mathematics

Article

The Sharp Upper Estimate Conjecture for the Dimension $\delta_k(V)$ of New Derivation Lie Algebra

Naveed Hussain [1,*,†], Ahmad N. Al-Kenani [2,†], Muhammad Arshad [1,†] and Muhammad Asif [3,†]

1 Department of Mathematics and Statistics, University of Agriculture, Faisalabad 38000, Pakistan; dr.marshad@uaf.edu.pk
2 Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia; analkenani@kau.edu.sa
3 Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore 54000, Pakistan; sp18-pmt-001@cuilahore.edu.pk
* Correspondence: dr.nhussain@uaf.edu.pk; Tel.: +92-3161698297
† These authors contributed equally to this work.

Abstract: Hussain, Yau, and Zuo introduced the Lie algebra $\mathcal{L}_k(V)$ from the derivation of the local algebra $M_k(V) := \mathcal{O}_n/(g + \mathbb{J}_1(g) + \cdots + \mathbb{J}_k(g))$. To find the dimension of a newly defined algebra is an important task in order to study its properties. In this regard, we compute the dimension of Lie algebra $\mathcal{L}_5(V)$ and justify the sharp upper estimate conjecture for fewnomial isolated singularities. We also verify the inequality conjecture: $\delta_5(V) < \delta_4(V)$ for a general class of singularities. Our findings are novel and an addition to the study of Lie algebra.

Keywords: singularities; isolated hypersurface singularity; Lie algebra; local algebra; fewnomial

MSC: 14B05; 32S05

Citation: Hussain, N.; Al-Kenani, A.N.; Arshad, M.; Asif, M. The Sharp Upper Estimate Conjecture for the Dimension $\delta_k(V)$ of New Derivation Lie Algebra. *Mathematics* **2022**, *10*, 2618. https://doi.org/10.3390/math10152618

Academic Editor: Askar Tuganbaev

Received: 29 June 2022
Accepted: 22 July 2022
Published: 27 July 2022

1. Introduction

It is commonly known that at the origin of $\mathbb{C}^n$, $\mathcal{O}_n$ are the germs of holomorphic functions. Naturally, the algebra of n indeterminate power series may be identified by the $\mathcal{O}_n$. Yau considered the Lie algebras of the derivation of moduli algebra $A(V) := \mathcal{O}_n/(g, \frac{\partial g}{\partial x_1}, \cdots, \frac{\partial g}{\partial x_n})$, where $L(V) = \text{Der}(A(V), A(V))$, and V denotes the isolated hypersurface singularity. $L(V)$ is well recognized as solvable finite dimensional Lie algebra ([1–3]). $L(V)$ distinguished from the other types of Lie algebra present in singularity theory ([4,5]) is known as the Yau algebra of V [6]. Several new natural connections have been developed in recent years by Hussain, Yau, Zuo, and their research fellows ([7–12]) between the finite set of solvable dimensional Lie algebras (nilpotent) and the complex analytical set of isolated hypersurface singularities. Three different ways have been introduced to associate isolated hypersurface singularities with Lie algebra. From a geometric point of view, these associations support understanding the solvable Lie algebra (nilpotent), [9]. Since the 1980s, Yau and their research fellows have provided much work on singularities [9,13–22].

Let a holomorphic function $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be defined by the isolated hypersurface singularity $(V, 0)$, with its multiplicity $mult(g)$. $mult(g)$ in the power series expansion is the order of the nonvanishing lowest term of g at o. In [23], the new derivation Lie algebras are defined in the following way:

Let $\mathbb{J}_k(g) = < \frac{\partial^k g}{\partial x_{i_1} \cdots \partial x_{i_k}} \mid 1 \leq i_1, \cdots, i_k \leq n >$ be an ideal. For $mult(g) = m$ and $1 \leq k \leq m$, $M_k(V) := \mathcal{O}_n/(g + \mathbb{J}_1(g) + \cdots + \mathbb{J}_k(g))$ are the new k-th local algebra and $\mathcal{L}_k(V)$ its new Lie algebras of derivations with dimension $\delta_k(V)$, which is a new numerical analytic invariant. $\mathcal{L}_k(V)$ is the generalization of Yau algebra. More details can be found in ([23]).

A conjecture for the analytic invariant $\delta_k(V)$ was proposed in [23] as:

Conjecture 1 ([23])**.** *Let $\delta_k(\{x_1^{b_1}+\cdots+x_n^{b_n}=0\})=h_k(b_1,\cdots,b_n)$, $0\leq k\leq n$ and $(V,0)=\{(x_1,x_2,\cdots,x_n)\in\mathbb{C}^n : g(x_1,x_2,\cdots,x_n)=0\},(n\geq 2)$ be an isolated singularity with weight type $(w_1,w_2,\cdots,w_n;1)$. Then, $\delta_k(V)\leq h_k(1/w_1,\cdots,1/w_n)$.*

In [23], the inequality conjecture for $\delta_k(V)$ was also proposed in following way:

Conjecture 2 ([23])**.** *With the above notations, let $(V,0)$ be defined by $g\in\mathcal{O}_n$, $n\geq 2$. Then,*

$$\delta_{(k+1)}(V)<\delta_k(V),k\geq 1.$$

For binomial and trinomial singularities, Conjecture 1 holds true when $k=1,2,3,4$ ([12,17,20,23,24]), and Conjecture 2 holds true for $k=1,2,3$ ([23,24]).

The main goal of this study is to confirm Conjecture 1 (resp. Conjecture 2) for binomial and trinomial singularities when $k=5$ (resp. $k=4$). The following are our key findings.

Theorem 1. *Let $(V(g),0)=\{(x_1,x_2,\cdots,x_n)\in\mathbb{C}^n : x_1^{b_1}+\cdots+x_n^{b_n}=0\},(n\geq 2;\ b_j\geq 7,\ 1\leq j\leq n)$, where b_j are fixed natural numbers. Then,*

$$\delta_5(V(g))=h_5(b_1,\cdots,b_n)=\sum_{i=1}^{n}\frac{b_i-6}{b_i-5}\prod_{j=1}^{n}(b_j-5).$$

Theorem 2. *Let $(V,0)$ be a binomial singularity, which is defined by $g(x_1,x_2)$, a weighted homogeneous polynomial with weight type $(w_1,w_2;1)$ and $mult(g)\geq 7$. Then,*

$$\delta_5(V)\leq h_5(\frac{1}{w_1},\frac{1}{w_2})=\sum_{j=1}^{2}\frac{\frac{1}{w_j}-6}{\frac{1}{w_j}-5}\prod_{i=1}^{2}(\frac{1}{w_i}-5).$$

Theorem 3. *Let $(V,0)$ be a binomial singularity, which is defined by $g(x_1,x_2)$, a weighted homogeneous polynomial with weight type $(w_1,w_2;1)$ and $mult(g)\geq 7$. Then,*

$$\delta_5(V)<\delta_4(V).$$

Theorem 4. *Let $(V,0)$ be a trinomial singularity, which is defined by $g(x_1,x_2,x_3)$, a weighted homogeneous polynomial with weight type $(w_1,w_2,w_3;1)$ and $mult(g)\geq 7$.*

Then,

$$\delta_5(V)\leq h_5(\frac{1}{w_1},\frac{1}{w_2},\frac{1}{w_3})=\sum_{j=1}^{3}\frac{\frac{1}{w_j}-6}{\frac{1}{w_j}-5}\prod_{i=1}^{3}(\frac{1}{w_i}-5).$$

Theorem 5. *Let $(V,0)$ be a trinomial singularity, which is defined by $g(x_1,x_2,x_3)$, a weighted homogeneous polynomial with weight type $(w_1,w_2,w_3;1)$ and $mult(g)\geq 7$.*

Then,

$$\delta_5(V)<\delta_4(V).$$

2. Preliminaries

Proposition 1.2 of [25] states: Let finite dimension associative algebras A and B have units for the tensor product,

$$\mathrm{Der}S\cong(\mathrm{Der}A)\otimes C(B)+C(A)\otimes(\mathrm{Der}B).$$

Theorem 6 ([25])**.** *For commutative associative algebras A,B,*

$$DerS\cong(DerA)\otimes B+A\otimes(DerB). \tag{1}$$

The following result is used in this work.

