

Article

The Statistical Estimation Averaging Method to Express the Effective Electromagnetic Parameters over a Planar Information Meta-Surface

Ting-Ting Gu ^{1,2,*}  and Tie-Jun Cui ^{3,4,*} 

¹ Department of Information Science and Electronic Engineering (ISEE), Zhejiang University, Hangzhou 310027, China

² State Key Laboratory of CAD&CG, Zhejiang University, Hangzhou 310058, China

³ Institute of Electromagnetic Space, Southeast University, Nanjing 210096, China

⁴ State Key Laboratory of Millimeter Waves, Southeast University, Nanjing 210096, China

* Correspondence: ttgu@zju.edu.cn (T.-T.G.); tjcui@seu.edu.cn (T.-J.C.)

† These authors contributed equally to this work.

Abstract: The electromagnetic scattering of a meta-surface, formed by a honeycomb substrate or periodical arranged meta-materials, has great meanings to communication technologies. In a conventional treatment to investigate the properties of these interfaces, either a variational approach or series expansions had been applied, instead of estimating the effective permittivity or permeability by the mean value of its spatial discretized statistics due to spatiotemporal fusion. Following this, this study has re-examined the problem by the Statistic Estimation Averaging method (SEAm), through the transferred conditional structural Probability Density Functions (PDFs) to realize the structural determinations by homogenization. The parameters estimated by SEAm, which exploited the concept of a homogenized medium to express properties of a structural complex medium, has been verified of validity and accuracy by comparing with the measured results of a honeycomb structure. The method can be extended to estimate the parameters of an equivalent surface, such as randomly scattering from information metamaterials. As a new wireless communication relay technology, considering that information metamaterials can modulate the electromagnetic characteristics of communication links and wireless channels simultaneously by means of spatiotemporal sequence coding, the study also gives a preliminary proposition on state estimation method of information meta-surface, which would interpret the modulation effect of wireless channels caused by its inhomogeneity of antenna wavefront by statistical estimation average information entropy.

Keywords: medium equivalent theory; estimation theory; SEAm; DMIA; electromagnetic scattering; wireless channel

MSC: 78A48



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1. Introduction

Recent progresses have been made on microwave wireless communication systems and radar technology by using digital meta-surfaces to realize space- and frequency-division multiplexing and smart Doppler cloaking [1,2]. With new features on intelligent beam control and simplified architecture, the time-domain digital coding meta-surface provides advantages of flexibility and adaptability in its reconfigurable radiating arrays controlled by programmable digital signals [3–5]. Yet, the investigations on the advanced devices are mostly developed on the designs of apparatuses for suiting their new wireless communication systems, which seems to be absent of theoretical knowledge on its effective modulation effect as its distinction to a conventional transmission network on wireless channels.

In a traditional view of field analysis, the scattering of a passive meta-surface array is similar to that of a complex structure material; some physical properties of whom presented anisotropic characteristics due to the periodical arrangement of units in cross section. Analytical techniques to model these periodical structures, arrays and meshes had been previously summarized in the book [6], including the field matching approach with Floquet series expansion, which had been applied to study the electromagnetic properties of periodical structure materials, such as honeycomb substrate and foam-based honeycomb sandwich structures [7,8].

As one of the most useful techniques to find the effective permittivity and permeability of such structural complex materials, the idea of homogenization had been established and its application in designing of millimeter-wave electromagnetic absorbers has developed for decades [9–12]. The original prototype was made by a variational approach in the study on the effective magnetic permeability of multiphase materials [13]. Later, this was developed into a generalized approach to establish multiphase dielectric mixture theory, which is derived from some dielectric mixture equations modified from the Landau–Lifshitz formula [14–16]. These early derivations had been addressed in the book for electromagnetic mixing formulas and applications [17]. Afterwards, the dielectric mixture equations have been applied for scenarios of porous materials and particles scattering [18,19].

In the present study, a new dielectric mixing equation is to be derived for a structural complex material to express the honeycomb-structured dielectric cross-section from a statistic estimation point of view. In mathematical formulation of the effective parameters, we presume a two-port network can be applied to represent the transmission and reflection of any meta-surface including random surfaces and some uncertain-state meta-surface or array. Undoubtedly, averaged statistical estimating parameters cannot yet be qualified to determine the scattering of an arbitrary-arranged meta-material array or a scattering wall modulated by irregularly excited signals; however, the method indeed has potentials in meta-surface array multi-hierarchy fast design, instead of integrated computation of large-scale array in wireless relay communication, which is not only used to produce multiple physical performances, but also to assess the information metamaterials on channel quality improvement due to the uncertain-state meta-elements on it. In these information meta-material systems, each meta-element is to be controlled by different time digital-coding sequence, by which it has have been realized from prototypes to implementation recently [20–23].

Encouraged by that, the method of SEA is proposed to study the effective electromagnetic parameters in two complete subspaces from a complex structural material defined by structural PDFs based on spatial discretization, which are expressed by the unknown priori probability distributions. We assume that the true estimators of permittivity from two independent subspaces share the same value. Considering of a honeycomb-structured meta-surface, the electromagnetic scattering of it can be interpreted by a homogenized medium if the operated frequencies are at subwavelength band. Thus, the equivalent medium is subjected to a posterior probability distribution with an uniformed distribution of PDF. Based on the law of conservation of statistical particles, the estimators can be evaluated according to the conditional probabilities, respectively.

The applied scenarios of SEAm are summarized as parameter estimation and state estimation, respectively. In the latter case, the SEAm is extended to predict the average Power Spectral Density (PSD) of an information meta-material where the structural PDFs are defined by spatial discretization of wave functions especially for a Digital-coding Meta-material Information-encrypted Antenna (DMIA). The digital-coding feeding elements are separated as electrically and magnetically excitation arrays due to different near-field radiation principles [24,25]. The aim is to estimate the average states of a PSD considering each excitation mode, respectively.

In this work, the computational analyses are given to express the effective permittivity of a honeycomb-structured meta-surface, which is compared to the values obtained by

conventional approaches, including Weighted Average method (WAm), Hashin–Shtrikman Variational formula (HSV), and Strong Fluctuation Theory (SFT). The result is validated by an experiment of aramid paper through Radar Cross-Section (RCS) performances. Other approach such as Full-Space Retrieval methods (FSRm) by reflection and transmission coefficients, transmission line matrix, etc., is not expanded upon herein [26–28].

2. Formulation

2.1. Spatio-Temporal Discretization of Complex Materials

Periodical arrangements of planar conductive elements of various shapes have much uses in microwave technology, as well as the structure of some artificial materials such as honeycomb to design frequency selective performances. To consider them, we will start with the spatial discretization of these materials in cross-section (lying on the *x-y* plane), including inhomogeneous medium, random medium and mixed materials, respectively.

- **Averaged material parameters**

For a inhomogeneous medium, the averaged parameters can be approximated by the mean values of permittivity and permeability in statistics by spatial differentiation, which are determined by the distribution function $f_{mn}(x, y)$, respectively. In a rectangular coordinate, the averaged permittivity is

$$\bar{\epsilon} = \lim_{M,N \rightarrow \infty} \sum_{m=-M}^M \sum_{n=-N}^N f_{mn}^{XY} \cdot \epsilon^{ref}, \tag{1}$$

in which ϵ^{ref} is the reference parameter of permittivity form the origin coordinates. For a random medium, the weighting function $f_{mn}(x_0, y_0, t)$ varies by time at any certain point.

