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Stationary Conditions and Characterizations of Solution Sets for Interval-Valued Tightened Nonlinear Problems

Kin Keung Lai [1,*,†], Shashi Kant Mishra [2,†], Sanjeev Kumar Singh [2,†] and Mohd Hassan [2,†]

[1] International Business School, Shaanxi Normal University, Xi'an 710119, China
[2] Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi 221005, India
* Correspondence: mskklai@outlook.com
† These authors contributed equally to this work.

Abstract: In this paper, we obtain characterizations of solution sets of the interval-valued mathematical programming problems with switching constraints. Stationary conditions which are weaker than the standard Karush–Kuhn–Tucker conditions need to be discussed in order to find the necessary optimality conditions. We introduce corresponding weak, Mordukhovich, and strong stationary conditions for the corresponding interval-valued mathematical programming problems with switching constraints (IVPSC) and interval-valued tightened nonlinear problems (IVTNP), because the W-stationary condition of IVPSC is equivalent to Karush–Kuhn–Tucker conditions of the IVTNP. Furthermore, we use strong stationary conditions to characterize the several solutions sets for IVTNP, in which the last ones are particular solutions sets for IVPSC at the same time, because the feasible set of tightened nonlinear problems (IVTNP) is a subset of the feasible set of the mathematical programs with switching constraints (IVPSC).

Keywords: nonlinear programming; switching constraints; stationary conditions; interval-valued optimization

MSC: 90C30; 90C33; 49K10

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1. Introduction

Mathematical programming problems with equilibrium constraints (MPEC) [1] and mathematical programming problems with vanishing constraints (MPVC) [2] have recently found considerable attention in the area of optimal control, mathematical equilibrium, truss topology, and other research fields [3] due to a wide range of applications in real-life problems.

Singh et al. [4] established Lagrange-type duality results and saddle point optimality criteria for mathematical programs with equilibrium constraints for differentiable functions. Pandey and Mishra [5] established Wolfe and Mond–Weir-type duality results for mathematical programs with equilibrium constraints using convexificators. Pandey and Mishra [6] obtained optimality and duality results for semi-infinite mathematical programs with equilibrium constraints using convexificators. Pandey and Mishra [7] established that the Mordukhovich (M) stationary conditions [7] are strong KKT-type sufficient optimality conditions for the nonsmooth multiobjective semi-infinite mathematical programs with equilibrium constraints. Mishra et al. [8] obtained duality results for mathematical programs with vanishing constraints for differentiable functions. Mishra et al. [9] showed that Cottle, Slater, and Mangasarian–Fromovitz constraint qualifications do not hold at an efficient solution under fairly mild assumptions, whereas the Guignard constraint qualification was satisfied sometimes for mathematical programs with vanishing constraints. Mishra et al. [9] introduced suitable modifications of said constraint qualifications, established relationships, and derived the KKT-type necessary optimality conditions. Guu et al. [10] established strong KKT-type sufficient optimality conditions for nonsmooth multiobjective semi-infinite programming problems with vanishing constraints. Lai et al. [11] established Fritz–John and KKT-type stationary points conditions for nonsmooth semi-definite multiobjective mathematical programs with vanishing constraints.

Mehlitz [12] introduced the mathematical program with switching constraints (MPSC). It is not surprising that the issues involving the usual constraint qualifications for MPEC and MPVC also exist for MPSC. Mehlitz [12] showed that if an MPSC is treated as a nonlinear program, the Mangasarian–Fromovitz constraint qualifications fail at any feasible point for which there is a pair of switching functions with a value equal to zero. As a result, he introduced the concepts of weak, Mordukhovich (M-), and strong (S-) stationarity for MPSC and presented some constraint qualifications. Kanzow et al. [13] provided several relaxation methods from the numerical solutions of MPEC to MPSC. Liang and Ye [14] obtained various optimality conditions and local error bounds for MPSC. Pandey and Singh [15] studied several constraint qualifications and stationarity for multiobjective mathematical programs with switching constraints.

Uncertainty in the real world is inevitable. Therefore, imposing uncertainty in optimization problems becomes an interesting research topic. Interval-valued nonlinear programming is one such research area; see [16–19]. Lai et al. [20] established sufficient optimality conditions and duality results for semidifferentiable mathematical programming problems. Sharma et al. [21] established the Hermite–Hadamard inequalities for preinvex interval-valued functions. Su and Dinh [22] established duality results for interval-valued pseudoconvex optimization problems with equilibrium constraints with applications. Wang and Wang [23] obtained duality results for nondifferentiable semi-infinite interval-valued optimization problems with vanishing constraints.

The characterization of solution sets in mathematical programming is useful in understanding the development of solution methods for solving the problem. Mangasarian [24] introduced the concept of the characterization of solutions sets for convex programs, and Burke and Ferris [25] provided several characterizations of solution sets for nonsmooth convex programs. Jeyakumar et al. [26] provided Lagrange multiplier-based characterizations of solution sets of cone-constrained convex programs and semidefinite programs. Dinh et al. [27] studied Lagrange multiplier characterizations of solution sets of constrained pseudolinear optimization problems. Furthermore, Jeyakumar et al. [28] gave a dual characterization of the weak and proper solution sets. Jeyakumar et al. [28] discussed Lagrange multiplier characterizations of the solutions sets under regularity conditions. Lalitha and Mehta [29] derived Lagrange multiplier characterizations of solution sets for nonlinear mathematical programs with an h-convex objective and h-pseudolinear constraints. Several Lagrange multiplier characterizations of solution sets for a convex infinite programming problems are obtained in [30]. Mishra et al. [31] established several Lagrange multiplier characterizations of solution sets for constrained nonsmooth pseudolinear optimization problems. Recently, Sisarat and Wangkeeree [32] provided some characterizations of solution sets of constant pseudo Lagrangian-type functions and established Lagrange multiplier characterizations. Some recent developments of significant research on characterizations of solution sets are in [33–43] and references therein. Recently, Treanta [44] provided several characterizations of solution sets of interval-valued variational control problems and discussed its relationship with variational control problems.

Motivated by the above-mentioned work, firstly, we consider interval-valued mathematical programming with switching constraints (IVPSC). We introduce corresponding weak, Mordukhovich, and strong stationary conditions (W-stationary, M-stationary and S-stationary for short). We propose an interval-valued tightened nonlinear problem (IVTNP) associated with IVPSC. We provide several characterizations of solution sets for IVPSC with the help of the S-stationary condition and IVTNP. We construct the corresponding Lagrangian function for IVPSC. We use semiconvex functions introduced by Mifflin [45], extend for interval-valued nonsmooth functions and provide the properties of interval-valued semiconvex functions. Furthermore, we prove that the associated Lagrangian is constant under the S-stationary and semiconvexity conditions with a Clarke subdifferential. We also provide an example to support the theoretical findings.

2. Preliminaries

2.1. Interval Analysis

We collect some basic concepts and essential definitions related to interval-valued functions from Moore [46] and Wu [18].

We denote by $\mathcal{I}(\mathbb{R})$ the class of all closed intervals in $\mathbb{R}$. Let $U = [u^L, u^U]$, where u^L and u^U denote the lower and upper bounds of U, respectively. Let $U = [u^L, u^U]$ and $V = [v^L, v^U]$ be in $\mathcal{I}(\mathbb{R})$; then, we have

(i) $U + V = \{u + v : u \in U,\ v \in V\} = [u^L + v^L, u^U + v^U]$,
(ii) $-U = \{-u : u \in U\} = [-u^U, -u^L]$,
(iii) $U - V = U + (-V) = [u^L - v^U, u^U - v^L]$,
(iv) $tU = \{tu : u \in U\} = \begin{cases} [tu^L, tu^U] & \text{if} \quad t \geq 0 \\ [tu^U, tu^L] & \text{for} \quad t < 0 \end{cases}$ where t is a real number.