Theorem 7 ([17]). *For ideal $\mathbb{J}$ in $R = \mathbb{C}\{x_1, \cdots, x_n\}$,*

$$(Der_{\mathbb{J}} R)/(\mathbb{J} \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/\mathbb{J}).$$

The linear endomorphism D of commutative associative algebra A with $D(ab) = D(a)b + aD(b)$ is called a derivation of A.

Proposition 1. *Analytically, a weighted homogeneous fewnomial singularity g with $mult(g) \geq 3$ is equivalent to a linear combination of the series:*

Type A. $x_1^{b_1} + x_2^{b_2} + \cdots + x_{n-1}^{b_{n-1}} + x_n^{b_n}$, $n \geq 1$,
Type B. $x_1^{b_1} x_2 + x_2^{b_2} x_3 + \cdots + x_{n-1}^{b_{n-1}} x_n + x_n^{b_n}$, $n \geq 2$,
Type C. $x_1^{b_1} x_2 + x_2^{b_2} x_3 + \cdots + x_{n-1}^{b_{n-1}} x_n + x_n^{b_n} x_1$, $n \geq 2$.

Corollary 1. *Analytically, each binomial isolated singularity is equivalent to one of the three series: A) $x_1^{b_1} + x_2^{b_2}$, B) $x_1^{b_1} x_2 + x_2^{b_2}$, C) $x_1^{b_1} x_2 + x_2^{b_2} x_1$.*

Proposition 2 ([26]). *Let $g(x_1, x_2, x_3)$ be a weighted homogeneous fewnomial isolated singularity with $mult(g) \geq 3$. Then, g is analytically equivalent to one of the five series:*

Type 1. $x_1^{b_1} + x_2^{b_2} + x_3^{b_3}$,
Type 2. $x_1^{b_1} x_2 + x_2^{b_2} x_3 + x_3^{b_3}$,
Type 3. $x_1^{b_1} x_2 + x_2^{b_2} x_3 + x_3^{b_3} x_1$,
Type 4. $x_1^{b_1} + x_2^{b_2} + x_3^{b_3} x_1$,
Type 5. $x_1^{b_1} x_2 + x_2^{b_2} x_1 + x_3^{b_3}$.

3. Proof of Theorems

The following propositions will be used to prove the main results of this paper.

Proposition 3. *Let $(V(g), 0)$ be an isolated singularity and $g = x_1^{b_1} + x_2^{b_2} + \cdots + x_n^{b_n}$ ($b_j \geq 7, j = 1, 2, \cdots, n$) be a weighted homogeneous polynomial with weight type $(\frac{1}{b_1}, \frac{1}{b_2}, \cdots, \frac{1}{b_n}; 1)$. Then,*

$$\delta_5(V(g)) = \sum_{i=1}^{n} \frac{b_i - 6}{b_i - 5} \prod_{j=1}^{n} (b_j - 5).$$

Proof. After simple calculation, the moduli algebra $M_5(V)$ has a monomial basis of the form

$$\{x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}, 0 \leq j_1 \leq b_1 - 6, 0 \leq j_2 \leq b_2 - 6, \cdots, 0 \leq j_n \leq b_n - 6\},$$

with the following relations:

$$x_1^{b_1-5} = 0, x_2^{b_2-5} = 0, x_3^{b_3-5} = 0, \cdots, x_n^{b_n-5} = 0. \tag{2}$$

Without loss of generality, one can write derivation D in terms of the monomial basis in the following way:

$$Dx_i = \sum_{j_1=0}^{b_1-6} \sum_{j_2=0}^{b_2-6} \cdots \sum_{j_n=0}^{b_n-6} c^i_{j_1, j_2, \cdots, j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}, \ i = 1, 2, \cdots, n.$$

The sufficient and necessary conditions may be found using the relations (2) to define a derivation of $M_5(V)$ in following way:

$$c^1_{0,j_2,j_3,,\cdots,j_n} = 0; 0 \le j_2 \le b_2 - 6, 0 \le j_3 \le b_3 - 6, \cdots, 0 \le j_n \le b_n - 6;$$
$$c^2_{j_1,0,j_3,,\cdots,j_n} = 0; 0 \le j_1 \le b_1 - 6, 0 \le j_3 \le b_3 - 6, \cdots, 0 \le j_n \le b_n - 6;$$
$$c^3_{j_1,j_2,0,\cdots,j_n} = 0; 0 \le j_1 \le b_1 - 6, 0 \le j_2 \le b_2 - 6, \cdots, 0 \le j_n \le b_n - 6;$$
$$\vdots$$
$$c^n_{j_1,j_2,j_3,\cdots,j_{n-1},0} = 0; 0 \le j_1 \le b_1 - 6, 0 \le j_2 \le b_2 - 6, \cdots, 0 \le j_{n-1} \le b_{n-1} - 6.$$

The Lie algebra $\mathcal{L}_5(V)$ has the following basis:

$$x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \partial_1,\ 1 \le j_1 \le b_1 - 6, 0 \le j_2 \le b_2 - 6, 0 \le j_3 \le b_3 - 6, \cdots, 0 \le j_n \le b_n - 6;$$
$$x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \partial_2,\ 0 \le j_1 \le b_1 - 6, 1 \le j_2 \le b_2 - 6, 0 \le j_3 \le b_3 - 6, \cdots, 0 \le j_n \le b_n - 6;$$
$$x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \partial_3,\ 0 \le j_1 \le b_1 - 6, 0 \le j_2 \le b_2 - 6, 1 \le j_3 \le b_3 - 6, 0 \le j_4 \le b_4 - 6,$$
$$0 \le j_5 \le b_5 - 6, 0 \le j_6 \le b_6 - 6, \cdots, 0 \le j_n \le b_n - 6;$$
$$\vdots$$
$$x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \partial_n,\ 0 \le j_1 \le b_1 - 6, 0 \le j_2 \le b_2 - 6, 0 \le j_3 \le b_3 - 6, \cdots, 1 \le j_n \le b_n - 6.$$

This implies

$$\delta_5(V(g)) = \sum_{i=1}^{n} \frac{b_i - 6}{b_i - 5} \prod_{j=1}^{n} (b_j - 5).$$