- **Mixed materials**

For a mixture by two different materials, the averaged permittivity value can be approximated by:

$$\epsilon^{mix} = \lim_{M,N \rightarrow \infty} \sum_{m=-M}^M \sum_{n=-N}^N [f_{mn}^a + f_{mn}^b \frac{\epsilon_b}{\epsilon_a}] \epsilon_a \tag{2}$$

with two different weighting coefficient functions f_{mn}^a and f_{mn}^b representing for the filling material (ϵ_a or ϵ_0 for air) and the skeleton material (ϵ_b) on the planar surface with periodical-arranged conductive elements, respectively.

- **Parameter Estimation**

Seen as a two-port network, the properties of electromagnetic scattering from a honeycomb-structured cross-section have been analyzed [7,8]. The reflection and transmission of electromagnetic field at the boundary of free-space to honeycomb-structured cross-section present anisotropic properties due to the discontinuity of the material parameters. Thus, the effective permittivity of this meta-surface should be considered as a 2×2 matrix ($\bar{\epsilon}_T$). In a rectangular coordinate system, it can be written as:

$$\bar{\epsilon}_T = \begin{bmatrix} \hat{\epsilon}_x & \\ & \hat{\epsilon}_y \end{bmatrix}, \tag{3}$$

in which the diagonal elements of the permittivity $\hat{\epsilon}_x$ and $\hat{\epsilon}_y$ are the parameters to be estimated.

Next, we consider the spatial expansion of the dielectric permittivity. With $\hat{\epsilon}_x = \hat{\epsilon}_y$ being valid for honeycomb-structured cross-section, the element of $\bar{\epsilon}_T$ can be express in the form of

$$\hat{\epsilon}_x \approx (1, 1, \dots, 1, \dots, 1, 1) \times \begin{bmatrix} f_{m,-n}^{XY}\{\epsilon\}_{m,-n} & \dots & f_{m,0}^{XY}\{\epsilon\}_{m,0} & \dots & f_{m,n}^{XY}\{\epsilon\}_{m,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ f_{0,-n}^{XY}\{\epsilon\}_{0,-n} & \dots & f_{0,0}^{XY}\{\epsilon\}_{0,0} & \dots & f_{0,n}^{XY}\{\epsilon\}_{0,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ f_{-m,-n}^{XY}\{\epsilon\}_{-m,-n} & \dots & f_{-m,0}^{XY}\{\epsilon\}_{-m,0} & \dots & f_{-m,n}^{XY}\{\epsilon\}_{-m,n} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad (4)$$

in which a basis function $f_{m,n}^{XY}$ is used to express the influences of height gain (caused by curvature change) and fluctuations in different locations (uncertainties of random medium), and the symbols $\{\epsilon\}_{i,j}$ ($i = -m, m; j = -n, n$, when m and n are large) are statistics for permittivity.

2.2. Structural PDF

From Equation (4), the value of $\hat{\epsilon}_x$ is given by spatial expansion of permittivity for meta-surface. This is valid for some regular structural cross-sections, which have explicit formulation of weighting functions $f_{i,j}^{XY}$ from spatial distribution. Then, we may have the effective parameters expressed by the use of basis functions f_i^X and f_j^Y , termed by **marginal structural PDFs** when Equation (4) can be derived into the following expression, written as:

$$\hat{\epsilon}_x = (f_{-m}^X, f_{1-m}^X, \dots, f_0^X, \dots, f_{m-1}^X, f_m^X) \times \begin{bmatrix} \{\epsilon\}_{m,-n} & \dots & \{\epsilon\}_{m,0} & \dots & \{\epsilon\}_{m,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\epsilon\}_{0,-n} & \dots & \{\epsilon\}_{0,0} & \dots & \{\epsilon\}_{0,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\epsilon\}_{-m,-n} & \dots & \{\epsilon\}_{-m,0} & \dots & \{\epsilon\}_{-m,n} \end{bmatrix} \begin{bmatrix} f_m^Y \\ \vdots \\ f_0^Y \\ \vdots \\ f_m^Y \end{bmatrix} \quad (5)$$

in which the basis functions f_i^X and f_j^Y are subjected to different distributions (f_X and f_Y). Thus, we may also define the basis function $f_{i,j}^{XY}$ which is determined by the **joint structural PDF** (f_{XY}). So that the value of $\hat{\epsilon}_x$ can be estimated by random variable ϵ and the PDFs, for which it is not only related to the averaged value of permittivity, but also related to the expectation of permittivity influenced by spatial distributions. This is also applied to the parameter estimation of permeability.

2.3. The Method of SEA for Effective Dielectric Parameters

The effective permittivity of a honeycomb-structured cross-section (as shown in Figure 1a) is to be derived. We assume that the electromagnetic scattering property of the cross-section is equivalent to the scattering of a dielectric surface at sub-wavelength. Let the effective permittivity be written as the estimation parameter ($\hat{\epsilon}_x$). In the statistical estimation theory, the parameter can be estimated by the averaged mean value of materials forming the dielectric surface. To consider the presented cross-section by a real honeycomb-structured material, we assume that the cross-section can be physically divided into two parts being composed of discrete elements by materials (A and B) with the permittivities ϵ_a and ϵ_b , respectively.

- (a) Statistical Permittivities in Two Subspaces

A priori structural PDF is considered to express the complex structural cross-section of a honeycomb-structured material. Through spatial discretization, the statistical permittivities in transverse direction are chosen in value of ϵ_a or ϵ_b ; therefore, the physical space can be divided into two subspaces A and B (as shown in Figure 1b). Since the physical structure is fixed, the values of statistics in each subspace are ϵ_a and ϵ_b , representing for air and skeleton material, respectively. These two groups of statistics for permittivity constitute an

ensemble statistical group in evaluation of the effective permittivity of the cross-section. The estimator is approximated by $\hat{\epsilon}_x = \hat{\epsilon}_y$ and $\mu_x = \mu_y = \mu_0$ considering it as isotropic and nonmagnetic.

