\alpha)]\mathcal{B}(n,\ell)}{\varrho\vartheta(M-L)(1-\alpha)}\right]^2, \quad n \geq 2.\end{aligned}$$

That is

$$\delta \leq 1 - \frac{2(n-1)(1-\varrho M)\varrho\vartheta(M-L)(1-\alpha)^2 - 2n[\varrho\vartheta(M-L)]^2(1-\alpha)^2 + 2[\varrho\vartheta(M-L)]^2(1-\alpha)^2}{(1+n\lambda-\lambda)[(n-1)(1-\varrho B)+\varrho\vartheta(M-L)(n-\delta)]^2\mathcal{B}(n,\ell) - 2[\varrho\vartheta(M-L)]^2(1-\alpha)^2}.$$

Since

$$\Psi(n) = 1 - \frac{2(n-1)(1-\varrho M)\varrho\vartheta(M-L)(1-\alpha)^2 - 2n[\varrho\vartheta(M-L)]^2(1-\alpha)^2 + 2[\varrho\vartheta(M-L)]^2(1-\alpha)^2}{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\delta)]^2\mathcal{B}(n,\ell) - 2[\varrho\vartheta(M-L)]^2(1-\alpha)^2}$$

is an increasing function of n, $(n \geq 2)$, we readily have

$$\delta \leq \Psi(2) = 1 - \frac{2(1-\alpha)^2[1-\varrho M+\varrho\vartheta(M-L)]\varrho\vartheta(M-L)}{(1+\lambda)[1-\varrho M+\varrho\vartheta(M-L)(2-\alpha)]^2\mathcal{B}(2,\ell) - 2(1-\alpha)^2[\varrho\vartheta(M-L)]^2}, \tag{25}$$

which completes the proof. □

Hölder-Type Inequalities

Recently, Nishiwaki, Owa, and Srivastava [36] have given some results of Hölder-type inequalities for a subclass of uniformly starlike functions. Lately, Choi, Kim, and Owa in [37] gave the following generalized convolution as

$$H_m(z) = z - \sum_{n=2}^{\infty}\left(\prod_{j=1}^{m} a_{n,j}^{p_j}\right)z^n \quad (p_j > 0, j = 1,2,\ldots,m) \tag{26}$$

For functions $f_j(z) \in \mathcal{W}_{\mathcal{B}}^{\ell}(L,M,\alpha,\varrho,\vartheta,\lambda)(j = 1,2,\ldots,m)$ given by (13), the familiar Hölder inequality assumes the following form

$$\sum_{n=2}^{\infty}\left(\prod_{j=1}^{m} a_{n,j}\right) \leq \prod_{j=1}^{m}\left(\sum_{n=2}^{\infty} a_{n,j}^{p_j}\right)^{\frac{1}{p_j}} \quad (p_j > 1, j = 1,2,\ldots,m; \sum_{j=i}^{m}\frac{1}{p_j} \geq 1). \tag{27}$$

Our first result for the generalized convolution $H_m(z)$ defined by (26) is contained in the theorem below.

Theorem 5 (Hölder's Inequality). *If $f_j(z) \in \mathcal{W}_{\mathcal{B}}^{\ell}(A,B,\xi_j,\varrho,\vartheta,\lambda), -1 \leq B < A \leq 1, 0 < \varrho \leq 1, 0 \leq \lambda \leq 1,\ j = 1,2,\ldots,m$, then $H_m(z) \in \mathcal{W}_{\mathcal{B}}^{\ell}(A,B,\xi,\varrho,\vartheta,\lambda)$ with*

$$\xi \leq 1 - \frac{\prod_{j=i}^{m}(1-\xi_j)^{p_j} - [(1-\varrho B)+\varrho\vartheta(M-L)][\varrho\vartheta(M-L)]^s}{(1+\lambda)^{s-1}\mathcal{B}(n,\ell)^{s-1}\prod_{j=i}^{m}[1-\varrho M+\varrho\vartheta(M-L)(2-\xi_j)]^{p_j} - [\varrho\vartheta(M-L)]^s\prod_{j=i}^{m}(1-\xi_j)^{p_j}}.$$

where $S = \sum_{j=i}^{m} p_j \geq 1; p_j \geq \frac{1}{q_j}(j = 1,2,\ldots,m), q_j > 1(j = 1,2,\ldots,m); \sum_{j=i}^{m}\frac{1}{q_j} \geq 1.$