□

Remark 1. *Let $(V(g), 0)$ be a fewnomial isolated singularity, where $g = x_1^{b_1} + x_2^{b_2}$ ($b_j \ge 7, j = 1,2$) is a weighted homogeneous polynomial with weight type $(\frac{1}{b_1}, \frac{1}{b_2}; 1)$. Then, from Proposition 3, we obtain*

$$\delta_5(V) = 2b_1b_2 - 11(b_1 + b_2) + 60.$$

Proposition 4. *Let $(V, 0)$ be a binomial singularity of type B defined by $g = x_1^{b_1} x_2 + x_2^{b_2}$ ($b_1 \ge 6$, $b_2 \ge 7$) with weight type $(\frac{b_2-1}{b_1b_2}, \frac{1}{b_2}; 1)$. Then,*

$$\delta_5(V) = 2b_1b_2 - 11(b_1 + b_2) + 63.$$

For $mult(g) \ge 7$, we conclude that

$$2b_1b_2 - 11(b_1 + b_2) + 63 \le \frac{2b_1b_2^2}{b_2 - 1} - 11(\frac{b_1b_2}{b_2 - 1} + b_2) + 60.$$

Proof. After simple calculation, the moduli algebra $M_5(V)$ defined as

$$M_5(V) = \mathbb{C}\{x_1, x_2\}/(g_{x_1x_1x_1x_1x_1}, g_{x_2x_2x_2x_2x_2}, g_{x_1x_2x_2x_2x_2}, g_{x_1x_1x_2x_2x_2}, g_{x_1x_1x_1x_2x_2}, g_{x_1x_1x_1x_1x_2})$$

has a monomial basis of the form

$$\{x_1^{j_1} x_2^{j_2}, 0 \le j_1 \le b_1 - 6; 0 \le j_2 \le b_2 - 6; x_1^{b_1-5}\}. \tag{3}$$

Without loss of generality, one can write derivation D in terms of the monomial basis in the following way:

$$Dx_i = \sum_{j_1=0}^{b_1-6}\sum_{j_2=0}^{b_2-6} c^i_{j_1,j_2} x_1^{j_1} x_2^{j_2} + c^i_{b_1-5,0} x_1^{b_1-5},\ i = 1,2.$$

The Lie algebra $\mathcal{L}_5(V)$ has the following basis:

$$x_1^{j_1} x_2^{j_2}\partial_1, 1 \leq j_1 \leq b_1-6, 0 \leq j_2 \leq b_2-6; x_1^{j_1} x_2^{j_2}\partial_2, 0 \leq j_1 \leq b_1-6, 1 \leq j_2 \leq b_2-6;$$

$$x_2^{b_2-6}\partial_1; x_1^{b_1-5}\partial_1; x_1^{b_1-5}\partial_2.$$

We obtain the following formula

$$\delta_5(V) = 2b_1b_2 - 11(b_1+b_2) + 63.$$

Finally, we need to show that

$$2b_1b_2 - 11(b_1+b_2) + 63 \leq \frac{2b_1b_2^2}{b_2-1} - 11\left(\frac{b_1b_2}{b_2-1} + b_2\right) + 60. \tag{4}$$

After solving 4, we have $b_1(b_2-9) + b_2(b_1-5) + 5 \geq 0$. □

Proposition 5. *Let $(V,0)$ be a binomial singularity of type C defined by $g = x_1^{b_1}x_2 + x_2^{b_2}x_1$ ($b_1 \geq 6, b_2 \geq 6$) with weight type $(\frac{b_2-1}{b_1b_2-1}, \frac{b_1-1}{b_1b_2-1}; 1)$. Then,*

$$\delta_5(V) = \begin{cases} 2b_1b_2 - 11(b_1+b_2) + 66; & b_1 \geq 7, b_2 \geq 7 \\ b_2 - 2; & b_1 = 6, b_2 \geq 6. \end{cases}$$

For $mult(g) \geq 7$, we conclude that

$$2b_1b_2 - 11(b_1+b_2) + 66 \leq \frac{2(b_1b_2-1)^2}{(b_1-1)(b_2-1)} - 11(b_1b_2-1)\left(\frac{b_1+b_2-2}{(b_1-1)(b_2-1)}\right) + 60.$$

Proof. After simple calculation, the following moduli algebra

$$M_5(V) = \mathbb{C}\{x_1,x_2\}/(g_{x_1x_1x_1x_1x_1}, g_{x_2x_2x_2x_2x_2}, g_{x_1x_2x_2x_2x_2}, g_{x_1x_1x_2x_2x_2}, g_{x_1x_1x_1x_2x_2}, g_{x_1x_1x_1x_1x_2})$$

has a monomial basis of the form

$$\{x_1^{j_1}x_2^{j_2}, 0 \leq j_1 \leq b_1-6; 0 \leq j_2 \leq b_2-6; x_1^{b_1-5}; x_2^{b_2-5}\}. \tag{5}$$

Without loss of generality, one can write derivation D in terms of the monomial basis in the following way:

$$Dx_i = \sum_{j_1=0}^{b_1-6}\sum_{j_2=0}^{b_2-6} c^i_{j_1,j_2} x_1^{j_1} x_2^{j_2} + c^i_{b_1-5,0} x_1^{b_1-5} + c^i_{0,b_2-5} x_2^{b_2-5},\ i = 1,2.$$

The Lie algebra $\mathcal{L}_5(V)$ has the following basis:

$$x_1^{j_1} x_2^{j_2}\partial_1, 1 \leq j_1 \leq b_1-6, 0 \leq j_2 \leq b_2-6; x_1^{j_1} x_2^{j_2}\partial_2, 0 \leq j_1 \leq b_1-6, 1 \leq j_2 \leq b_2-6;$$

$$x_2^{b_2-6}\partial_1; x_2^{b_2-5}\partial_1; x_1^{b_1-5}\partial_1; x_2^{b_2-5}\partial_2; x_1^{b_1-6}\partial_2; x_1^{b_1-5}\partial_2.$$

Therefore, we obtain

$$\delta_5(V) = 2b_1b_2 - 11(b_1+b_2) + 66.$$

For $b_1 = 6, b_2 \geq 6$, we obtain the following bases of Lie algebra $\mathcal{L}_5(V)$:

$$x_2^{j_2}\partial_2, 1 \leq j_2 \leq b_2 - 5; x_2^{b_2-5}\partial_1; x_1\partial_1; x_1\partial_2.$$

We also need to show that

$$2b_1b_2 - 11(b_1+b_2) + 66 \leq \frac{2(b_1b_2-1)^2}{(b_1-1)(b_2-1)} - 11(b_1b_2-1)(\frac{b_1+b_2-2}{(b_1-1)(b_2-1)}) + 60. \quad (6)$$

After solving 6, we have
$b_1b_2^2[(b_2-4)(b_1-4) - b_1(b_2-7)] + b_2^3 + 4b_1^2b_2 + 10b_2^2(b_1-5) + 6b_1b_2(b_1-5)$
$+3b_1^2(b_2-5) + b_1b_2(b_1-5) + 15b_1 + 2(b_2-5) \geq 0.$

Similarly, we can check that Conjecture 1 holds true for $b_1 = 6, b_2 \geq 6$. □