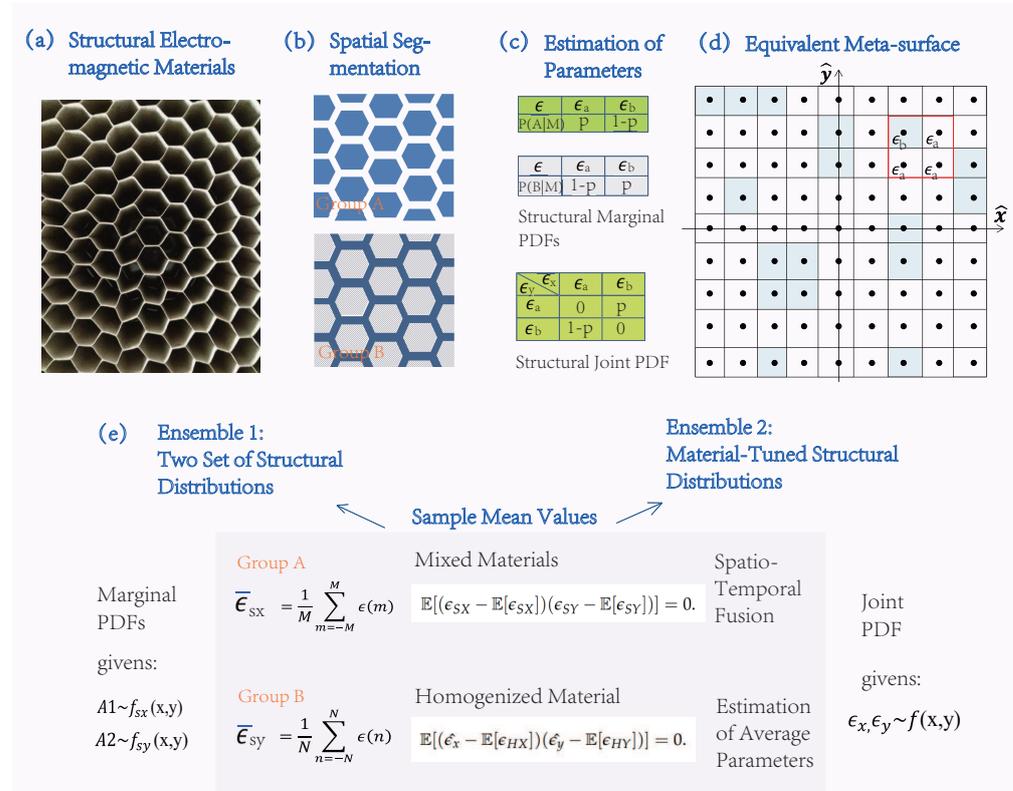


Figure 1. The method of SEA to estimate the effective permittivity of a honeycomb-structured meta-surface: (a) the cross-section of aramid paper sample with slightly distortion; (b) modeling of materials in two subspaces; (c) the structural PDFs; (d) the digital-coding information meta-surface; (e) parameter estimation.

• (b) Estimation of Permittivity by Two Subspaces

To estimate the value of $\hat{\epsilon}_x$, the estimation theory is adopted in the next. We assume two random variables ϵ_{x1} and ϵ_{x2} are the effective permittivities in subspaces A and B, respectively. The expectation values of them are statistically independent. Thus, we can write

$$\mathbb{E}[(\epsilon_{x1} - \mathbb{E}[\epsilon_{x1}])(\epsilon_{x2} - \mathbb{E}[\epsilon_{x2}])] = 0. \tag{6}$$

With substitutions of the random variables and their expectations by the estimators ($\hat{\epsilon}_{x1}$ and $\hat{\epsilon}_{x2}$) and the mean values of statistics, respectively, Equation (6) can be derived into:

$$\mathbb{E} \left[\left(\hat{\epsilon}_{x1} - \frac{1}{M} \sum_{m=-M}^M \epsilon_m \Big|_{\epsilon_m \in A} \right) \left(\hat{\epsilon}_{x2} - \frac{1}{N} \sum_{n=-N}^N \epsilon_n \Big|_{\epsilon_n \in B} \right) \right] = 0, \tag{7}$$

which equals to

$$\mathbb{E}[\hat{\epsilon}_{x1}\hat{\epsilon}_{x2}] - \frac{1}{M} \sum_{m=-M}^M \epsilon_m \cdot \mathbb{E}[\hat{\epsilon}_{x2}] - \frac{1}{N} \sum_{n=-N}^N \epsilon_n \cdot \mathbb{E}[\hat{\epsilon}_{x1}] + \frac{1}{MN} \sum_{m=-M}^M \sum_{n=-N}^N \epsilon_m \epsilon_n = 0. \tag{8}$$

In Equation (7), the mean values of statistical groups (A and B) are determined by $\hat{\epsilon}_{x1} = \epsilon_a$ and $\hat{\epsilon}_{x2} = \epsilon_b$, respectively, when it is subjected to the prior distribution (with the discretized permittivities $\epsilon_m = \epsilon_a$ and $\epsilon_n = \epsilon_b$ for $\epsilon_m \in A, \epsilon_n \in B$). Thus, we have $\mathbb{E}[\hat{\epsilon}_x] = P(A)\epsilon_a + P(B)\epsilon_b$, considering $\hat{\epsilon}_{x1}$ and $\hat{\epsilon}_{x2}$ are independently subevents of the space and the probabilities $P(A)$ and $P(B)$ equal to the spatial ratio of subspaces A and B , respectively. When it being subjected to a posterior distribution, the effective permittivity can be obtained from the structural PDFs of a uniformly distributed medium (in Figure 1c). It has:

$$\mathbb{E}[(\hat{\epsilon}_{x1} - \mathbb{E}[\epsilon_{x1}^h])(\hat{\epsilon}_{x2} - \mathbb{E}[\epsilon_{x2}^h])] = 0, \tag{9}$$

with ϵ_{x1}^h and ϵ_{x2}^h denoting for the random variables defined by the uniform-distributed structural PDFs in subspaces A and B , respectively. In this case, it has $\hat{\epsilon}_{x1} = P(A)\epsilon_a + P(B)\epsilon_b$ and $\hat{\epsilon}_{x2} = P(A)\epsilon_a + P(B)\epsilon_b$, so that

$$\mathbb{E}[\hat{\epsilon}_x] = P(A)\hat{\epsilon}_{x1} + P(B)\hat{\epsilon}_{x2} = P(A)\epsilon_a + P(B)\epsilon_b \tag{10}$$

with $P(A) = 1 - P(B)$.

It is concluded that the predicted values of effective permittivities in subspaces are subjected to various distributions of the material defined by different structural PDFs. In Equation (10), the estimated value of permittivity can only be applied to express the longitudinal component of the porous material $\hat{\epsilon}_z$. For a periodical-arranged cross-section or a digital-coding meta-surface designed by $N \times N$ -port independent randomly switched by "on-off" states (illustrated in Figure 1d), the scattering of these structures cannot be ignored.

• **(c) Averaging Distribution by Homogenization**

With variables $\Delta\epsilon_A$ and $\Delta\epsilon_B$ expressing the material change in each subspace, the resulting estimators can be equal by interchange of material constructions in subspaces A and B . It has

$$\mathbb{E} \left[\left(\hat{\epsilon}_x^o - \mathbb{E} \left[\frac{1}{M} \sum_{m=-M}^M \epsilon_m \Big|_{\{\epsilon_m \in A\}} - \Delta\epsilon_A \right] \right) \left(\hat{\epsilon}_x^o - \mathbb{E} \left[\frac{1}{N} \sum_{n=-N}^N \epsilon_n \Big|_{\{\epsilon_n \in B\}} - \Delta\epsilon_B \right] \right) \right] = 0 \tag{11}$$

by letting $\hat{\epsilon}_x^o = \hat{\epsilon}_{x1} = \hat{\epsilon}_{x2}$ (the superscript "o" denotes for the optimal estimator).