Proof. For $f_j(z) \in \mathcal{W}_{\mathcal{B}}^{\ell}(A,B,\xi_j,\varrho,\vartheta,\lambda)(j = 1,2,\ldots,m)$, Theorem 2 gives us that

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)]}{\varrho\vartheta(M-L)(1-\xi_j)}\mathcal{B}(n,\ell)a_{n,j} \leq 1$$

which in turn implies

$$\left(\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)]}{\varrho\vartheta(M-L)(1-\xi_j)}\mathcal{B}(n,\ell)a_{n,j}\right)^{\frac{1}{q_j}} \leq 1$$

$$\textit{with} \quad q_j > 1 \quad (j = 1,2,\ldots,m); \sum_{j=i}^{m}\frac{1}{q_j} = 1.$$

By applying the Hölder inequality (27), we have

$$\sum_{n=2}^{\infty}\left[\prod_{j=i}^{m}\left(\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)]}{\varrho\vartheta(M-L)(1-\xi_j)}\mathcal{B}(n,\ell)\right)^{\frac{1}{q_j}}a_{n,j}^{\frac{1}{q_j}}\right]\leq 1.$$

Thus,

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi)]}{\varrho\vartheta(M-L)(1-\xi)}\mathcal{B}(n,\ell)\left(\prod_{j=1}^{m}a_{n,j}^{p_j}\right)\leq 1.$$

That is,

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi)]}{\varrho\vartheta(M-L)(1-\xi)}\mathcal{B}(n,\ell)\left(\prod_{j=1}^{m}a_{n,j}^{p_j}\right)\\ \leq\;&\sum_{n=2}^{\infty}\left[\prod_{j=i}^{m}\left(\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)]}{\varrho\vartheta(M-L)(1-\xi_j)}\mathcal{B}(n,\ell)\right)^{\frac{1}{q_j}}a_{n,j}^{\frac{1}{q_j}}\right].\end{aligned}$$

Note that we have to find the largest ξ, such that

$$\begin{aligned}&\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi)]}{\varrho\vartheta(M-L)(1-\xi)}\mathcal{B}(n,\ell)\left(\prod_{j=1}^{m}a_{n,j}^{p_j}\right)\\ \leq\;&\prod_{j=i}^{m}\left(\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)]}{\varrho\vartheta(M-L)(1-\xi_j)}\mathcal{B}(n,\ell)\right)^{\frac{1}{q_j}}a_{n,j}^{\frac{1}{q_j}}\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi)]}{\varrho\vartheta(M-L)(1-\xi)}\mathcal{B}(n,\ell)\left(\prod_{j=1}^{m}a_{n,j}^{p_j-\frac{1}{q_j}}\right)\\ \leq\;&\prod_{j=i}^{m}\left(\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)]}{\varrho\vartheta(M-L)(1-\xi_j)}\mathcal{B}(n,\ell)\right)^{\frac{1}{q_j}},\quad for\quad all\quad n\geq 2.\end{aligned}$$

Since,

$$\prod_{j=i}^{m}\left(\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)]}{\varrho\vartheta(M-L)(1-\xi_j)}\mathcal{B}(n,\ell)\right)^{p_j-\frac{1}{q_j}}a_{n,j}^{p_j-\frac{1}{q_j}}\leq 1$$

$(p_j-\frac{1}{q_j}\geq 0, j=1,2,\ldots,m)$ we see that,

$$\prod_{j=i}^{m}a_{n,j}^{p_j-\frac{1}{q_j}}\leq\frac{1}{\prod_{j=i}^{m}\left(\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)]}{\varrho\vartheta(M-L)(1-\xi_j)}\mathcal{B}(n,\ell)\right)^{p_j-\frac{1}{q_j}}}.\tag{28}$$