Let $U = [u^L, u^U]$ and $V = [v^L, v^U]$ be two closed intervals in $\mathbb{R}$. We write $U \preceq V$ if and only if $u^L \leq v^L$ and $u^U \leq v^U$. It means that U is inferior to V, or V is superior to U. It is easy to see that " $\preceq$ " is a partial ordering on $\mathcal{I}(\mathbb{R})$.

The function $f : \mathbb{R}^n \to \mathcal{I}$ is called an interval valued function; this means $f(u) = f(u_1, \cdots, u_n)$ is a closed interval in $\mathbb{R}$ for each $u \in \mathbb{R}^n$. f can be written as $f(u) = [f^L(u), f^U(u)]$, where f^L and f^U are two real valued functions defined on $\mathbb{R}^n$ such that $f^L(u) \leq f^U(u)$, $\forall u \in \mathbb{R}^n$.

We write $U \prec_{LU} V$ if and only if $U \preceq_{LU} V$ and $U \neq V$. We say $U = (U_1, \cdots, U_p)$ is an interval valued vector if each component $U_k = [u_k^L, u_k^U]$ is a closed interval for $k = 1, \cdots, p$. Suppose $U = (U_1, \cdots, U_p)$ and $V = (V_1, \cdots, V_p)$ are two interval valued vectors. We write $U \preceq_{LU} V$ if and only if $U_k \preceq_{LU} V_k$ $\forall k = 1, \cdots, p$, and $U \prec_{LU} V$ if and only if $U_k \preceq_{LU} V_k$, $\forall k = 1, \cdots, p$ and $U_q \prec_{LU} V_q$ for at least one q.

Definition 1 ([17])**.** *An interval-valued function $f(u) = [f^L(u), f^U(u)]$ defined on $X \subseteq \mathbb{R}^n$ is said to be LU-convex if $\forall u, v \in X,\ \lambda \in (0,1)$,*

$$f(\lambda u + (1-\lambda)v) \preceq_{LU} \lambda f(u) + (1-\lambda) f(v).$$

2.2. Generalized Derivatives

We collect the definitions and properties of generalized derivatives from Clarke [47]. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function at $u \in \mathbb{R}^n$. The generalized directional derivative of f at u in the direction $d \in \mathbb{R}^n$ is denoted by $f^c(u; d)$ and is defined by

$$f^c(u; d) := \limsup_{\substack{h \to 0 \\ t \downarrow 0}} \frac{f(u + h + td) - f(u + h)}{t}$$

and the Clarke's subdifferential of f at u, denoted by $\partial^c f(u)$, is defined by

$$\partial^c f(u) := \{u \in \mathbb{R}^n :\ f^c(u; d) \geq \langle u, d \rangle,\ \forall d \in \mathbb{R}^n\}.$$

We denote by $\langle u, v \rangle$ the usual inner product in n-dimensional real Euclidean space $\mathbb{R}^n$, i.e.,

$$\langle u, v \rangle = u^T v, \text{ for } u, v \in \mathbb{R}^n.$$

The directional derivatives of f at u in the direction of d, denoted by $f'(u; d)$, are defined by

$$\lim_{t \downarrow 0} \frac{f(u + td) - f(u)}{t} \text{ provided the limit exists.}$$

f is said to be regular at u in the Clarke sense if $f^{'}(u;d)$ exists and is equal to $f^{c}(u;d)$ for every $d \in \mathbb{R}^n$ [48].

Consider $f : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is an interval-valued function; then, $f(u) = [f^L(u), f^U(u)]$ is regular if both the upper and lower bound functions f^L and f^U are regular.

Suppose M is the closed convex subset of $\mathbb{R}^n$. The normal cone [49] to M at u is

$$N(M,u) = \{\eta \in \mathbb{R}^n \ : \ \langle \eta, v-u \rangle \leq 0, \ \forall v \in M\}.$$

Definition 2 ([45]). *Suppose X is a nonempty subset of $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be semiconvex at $u \in X$ if f is locally Lipschitz at u and regular at u, and it satisfies the following condition*

$$u + d \in X, d \in \mathbb{R}^n, f^{'}(u;d) \geq 0 \implies f(u+d) \geq f(u).$$

The interval-valued function $f : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is said to be semiconvex on X if f^L and f^U are semiconvex at every $u \in X$.

We can easily see from the above definition that f is semiconvex at u if $\exists \, u \in \partial^c f(u)$: $\langle \eta, v-u \rangle \geq 0 \implies f(v) \geq f(u)$.

Mifflin [45] provided an important result on semiconvex functions, which can be further generalized for interval-valued functions.

Lemma 1. *Let the function f be semiconvex on a convex set $X \subset \mathbb{R}^n$. Then, for $u \in X, d \in \mathbb{R}^n$ with $u + d \in X$, we have*

$$f(u+d) \leq f(u) \implies f^{'}(u;d) \leq 0.$$

The interval-valued function $f : \mathbb{R}^n \to \mathcal{I}(\mathbb{R})$ is semiconvex; then, for $u \in X \subset \mathbb{R}^n$, $d \in \mathbb{R}^n$ with $u + d \in X$, we have

$$f(u+d) \preceq_{LU} f(u) \implies f^{'}(u;d) \preceq_{LU} 0.$$

This means that

$$\begin{aligned} f^L(u+d) \leq f^L(u) &\implies f^{L'}(u;d) \leq 0 \\ \text{and } f^U(u+d) \leq f^U(u) &\implies f^{U'}(u;d) \leq 0. \end{aligned}$$

2.3. Interval-Valued Mathematical Programs with Switching Constraints (IVPSC)

We consider the following interval-valued mathematical programs with switching constraints (IVPSC)

$$\begin{aligned} \min f(u) &= [f^L(u), f^U(u)] \\ \text{subject to } \ g_i(u) &\leq 0, \ \forall \, i = 1, \cdots, p, \\ h_j(u) &= 0, \ \forall \, j = 1, \cdots, q, \\ G_k(u)H_k(u) &= 0, \ \forall \, k = 1, \cdots r, \end{aligned} \tag{1}$$

where the functions $f^L, \ g_i, \ h_j, \ G_k, \ H_k : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable on $\mathbb{R}^n$. We say $G_k(u)H_k(u) = 0$, while the switching constraint since functions $G_k(u), \ H_k(u)$ are active is at least one, $G_k(u) = 0$ or $H_k(u) = 0$ for all $k = 1, \cdots, r$, at any feasible point of IVPSC.

We denote the solution set of IVPSC by S.

$$\begin{aligned} S = \{u \in M : f^L(u) \leq f^L(v), \ f^U(u) \leq f^U(v), \ g(u) \leq 0, \\ h(u) = 0, \ G_k(u)H_k(u) = 0, \forall v \in M\}. \end{aligned}$$

2.4. Stationary Conditions

We need to mention some index sets to define stationary conditions at the feasible point $\bar{u}$ for IVPSC.

$$
\begin{aligned}
I_g(\bar{u}) &:= \{i \in \{1, \cdots, p\} : g_i(\bar{u}) = 0\}, \\
I^G(\bar{u}) &:= \{k \in \{1, \cdots, r\} : G_k(\bar{u}) = 0 \text{ and } H_k(\bar{u}) \neq 0\}, \\
I^H(\bar{u}) &:= \{k \in \{1, \cdots, r\} : G_k(\bar{u}) \neq 0 \text{ and } H_k(\bar{u}) = 0\}, \\
I^{GH}(\bar{u}) &:= \{k \in \{1, \cdots, r\} : G_k(\bar{u}) = 0 \text{ and } H_k(\bar{u}) = 0\}.
\end{aligned}
$$

We establish some stationary conditions in the Clarke subdifferential form motivated by Mehlitz [12]. In order to define the stationary conditions, we need to introduce the KKT system of IVPSC, which is as follows.