For consistency of Equation (7), the domain of the functions remains unchanged, which can be written as:

$$\epsilon_{x1}^o = \epsilon_a - \Delta\epsilon_A = \frac{1}{N_1} \sum_{i=1}^{N_1} \epsilon_i, \quad \epsilon_{x2}^o = \epsilon_b - \Delta\epsilon_B = \frac{1}{N_2} \sum_{j=1}^{N_2} \epsilon_j, \tag{12}$$

with $\epsilon_m = \epsilon_a$ and $\epsilon_n = \epsilon_b$. The estimated values $\hat{\epsilon}_{x1}$ and $\hat{\epsilon}_{x2}$ have been changed from ϵ_a and ϵ_b into $\hat{\epsilon}_x^o$, as seen from Figure 2. In this case, the value of $\hat{\epsilon}_x^o$ can be approximated by the mean values of statistics in subspaces A and B (ϵ_i and ϵ_j), respectively.

Consider that random variables ϵ_{x1}^o and ϵ_{x2}^o are subjected to two additional distributions defined by marginal structural PDFs as $f_{x1}^o(x, y)$ and $f_{x2}^o(x, y)$. For N' is a large number, it is assumed that

$$\lim_{N_1 \gg N'} \left| \frac{1}{N_1} \sum_{i=1}^{N_1} \epsilon_i - \mathbb{E}[\epsilon_{x1}^o] \right| < \delta_1; \quad \lim_{N_2 \gg N'} \left| \frac{1}{N_2} \sum_{j=1}^{N_2} \epsilon_j - \mathbb{E}[\epsilon_{x2}^o] \right| < \delta_2, \tag{13}$$

in which δ_1 and δ_2 have little values. Due to the spatiotemporal fusion, the expectation of joint random variables $\epsilon_{x_1}^o \epsilon_{x_2}^o$ are defined by

$$\lim_{N_1 N_2 \gg N^2} \left| \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \epsilon_i \epsilon_j - \mathbb{E}[\epsilon_{x_1}^o \epsilon_{x_2}^o] \right| < \delta_1 \delta_2. \tag{14}$$

By the uses of equations from (11) to (14), we can write:

$$\mathbb{E}[(\hat{\epsilon}_x^o - \mathbb{E}[\epsilon_{x_1}^o])(\hat{\epsilon}_x^o - \mathbb{E}[\epsilon_{x_2}^o])] = 0. \tag{15}$$

in which the random variable $\hat{\epsilon}_x^o$ is defined in the complete set of two subspaces. For simplicity, Equation (15) is written as $(\hat{\epsilon}_x^o)^2 - 2u\hat{\epsilon}_x^o + v = 0$ with $u = \frac{1}{2}(\mathbb{E}[\epsilon_{x_1}^o] + \mathbb{E}[\epsilon_{x_2}^o])$ and $v = \mathbb{E}[\epsilon_{x_1}^o \epsilon_{x_2}^o]$. The coefficients u and v are to be evaluated.

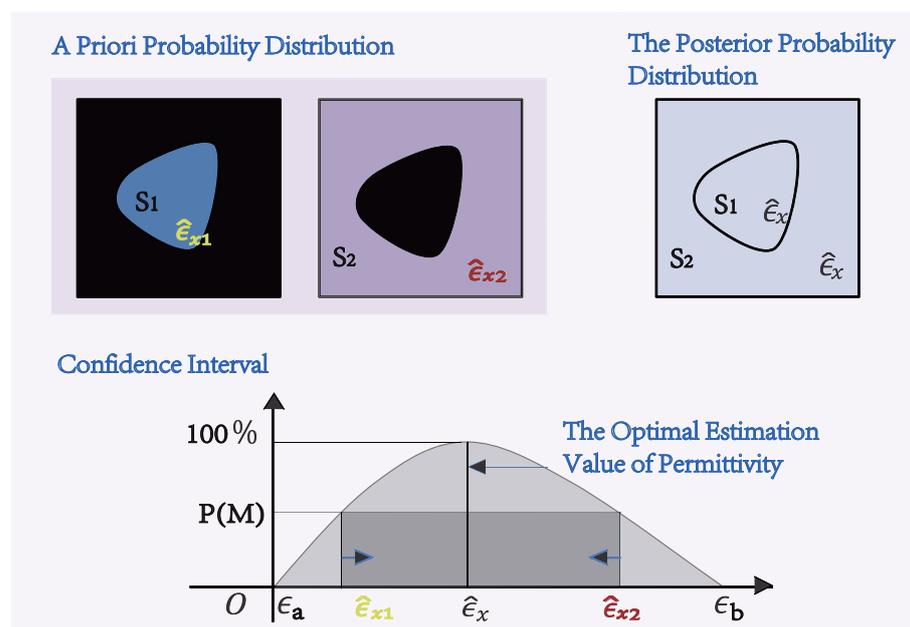


Figure 2. Distributions of PDFs for a priori probability, the posterior probability, and confidence interval, respectively.

Consider the electromagnetic scattering of a honeycomb-structured cross-section. An effective permittivity of the meta-surface can be expressed at sub-wavelength band, by the mixing formula:

$$\hat{\epsilon}_x^o = u \pm \sqrt{|u^2 - v|}, \tag{16}$$

which is subjected to the spatial duty ratio of materials (with p and $q = 1 - p$ for filling material and skeleton material, respectively). The parameters are defined by $u = \frac{1}{2}[\epsilon_a(p - q) + \epsilon_b(q - p)]$, and $v = -\epsilon_a \epsilon_b$, respectively.

3. Derivation

3.1. Variables and Parameters

Some of the variables and functions used are defined in Table 1.

Table 1. Definitions of the variables and functions.

Symbol	Type	Definition	Equation
$\epsilon_{x_1}, \epsilon_{x_2}$	Random variables	The mean values of permittivity in statistical subspaces (A, B)	$\epsilon_{x_1=2} \sim f_{x_2}$ $\epsilon_{x_2} \sim f_{x_2}$
f_{x_1}, f_{x_2}	Marginal structural PDFs	The assumed distributions in statistical subspaces (A, B)	$f_{x_1} \sim B(0,1)$ $f_{x_2} \sim B(0,1)$
$\Delta\epsilon_A$	Random variable	Deviation of ϵ_{x_1} from $\hat{\epsilon}_x^o$	$\Delta\epsilon_A \sim f_{X Y}$
$\Delta\epsilon_B$	Random variable	Deviation of ϵ_{x_2} from $\hat{\epsilon}_x^o$	$\Delta\epsilon_B \sim f_{Y X}$
$f_{x_1x_2}^h$	Joint structural PDF	A posterior distribution of a homogenized medium	$f_{x_1x_2} \sim B(0,1)$
ϵ_L	Estimator	In longitude	Obtained by WAm
$\bar{\epsilon}_T$	2×2 Matrix	In transverse	$diag(\hat{\epsilon}_x, \hat{\epsilon}_y)$
$\hat{\epsilon}_x, \hat{\epsilon}_y$	Estimators	Effective parameters	By Equation (13)