This last inequality (28) implies that

$$\begin{aligned}&\frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi)]}{\varrho\vartheta(M-L)(1-\xi)}\mathcal{B}(n,\ell)\\ &\qquad\leq\frac{\prod_{j=i}^{m}(1+n\lambda-\lambda)^{p_j}[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)\mathcal{B}(n,\ell)]^{p_j}}{\prod_{j=i}^{m}[\varrho\vartheta(M-L)(1-\xi_j)]^{p_j}}\end{aligned}$$

which implies

$$[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi)]\prod_{j=i}^{m}[\varrho\vartheta(M-L)]^{p_j-1}(1-\xi_j)^{p_j}$$
$$\leq \prod_{j=i}^{m}(1+n\lambda-\lambda)^{p_j-1}\mathcal{B}(n,\ell)^{p_j-1}[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)]^{p_j}(1-\xi_j)^{p_j}.$$

Therefore, ξ should be

$$\xi \leq 1-\frac{n\Upsilon_j+\Upsilon_j-(n-1)(1-\varrho B)\prod_{j=i}^{m}[\varrho\vartheta(M-L)]^{p_j-1}(1-\xi_j)^{p_j}}{\prod_{j=i}^{m}(1+n\lambda-\lambda)^{p_j-1}\mathcal{B}(n,\ell)^{p_j-1}[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)]^{p_j}-\Upsilon_j}$$

and

$$\Upsilon_j=\prod_{j=i}^{m}[\varrho\vartheta(M-L)]^{p_j}(1-\xi_j)^{p_j}.$$

Let

$$\Phi(n) \leq 1-\frac{n\Upsilon_j+\Upsilon_j-(n-1)(1-\varrho B)\prod_{j=i}^{m}[\varrho\vartheta(M-L)]^{p_j-1}(1-\xi_j)^{p_j}}{\prod_{j=i}^{m}(1+n\lambda-\lambda)^{p_j-1}\mathcal{B}(n,\ell)^{p_j-1}[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\xi_j)]^{p_j}-\Upsilon_j}$$

which is an increasing function of n; hence, we have

$$\xi \leq \Phi(2)=1-\frac{\prod_{j=i}^{m}(1-\xi_j)^{p_j}-[(1-\varrho M)+\varrho\vartheta(M-L)][\varrho\vartheta(M-L)]^{s}}{(1+\lambda)^{s-1}\mathcal{B}(n,\ell)^{s-1}\prod_{j=i}^{m}[1-\varrho M+\varrho\vartheta(M-L)(2-\xi_j)]^{p_j}-[\varrho\vartheta(M-L)]^{s}\prod_{j=i}^{m}(1-\xi_j)^{p_j}}.$$

This completes the proof of the theorem. □

4. Closure Properties under Integral Transform

Fournier and Ruscheweyh in [38] introduced the operator for a function $f \in \mathcal{S}$ and defined the integral transform

$$\mathcal{V}_\eta(f)(z)=\int_0^1 \eta(t)\frac{f(tz)}{t}dt,$$

where $\eta(t)$ a non-negative real-valued integrable function(weight function) satisfying the normalizing condition $\int_0^1 \eta(t)dt = 1$. Interestingly, the general integral transform $\mathcal{V}_\eta(f)(z)$ reduces to various well-known integral operators for specific choices of $\eta(t)$. For example, fixing

$$\eta(t)=(1+c)t^c, \quad c>-1,$$

$\mathcal{V}_\eta$ gives the Bernardi operator [39].
While taking

$$\eta(t)=\frac{(c+1)^\delta}{\Gamma(\delta)}t^c\left(log\frac{1}{t}\right)^{\delta-1}, \quad (c>-1;\delta\geq 0)$$

$\mathcal{V}_\eta$ gives Komatu operator (for more details, see [40]).
By definition, we have

$$\mathcal{V}_\eta(f)(z)=\frac{(c+1)^\delta}{\Gamma(\delta)}\int_0^1(-1)^{\delta-1}t^c(logt)^{\delta-1}\left[z-\sum_{n=2}^{\infty}a_nz^nt^{n-1}\right]dt$$

$$\mathcal{V}_\eta(f)(z)=\frac{(-1)^{\delta-1}(c+1)^\delta}{\Gamma(\delta)}\lim_{r\to o^+}\left[\int_r^1 t^c(logt)^{\delta-1}\left[z-\sum_{n=2}^{\infty}a_nz^nt^{n-1}\right]dt\right].$$

A simple calculation gives

$$\mathcal{V}_\eta(f)(z) = z - \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} a_n z^n.$$

We see that, for $\delta = 1$, the Komatu operator reduces to the Bernardi operator. In this section, we discuss the closure properties for $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$.