Definition 3. *(KKT-type conditions): A feasible point $\bar{u}$ of IVPSC is said to satisfy KKT-type conditions if there exist multipliers $\lambda^L, \lambda^U, \lambda_i(i \in \{1, \cdots, p\}), \lambda_j(j \in \{1, \cdots, q\}), \lambda_k, \mu_k(k \in \{1, \cdots, r\})$ such that the following conditions hold*

$$
\begin{aligned}
0 \in \lambda^L \partial^c f^L(\bar{u}) + \lambda^U \partial^c f^U(\bar{u}) + \sum_{i=1}^{p} \lambda_i \partial^c g_i(\bar{u}) + \sum_{j=1}^{q} \lambda_j \partial^c h_i(\bar{u}) \\
+ \sum_{k=1}^{r} [\lambda_k \partial^c G_k(\bar{u}) + \mu_k \partial^c H_k(\bar{u})], \\
\lambda_i \geq 0 \, \forall \, i \in I_g(\bar{u}).
\end{aligned}
$$

1. Weakly stationary point (W-stationary point): A feasible point $\bar{u}$ of IVPSC is called W-stationary if there exist multipliers $\lambda^L, \lambda^U, \lambda_i(i \in \{1, \cdots, p\}), \lambda_j(j \in \{1, \cdots, q\}), \lambda_k$, $\mu_k(k \in \{1, \cdots, r\})$ such that the following conditions hold

$$
\begin{aligned}
0 \in \lambda^L \partial^c f^L(\bar{u}) + \lambda^U \partial^c f^U(\bar{u}) + \sum_{i=1}^{p} \lambda_i \partial^c g_i(\bar{u}) + \sum_{j=1}^{q} \lambda_j \partial^c h_i(\bar{u}) \\
+ \sum_{k=1}^{r} [\lambda_k \partial^c G_k(\bar{u}) + \mu_k \partial^c H_k(\bar{u})], \\
\lambda_i \geq 0 \, \forall \, i \in I_g(\bar{u}), \quad \lambda_k = 0 \, \forall \, k \in I^G(\bar{u}), \quad \mu_k = 0 \, \forall \, k \in I^H(\bar{u}).
\end{aligned}
$$

2. Mordukhovich stationary point (M-stationary point): A feasible point $\bar{u}$ of IVPSC is called M-stationary if there exist multipliers $\lambda^L, \lambda^U, \lambda_i(i \in \{1, \cdots, p\}), \lambda_j(j \in \{1, \cdots, q\}), \lambda_k, \mu_k(k \in \{1, \cdots, r\})$ such that the following conditions hold

$$
\begin{aligned}
0 \in \lambda^L \partial^c f^L(\bar{u}) + \lambda^U \partial^c f^U(\bar{u}) + \sum_{i=1}^{p} \lambda_i \partial^c g_i(\bar{u}) + \sum_{j=1}^{q} \lambda_j \partial^c h_i(\bar{u}) \\
+ \sum_{k=1}^{r} [\lambda_k \partial^c G_k(\bar{u}) + \mu_k \partial^c H_k(\bar{u})], \\
\lambda_i \geq 0 \, \forall \, i \in I_g(\bar{u}), \quad \lambda_k = 0 \, \forall \, k \in I^G(\bar{u}), \quad \mu_k = 0 \, \forall \, k \in I^H(\bar{u}), \\
\text{and } \lambda_k \mu_k = 0 \quad \forall \, k \in I^{GH}(\bar{u}).
\end{aligned}
$$

3. Strong stationary point (S-stationary point): A feasible point $\bar{u}$ of IVPSC is called S-stationary if there exist multipliers $\lambda^L, \lambda^U, \lambda_i(i \in \{1, \cdots, p\}), \lambda_j(j \in \{1, \cdots, q\}), \lambda_k$, $\mu_k(k \in \{1, \cdots, r\})$ such that the following conditions hold

$$
0 \in \lambda^L \partial^c f^L(\bar{u}) + \lambda^U \partial^c f^U(\bar{u}) + \sum_{i=1}^{p} \lambda_i \partial^c g_i(\bar{u}) + \sum_{j=1}^{q} \lambda_j \partial^c h_i(\bar{u})
$$

$$+\sum_{k=1}^{r}[\lambda_k\partial^c G_k(\bar{u})+\mu_k\partial^c H_k(\bar{u})],$$
$$\lambda_i \geq 0\ \forall\, i\in I_g(\bar{u}),\quad \lambda_k=0\ \forall\, k\in I^G(\bar{u}),\quad \mu_k=0\ \forall\, k\in I^H(\bar{u}),$$
$$\text{and } \lambda_k=0,\ \mu_k=0\quad \forall\, k\in I^{GH}(\bar{u}).$$

We can easily see that the following relationship holds between the above stationary conditions.

S-stationary condition $\Longrightarrow$ M-stationary condition $\Longrightarrow$ W-stationary condition.

The W-stationary condition of IVPSC at one of its feasible points $\bar{u}$ is equivalent to KKT conditions of the following tightened nonlinear problem.

We consider the interval-valued tightened nonlinear problem (IVTNP) at $\bar{u}$.

$$\begin{aligned}(IVTNP)\qquad & \min f(\bar{u})=[f^L(\bar{u}),f^U(\bar{u})]\\ \text{subject to } & g_i(\bar{u})\leq 0,\ \forall\, i=1,\cdots,p,\\ & h_j(\bar{u})=0,\ \forall\, j=1,\cdots,q,\\ & G_k(\bar{u})=0,\ \forall\, k\in I^G(\bar{u})\cup I^{GH}(\bar{u}),\\ & H_k(\bar{u})=0,\ \forall\, k\in I^H(\bar{u})\cup I^{GH}(\bar{u}).\end{aligned}\tag{2}$$

The feasible set of IVTNP is a subset of the feasible set of IVPSC.

3. Lagrange Multiplier Characterization

We suppose that there exist multipliers $\lambda^L,\lambda^U,\lambda_i(i\in\{1,\cdots,p\}),\lambda_j(j\in\{1,\cdots,q\})$, $\lambda_k,\mu_k(k\in\{1,\cdots,r\})$ such the the following optimality conditions hold

$$\begin{aligned}0\in\lambda^L\partial^c f^L(u)+\lambda^U\partial^c f^U(u)+\sum_{i=1}^{p}\lambda_i\partial^c g_i(u)+\sum_{j=1}^{q}\lambda_j\partial^c h_i(u)&\\ +\sum_{k=1}^{r}\Big(\lambda_k\partial^c G_k(u)+\mu_k\partial^c H_k(u)\Big)+N(M,u)&,\\ \lambda_i g_i(u)=0,\forall i\in\{1,\cdots,p\},\ \lambda_j h_j(u)=0,\forall j\in\{1,\cdots,q\}&,\\ \lambda_k G_k(u)=0,\forall k\in I^G(\bar{u})\cup I^{GH}(\bar{u}),\ \mu_k H_k(u)=0,\forall k\in I^H(\bar{u})\cup I^{GH}(\bar{u})&.\end{aligned}\tag{3}$$

The addition of normal cone $N(M,u)$ in the above optimality condition is motivated by Theorem 5.1.6 of [50].