Remark 2. *Let $(V,0)$ be a trinomial singularity of type 1 defined by $g = x_1^{b_1} + x_2^{b_2} + x_3^{b_3}$ ($b_1 \geq 7, b_2 \geq 7, b_3 \geq 7$) with weight type $(\frac{1}{b_1}, \frac{1}{b_2}, \frac{1}{b_3}; 1)$. Then, from Proposition 3, we obtain*

$$\delta_5(V) = 3b_1b_2b_3 + 85(b_1+b_2+b_3) - 16(b_1b_2+b_1b_3+b_2b_3) - 450.$$

Proposition 6. *Let $(V,0)$ be a trinomial singularity of type 2 defined by $g = x_1^{b_1}x_2 + x_2^{b_2}x_3 + x_3^{b_3}$ ($b_1 \geq 6, b_2 \geq 6, b_3 \geq 7$) with weight type $(\frac{1-b_3+b_2b_3}{b_1b_2b_3}, \frac{b_3-1}{b_2b_3}, \frac{1}{b_3}; 1)$. Then,*

$$\delta_5(V) = \begin{cases} 3b_1b_2b_3 - 16(b_1b_2+b_1b_3+b_2b_3) + 89(b_1+b_3) & \\ +85b_2 - 493; & b_1 \geq 6, b_2 \geq 7, b_3 \geq 7 \\ 2b_1b_3 - 7b_1 - 9b_3 + 29; & b_1 \geq 6, b_2 = 6, b_3 \geq 7. \end{cases}$$

For $b_1 \geq 6, b_2 \geq 7, b_3 \geq 7$, we conclude that:

$$3b_1b_2b_3 - 16(b_1b_2+b_1b_3+b_2b_3) + 89(b_1+b_3) + 85b_2 - 493 \leq \frac{3b_1b_2^2b_3^3}{(1-b_3+b_2b_3)(b_3-1)}$$
$$-16(\frac{b_1b_2^2b_3^2}{(1-b_3+b_2b_3)(b_3-1)} + \frac{b_1b_2b_3^2}{1-b_3+b_2b_3} + \frac{b_2b_3^2}{b_3-1}) + 85(\frac{b_1b_2b_3}{1-b_3+b_2b_3}$$
$$+\frac{b_2b_3}{b_3-1} + b_3) - 450.$$

Proof. After simple calculation, the moduli algebra $M_5(V)$ has the following basis:

$$\{x_1^{j_1}x_2^{j_2}x_3^{j_3}, 0 \leq j_1 \leq b_1-6; 0 \leq j_2 \leq b_2-6; 0 \leq j_3 \leq b_3-6; x_1^{b_1-5}x_3^{j_3}, 0 \leq j_3 \leq b_3-6;$$
$$x_1^{j_1}x_3^{b_3-5}, 0 \leq j_1 \leq b_1-6\}.$$

Without loss of generality, one can write derivation D in terms of the monomial basis in the following way:

$$Dx_i = \sum_{j_1=0}^{b_1-6}\sum_{j_2=0}^{b_2-6}\sum_{j_3=0}^{b_3-6} c^i_{j_1,j_2,j_3}x_1^{j_1}x_2^{j_2}x_3^{j_3} + \sum_{j_1=0}^{b_1-6} c^i_{j_1,0,b_3-5}x_1^{j_1}x_3^{b_3-5} + \sum_{j_3=0}^{b_3-6} c^i_{b_1-5,0,j_3}x_3^{j_3}x_1^{b_1-5}, \ i = 1,2,3.$$

The Lie algebra $\mathcal{L}_5(V)$ has following basis:

$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_1,\ 1\leq j_1\leq b_1-6, 0\leq j_2\leq b_2-6, 0\leq j_3\leq b_3-6; x_1^{b_1-5}x_3^{j_3}\partial_1,\ 0\leq j_3\leq b_3-6,$$
$$x_2^{b_2-6}x_3^{j_3}\partial_1,\ 1\leq j_3\leq b_3-6; x_1^{j_1}x_2^{b_2-5}\partial_1,\ 0\leq j_1\leq b_1-6,$$
$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_2,\ 0\leq j_1\leq b_1-6, 1\leq j_2\leq b_2-6, 0\leq j_3\leq b_3-6; x_1^{b_1-5}x_3^{j_3}\partial_2,\ 0\leq j_3\leq b_3-6,$$
$$x_1^{j_1}x_2^{b_2-5}\partial_2,\ 0\leq j_1\leq b_1-6; x_1^{j_1}x_3^{b_3-6}\partial_2,\ 1\leq j_1\leq b_1-6,$$
$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_3,\ 0\leq j_1\leq b_1-6, 0\leq j_2\leq b_2-6, 1\leq j_3\leq b_3-6, x_1^{j_1}x_2^{b_2-5}\partial_3,\ 0\leq j_1\leq b_1-6,$$
$$x_1^{b_1-5}x_3^{j_3}\partial_3,\ 1\leq j_3\leq b_3-6.$$

We obtain

$$\delta_5(V)=3b_1b_2b_3-16(b_1b_2+b_1b_3+b_2b_3)+89(b_1+b_3)+85b_2-493.$$

For $b_1\geq 6, b_2=6, b_3\geq 7$, we obtain the following basis:

$$x_1^{j_1}x_3^{j_3}\partial_1,\ 1\leq j_1\leq b_1-5, 0\leq j_3\leq b_3-6; x_1^{j_1}x_2\partial_1,\ 0\leq j_1\leq b_1-6,$$
$$x_1^{j_1}x_2\partial_2,\ 0\leq j_1\leq b_1-6; x_1^{j_1}x_3^{b_3-6}\partial_2,\ 1\leq j_1\leq b_1-5,$$
$$x_1^{j_1}x_3^{j_3}\partial_3,\ 0\leq j_1\leq b_1-5, 1\leq j_3\leq b_3-6; x_1^{j_1}x_2\partial_3,\ 0\leq j_1\leq b_1-6.$$

We obtain

$$\delta_5(V)=2b_1b_3-7b_1-9b_3+29.$$

For $b_1\geq 6, b_2\geq 7, b_3\geq 7$, we need to prove following inequality:

$$3b_1b_2b_3-16(b_1b_2+b_1b_3+b_2b_3)+89(b_1+b_3)+85b_2-493\leq\frac{3b_1b_2^2b_3^3}{(1-b_3+b_2b_3)(b_3-1)}$$
$$-16\Big(\frac{b_1b_2^2b_3^2}{(1-b_3+b_2b_3)(b_3-1)}+\frac{b_1b_2b_3^2}{1-b_3+b_2b_3}+\frac{b_2b_3^2}{b_3-1}\Big)+85\Big(\frac{b_1b_2b_3}{1-b_3+b_2b_3}$$
$$+\frac{b_2b_3}{b_3-1}+b_3\Big)-450.$$

After solving the above inequality, we obtain
$(b_1-4)^3(b_2-6)b_3+(b_2-5)b_1b_3((b_3-4)(b_1-6)+(b_2-4)(b_3-4))+b_2(3b_3-5)(b_1-4)+b_2(b_1-3)+6\geq 0$.