Theorem 6. *Let $f(z) \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$. Then, $\mathcal{V}_\eta(f)(z) \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$.*

Proof. We need to prove that

$$\sum_{n=2}^{\infty} \frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M) + \varrho\vartheta(M-L)(n-\alpha)]}{\varrho\vartheta(M-L)(1-\alpha)} \left(\frac{c+1}{c+n}\right)^{\delta} \mathcal{B}(n,\ell) a_n \leq 1. \tag{29}$$

On the other hand, by Theorem 2, $f(z) \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$, if and only if

$$\sum_{n=2}^{\infty} \frac{(1+n\lambda-\lambda)[(n-1)(1-\varrho M) + \varrho\vartheta(M-L)(n-\alpha)]}{\varrho\vartheta(M-L)(1-\alpha)} \mathcal{B}(n,\ell) a_n \leq 1.$$

Hence, $\frac{c+1}{c+n} < 1$. Therefore, (29) holds and the proof is complete. □

The above theorem yields the following two results:

Theorem 7.

(1) *If $f(z)$ is starlike of order ϑ, then $\mathcal{V}_\eta f(z)$ is also starlike of order α.*
(2) *If $f(z)$ is convex of order ϑ, then $\mathcal{V}_\eta f(z)$ is also convex of order α.*

Theorem 8. *Let $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$. Then, $\mathcal{V}_\eta f(z)$ is starlike of order $0 \leq \zeta < 1$ in $|z| < R_1$, that is, $\mathcal{R}\left(\frac{zf'(z)}{f(z)}\right) > \zeta \quad ,(|z| < R_1; 0 \leq \zeta < 1)$, where*

$$R_1 = \inf_{n \geq 2}\left\{\frac{(1-\zeta)\mathcal{M}_n}{(n-\zeta)(M-L)(1-\alpha)\varrho\vartheta}\right\}^{\frac{1}{n-1}} \tag{30}$$

and $\mathcal{M}_n$ is given by (10).

Proof. For $0 \leq \zeta < 1$, we have to prove that

$$\left|\frac{z(\mathcal{V}_\eta f(z))'}{\mathcal{V}_\eta f(z)} - 1\right| < 1 - \zeta \quad , \quad |z| < R_1,$$

where R_1 is given by (30). Thus, we readily obtain

$$\begin{aligned}\left|\frac{z(\mathcal{V}_\eta f(z))'}{\mathcal{V}_\eta f(z)} - 1\right| &= \left|\frac{-\sum_{n=2}^{\infty}(n-1)\left(\frac{c+1}{c+n}\right)^{\delta} a_n z^{n-1}}{1 - \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} a_n z^{n-1}}\right| \\ &\leq \frac{\sum_{n=2}^{\infty}(n-1)\left(\frac{c+1}{c+n}\right)^{\delta} a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} a_n |z|^{n-1}}\end{aligned}$$

which is less than $1 - \zeta$ if

$$\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta}\left(\frac{n-\zeta}{1-\zeta}\right) a_n |z|^{n-1} \leq 1. \tag{31}$$

However, $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$, if and only if (by Theorem 2),

$$\sum_{n=2}^{\infty} \frac{\mathcal{M}_n}{(M-L)(1-\alpha)\varrho\vartheta} a_n \leq 1. \tag{32}$$

Relation (31) holds if:

$$\left(\frac{c+1}{c+n}\right)^{\delta}\left(\frac{n-\zeta}{1-\zeta}\right)|z|^{n-1} \leq \frac{\mathcal{M}_n}{(M-L)(1-\alpha)\varrho\vartheta}$$

equivalently,

$$|z| \leq \left\{\left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\zeta)\mathcal{M}_n}{(n-\zeta)(M-L)(1-\alpha)\varrho\vartheta}\right\}^{\frac{1}{n-1}}$$

which yields the starlikeness of the family. □

Using the fact that f is convex if and only if zf' is starlike, we state the following theorem without proof.