The Lagrangian function is defined by

$$\begin{aligned}L(u,\lambda,\mu)=\lambda^L f^L(u)+\lambda^U f^U(u)+\sum_{i=1}^{p}\lambda_i g_i(u)+\sum_{j=1}^{q}\lambda_j h_i(u)&\\ +\sum_{k=1}^{r}\Big(\lambda_k G_k(u)+\mu_k H_k(u)\Big)&.\end{aligned}\tag{4}$$

Lemma 2. *Let $\bar{u}$ be the solution to the problem (IVTNP) such that the condition (3) and S-stationary condition hold. Suppose that the functions $f^L,f^U,g_i(i\in\{1,\cdots,p\}),h_j(j\in\{1,\cdots,q\})$, $G_k,H_k(k\in\{1,\cdots,r\})$ are regular at $\bar{u}$ and the Lagrangian function $L(\cdot,\lambda,\mu)$ is semiconvex at $\bar{u}$; then, $L(\cdot,\lambda.\mu)$ is constant on S.*

Proof. Let $\bar{u}\in S$, and there exist multipliers $\lambda^g,\lambda^h,\lambda^G,\lambda^H$ such that condition (3) holds. Then, there exist $u^L\in\partial^c f^L(\bar{u}),u^U\in\partial^c f^U(\bar{u}),w\in N(M,\bar{u}),\nu_g\in\partial^c g_i(\bar{u})(i\in\{1,\cdots,p\})$, $\nu_h\in\partial^c h_j(\bar{u})(j\in\{1,\cdots,q\}),\nu_G\in\partial^c G_k(\bar{u}),\nu_H\in\partial^c H_k(\bar{u})(k\in\{1,\cdots,r\})$, such that

$$\lambda^L u^L+\lambda^U u^U+\sum_{i=1}^{p}\lambda_i\nu_g+\sum_{j=1}^{q}\lambda_j\nu_h+\sum_{k=1}^{r}\Big(\lambda_k\nu_G+\mu_k\nu_H\Big)=-w.$$

As M is a closed convex subset of X, $\langle w, v-\bar{u}\rangle \leq 0\ \forall v \in M$, hence, we have

$$\left\langle \lambda^L u^L + \lambda^U u^U + \sum_{i=1}^{p} \lambda_i \nu_g + \sum_{j=1}^{q} \lambda_j \nu_h + \sum_{k=1}^{r} \Big(\lambda_k \nu_G + \mu_k \nu_H\Big), v-\bar{u} \right\rangle \geq 0. \tag{5}$$

Now, since $L(\cdot,\lambda,\mu)$ is regular at $\bar{u}$, we have

$$\begin{aligned}&\Big[\lambda^L f^L + \lambda^U f^U + \sum_{i\in I_g(\bar{u})} \lambda_i g_i + \sum_{j=1}^{q} \lambda_j h_j + \sum_{k=1}^{r} \Big(\lambda_k G_k + \mu_k H_k\Big)\Big]^c (\bar{u}, v-\bar{u})\\ &= \Big[\lambda^L f^L + \lambda^U f^U + \sum_{i\in I_g(\bar{u})} \lambda_i g_i + \sum_{j=1}^{q} \lambda_j h_j + \sum_{k=1}^{r} \Big(\lambda_k G_k + \mu_k H_k\Big)\Big]' (\bar{u}, v-\bar{u}).\end{aligned} \tag{6}$$

Using the regularity of $f^L, f^U, g_i(i \in \{1,\cdots,p\}), h_j(j \in \{1,\cdots,q\}), G_k$, $H_k(k\in\{1,\cdots,r\})$ and from (5) and (6), we obtain

$$\Big[\lambda^L f^L + \lambda^U f^U + \sum_{i\in I_g(\bar{u})} \lambda_i g_i + \sum_{j=1}^{q} \lambda_j h_j + \sum_{k=1}^{r} \Big(\lambda_k G_k + \mu_k H_k\Big)\Big]' (\bar{u}, v-\bar{u}) \geq 0.$$

Since $L(\cdot,\lambda,\mu)$ is semiconvex at $\bar{u}$, we have

$$\begin{aligned}&\lambda f(\bar{u}) + \sum_{i\in I_g(\bar{u})} \lambda_i g_i(\bar{u}) + \sum_{j=1}^{q} \lambda_j h_j(\bar{u}) + \sum_{k=1}^{r} \Big(\lambda_k G_k(\bar{u}) + \mu_k H_k(\bar{u})\Big)\\ &\preceq_{LU} \lambda f(v)) + \sum_{i\in I_g(\bar{u})} \lambda_i g_i(v) + \sum_{j=1}^{q} \lambda_j h_j(v) + \sum_{k=1}^{r} \Big(\lambda_k G_k(v) + \mu_k H_k(v)\Big).\end{aligned}$$

This means

$$\begin{aligned}&\lambda^L f^L(v) + \lambda^U f^U(v) + \sum_{i\in I_g(\bar{u})} \lambda_i g_i(v) + \sum_{j=1}^{q} \lambda_j h_j(v) + \sum_{k=1}^{r} \Big(\lambda_k G_k(v) + \mu_k H_k(v)\Big)\\ &\geq \lambda^L f^L(\bar{u}) + \lambda^U f^U(\bar{u}) + \sum_{i\in I_g(\bar{u})} \lambda_i g_i(\bar{u}) + \sum_{j=1}^{q} \lambda_j h_j(\bar{u}) + \sum_{k=1}^{r} \Big(\lambda_k G_k(\bar{u}) + \mu_k H_k(\bar{u})\Big).\end{aligned} \tag{7}$$

Since condition (3) and S-stationary condition hold at $\bar{u}$, so

$$\begin{aligned}\lambda_i g_i(\bar{u}) = 0, \forall i \in \{1,\cdots,p\},\ \lambda_j h_j(\bar{u}) = 0, \forall j \in \{1,\cdots,q\},\\ \lambda_k G_k(\bar{u}) = 0, \forall k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}),\\ \mu_k H_k(\bar{u}) = 0, \forall k \in I^H(\bar{u}) \cup I^{GH}(\bar{u}).\end{aligned}$$

Hence, (7) becomes

$$\begin{aligned}\lambda^L f^L(v) + \lambda^U f^U(v) + \sum_{i\in I_g(\bar{u})} \lambda_i g_i(v) + \sum_{j=1}^{q} \lambda_j h_j(v) + \sum_{k=1}^{r} \Big(\lambda_k G_k(v) + \mu_k H_k(v)\Big)\\ \geq \lambda^L f^L(\bar{u}) + \lambda^U f^U(\bar{u}).\end{aligned} \tag{8}$$

When $v \in S$, this means $v \in M$, $g_i(v) = 0\ \forall i \in I_g(\bar{u})$ and $\lambda^L f^L(v) + \lambda^U f^U(v) = \lambda^L f^L(\bar{u}) + \lambda^U f^U(\bar{u})$. Hence,

$$\lambda^L f^L(\bar{u}) + \lambda^U f^U(\bar{u}) = \lambda^L f^L(v) + \lambda^U f^U(v)$$

$$
\begin{aligned}
&\geq \lambda^L f^L(v) + \lambda^U f^U(v) + \sum_{i \in I_g(\bar{u})} \lambda_i g_i(v) + \sum_{j=1}^{q} \lambda_j h_j(v) \\
&+ \sum_{k=1}^{r} \Big(\lambda_k G_k(v) + \mu_k H_k(v) \Big) \\
&\geq \lambda^L f^L(\bar{u}) + \lambda^U f^U(\bar{u}).
\end{aligned} \tag{9}
$$

Then, it follows from (8) and (9) that

$$
\begin{aligned}
&\sum_{i \in I_g(\bar{u})} \lambda_i g_i(v) = 0 \text{ i.e., } g_i = 0 \ (i \in I_g(\bar{u})), \\
&\sum_{j=1}^{q} \lambda_j h_j(v) = 0 \text{ i.e., } h_j = 0 \ (j \in \{1, \cdots, q\}), \\
&\sum_{k=1}^{r} \Big(\lambda_k G_k(v) + \mu_k H_k(v) \Big) = 0 \text{ i.e., } G_k = 0 = H_k \ (k \in \{1, \cdots, r\}.
\end{aligned}
$$