3.2. Distributions and Probabilities

Considering the material as uniformly distributed, a joint posterior structural PDF can be written by the following expression:

$$f_{x_1x_2}^h(x, y) = \begin{cases} \frac{1}{S_1+S_2}, & (x, y) \in D_1 \cup D_2 \\ 0, & else \end{cases} \tag{17}$$

with the parameters S_1 and S_2 expressing the areas of D_1 and D_2 for materials with ϵ_a and ϵ_b , respectively. Thus, the expectations of joint random variables $\epsilon_{x_1}^h, \epsilon_{x_2}^h$ and random variables $\epsilon_{x_1}^h$ and $\epsilon_{x_2}^h$ are given by

$$\mathbb{E}[\epsilon_{x_1}^h, \epsilon_{x_2}^h] = \iint_{D_1+D_2} \epsilon_a \epsilon_b f_{x_1x_2}^h(x, y) ds = \epsilon_a \epsilon_b, \tag{18}$$

and

$$\mathbb{E}[\epsilon_{x_1}^h] = \iint_{D_1} \bar{\epsilon}_1 f_{x_1x_2}^h(x, y) ds = \bar{\epsilon}_1 \cdot P(A), \quad \mathbb{E}[\epsilon_{x_2}^h] = \iint_{D_2} \bar{\epsilon}_2 f_{x_1x_2}^h(x, y) ds = \bar{\epsilon}_2 \cdot P(B), \tag{19}$$

in which the variables $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ are the expectations in subspaces A and B , respectively. The probabilities $P(A)$ and $P(B)$ are defined by (10), being equivalent to the duty ratio of filling material and skeleton material, respectively. The probabilities $P(AM)$ and $P(M)$ are the spatial ratio of one unit for material with ϵ_a and the spatial ratio of the unit, respectively. By the expansion of full probability formula, it has

$$P(A) = \frac{P(AM)}{P(M)} = P(A|M) + P(\bar{A}|M); \quad P(B) = \frac{P(BM)}{P(M)} = P(B|M) + P(\bar{B}|M) \tag{20}$$

respectively.

By the use of Equation (15), an optimal estimator $\hat{\epsilon}_x^o$ can be determined from the statistical distributions of random variables $\epsilon_{x_1}^o$ and $\epsilon_{x_2}^o$, so that

$$\mathbb{E}[\epsilon_{x_1}^o] = \iint_{D_1} \epsilon_a f_{x_1x_2}^o(x, y) ds = \epsilon_a P(A|M), \quad \mathbb{E}[\epsilon_{x_2}^o] = \iint_{D_2} \epsilon_b f_{x_1x_2}^o(x, y) ds = \epsilon_b P(B|M) \tag{21}$$

in which the conditional probabilities $P(A|M)$ and $P(B|M)$ are used to express the material interchange. We have $P(AM) = [P(A|M) + P(\bar{A}|M)]P(M)$. In Figure 3, we have $\epsilon_a P(A|M) = \epsilon_a P(A) - \epsilon_b P(BM)/P(M)$.

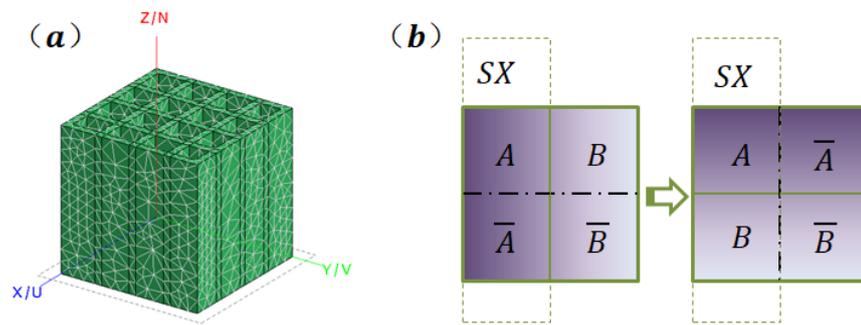


Figure 3. Physical model of a periodic-arranged material with (a) periodic-arranged cross-section and (b) composition of subspace SX with materials A and \bar{A} for a priori distribution and with materials A and B for the posterior distribution, respectively.

3.3. Computations of u and v

By the use of equation (21), the expectations can be evaluated by:

$$\mathbb{E}[\epsilon_{x_1}^o] = \epsilon_a \cdot [P(A) - P(B)], \quad \mathbb{E}[\epsilon_{x_2}^o] = \epsilon_b \cdot [P(B) - P(A)], \quad \mathbb{E}[\epsilon_{x_1}^o \epsilon_{x_2}^o] = -\epsilon_a \epsilon_b \quad (22)$$

respectively; therefore, we have $2u = \epsilon_a p - \epsilon_a(1 - p) + \epsilon_b q + \epsilon_b(1 - q)$, and $v = -\epsilon_a \epsilon_b$, where parameters p and q are the spatial ratio of two constructive materials on the honeycomb-structure cross-section (with p and $q = 1 - p$ for filling material and skeleton material, respectively).

4. Numerical Results

The absolute values of relative permittivity estimated by SEAm are computed for different spatial conditions and material parameters in Figure 4 to express an equivalent cross-section of a honeycomb-structured material defined by Equation (11). The permittivities of air and material are taken as $\epsilon_0 = 8.854187817 \times 10^{-12}$ F/m (the permittivity in free space), and ϵ_a is chosen by a wave-absorbing material with relative permittivity $\epsilon_a/\epsilon_0 = 8.7868 + 9.2263i$. The comparing results are obtained by WAM, HSVm, and SFT, respectively, for which, the formulas are summarized in Appendix A. The round dot and squared dot symbols highlighted in Figure 4 are defined by the absolute value of relative permittivities under the same duty ratio of material-to-air with $\nu_0 = \nu_1 = 0.5$, where each of them reads: $|\epsilon_1|/\epsilon_0 = 6.8714$, $|\epsilon_2|/\epsilon_0 = 5.1141$, $|\epsilon_3|/\epsilon_0 = 3.5704$, and $|\epsilon_4|/\epsilon_0 = 2.4922$, respectively. The shaded area is surrounded by two curves for WAM and square root values of two results by HSVm, respectively.

The computed applicable values of effective permittivity are non-unique for an arbitrary-shaped, periodical-arranged meta-surface, which are taken from the available region of HSVm, derived from Maxwell’s equations by means of variational methods. In the example of a honeycomb-structured cross-section, the simplest periodic unit (shaped by concentric hexagons) can be seen as isotropy on the cross-section (approximated by concentric circles), which are occupied by homogeneous materials ϵ_0 and ϵ_a , respectively. When the maximum radius is at sub-wavelength, the scattering of a honeycomb-structured cross-section can be equivalent to the scattering of the random medium for which the probabilities of particle existences are equal to the spatial duty ratio of materials. From these computational results, it is seen that the nonlinear approximation mixing formulas obtained by SEAm is in consistent with conventional approaches by HSVm and SFT.

With the same computational parameters, the second-order derivative functions of these formulas by different approaches are plotted in Figure 5, accordingly. Through the definition of function’s convexities, the inflection points of curves by SEAm and SFT formulas in Figure 4 are determined (at spatial duty ratio $\nu_0 = \nu_1 = 0.5$), whereas the

approach by HSVm gives a range of available values for a specific spatial duty ratio to equivalent the permittivity for periodical-arranged meta-surface.