Theorem 9. *Let $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$. Then, $\mathcal{V}_\eta f(z)$ is convex of order $0 \leq \zeta < 1$ in $|z| < R_2$, that is, $\mathcal{R}\left(\frac{(zf'(z))'}{f'(z)}\right) > \zeta \quad ,(|z| < R_2; 0 \leq \zeta < 1)$, where*

$$R_2 = \inf_{n \geq 2}\left\{\left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\zeta)\mathcal{M}_n}{n(n-\zeta)(M-L)(1-\alpha)\varrho\vartheta}\right\}^{\frac{1}{n-1}}. \tag{33}$$

Proof. It is sufficient to prove

$$\left|\frac{z(\mathcal{V}_\eta f(z))''}{(\mathcal{V}_\eta f(z))'}\right| < 1-\zeta \quad for \quad |z| < R_2,$$

where R_2 is given by (33).

Thus, we obtain

$$\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta}\left(\frac{n(n-\zeta)}{1-\zeta}\right) a_n |z|^{n-1} \leq 1.$$

Proof follows on lines similar to the above theorem; hence, we omit it. □

5. Subordination Results

For two analytic functions $f_1, f_2 \in \mathcal{A}$, we say that f_1 is subordinate to f_2, and we denote by $f_1 \prec f_2$, if there is a Schwarz function $\omega(z)$ which is analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$, for all $z \in \mathbb{U}$, such that $f_1(z) = f_2(\omega(z))$ for $z \in \mathbb{U}$. Note that, if the function f_2 is univalent in $\mathbb{U}$, due to Miller and Mocanu [41] (see [42]), we have

$$f_1(z) \prec f_2(z) \Longleftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\mathbb{U}) \subset f_2(\mathbb{U}).$$

Now, we recall the following results due to Wilf [43], which are much more useful in the sequel.

Definition 2 (Subordinating Factor Sequence)**.** *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence whenever f is analytic, univalent, and convex in $\mathbb{U}$. We have the subordination given by*

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), \; a_1 = 1, z \in \mathbb{U}. \tag{34}$$

Lemma 1. *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\mathcal{R}\left\{1+2\sum_{n=1}^{\infty} b_n z^n\right\} > 0, \quad z \in \mathbb{U}. \tag{35}$$

Theorem 10. *Let $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$ and $g \in \mathcal{C}$ the class of convex functions, then*

$$\frac{\mathcal{M}_2}{2[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}(f*g)(z) \prec g(z) \tag{36}$$

where

$$\mathcal{M}_2 = (1+\lambda)[(1-\varrho M)+\varrho\vartheta(M-L)(2-\alpha)]\mathcal{B}(2,\ell) \tag{37}$$

and

$$\mathcal{R}\{f(z)\} > -\frac{[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}{\mathcal{M}_2}, \quad z \in \mathbb{U}. \tag{38}$$

The constant factor $\frac{\mathcal{M}_2}{2[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}$ in (36) cannot be replaced by a larger number.

Proof. Let $f \in \mathcal{W}_{\mathcal{B}}^{\ell}(L, M, \alpha, \varrho, \vartheta, \lambda)$ and suppose that $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n \in \mathcal{C}$. Then,

$$\begin{aligned}&\frac{\mathcal{M}_2}{2[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}(f*g)(z)\\ &\qquad = \frac{\mathcal{M}_2}{2[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}\left(z+\sum_{n=2}^{\infty} b_n a_n z^n\right).\end{aligned}$$

Thus, by Definition 2, the subordination result holds true if

$$\left\{\frac{\mathcal{M}_2}{2[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality

$$\mathcal{R}\left\{1+\sum_{n=1}^{\infty}\frac{\mathcal{M}_2}{[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}a_n z^n\right\} > 0, \quad z \in \mathbb{U}. \tag{39}$$