Therefore, $L(\cdot, \lambda, \mu)$ is constant on S. □

Theorem 1. *Let $\bar{u}$ be the solution to the problem (IVTNP), such that the condition (3) and S-stationary condition hold. Suppose that the functions f^L, f^U are semiconvex on M and the Lagrangian function $L(\cdot, \lambda, \mu)$ is semiconvex at $\bar{u}$, and suppose that the functions f^L, f^U, $g_i(i \in \{1, \cdots, p\}), h_j(j \in \{1, \cdots, q\}), G_k, H_k(k \in \{1, \cdots, r\})$ are regular at $\bar{u}$. Then, $S = S_1 = S_1^{'}$, where*

$$
\begin{aligned}
S_1 = \Big\{ & v \in M : \exists \eta \in \{\lambda^L \partial^c f^L(\bar{u}) + \lambda^U \partial^c f^U(\bar{u})\} \cap \{\lambda^L \partial^c f^L(v) + \lambda^U \partial^c f^U(v)\}, \\
& \langle \eta, \bar{u} - v \rangle = 0, g_i(v) = 0 \ \forall\, i \in I_g(\bar{u}),\ g_i(v) \leq 0 \ \forall\, i \in \{1, \cdots, p\} \setminus I_g(\bar{u}), \\
& h_j(v) = 0 \ \forall\, j \in \{1, \cdots, q\}, G_k(v) = 0, \forall k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}), \\
& H_k(v) = 0, \forall k \in I^H(\bar{u}) \cup I^{GH}(\bar{u}) \Big\}, \\
S_1^{'} = \Big\{ & v \in M : \exists \eta \in \lambda^L \partial^c f^L(v) + \lambda^U \partial^c f^U(v), \langle \eta, \bar{u} - v \rangle = 0, \\
& g_i(v) = 0 \ \forall\, i \in I_g(\bar{u}),\ g_i(v) \leq 0 \ \forall\, i \in \{1, \cdots, p\} \setminus I_g(\bar{u}), \\
& h_j(v) = 0 \ \forall\, j \in \{1, \cdots, q\}, G_k(v) = 0, \forall k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}), \\
& H_k(v) = 0, \forall k \in I^H(\bar{u}) \cup I^{GH}(\bar{u}) \Big\}.
\end{aligned}
$$

Proof. Clearly, $S_1 \subset S_1^{'}$, we claim that $S \subset S_1$ and $S_1^{'} \subset S$.

Let us suppose that $v \in S_1^{'}$, then $\exists\ \eta \in \lambda^L \partial^c f^L(v) + \lambda^U \partial^c f^U(v)$, such that $\langle \eta, \bar{u} - v \rangle = 0, g_i(v) = 0 \ \forall\ i \in I_g(\bar{u}),\ g_i(v) \leq 0 \ \forall\ i \in \{1, \cdots, p\} \setminus I_g(\bar{u}), h_j(v) = 0 \ \forall\ j \in \{1, \cdots, q\}, G_k(v) = 0, \forall k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}), H_k(v) = 0, \forall k \in I^H(\bar{u}) \cup I^{GH}(\bar{u})$.

Since f^L and f^U are semiconvex on X, $f^L(\bar{u}) \geq f^L(v)$ and $f^U(\bar{u}) \geq f^U(v)$.

In addition, since $\bar{u}, v \in M$ and $\bar{u}$ is a solution to (IVPSC), $v \in S$.

Now, we claim that $S \subset S_1$. Suppose $v \in S$, it follows from Lemma 2 that we have $g_i(v) = 0 \ \forall\, i \in I_g(\bar{u}),\ g_i(v) \leq 0 \ \forall\, i \in \{1, \cdots, p\} \setminus I_g(\bar{u})$.
As $\bar{u}$ satisfies condition (3) with $\lambda_i \in \mathbb{R}_+$ and the S-stationary condition holds at $\bar{u}$, then there exists $u^L \in \partial^c f^L(\bar{u}), u^U \in \partial^c f^U(\bar{u}), w \in N(M, \bar{u}), \nu_g \in \partial^c g_i(\bar{u})(i \in \{1, \cdots, p\})$, $\nu_h \in \partial^c h_j(\bar{u})(j \in \{1, \cdots, q\}), \nu_G \in \partial^c G_k(\bar{u}), \nu_H \in \partial^c H_k(\bar{u})(k \in \{1, \cdots, r\})$, such that

$$
\lambda^L u^L + \lambda^U u^U + \sum_{i=1}^{p} \lambda_i \nu_g + \sum_{j=1}^{q} \lambda_j \nu_h + \sum_{k=1}^{r} \Big(\lambda_k \nu_G + \mu_k \nu_H \Big) = -w.
$$

As M is a closed convex subset of X, $\langle w, v-\bar{u}\rangle \leq 0\ \forall v \in M$, therefore, for $v \in S \subseteq M$, we obtain

$$\left\langle \lambda^L u^L + \lambda^U u^U + \sum_{i=1}^{p} \lambda_i \nu_g + \sum_{j=1}^{q} \lambda_j \nu_h + \sum_{k=1}^{r} \Big(\lambda_k \nu_G + \mu_k \nu_H\Big), v-\bar{u} \right\rangle \geq 0.$$

i.e.,

$$\begin{aligned}\langle \lambda^L u^L + \lambda^U u^U, v-\bar{u}\rangle &\geq -\Big\langle \sum_{i=1}^{p} \lambda_i \nu_g + \sum_{j=1}^{q} \lambda_j \nu_h + \sum_{k=1}^{r} \Big(\lambda_k \nu_G + \mu_k \nu_H\Big), v-\bar{u}\Big\rangle \\ &= -\Big\langle \sum_{i\in I_g(\bar{u})} \lambda_i \nu_g + \sum_{j=1}^{q} \lambda_j \nu_h + \sum_{k=1}^{r} \Big(\lambda_k \nu_G + \mu_k \nu_H\Big), v-\bar{u}\Big\rangle. \end{aligned} \tag{10}$$

Since $\lambda_i g_i(\bar{u}) = 0, \forall i \in \{1,\cdots,p\}$, $\lambda_j h_j(\bar{u}) = 0, \forall j \in \{1,\cdots,q\}$ and S-stationary holds at $\bar{u}$,

$$(\lambda_i g_i)'(\bar{u}, v-\bar{u}) = \lim_{t\downarrow 0} \frac{\lambda_i g_i(\bar{u}+t(v-\bar{u})) - \lambda_i g_i(\bar{u})}{t} = \lim_{t\downarrow 0} \frac{\lambda_i g_i(\bar{u}+t(v-\bar{u}))}{t}, \tag{11}$$

$$(\lambda_j^h h_j)'(\bar{u}, v-\bar{u}) = \lim_{t\downarrow 0} \frac{\lambda_j^h h_j(\bar{u}+t(v-\bar{u})) - \lambda_j h_j(\bar{u})}{t} = \lim_{t\downarrow 0} \frac{\lambda_j^h h_j(\bar{u}+t(v-\bar{u}))}{t}, \tag{12}$$

$$(\lambda_k G_k)'(\bar{u}, v-\bar{u}) = \lim_{t\downarrow 0} \frac{\lambda_k G_k(\bar{u}+t(v-\bar{u})) - \lambda_k G_k(\bar{u})}{t} = \lim_{t\downarrow 0} \frac{\lambda_k G_k(\bar{u}+t(v-\bar{u}))}{t}, \tag{13}$$

$$(\mu_k H_k)'(\bar{u}, v-\bar{u}) = \lim_{t\downarrow 0} \frac{\mu_k H_k(\bar{u}+t(v-\bar{u})) - \mu_k H_k(\bar{u})}{t} = \lim_{t\downarrow 0} \frac{\mu_k H_k(\bar{u}+t(v-\bar{u}))}{t}. \tag{14}$$

Since M is a convex subset of M, we have $\bar{u}+t(v-\bar{u}) \in M$, provided $\bar{u}, v \in M$ and $t \in (0,1)$.