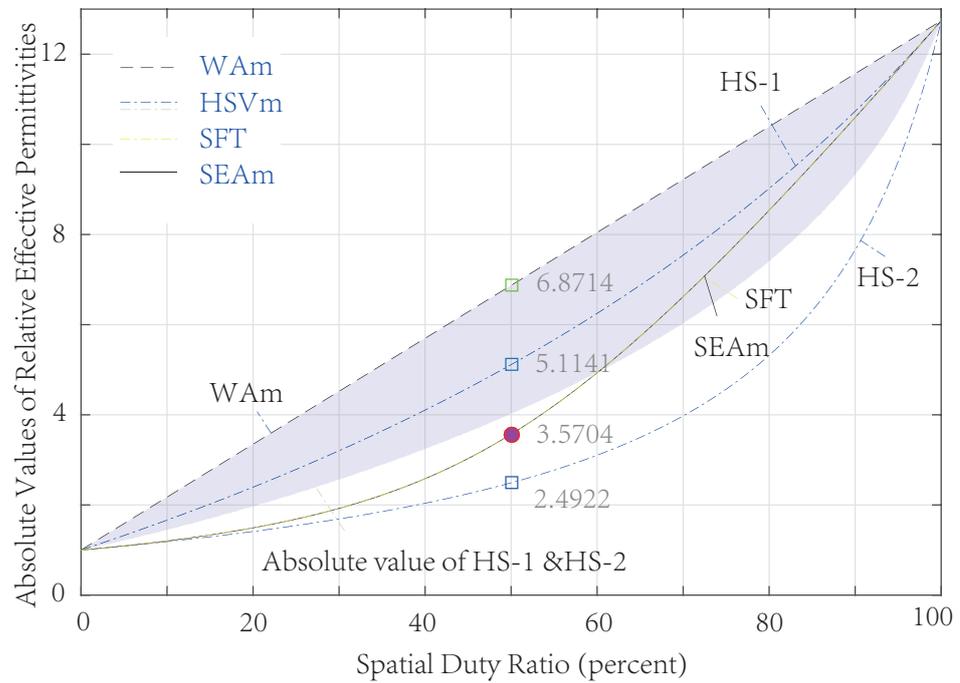


Figure 4. Absolute values of effective relative permittivities with dependencies on spatial duty ratio and material relative permittivities by various formulas to express a wave-absorbing honeycomb-structured cross-section. Round dot and squared dot symbols are highlighted for different approaches by same duty ratio of 50%. Shaded area is surrounded by two curves for WAm and square root values of two results by HSVm.

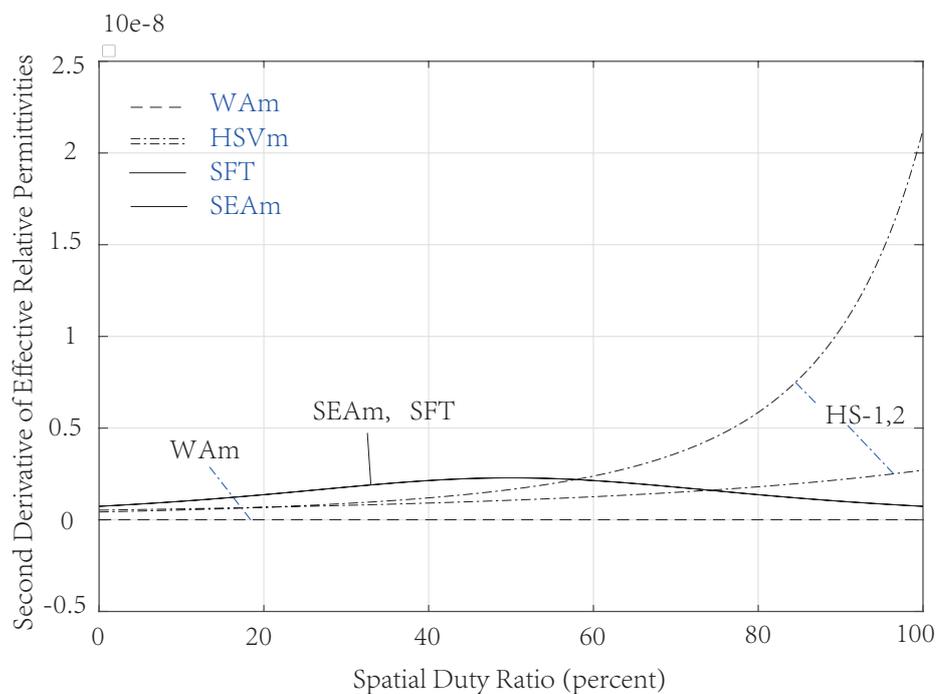


Figure 5. Absolute values of the second-order derivative functions of the effective relative permittivities by WAm, SEAm, SFT, HS-1 (the upper boundary by HSVm), and HS-2 (the lower boundary by HSVm), respectively.

The contour lines of the absolute values of effective relative permittivity are plotted by SEAm in Figure 6, as the functions of material ratio and duty ratio by increment of skeleton material parameter and spatial duty percentage, respectively. It is observed that the change of parameter ratio (≥ 5) of materials has little influence to the values of $\hat{\epsilon}_x$ when the spatial duty ratio is less than 20 percent. In these cases, the information entropy carried by geometric structures is almost negligible from the periodic honeycomb-shapes meta-surface.

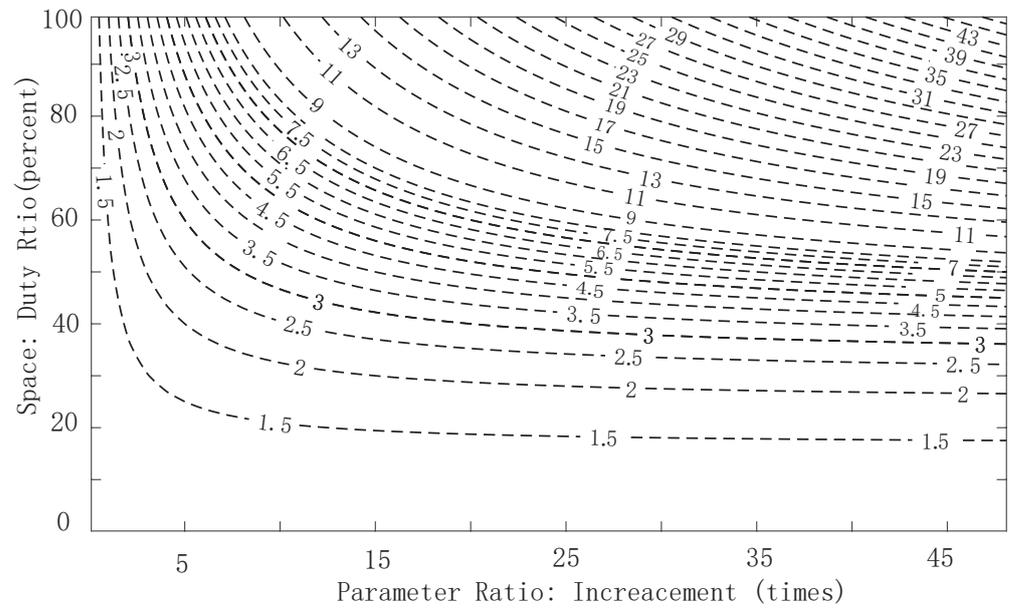


Figure 6. Contour lines for covariant values of effective relative permittivity tuned by spatial duty ratio and material doping ratio.