Similarly, one can prove that for $b_1\geq 6, b_2=6, b_3\geq 7$ Conjecture 1 holds true. □

Proposition 7. *Let $(V,0)$ be a trinomial singularity of type 3 defined by $g=x_1^{b_1}x_2+x_2^{b_2}x_3+x_3^{b_3}x_1$ ($b_1\geq 6, b_2\geq 6, b_3\geq 6$) with weight type*

$$\Big(\frac{1-b_3+b_2b_3}{1+b_1b_2b_3},\frac{1-b_1+b_1b_3}{1+b_1b_2b_3},\frac{1-b_2+b_1b_2}{1+b_1b_2b_3};1\Big).$$

Then,

$$\delta_5(V)=\begin{cases}3b_1b_2b_3+89(b_1+b_2+b_3)-16(b_1b_2+b_1b_3+b_2b_3) & \\ -543; & b_1\geq 7, b_2\geq 7, b_3\geq 7\\ 2b_2b_3-9b_2-7b_3+33; & b_1=6, b_2\geq 7, b_3\geq 6\\ 2b_1b_3-7b_1-9b_3+33; & b_1\geq 6, b_2=6, b_3\geq 6\\ 2b_1b_2-9b_1-7b_2+33; & b_1\geq 7, b_2\geq 7, b_3=6\end{cases}$$

For $b_1 \geq 7, b_2 \geq 7, b_3 \geq 7$, we conclude that:
$3b_1b_2b_3 + 89(b_1 + b_2 + b_3) - 16(b_1b_2 + b_1b_3 + b_2b_3) - 543 \leq \frac{3(1+b_1b_2b_3)^3}{(1-b_3+b_2b_3)(1-b_1+b_1b_3)(1-b_2+b_1b_2)}$
$+ 85(\frac{1+b_1b_2b_3}{1-b_3+b_2b_3} + \frac{1+b_1b_2b_3}{1-b_1+b_1b_3} + \frac{1+b_1b_2b_3}{1-b_2+b_1b_2}) - 16(\frac{(1+b_1b_2b_3)^2}{(1-b_3+b_2b_3)(1-b_1+b_1b_3)} + \frac{(1+b_1b_2b_3)^2}{(1-b_1+b_1b_3)(1-b_2+b_1b_2)}$
$+ \frac{(1+b_1b_2b_3)^2}{(1-b_3+b_2b_3)(1-b_2+b_1b_2)}) - 450.$

Proof. The moduli algebra $M_5(V)$ has the following monomial basis

$$\{x_1^{j_1}x_2^{j_2}x_3^{j_3}, 0 \leq j_1 \leq b_1 - 6; 0 \leq j_2 \leq b_2 - 6; 0 \leq j_3 \leq b_3 - 6; x_1^{b_1-5}x_3^{j_3}, 0 \leq j_3 \leq b_3 - 6;$$
$$x_2^{j_2}x_3^{b_3-5}, 0 \leq j_2 \leq b_2 - 6; x_1^{j_1}x_2^{b_2-5}, 0 \leq j_1 \leq b_1 - 6\}.$$

Without loss of generality, one can write derivation D in terms of the monomial basis in the following way:

$$Dx_i = \sum_{j_1=0}^{b_1-6}\sum_{j_2=0}^{b_2-6}\sum_{j_3=0}^{b_3-6} c^i_{j_1,j_2,j_3}x_1^{j_1}x_2^{j_2}x_3^{j_3} + \sum_{j_1=0}^{b_1-6} c^i_{j_1,b_2-5,0}x_1^{j_1}x_2^{b_2-5} + \sum_{j_3=0}^{b_3-6} c^i_{b_1-5,0,j_3}x_1^{b_1-5}x_3^{j_3}$$
$$+ \sum_{j_2=0}^{b_2-6} c^i_{0,j_2,b_3-5}x_2^{j_2}x_3^{b_3-5},\ i = 1,2,3.$$

The Lie algebras $\mathcal{L}_5(V)$ have the following bases:

$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_1,\ 1 \leq j_1 \leq b_1 - 6, 0 \leq j_2 \leq b_2 - 6, 0 \leq j_3 \leq b_3 - 6; x_2^{j_2}x_3^{b_3-5}\partial_1,\ 0 \leq j_2 \leq b_2 - 7,$$
$$x_2^{b_2-6}x_3^{j_3}\partial_1,\ 1 \leq j_3 \leq b_3 - 5; x_1^{j_1}x_2^{b_2-5}\partial_1,\ 0 \leq j_1 \leq b_1 - 6; x_1^{b_1-5}x_3^{j_3}\partial_1,\ 0 \leq j_3 \leq b_3 - 6,$$
$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_2,\ 0 \leq j_1 \leq b_1 - 6, 1 \leq j_2 \leq b_2 - 6, 0 \leq j_3 \leq b_3 - 6; x_1^{b_1-5}x_3^{j_3}\partial_2,\ 0 \leq j_3 \leq b_3 - 6,$$
$$x_1^{j_1}x_2^{b_2-5}\partial_2,\ 0 \leq j_1 \leq b_1 - 6; x_1^{j_1}x_3^{b_3-6}\partial_2,\ 1 \leq j_1 \leq b_1 - 6; x_2^{j_2}x_3^{b_3-5}\partial_2,\ 0 \leq j_2 \leq b_2 - 6,$$
$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_3,\ 0 \leq j_1 \leq b_1 - 6, 0 \leq j_2 \leq b_2 - 6, 1 \leq j_3 \leq b_3 - 6; x_1^{j_1}x_2^{b_2-5}\partial_3,\ 0 \leq j_1 \leq b_1 - 6,$$
$$x_1^{b_1-6}x_2^{j_2}\partial_3,\ 1 \leq j_2 \leq b_2 - 6; x_2^{j_2}x_3^{b_3-5}\partial_3,\ 0 \leq j_2 \leq b_2 - 6; x_1^{b_1-4}x_3^{j_3}\partial_3,\ 0 \leq j_3 \leq b_3 - 6.$$

Therefore, we have

$$\delta_5(V) = 3b_1b_2b_3 + 89(b_1 + b_2 + b_3) - 16(b_1b_2 + b_1b_3 + b_2b_3) - 543.$$

In case of $b_1 = 6, b_2 \geq 7, b_3 \geq 6$, we obtain the following basis:

$$x_2^{b_2-6}x_3^{j_3}\partial_1,\ 1 \leq j_3 \leq b_3 - 5; x_1x_3^{j_3}\partial_1,\ 0 \leq j_3 \leq b_3 - 6; x_2^{b_2-5}\partial_1; x_3^{b_3-5}\partial_2,$$
$$x_1x_3^{j_3}\partial_2,\ 0 \leq j_3 \leq b_3 - 6; x_2^{b_2-5}\partial_2; x_2^{j_2}x_3^{j_3}\partial_2,\ 1 \leq j_2 \leq b_2 - 6, 0 \leq j_3 \leq b_3 - 5,$$
$$x_2^{j_2}x_3^{j_3}\partial_3,\ 0 \leq j_2 \leq b_2 - 6, 1 \leq j_3 \leq b_3 - 5, x_1x_3^{j_3}\partial_3,\ 0 \leq j_3 \leq b_3 - 6; x_2^{b_2-5}\partial_3.$$

Therefore, we have

$$\delta_5(V) = 2b_2b_3 - 9b_2 - 7b_3 + 33.$$

Similarly, we can obtain bases for $b_1 \geq 7, b_2 \geq 7, b_3 = 6$ and $b_1 \geq 6, b_2 = 6, b_3 \geq 6$. For $b_1 \geq 7, b_2 \geq 7, b_3 \geq 7$, we need to prove following inequality:

$3b_1b_2b_3 + 89(b_1 + b_2 + b_3) - 13(b_1b_2 + b_1b_3 + b_2b_3) - 543 \leq \frac{3(1+b_1b_2b_3)^3}{(1-b_3+b_2b_3)(1-b_1+b_1b_3)(1-b_2+b_1b_2)}$
$+ 85(\frac{1+b_1b_2b_3}{1-b_3+b_2b_3} + \frac{1+b_1b_2b_3}{1-b_1+b_1b_3} + \frac{1+b_1b_2b_3}{1-b_2+b_1b_2}) - 16(\frac{(1+b_1b_2b_3)^2}{(1-b_3+b_2b_3)(1-b_1+b_1b_3)} + \frac{(1+b_1b_2b_3)^2}{(1-b_1+b_1b_3)(1-b_2+b_1b_2)}$
$+ \frac{(1+b_1b_2b_3)^2}{(1-b_3+b_2b_3)(1-b_2+b_1b_2)}) - 450.$

After solving the above inequality, we obtain
$4(b_1b_2+b_2b_3+b_1b_3)+b_1(b_2-6)+b_2(b_3-6)+b_3(b_1-6)+4b_1^2[b_2(b_3-6)+b_3(b_2-6)]$
$+3b_2^2[b_1(b_3-5)+b_3(b_1-6)]+5b_3^2[b_1(b_2-6)+b_2(b_1-5)]+2(b_1^2+b_2^2+b_3^2)+3(b_1^3b_2+$
$b_2^3b_3$
$+b_3^3b_1)+2b_1^2b_2^2b_3^2+5(b_1b_2^2b_3+b_1b_2b_3^2)+2b_1^2b_2b_3+b_1b_2b_3[2b_1-10]+b_1^3b_2b_3^2(b_3-6)(b_2-6)$
$+b_1^2b_3^2(b_3-6)(b_1b_2-6)+b_1^2b_2b_3^2(b_3+b_2-7)+3b_1b_2b_3^3(b_1-6)+b_1^2b_2^3b_3(b_3-6)(b_1-5)$
$+b_1^2b_2^2(b_1-6)(b_2a_3-5)+b_1^3b_2b_3(b_2-6)+b_1^2b_2^2b_3(b_1-5+(b_3-6))+b_1b_2^2b_3^3(b_2-6)(b_1$
$-5)+b_2^2b_3^2(b_2-6)(b_1b_3-6)+11\geq 0.$

Similarly, we can check that Conjecture 1 holds true for 1): $b_1,b_3\geq 6, b_2=6$; 2): $b_1\geq 7, b_2\geq 7, b_3=6$; and 3): $b_1=6, b_2\geq 7, b_3\geq 6$. □

Proposition 8. *Let $(V,0)$ be a trinomial singularity of type 4 defined by $g=x_1^{b_1}+x_2^{b_2}+x_3^{b_3}x_2$ ($b_1\geq 7, b_2\geq 7, b_3\geq 6$) with weight type $(\frac{1}{b_1},\frac{1}{b_2},\frac{b_2-1}{b_2b_3};1)$. Then,*

$$\delta_5(V)=3b_1b_2b_3+89b_1+85(b_2+b_3)-16(b_1b_2+b_1b_3+b_2b_3)-471.$$

For $mult(g)\geq 7$, we conclude that:
$3b_1b_2b_3+89b_1+85(b_2+b_3)-16(b_1b_2+b_1b_3+b_2b_3)-471\leq \frac{3b_2^2b_1b_3}{b_2-1}+85(b_1+b_2+\frac{b_2b_3}{b_2-1})$
$-16(b_1b_2+\frac{b_1b_2b_3}{b_2-1}+\frac{b_2^2b_3}{b_2-1})-450.$

Proof. The moduli algebra $M_5(V)$ has the following monomial basis

$$\{x_1^{j_1}x_2^{j_2}x_3^{j_3}, 0\leq j_1\leq b_1-6; 0\leq j_2\leq b_2-6; 0\leq j_3\leq b_3-6; x_1^{j_1}x_3^{b_3-5}, 0\leq j_2\leq b_1-6\}.$$

Without loss of generality, one can write derivation D in terms of the monomial basis in the following way:

$$Dx_i=\sum_{j_1=0}^{b_1-6}\sum_{j_2=0}^{b_2-6}\sum_{j_3=0}^{b_3-6}c^i_{j_1,j_2,j_3}x_1^{j_1}x_2^{j_2}x_3^{j_3}+\sum_{j_1=0}^{b_1-6}c^i_{j_1,0,b_3-5}x_1^{j_1}x_3^{b_3-5},\ i=1,2,3.$$

The Lie algebras $\mathcal{L}_5(V)$ have the following bases:

$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_1,\ 1\leq j_1\leq b_1-6, 0\leq j_2\leq b_2-6, 0\leq j_3\leq b_3-6; x_1^{j_1}x_3^{b_3-5}\partial_1,\ 1\leq j_1\leq b_1-6,$$
$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_2,\ 1\leq j_1\leq b_1-6, 1\leq j_2\leq b_2-6, 0\leq j_3\leq b_3-6; x_1^{j_1}x_3^{b_3-5}\partial_2,\ 0\leq j_1\leq b_1-6,$$
$$x_2^{j_2}x_3^{j_3}\partial_2,\ 1\leq j_2\leq b_2-6, 0\leq j_3\leq b_3-6, x_1^{j_1}x_2^{b_2-6}\partial_3,\ 0\leq j_1\leq b_1-6$$
$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_3,\ 0\leq j_1\leq b_1-6, 0\leq j_2\leq b_2-6, 1\leq j_3\leq b_3-6, x_1^{j_1}x_3^{b_3-5}\partial_3,\ 0\leq j_1\leq b_1-6.$$

Therefore, we have

$$\delta_5(V)=3b_1b_2b_3+89b_1+85(b_2+b_3)-16(b_1b_2+b_1b_3+b_2b_3)-471.$$

Next, we also need to show that when $b_1\geq 7, b_2\geq 7, b_3\geq 6$,

$3b_1b_2b_3+89b_1+85(b_2+b_3)-16(b_1b_2+b_1b_3+b_2b_3)-471\leq \frac{3b_2^2b_1b_3}{b_2-1}+85(b_1+b_2+\frac{b_2b_3}{b_2-1})$
$-16(b_1b_2+\frac{b_1b_2b_3}{b_2-1}+\frac{b_2^2b_3}{b_2-1})-450.$

From the above inequality, we obtain

$$\frac{b_1b_3(2b_2-11)}{b_2-6}+b_2b_3+b_3(b_2-4)+\frac{6b_3}{b_2-5}+\frac{b_1[b_2(b_3-5)+6]}{b_2-5}\geq 0.$$