Hence,

$$\begin{aligned} \lambda_i g_i(\bar{u}+t(v-\bar{u})) &\leq 0, \forall i \in \{1,\cdots,p\}, \\ \lambda_j h_j(\bar{u}+t(v-\bar{u})) &= 0, \forall j \in \{1,\cdots,q\}, \\ \lambda_k G_k(\bar{u}+t(v-\bar{u})) &= 0, \forall k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}), \\ \mu_k H_k(\bar{u}+t(v-\bar{u})) &= 0, \forall k \in I^H(\bar{u}) \cup I^{GH}(\bar{u}). \end{aligned}$$

From (11)–(14) and the above argument, we obtain

$$\begin{aligned} (\lambda_i g_i)'(\bar{u}, v-\bar{u}) &\leq 0, i \in \{1,\cdots,p\}, \\ (\lambda_j^h h_j)'(\bar{u}, v-\bar{u}) &= 0, j \in \{1,\cdots,q\}, \\ (\lambda_k G_k)'(\bar{u}, v-\bar{u}) &= 0, k \in \{1,\cdots,r\}, \\ (\mu_k H_k)'(\bar{u}, v-\bar{u}) &= 0, k \in \{1,\cdots,r\}. \end{aligned}$$

Since, g_i, h_j, G_k, H_k are regular at $\bar{u}$, i.e.,

$$\begin{aligned} (\lambda_i g_i)'(\bar{u}, v-\bar{u}) &= (\lambda_i g_i)^c(\bar{u}, v-\bar{u}), \\ (\lambda_j^h h_j)'(\bar{u}, v-\bar{u}) &= (\lambda_j^h h_j)^c(\bar{u}, v-\bar{u}), \\ (\lambda_k G_k)'(\bar{u}, v-\bar{u}) &= (\lambda_k G_k)^c(\bar{u}, v-\bar{u}), \end{aligned}$$

$$(\mu_k H_k)'(\bar{u}, v-\bar{u}) = (\mu_k H_k)^c(\bar{u}, v-\bar{u}).$$

Let $\nu_g \in \partial^c g_i(\bar{u})(i \in \{1, \cdots, p\}), \nu_h \in \partial^c h_j(\bar{u})(j \in \{1, \cdots, q\}), \nu_G \in \partial^c G_k(\bar{u}), \nu_H \in \partial^c H_k(\bar{u})(k \in \{1, \cdots, r\})$, such that

$$\begin{aligned} \langle \lambda_i \nu_g, v-\bar{u} \rangle &\leq 0, \forall\, i \in \{1, \cdots, p\}, \\ \langle \lambda_j \nu_h, v-\bar{u} \rangle &= 0, \forall\, j \in \{1, \cdots, q\}, \\ \langle \lambda_k \nu_G, v-\bar{u} \rangle &= 0, \forall\, k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}), \\ \langle \mu_k \nu_H, v-\bar{u} \rangle &= 0, \forall\, k \in I^H(\bar{u}) \cup I^{GH}(\bar{u}). \end{aligned} \tag{15}$$

From (15) and (10), we obtain $\langle \lambda^L u^L + \lambda^U u^U, v-\bar{u} \rangle \geq 0$.
Now, since $f^L(v) = f^L(\bar{u})$ and $f^U(v) = f^U(\bar{u})$, and f^L, f^U are semiconvex at $\bar{u}$. Lemma 1 implies that $f'(\bar{u}, v-\bar{u}) \preceq_{LU} 0$; this means $(\lambda^L f^L + \lambda^U f^U)'(\bar{u}, v-\bar{u}) \leq 0$. Therefore,

$$\begin{aligned} \langle \lambda^L u^L + \lambda^U u^U, v-\bar{u} \rangle &\leq (\lambda^L f^L + \lambda^U f^U)^c(\bar{u}, v-\bar{u}) \\ &= (\lambda^L f^L + \lambda^U f^U)'(\bar{u}, v-\bar{u}) \leq 0, \\ \text{where } u^L &\in \partial^c f^L(\bar{u}), u^U \in \partial^c f^U(\bar{u}). \end{aligned}$$

Hence, $\langle \lambda^L u^L + \lambda^U u^U, v-\bar{u} \rangle = 0$.
Now, we have to prove that $\lambda^L u^L + \lambda^U u^U \in \lambda^L \partial f^L(\bar{u}) + \lambda^U \partial f^U(\bar{u}) \cap \lambda^L \partial f^L(v) + \lambda^U \partial f^U(v)$.
Since $\lambda^L u^L + \lambda^U u^U \in \lambda^L \partial f^L(\bar{u}) + \lambda^U \partial f^U(\bar{u})$, it remains to prove that $\lambda^L u^L + \lambda^U u^U \in \lambda^L \partial f^L(v) + \lambda^U \partial f^U(v)$.
f^L and f^U are regular at $\bar{u}$ and v, so we have

$$\begin{aligned} (\lambda^L f^L + \lambda^U f^U)^c(\bar{u}, d) &= (\lambda^L f^L + \lambda^U f^U)'(\bar{u}, d), \\ (\lambda^L f^L + \lambda^U f^U)^c(v, d) &= (\lambda^L f^L + \lambda^U f^U)'(v, d), \forall d \in \mathbb{R}^n. \end{aligned}$$

Now, we claim that there does not exist any $d_0 \in \mathbb{R}^n$ such that $(\lambda^L f^L + \lambda^U f^U)'(\bar{u}, d_0) < (\lambda^L f^L + \lambda^U f^U)'(v, d_0)$.
Suppose on contrary, there exists $d_0 \in \mathbb{R}^n$, such that $(\lambda^L f^L + \lambda^U f^U)'(\bar{u}, d_0) < (\lambda^L f^L + \lambda^U f^U)'(v, d_0)$, i.e.,

$$\begin{aligned} &\lim_{t_1 \downarrow 0} \frac{(\lambda^L f^L + \lambda^U f^U)(v + t_1 d_0) - (\lambda^L f^L + \lambda^U f^U)(v)}{t_1} \\ &- \lim_{t_2 \downarrow 0} \frac{(\lambda^L f^L + \lambda^U f^U)(\bar{u} + t_2 d_0) - (\lambda^L f^L + \lambda^U f^U)(\bar{u})}{t_2} < 0. \end{aligned}$$

Then

$$\begin{aligned} &\lim_{t \downarrow 0} \Bigg[\frac{(\lambda^L f^L + \lambda^U f^U)(v + t d_0) - (\lambda^L f^L + \lambda^U f^U)(v)}{t} \\ &- \frac{(\lambda^L f^L + \lambda^U f^U)(\bar{u} + t d_0) - (\lambda^L f^L + \lambda^U f^U)(\bar{u})}{t} \Bigg] < 0. \end{aligned}$$

Since $(\lambda^L f^L + \lambda^U f^U)(v) = (\lambda^L f^L + \lambda^U f^U)(\bar{u})$, we have

$$\lim_{t \downarrow 0} \frac{(\lambda^L f^L + \lambda^U f^U)(v + t d_0) - (\lambda^L f^L + \lambda^U f^U)(\bar{u} + t d_0)}{t} < 0.$$

Thus, $\exists t_0 \in (0,1)$ and $\epsilon > 0$ small enough such that

$$(\lambda^L f^L + \lambda^U f^U)(v + td_0) - (\lambda^L f^L + \lambda^U f^U)(\bar{u} + td_0) < -\epsilon < 0 \forall t \in (0, t_0). \tag{16}$$

Easily, we can see that $F(t) = (\lambda^L f^L + \lambda^U f^U)(v + td_0) - (\lambda^L f^L + \lambda^U f^U)(\bar{u} + td_0)$ is continuous at $t = 0$.