5. Experimental Validation

In the following example, we developed an experiment on electromagnetic scattering of a cuboid-shaped aramid paper in order to verify the effectiveness of SEAm to simulate results on a testing cuboid, which is occupied by a homogeneous medium with the computed parameters. On the back side, the aramid paper is covered by tinfoil. Considering the materials are nonmagnetic (with $\mu = \mu_0$), we take the parameter for skeleton material of aramid paper with relative permittivity of $\epsilon/\epsilon_0 = 3.5$. Thus, the filling material of the testing cuboid is characterized by effective permittivity of $\epsilon_x/\epsilon_0 = 1.0826$, $\epsilon_y/\epsilon_0 = 1.0826$, $\epsilon_z/\epsilon_0 = 1.1226$, and permeability of $\mu = \mu_0$, respectively. The antenna device is set to be placed in a microwave darkroom for testing, where the measured data for Radar Cross-Section (RCS) have been collected by Vertical-transmit Vertical-receive (VV) and Horizontal-transmit Horizontal-receive (HH) polarizations, respectively.

The absolute values of RCS are plotted in Figure 7 for simulated results (in black and with solid lines) and measurement data (in red and with dashed lines), respectively. The operated frequencies are chosen as frequencies: $f = 5.4$ GHz and $f = 3.2$ GHz, respectively. From the full-wave simulations of a homogenized medium and measurement on a real aramid paper (shaped by $300 \text{ mm} \times 300 \text{ mm} \times 66 \text{ mm}$, unit sized by hexagon unit side length 1.6 mm and inscribed circle diameter 3.2 mm , respectively), it is concluded that the proposed method is valid to simulate the scattering of a honeycomb-structured meta-surface, but it is also restricted to operating frequencies (valid for $\lambda_{\min} \geq 2\sqrt{3}r_0$) and incident angles.

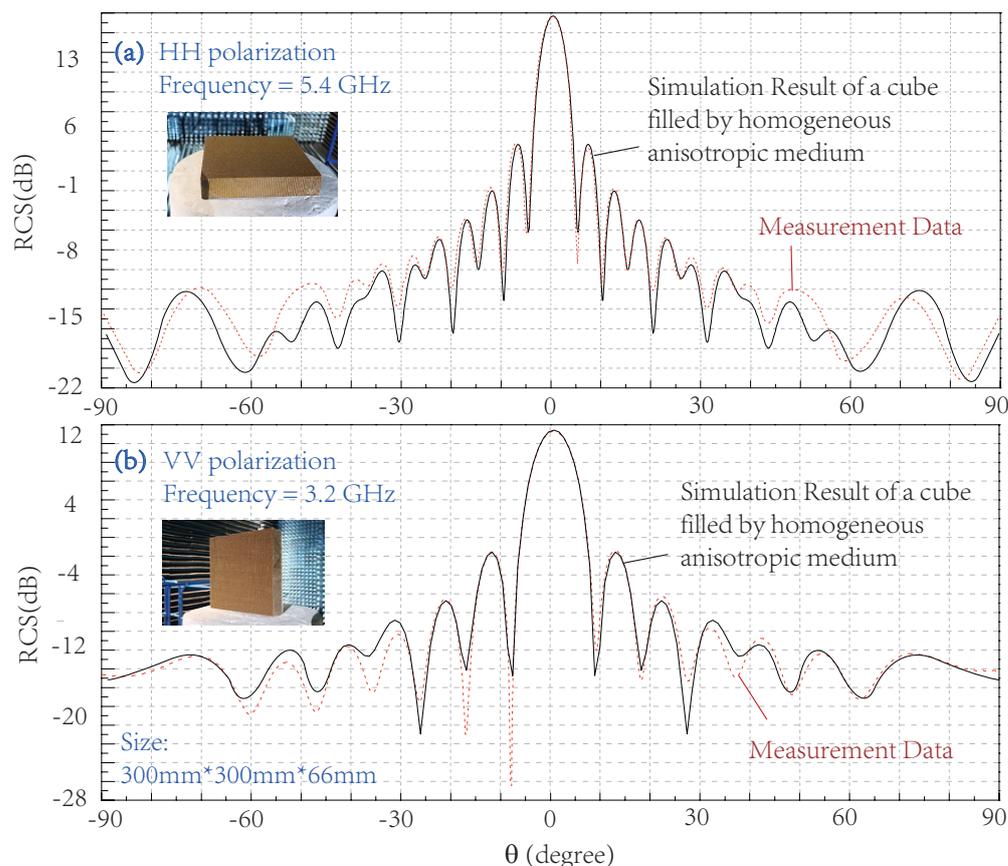


Figure 7. The RCS values of a cuboid-shaped aramid material measured by an experiment which is compared to the full-wave simulation results of a cuboid object occupied by homogeneous anisotropic medium with (a) HH polarization at $f = 5.4$ GHz; and (b) VV polarization at $f = 3.2$ GHz, respectively.

6. Conclusions

A methodology has been proposed to consider the electromagnetic equivalence problem of a structural complex material. We exploited the concept of a homogenized medium to express the electromagnetic features of a periodical-arranged cross-section or information meta-surface. The method of SEA is applied to estimate the effective parameters of such structures.

(i) The estimation theory is applied by second-order spatial discretized statistics to estimate the effective permittivity or permeability of a honeycomb cross-section. By interchange of material compositions, the homogenized medium can be used to express the electromagnetic property of a real structure.

(ii) The electromagnetic scattering of a periodically arranged meta-surface is influenced by the unit length of the structure due to spatiotemporal fusion. This article also gives a preliminary investigation on the equivalent of an information meta-surface in wireless communication applications.

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Abbreviations

The following abbreviations are used in this manuscript:

DMIA	Digital-coding Meta-material Information-encrypted Antenna
FSRm	Full-Space Retrieval method
HH	Horizontal-transmit, Horizontal-receive
HSVm	Hashin-Shtrikman Variational method
PDF	Probability Density Function
PSD	Power Spectral Density
RCS	Radar Cross Section
SEAm	Statistical Estimation Averaging method
SFT	Strong Fluctuation Theory
VV	Vertical-transmit, Vertical-receive
WAm	Weighted Average method

Appendix A. The Formulas for WAm, HSVm, and SFR

The applied formulas in Figure 4 are given for computing the effective permittivities of a structural complex material:

(i) WAm: an averaged method weighted by spatial duty ratio of ϵ_a and ϵ_0 (see Table 1), defined by $\epsilon_L = (1 - p)\epsilon_0 + p\epsilon_a$;

(ii) HSVm: a variational method derived from Maxwell's equations. It provides with a region of available values for one specific parameter condition [15–17].

(iii) SFT: the derivation from the microscopic Maxwell's Equations for random media;

(iv) Other: i.e., FSRm, theoretically being suitable for any problem. The retrieval methods are applied to obtain effective parameters based on simulation results by software or measurement data.

$$\epsilon = \frac{\gamma_1(1 - \Gamma)}{\gamma_0(1 + \Gamma)}; \quad \mu = \frac{\gamma_1(1 + \Gamma)}{\gamma_0(1 - \Gamma)}; \quad (\text{A1})$$

with propagation parameters γ_0 and γ_1 should be measured for any incidental angel in free space and dielectric region, and variable Γ stands for the reflection coefficient, respectively.