□

Proposition 9. *Let $(V,0)$ be a trinomial singularity of type 5 defined by $g = x_1^{b_1}x_2 + x_2^{b_2}x_1 + x_3^{b_3}$ ($b_1 \geq 6, b_2 \geq 6, b_3 \geq 7$) with weight type $(\frac{b_2-1}{b_1b_2-1}, \frac{b_1-1}{b_1b_2-1}, \frac{1}{b_3}; 1)$. Then,*

$$\delta_5(V) = \begin{cases} 3b_1b_2b_3 + 85(b_1+b_2) + 93b_3 - 16(b_1b_2 + b_1b_3 + b_2b_3) \\ -492; & b_1 \geq 7, b_2 \geq 7, b_3 \geq 7 \\ 2b_2b_3 - 11b_2 - 6b_3 + 34; & b_1 = 6, b_2 \geq 6, b_3 \geq 7 \end{cases}$$

For $b_1 \geq 7, b_2 \geq 7, b_3 \geq 7$, we conclude that:
$3b_1b_2b_3 + 85(b_1+b_2) + 93b_3 - 16(b_1b_2 + b_1b_3 + b_2b_3) - 492 \leq \frac{3b_3(b_1b_2-1)^2}{(b_2-1)(b_1-1)} + 85(\frac{b_1b_2-1}{b_2-1} + \frac{b_1b_2-1}{b_1-1}$
$+b_3) - 16(\frac{(b_1b_2-1)^2}{(b_2-1)(b_1-1)} + \frac{b_3(b_1b_2-1)}{b_1-1} + \frac{b_3(b_1b_2-1)}{b_2-1}) - 450.$

Proof. The moduli algebra $M_5(V)$ has the following monomial basis

$$\{x_1^{j_1}x_2^{j_2}x_3^{j_3}, 0 \leq j_1 \leq b_1 - 6; 0 \leq j_2 \leq b_2 - 6; 0 \leq j_3 \leq b_3 - 6; x_1^{b_1-5}x_3^{j_3}, 0 \leq j_3 \leq b_3 - 6;$$
$$x_2^{b_2-5}x_3^{j_3}, 0 \leq j_3 \leq b_3 - 6\},$$

Without loss of generality, one can write derivation D in terms of the monomial basis in the following way:

$$Dx_i = \sum_{j_1=0}^{b_1-6}\sum_{j_2=0}^{b_2-6}\sum_{j_3=0}^{b_3-6} c^i_{j_1,j_2,j_3} x_1^{j_1}x_2^{j_2}x_3^{j_3} + \sum_{j_3=0}^{b_3-6} c^i_{b_1-5,0,j_3} x_1^{b_1-5}x_3^{j_3} + \sum_{j_3=0}^{b_3-6} c^i_{0,b_2-5,j_3} x_2^{b_2-5}x_3^{j_3},\ i = 1,2,3.$$

The Lie algebras $\mathcal{L}_5(V)$ have the following bases:

$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_1,\ 1 \leq j_1 \leq b_1 - 6, 0 \leq j_2 \leq b_2 - 6, 0 \leq j_3 \leq b_3 - 6; x_1^{b_1-5}x_3^{j_3}\partial_1,\ 0 \leq j_3 \leq b_3 - 6,$$
$$x_2^{b_2-5}x_3^{j_3}\partial_1,\ 0 \leq j_3 \leq b_3 - 6; x_2^{b_2-6}x_3^{j_3}\partial_1,\ 0 \leq j_3 \leq b_3 - 6,$$
$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_2,\ 0 \leq j_1 \leq b_1 - 6, 1 \leq j_2 \leq b_2 - 6, 0 \leq j_3 \leq b_3 - 6; x_1^{b_1-5}x_3^{j_3}\partial_2,\ 0 \leq j_3 \leq b_3 - 6,$$
$$x_2^{b_2-5}x_3^{j_3}\partial_2,\ 0 \leq j_3 \leq b_3 - 6; x_1^{b_1-6}x_3^{j_3}\partial_2,\ 0 \leq j_3 \leq b_3 - 6,$$
$$x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_3,\ 0 \leq j_1 \leq b_1 - 6, 0 \leq j_2 \leq b_2 - 6, 1 \leq j_3 \leq b_3 - 6; x_1^{b_1-5}x_3^{j_3}\partial_3,\ 1 \leq j_3 \leq b_3 - 6,$$
$$x_2^{b_2-5}x_3^{j_3}\partial_3,\ 1 \leq j_3 \leq b_3 - 6.$$

Therefore, we have

$$\delta_5(V) = 3b_1b_2b_3 + 85(b_1+b_2) + 93b_3 - 16(b_1b_2 + b_1b_3 + b_2b_3) - 492.$$

For $b_1 = 6, b_2 \geq 6, b_3 \geq 7$, we obtain the following basis:

$$x_2^{j_2}x_3^{j_3}\partial_2,\ 1 \leq j_2 \leq b_2 - 5, 0 \leq j_3 \leq b_3 - 5; x_2^{b_2-4}x_3^{j_3}\partial_1,\ 0 \leq j_3 \leq b_3 - 5,$$
$$x_1x_3^{j_3}\partial_1,\ 0 \leq j_3 \leq b_3 - 5; x_2^{b_2-4}x_3^{j_3}\partial_2,\ 0 \leq j_3 \leq b_3 - 5,$$
$$x_2^{j_2}x_3^{j_3}\partial_3,\ 0 \leq j_2 \leq b_2 - 5, 1 \leq j_3 \leq b_3 - 5; x_1x_3^{j_3}\partial_2,\ 0 \leq j_3 \leq b_3 - 5,$$
$$x_1x_3^{j_3}\partial_3,\ 1 \leq j_3 \leq b_3 - 5.$$

We have
$$\delta_5(V) = 2b_2b_3 - 11b_2 - 6b_3 + 34.$$

Next, we need to show that when $b_1 \geq 7, b_2 \geq 7, b_3 \geq 7$, then
$3b_1b_2b_3 + 85(b_1+b_2) + 93b_3 - 16(b_1b_2 + b_1b_3 + b_2b_3) - 492 \leq \frac{3b_3(b_1b_2-1)^2}{(b_2-1)(b_1-1)} + 85(\frac{b_1b_2-1}{b_2-1} +$
$\frac{b_1b_2-1}{b_1-1} + b_3) - 16(\frac{(b_1b_2-1)^2}{(b_2-1)(b_1-1)} + \frac{b_3(b_1b_2-1)}{b_1-1} + \frac{b_3(b_1b_2-1)}{b_2-1}) - 450.$

After solving the above inequality, we obtain

$$b_1(b_1-6)(b_2-5)(b_3+(b_1-4)b_2(b_2-6)b_3)+b_1^2(b_3-5)(b_2-4)+b_2^2b_1+4b_1(b_2-5)$$
$$+4b_2(b_1-5)+4b_3(b_1-4)+11b_1b_2+13b_1b_3+4b_2b_3+21b_2+b_1b_2(b_1-5)$$
$$+(b_1-4)b_2(b_2-5)(b_3-4)+(b_1-5)(b_3-6)+21\geq 0.$$

Similarly, for $b_1=6, b_2\geq 6, b_3\geq 7$, Conjecture 1 also holds true. □

Proof of Theorem 1.

Proof. Proposition 3 implies the proof of Theorem 1. □

Proof of Theorem 2.

Proof. Theorem 2 is an immediate corollary of Remark 1, Proposition 4, and Proposition 5. □

Proof of Theorem 3.