Letting $t \to 0$, we have $(\lambda^L f^L + \lambda^U f^U)(v) - (\lambda^L f^L + \lambda^U f^U)(\bar{u}) < 0$, which is a contradiction, hence, if

$$\lambda^L u^L(d) \leq (\lambda^L f^L + \lambda^U f^U)'(\bar{u}, d) = (\lambda^L f^L + \lambda^U f^U)^c(\bar{u}, d) \ \forall\, d \in \mathbb{R}^n,$$
$$\text{and } \lambda^U u^U(d) \leq (\lambda^L f^L + \lambda^U f^U)'(\bar{u}, d) = (\lambda^L f^L + \lambda^U f^U)^c(\bar{u}, d) \ \forall\, d \in \mathbb{R}^n.$$

This proves that $\lambda^L u^L + \lambda^U u^U \in \lambda^L \partial^c f^L(\bar{u}) + \lambda^U \partial^c f^U(\bar{u})$ implies $\lambda^L u^L + \lambda^U u^U \in \lambda^L \partial^c f^L(v)$ $+\lambda^U \partial^c f^U(v)$. We have $\lambda^L u^L + \lambda^U u^U \in \lambda^L \partial f^L(\bar{u}) + \lambda^U \partial f^U(\bar{u}) \cap \lambda^L \partial f^L(v) + \lambda^U \partial f^U(v)$. Hence, $v \in S_1$. This completes the proof. □

Example 1. *Consider an interval-valued optimization problem (IVPSC)*

$$\begin{aligned} &\min f(u) \\ \textit{subject to } &u_1 - u_2 \leq 0, \\ &u_1 u_2 = 0. \end{aligned}$$

where $f : \mathbb{R}^2 \to \mathcal{I}(\mathbb{R})$ *is defined by*

$$f(u_1, u_2) = \left[u_2^2 - u_1^2, u_2^2\right].$$

As $f^L(u) = u_2^2 - u_1^2$ *and* $f^U(u) = u_2^2$ *are differentiable convex functions so the coresponding subgradient and gradient are the same.*
$\nabla f^L(u) = (-2u_1, 2u_2)^T$ *and* $\nabla f^U(u) = (0, 2u_2)^T$.

Consider a set $M = \{u = (u_1, u_2) : u_1 - u_2 \leq 0, u_1 u_2 = 0\}$. f *is a LU-convex on the set*

$$M = \{u = (u_1, u_2) : u_1 - u_2 \leq 0, u_1 u_2 = 0\}.$$

Lagrangian $L(\cdot, \lambda, \mu)(u) = \lambda^L(u_2^2 - u_1^2) + \lambda^U(u_2^2) + \lambda^g(u_1 - u_2) + \lambda u_1 + \mu u_2$.

Here, the solution set is $S = \{(0,0)\}$. *Let* $\bar{u} = (0,0)$ *hold the condition* (3) *and* $L(\cdot, \lambda, \mu)$ *is convex.*

We can easily see that the condition (3) *holds for the above interval-valued problem*

$$\lambda^L \begin{bmatrix} -2u_1 \\ 2u_2 \end{bmatrix} + \lambda^U \begin{bmatrix} 0 \\ 2u_2 \end{bmatrix} + \lambda_g \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \lambda_G \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (0,0),$$

with $\lambda_g = \lambda_G = \lambda_H = 0$ *and for any values of* λ^L *and* λ^U *at point* $\bar{u} = (0,0)$. *We can also see that the S-stationary condition holds for IVPSC.*

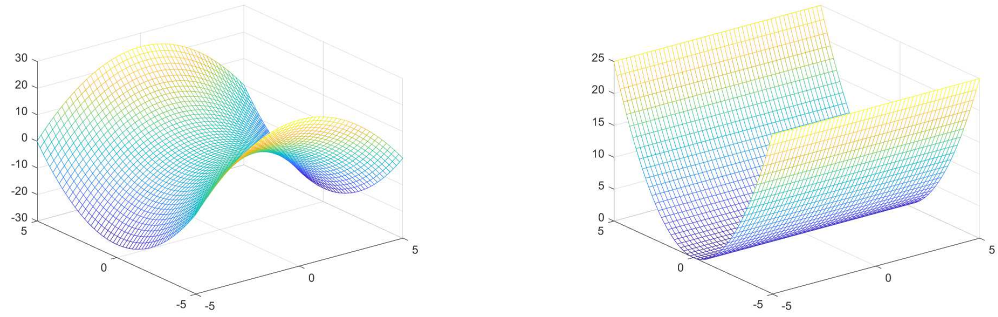
Choosing $\eta = 0 \in \lambda^L \partial f^L(\bar{u}) + \lambda^U \partial f^U(\bar{u})$ *such that* $\langle \eta, \bar{u} - v \rangle = 0 \Leftrightarrow v = 0$.

Hence,

$$\begin{aligned} S_1' = \Big\{ &v \in M : \exists \eta \in \lambda^L \partial^c f^L(v) + \lambda^U \partial^c f^U(v), \langle \eta, \bar{u} - v \rangle = 0, \\ &g_i(v) = 0 \ \forall\, i \in I_g(\bar{u}),\ g_i(v) \leq 0 \ \forall\, i \in \{1, \cdots, p\} \setminus I_g(\bar{u}), \\ &h_j(v) = 0 \ \forall\, j \in \{1, \cdots, q\}, G_k(v) = 0, \forall k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}), \\ &H_k(v) = 0, \forall k \in I^H(\bar{u}) \cup I^{GH}(\bar{u}) \Big\} = \{(0,0)\} = S. \end{aligned}$$

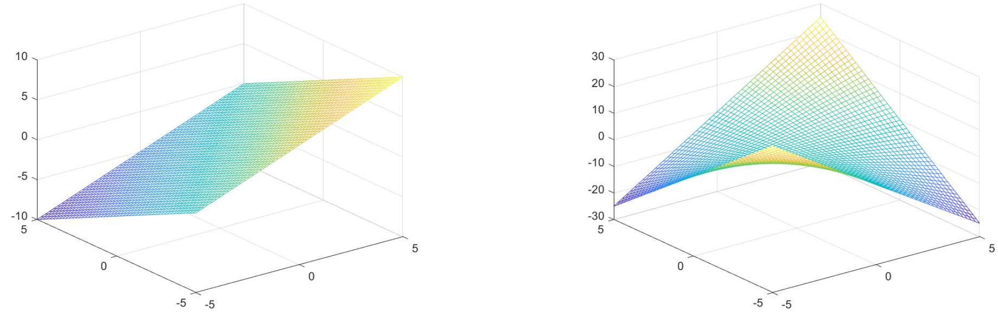
This verifies the above result.

Figure 1 represents the lower and upper bound function $f^L(u)$ and $f^U(u)$ of interval-valued objective function $f(u)$. Figure 2 shows the constraint functions $g_i(u)$ and switching constraints $G_k(u)H_k(u)$ for Example 1.



(**a**) $f^L(u)$ (**b**) $f^U(u)$

Figure 1. The lower and upper bound objective functions.



(**a**) $g_i(u)$ (**b**) $G_k(u)H_k(u)$

Figure 2. Constraints $g_i(u)$ and $G_k(u)H_k(u)$.