Appendix B. The Potential Application Scenarios of SEAm

Some application scenarios are given in Table A1.

Table A1. Application Scenarios of SEAm.

Scenario	Mathematical Formulation	Estimator	Case
Fixed, structured material Infinite-sized plate; Two-port networks.	Type I $\tilde{\epsilon} = (f_{-m}^X, f_{1-m}^X, \dots, f_0^X, \dots, f_{m-1}^X, f_m^X) \leftrightarrow (1, 1, \dots, 1, \dots, 1, 1) \times$ $\times \begin{bmatrix} 1 & \dots & 1 & \dots & 1 \\ \vdots & \dots & \vdots & \dots & \vdots \\ 1 & \dots & 1 & \dots & 1 \\ \vdots & \dots & \vdots & \dots & \vdots \\ 1 & \dots & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} f_m^Y \\ \vdots \\ f_0^Y \\ \vdots \\ f_{-m}^Y \end{bmatrix} \begin{bmatrix} f_{m,-n}^{XY} & \dots & f_{m,0}^{XY} & \dots & f_{m,n}^{XY} \\ \vdots & \dots & \vdots & \dots & \vdots \\ f_{0,-n}^{XY} & \dots & f_{0,0}^{XY} & \dots & f_{0,n}^{XY} \\ \vdots & \dots & \vdots & \dots & \vdots \\ f_{-m,-n}^{XY} & \dots & f_{-m,0}^{XY} & \dots & f_{-m,n}^{XY} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}$	Paras. ϵ_x, ϵ_y	I f^{SX}, f^{SY} $\sim B(0, 1);$ $\mathbb{E}[SX]$ $\mathbb{E}[SY]$
Random material Infinite-sized plate; with independent-identically distributed feeding-arrays.	Type II $\tilde{\epsilon} = (1, 1, \dots, 1, \dots, 1, 1)$ $\times \begin{bmatrix} \{\epsilon\}_{m,-n} & \dots & \{\epsilon\}_{m,0} & \dots & \{\epsilon\}_{m,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\epsilon\}_{0,-n} & \dots & \{\epsilon\}_{0,0} & \dots & \{\epsilon\}_{0,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\epsilon\}_{-m,-n} & \dots & \{\epsilon\}_{-m,0} & \dots & \{\epsilon\}_{-m,n} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}$	Paras. ϵ_x, ϵ_y	II $f^{XY} \sim B(0, 1);$ Transform f^{XY} into f^{SX}, f^{SY} \leftrightarrow Case I.
Regular antenna Hertzian dipole antenna fed by \mathcal{J}_e or \mathcal{J}_m returns. $n \times n$ - port & $MN \rightarrow \infty$	Type III $\tilde{\Phi}_1 = (1, 1, \dots, 1, \dots, 1, 1) \times$ $\begin{bmatrix} \{\mathcal{J}_e\}_{m,-n} \Phi_{m,-n} & \dots & \{\mathcal{J}_e\}_{m,0} \Phi_{m,0} & \dots & \{\mathcal{J}_e\}_{m,n} \Phi_{m,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\mathcal{J}_e\}_{0,-n} \Phi_{0,-n} & \dots & \{\mathcal{J}_e\}_{0,0} \Phi_{0,0} & \dots & \{\mathcal{J}_e\}_{0,n} \Phi_{0,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\mathcal{J}_e\}_{-m,-n} \Phi_{-m,-n} & \dots & \{\mathcal{J}_e\}_{-m,0} \Phi_{-m,0} & \dots & \{\mathcal{J}_e\}_{-m,n} \Phi_{-m,n} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix};$ Also by : $\tilde{\Phi}_2 = (1, 1, \dots, 1, \dots, 1, 1) \times$ $\begin{bmatrix} \{\mathcal{J}_m\}_{m,-n} \Phi_{m,-n} & \dots & \{\mathcal{J}_m\}_{m,0} \Phi_{m,0} & \dots & \{\mathcal{J}_m\}_{m,n} \Phi_{m,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\mathcal{J}_m\}_{0,-n} \Phi_{0,-n} & \dots & \{\mathcal{J}_m\}_{0,0} \Phi_{0,0} & \dots & \{\mathcal{J}_m\}_{0,n} \Phi_{0,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\mathcal{J}_m\}_{-m,-n} \Phi_{-m,-n} & \dots & \{\mathcal{J}_m\}_{-m,0} \Phi_{-m,0} & \dots & \{\mathcal{J}_m\}_{-m,n} \Phi_{-m,n} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}$	State $\hat{P}_0,$ by $ E ^2$ or $ H ^2$	III f^{SXSY} $\sim N(0, 1);$ $f^{X_1Y_1}, f^{X_2Y_2}$ additive.
DMI Antenna Meta-materials fed by \mathcal{J}_e and \mathcal{J}_m simultaneously New channel systems Independent-identically distributed $N \times N$ - port signal supplies.	Type IV $\tilde{\Phi} = (1, 1, \dots, 1, \dots, 1, 1) \times$ $\left\{ \begin{bmatrix} \{\mathcal{J}_{M,-N}^e\} \Phi_{M,-N} & \dots & \{\mathcal{J}_{M,0}^e\} \Phi_{M,0} & \dots & \{\mathcal{J}_{M,N}^e\} \Phi_{M,N} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\mathcal{J}_{0,-N}^e\} \Phi_{0,-N} & \dots & \{\mathcal{J}_{0,0}^e\} \Phi_{0,0} & \dots & \{\mathcal{J}_{0,N}^e\} \Phi_{0,N} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\mathcal{J}_{-M,-N}^e\} \Phi_{-M,-N} & \dots & \{\mathcal{J}_{-M,0}^e\} \Phi_{-M,0} & \dots & \{\mathcal{J}_{-M,N}^e\} \Phi_{-M,N} \end{bmatrix} + \begin{bmatrix} \{\mathcal{J}_{M,-N}^m\} \Phi_{M,-N} & \dots & \{\mathcal{J}_{M,0}^m\} \Phi_{M,0} & \dots & \{\mathcal{J}_{M,N}^m\} \Phi_{M,N} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\mathcal{J}_{0,-N}^m\} \Phi_{0,-N} & \dots & \{\mathcal{J}_{0,0}^m\} \Phi_{0,0} & \dots & \{\mathcal{J}_{0,N}^m\} \Phi_{0,N} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \{\mathcal{J}_{-M,-N}^m\} \Phi_{-M,-N} & \dots & \{\mathcal{J}_{-M,0}^m\} \Phi_{-M,0} & \dots & \{\mathcal{J}_{-M,N}^m\} \Phi_{-M,N} \end{bmatrix} \right\} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}$	State $(\hat{P}_1, \hat{P}_2),$ by $ E ^2$ and $ H ^2$	IV $f^{SX_1SY_1}$ $\sim P(\lambda_1);$ $f^{SX_2SY_2}$ $\sim P(\lambda_2);$ $f^{X_1Y_1}, f^{X_2Y_2}$ mixed.

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