Proof. It follows from Propositions 4–5, Remark 1 and Propositions 4–5, Remark 3 of [23] that the inequality $\delta_5(V)<\delta_4(V)$ holds true. □

Proof of Theorem 4.

Proof. Propositions 6–9 and Remark 2 imply the proof of Theorem 4. □

Proof of Theorem 5.

Proof. It is follows from Propositions 6–9, Remark 2 and Propositions 6–9, Remark 4 of [23] that the inequality $\delta_5(V)<\delta_4(V)$ holds true. □

4. Conclusions

The $\delta_k(V)$ is a new analytic invariant of singularities. To find the dimension of a newly defined algebra is an important task in order to study its applications. In this paper, we computed the dimension of the Lie algebra $\mathcal{L}_5(V)$ and proved the sharp upper estimate conjecture partially for $\delta_k(V)$ of fewnomial isolated singularities (binomial and trinomial). We also proved the inequality conjecture: $\delta_5(V)<\delta_4(V)$ for a general class of singularities. The main results of this paper are the extension of previous results published in [23]. The novelty of this paper is the validity of Conjectures 1 and 2 regarding a large class of singularities, for higher values of k. The present work may also help to verify the two inequality conjectures for the general k.

Author Contributions: Conceptualization, N.H., M.A. (Muhammad Arshad) and M.A. (Muhammad Asif); methodology, N.H., M.A. (Muhammad Arshad), A.N.A.-K. and M.A. (Muhammad Asif); validation, N.H., M.A. (Muhammad Arshad), A.N.A.-K. and M.A. (Muhammad Asif); writing—original draft preparation, review, and editing, N.H., M.A. (Muhammad Arshad), and M.A. (Muhammad Asif); supervision, N.H.; funding acquisition, A.N.A.-K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the editor and the anonymous reviewers.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Mather, J.; Yau, S.S.-T. Classification of isolated hypersurface singularities by their moduli algebras. *Invent. Math.* **1982**, *69*, 243–251. [CrossRef]
2. Yau, S.S.-T. Solvable Lie algebras and generalized Cartan matrices arising from isolated singularities. *Math. Z.* **1986**, *191*, 489–506. [CrossRef]
3. Yau, S.S.-T. Solvability of Lie algebras arising from isolated singularities and nonisolatedness of singularities defined by $sl(2, \mathbb{C})$ invariant polynomials. *Am. J. Math.* **1991**, *113*, 773–778. [CrossRef]
4. Aleksandrov, A.G.; Martin, B. Derivations and deformations of Artin algebras. *Beitr. Zur Algebra Geom.* **1992**, *33*, 115–130.
5. Arnold, V.; Varchenko, A.; Gusein-Zade, S. *Singularities of Differentiable Mappings*, 2nd ed.; MCNMO: Moskva, Russia, 2004.
6. Khimshiashvili, G. Yau Algebras of Fewnomial Singularities. Preprint. Available online: http://www.math.uu.nl/publications/preprints/1352.pdf (accessed on 26 May 2021).
7. Seeley, C.; Yau, S.S.-T. Variation of complex structure and variation of Lie algebras. *Invent. Math.* **1990**, *99*, 545–565. [CrossRef]
8. Elashvili, A.; Khimshiashvili, G. Lie algebras of simple hypersurface singularities. *J. Lie Theory* **2006**, *16*, 621–649.
9. Chen, B.; Hussain, N.; Yau, S.S.-T.; Zuo, H. Variation of complex structures and variation of Lie algebras II: New Lie algebras arising from singularities. *J. Differ. Geom.* **2020**, *115*, 437–473. [CrossRef]
10. Hussain, N.; Yau, S.S.-T.; Zuo, H. On the new k-th Yau algebras of isolated hypersurface singularities. *Math. Z.* **2020**, *294*, 331–358. [CrossRef]
11. Hussain, N.; Yau, S.S.-T.; Zuo, H. k-th Yau number of isolated hypersurface singularities and an inequality conjecture. *J. Aust. Math. Soc.* **2021**, *110*, 94–118. [CrossRef]
12. Hussain, N.; Yau, S.S.-T.; Zuo, H. Inequality conjectures on derivations of Local k-th Hessain algebras associated to isolated hypersurface singularities. *Math. Z.* **2021**, *298*, 1813–1829. [CrossRef]
13. Yau, S.S.-T. Continuous family of finite-dimensional representations of a solvable Lie algebra arising from singularities. *Proc. Natl. Acad. Sci. USA* **1983**, *80*, 7694–7696. [CrossRef] [PubMed]
14. Benson, M.; Yau, S.S.-T. Lie algebra and their representations arising from isolated singularities: Computer method in calculating the Lie algebras and their cohomology. *Adv. Stud. Pure Math. Complex Anal. Singul.* **1986**, *8*, 3–58.
15. Xu, Y.-J.; Yau, S.S.-T. Micro-local characterization quasi-homogeneous singularities. *Am. J. Math.* **1996**, *118*, 389–399.
16. Yau, S.S.-T.; Zuo, H. Derivations of the moduli algebras of weighted homogeneous hypersurface singularities. *J. Algebra* **2016**, *457*, 18–25. [CrossRef]
17. Yau, S.S.-T.; Zuo, H. A Sharp upper estimate conjecture for the Yau number of weighted homogeneous isolated hypersurface singularity. *Pure Appl. Math. Q.* **2016**, *12*, 165–181. [CrossRef]
18. Chen, H.; Yau, S.S.-T.; Zuo, H. Non-existence of negative weight derivations on positively graded Artinian algebras. *Trans. Am. Math. Soc.* **2019**, *372*, 2493–2535. [CrossRef]
19. Chen, B.; Chen, H.; Yau, S.S.-T.; Zuo, H. The non-existence of negative weight derivations on positive dimensional isolated singularities: Generalized Wahl conjecture. *J. Differ. Geom.* **2020**, *115*, 195–224. [CrossRef]
20. Hussain, N.; Yau, S.S.-T.; Zuo, H. On the derivation Lie algebras of fewnomial singularities. *Bull. Aust. Math. Soc.* **2018**, *98*, 77–88. [CrossRef]
21. Hussain, N.; Yau, S.S.-T.; Zuo, H. An inequality conjecture and a weak Torelli-type theorem for isolated complete intersection singularities. *J. Geom. Phys.* **2022**, *178*, 104542. [CrossRef]
22. Ma, G.; Yau, S.S.-T.; Zuo, H. On the non-existence of negative weight derivations of the new moduli algebras of singularities. *J. Algegbra* **2020**, *564*, 199–246. [CrossRef]
23. Hussain, N.; Yau, S.S.-T.; Zuo, H. On the Dimension of a New Class of Derivation Lie Algebras Associated to Singularities. *Mathematics* **2021**, *9*, 1650. [CrossRef]
24. Hussain, N.; Yau, S.S.-T.; Zuo, H. Derivation Lie algebras of new k-th local algebras of isolated hypersurface singularities. *Pac. J. Math.* **2021**, *314*, 311–331. [CrossRef]
25. Block, R. Determination of the differentiably simple rings with a minimal ideal. *Ann. Math.* **1969**, *90*, 433–459. [CrossRef]
26. Ebeling, W.; Takahashi, A. Strange duality of weighted homogeneous polynomial. *J. Compos. Math.* **2011**, *147*, 1413–1433. [CrossRef]