Corollary 1. *Let $\bar{u}$ be the solution to the problem (IVTNP) such that the condition (3) and S-stationary condition hold. Suppose that the functions $f^L, f^U, g_i(i \in \{1, \cdots, p\}), h_j(j \in \{1, \cdots, q\}), G_k, H_k(k \in \{1, \cdots, r\})$ are semiconvex on M and if the Lagrangian function $L(\cdot, \lambda, \mu)$ is semiconvex at $\bar{u}$, then $S = S_1 = S_1^{'}$, where*

$$
\begin{aligned}
S_1 = \Big\{ & v \in M : \exists \eta \in \{\lambda^L \partial^c f^L(\bar{u}) + \lambda^U \partial^c f^U(\bar{u})\} \cap \{\lambda^L \partial^c f^L(v) + \lambda^U \partial^c f^U(v)\}, \\
& \langle \eta, \bar{u} - v \rangle = 0, g_i(v) = 0 \; \forall \, i \in I_g(\bar{u}), \; g_i(v) \leq 0 \; \forall \, i \in \{1, \cdots, p\} \setminus I_g(\bar{u}), \\
& h_j(v) = 0 \; \forall \, j \in \{1, \cdots, q\}, G_k(v) = 0, \forall k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}), \\
& H_k(v) = 0, \forall k \in I^H(\bar{u}) \cup I^{GH}(\bar{u}) \Big\}, \\
S_1^{'} = \Big\{ & v \in M : \exists \eta \in \lambda^L \partial^c f^L(v) + \lambda^U \partial^c f^U(v), \langle \eta, \bar{u} - v \rangle = 0, \\
& g_i(v) = 0 \; \forall \, i \in I_g(\bar{u}), \; g_i(v) \leq 0 \; \forall \, i \in \{1, \cdots, p\} \setminus I_g(\bar{u}), \\
& h_j(v) = 0 \; \forall \, j \in \{1, \cdots, q\}, G_k(v) = 0, \forall k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}), \\
& H_k(v) = 0, \forall k \in I^H(\bar{u}) \cup I^{GH}(\bar{u}) \Big\}.
\end{aligned}
$$

We know that every convex function is semiconvex [51]. In the case where $f^L, f^U, g_i(i \in \{1, \cdots, p\}), h_j(j \in \{1, \cdots, q\}), G_k, H_k(k \in \{1, \cdots, r\})$ are convex functions, the Clarke subdifferential coincides with the subdifferential in the convex analysis.

Corollary 2. *Let $\bar{u}$ be the solution to the problem (IVTNP) such that the condition (3) and S-stationary condition hold. Suppose that the functions $f^L, f^U, g_i(i \in \{1, \cdots, p\}), h_j(j \in \{1, \cdots, q\}), G_k, H_k(k \in \{1, \cdots, r\})$ are convex; then, $S = S_2 = S_2'$, where*

$$
\begin{aligned}
S_2 = \Big\{ & v \in M : \exists \eta \in \{\lambda^L \partial f^L(\bar{u}) + \lambda^U \partial f^U(\bar{u})\} \cap \{\lambda^L \partial f^L(v) + \lambda^U \partial f^U(v)\}, \\
& \langle \eta, \bar{u} - v \rangle = 0, g_i(v) = 0 \; \forall \, i \in I_g(\bar{u}), \; g_i(v) \leq 0 \; \forall \, i \in \{1, \cdots, p\} \setminus I_g(\bar{u}), \\
& h_j(v) = 0 \; \forall \, j \in \{1, \cdots, q\}, G_k(v) = 0, \forall k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}), \\
& H_k(v) = 0, \forall k \in I^H(\bar{u}) \cup I^{GH}(\bar{u}) \Big\}, \\
S_2' = \Big\{ & v \in M : \exists \eta \in \lambda^L \partial f^L(v) + \lambda^U \partial f^U(v), \langle \eta, \bar{u} - v \rangle = 0, \\
& g_i(v) = 0 \; \forall \, i \in I_g(\bar{u}), \; g_i(v) \leq 0 \; \forall \, i \in \{1, \cdots, p\} \setminus I_g(\bar{u}), \\
& h_j(v) = 0 \; \forall \, j \in \{1, \cdots, q\}, G_k(v) = 0, \forall k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}), \\
& H_k(v) = 0, \forall k \in I^H(\bar{u}) \cup I^{GH}(\bar{u}) \Big\}.
\end{aligned}
$$

We can easily see that

$$
\begin{aligned}
& \Big[\bar{u} \in M, \sum_{i \in I_g(\bar{u})} \lambda_i g_i(v) + \sum_{j=1}^{q} \lambda_j h_j(v) + \sum_{k=1}^{r} \Big(\lambda_k G_k(v) + \mu_k H_k(v) \Big) = 0 \Big] \\
& \Leftrightarrow \Big[\bar{u} \in M, g_i(v) = 0 \; \forall \, i \in I_g(\bar{u}), \; g_i(v) \leq 0 \; \forall \, i \in \{1, \cdots, p\} \setminus I_g(\bar{u}), \\
& \qquad h_j(v) = 0 \; \forall \, j \in \{1, \cdots, q\}, G_k(v) = 0, \forall k \in I^G(\bar{u}) \cup I^{GH}(\bar{u}), \\
& \qquad H_k(v) = 0, \forall k \in I^H(\bar{u}) \cup I^{GH}(\bar{u}), \Big],
\end{aligned}
$$

and by Lemma 2, $L(v, \lambda, \mu) = \lambda^L f^L(v) + \lambda^U f^U(v) \; \forall \, v \in S$.

Corollary 3. *Suppose that the functions $f^L, f^U, g_i(i \in \{1, \cdots, p\}), h_j(j \in \{1, \cdots, q\}), G_k, H_k(k \in \{1, \cdots, r\})$ and $L(\cdot, \lambda, \mu)$ are semiconvex on M, then $S = S_3 = S_3'$, where*

$$
\begin{aligned}
S_3 = \Big\{ & v \in M : \sum_{i \in I_g(\bar{u})} \lambda_i g_i(v) + \sum_{j=1}^{q} \lambda_j h_j(v) + \sum_{k=1}^{r} \Big(\lambda_k G_k(v) + \mu_k H_k(v) \Big) = 0, \\
& \exists \eta \in \partial^c L(\cdot, \lambda^g, \lambda^h, \lambda^G, \lambda^H)(v), \; \langle \eta, \bar{u} - v \rangle = 0 \Big\}, \\
S_3' = \Big\{ & v \in M : \sum_{i \in I_g(\bar{u})} \lambda_i g_i(v) + \sum_{j=1}^{q} \lambda_j h_j(v) + \sum_{k=1}^{r} \Big(\lambda_k G_k(v) + \mu_k H_k(v) \Big) = 0, \\
& \exists \eta \in \partial^c L(\cdot, \lambda^g, \lambda^h, \lambda^G, \lambda^H)(\bar{u}) \cap \partial^c L(\cdot, \lambda^g, \lambda^h, \lambda^G, \lambda^H)(v), \; \langle \eta, \bar{u} - v \rangle = 0 \Big\}.
\end{aligned}
$$

4. Conclusions and Future Remarks

We have considered the interval-valued mathematical programming problem with switching constraints (IVPSC) and studied the Lagrange multiplier characterizations of solution sets with the help of a semiconvex function and S-stationary condition. The S-stationary condition is stronger than the W-stationary and M-stationary conditions. We have proved that the associated Lagrangian function is constant for IVTNP withholding of the S-stationary condition. Thus, the W-stationary condition holds, too. Based on the proved by Mehlitz [12] condition, for the W-stationary condition, the feasible set of a tightened nonlinear problem (IVTNP) is a subset of the feasible set of the mathematical programs with switching constraints (IVPSC). Therefore, we have characterized the particular solutions sets for IVTNP. The IVPSC is a new class of optimization problems with significant applications. MPSC can be used for the discretized version of the optimal control problem with switching structure [12], and we can extend the results to interval-valued optimization problems from a practical point of view. The IVPSC can be reformulated as a mathematical program with disjunctive constraints (MPDC) [14]. We can introduce an alternative approach to LICQ and establish the first and second order optimality conditions for MPDC with interval-valued objective functions. To the best of our knowledge, there are a few papers related to characterizations of solution sets and interval-valued nonlinear optimization. This article can be extended for various nonlinear programming problems such as MPEC, MPVC, MPDC, and many more from the application point of view.

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