By noting the fact that $\frac{\mathcal{M}_n}{((1-\alpha)(M-L)\varrho)}$ is increasing function for $n \geq 2$ and in particular

$$\frac{\mathcal{M}_2}{(1-\alpha)(M-L)\varrho\vartheta} \leq \frac{\mathcal{M}_n}{(1-\alpha)(M-L)\varrho\vartheta}, \quad n \geq 2,$$

therefore, for $|z| = r < 1$, we have

$$\begin{aligned}&\mathcal{R}\left\{1+\frac{\mathcal{M}_2}{[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}\sum_{n=1}^{\infty} a_n z^n\right\}\\ &= \mathcal{R}\left\{1+\frac{\mathcal{M}_2}{[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}z+\frac{\sum\limits_{n=2}^{\infty}\mathcal{M}_2 a_n z^n}{[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}\right\}\\ &\geq 1-\frac{\mathcal{M}_2}{[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}r-\frac{\sum\limits_{n=2}^{\infty}|\mathcal{M}_n a_n| r^n}{[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}\\ &> 0, \quad |z| = r < 1.\end{aligned}$$

Notice that the last but one inequality follows from the fact that

$$\mathcal{B}(2,\ell)\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(n-1)(1-\varrho M)+\varrho\vartheta(M-L)(n-\alpha)],$$

is an increasing function in n, $(n\geq 2)$. Thus, (39) holds true for $z\in\mathbb{U}$. This proves the inequality (36). The equality (38) follows by taking the convex function

$$g(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n\in\mathcal{C}$$

in (36). Next, to prove the sharpness of the constant $\frac{\mathcal{M}_2}{2[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}$.

We consider the function $\mathcal{F}_2\in\mathcal{W}_{\mathcal{B}}^{\ell}(L,M,\alpha,\varrho,\vartheta,\lambda)$, given by

$$\mathcal{F}_2(z):=z-\frac{(1-\alpha)(M-L)\varrho}{\mathcal{M}_2}z^2$$

where $\mathcal{M}_2$ is given by (37). For this $\mathcal{F}_2(z)$, (36) becomes

$$\frac{\mathcal{M}_2}{2[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}\mathcal{F}_2(z)\prec\frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\mathcal{R}\left(\frac{\mathcal{M}_2}{2[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}\mathcal{F}_2(z)\right)\right\}=-\frac{1}{2},\quad z\in\mathbb{U}.$$

This shows that the constant $\frac{\mathcal{M}_2}{2[(1-\alpha)(M-L)\varrho+\mathcal{M}_2]}$ is the best possible scenario. □

6. Conclusions

The study presented in this paper followed the line of research which introduces new classes of univalent functions based on the well-known Borel series. Then, for this newly defined function class, we presented the results of the studies carried out on coefficient estimates, the modified Hadamard product, Hölder inequality results, closure properties, and subordination results. Furthermore, we believe that this study will motivate a number of researchers to extend this idea for meromorphic functions, associated with q-calculus and (p,q)-calculus (see [44–46]), also based on Borel distributions with special functions [30,31,47]. Moreover, one can consider a function class $\mathcal{A}_p$, comprising functions of the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\qquad(p\in\mathbb{N}=\{1,2,3,\ldots\})$$

which are commonly known as $p-$ the valent or multivalent functions of order p if $f(z)=w$ has at most $p-$roots in $\mathbb{U}$ and at least one value of the function is taken exactly p times. We discuss the above results given in Theorems 1–10 . Further, by specializing the parameters, our new subclass $\mathcal{W}_{\mathcal{B}}^{\ell}(L,M,\alpha,\varrho,\vartheta,\lambda)$ yields many subclasses of analytic functions which have not been studied yet in association with Borel distributions. It also consists of many subclass analogues to classes studied [7,22,27,33,34] (see references cited therein) in association with Borel distributions.

Author Contributions: The authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are grateful to the referees of this article who gave insightful comments and advice that allowed us to revise and improve the content of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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