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Improved Stability Criteria for Delayed Neural Networks via a Relaxed Delay-Product-Type Lyapunov–Krasovskii Functional

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Abstract: In this paper, the asymptotic stability problem of neural networks with time-varying delays is investigated. First, a new sufficient and necessary condition on a general polynomial inequality was developed. Then, a novel augmented Lyapunov–Krasovskii functional (LKF) was constructed, which efficiently introduces some new terms related to the previous information of neuron activation function. Furthermore, based on the suitable LKF and the stated negative condition of the general polynomial, two criteria with less conservatism were derived in the form of linear matrix inequalities. Finally, two numerical examples were carried out to confirm the superiority of the proposed criteria, and a larger allowable upper bound of delays was achieved.

Keywords: neural networks; asymptotic stability; polynomial inequalities; time-varying delays; Lyapunov–Krasovskii functional (LKF)

MSC: 93D20



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1. Introduction

Neural networks (NNs) are generally recognized as one of the mathematical models that simulates the intelligent activities of the human brain [1]. Because of its strong ability to model complex and nonlinear relationships between the input and output, NNs have attracted widespread attention over the past decades and have been applied in many areas, such as image processing [2], optimization problems [3], signal processing [4], parallel computation [5] and other scientific fields. However, due to the limited switching frequency of amplifiers and the inherent communication time among neurons, time delays are inevitable, which may affect the performance of NNs and lead to undesired dynamics, such as oscillation or even instability [6–8]. Therefore, it is essential to consider an allowable upper bound of delays such that the NNs with a delay less than this bound remain stable, and many excellent results have been achieved to reduce the conservatism of the stability conditions [9–12].

It is well recognized that the Lyapunov–Krasovskii functional (LKF) method is powerful in analyzing such delayed NNs. The key point is concerned with the choice of LKF [13,14]. It has been verified that an LKF with delay-dependent matrices can yield less conservatism. A delay-product-type LKF is proposed in [15], which contains a matrix-valued polynomial of any chosen degree in time-varying delay. It shows that the conservatism is affected by the maximal degree of time-varying delay included in the delay-product-type terms of the LKF. Additionally, by taking into consideration more information on state-related vectors and the activation function, some newly augmented LKFs are presented in [13,14,16,17]. As a result, less conservative conditions are predicted, as more information about the state, delay or neuron activation functions are beneficial for reducing conservatism. Therefore, one of the research contents of this work is to build a

delay-product-type LKF that contains more information on state-related vectors and the activation function, and hence is able to lead to a less conservative stability criterion.

Besides the construction of a proper LKF, the estimation of an integral term originating in the derivative of LKF is strongly related to the conservatism of criteria [18]. A common term with the form $\int_{t-h}^t \int_{\theta}^t z^T(s) T z(s) ds d\theta$ is always included in different LKFs. Then, its derivative contains the term as follows: $-\int_{t-h}^t z^T(t) T z(t) ds$. Various integral inequalities techniques have been proposed to bound such integral terms. The Jensen inequality [19] is known as an efficient tool to estimate the integral terms, but the reciprocally convex combination terms are usually encountered with respect to the time-varying delays, and the reciprocally convex lemma must be used to handle them [20,21]. Based on this work, many researchers have focus on the investigation of novel integral inequalities to obtain less conservative results of stability for the systems. Wirtinger-based integral inequality is investigated in [18], which encompasses Jensen inequality and obtains better results than Jensen inequality. Free-matrix-based integral inequality with a tighter bound has been proposed, while the reciprocally convex lemma is no longer required, but this increased the computational complexity [22]. Several zero-value equalities that add cross terms to the time derivative LKF terms are proposed, and improved results are derived. A novel estimating method called the general free-weighting-matrix approach is created in [17]. It is shown that this approach can incorporate Wirtinger-based inequality and free-matrix-based integral inequality. It has to be pointed out that the construction of an LKF should take integral inequalities into account to ensure that the derivative of the LKF includes those vectors appearing in the integral inequality used [23]. Therefore, utilizing a valid integral inequality technique to bound the integral terms is strongly connected with the conservatism of stability criteria. This paper investigates the stability of NNs following this direction.

In this paper, we further investigate the stability problem for NNs with time-varying delays. The contributions of this paper are summarized as follows:

- (i) A new sufficient and necessary condition for the general polynomial inequalities was developed by introducing some slack variables. The negative definiteness determination method was applied to derive less conservative stability criteria.
- (ii) A suitable delay-product-type LKF was constructed by efficiently introducing some new terms relating to the previous information of neuron activation function. As a result, the stability criterion based on the established LKF becomes more dependent on the technique for dealing with the single integrations of activation function.
- (iii) Based on the LKF and the stated negative condition of the general polynomial, two less conservative criteria were derived, which are demonstrated through two numerical examples, and a larger allowable upper bound of delays is achieved.

Notations: Throughout this paper, \mathbb{N}_+ represents the set of positive integers. \mathbb{R}^n and $\mathbb{R}^{m \times n}$ denote the n -dimensional vector space and the set of all $m \times n$ real matrices, respectively. Z^{-1} and Z^T stand for the inverse and the transpose of the matrix Z , respectively. $P > 0$ ($P \geq 0$) means that P is a symmetric and positive definite (semi-definite) matrix. I and 0 denote the identity matrix and zero matrix. $\text{diag}(x_1, \dots, x_n)$ represents a diagonal matrix, in which, its diagonal elements are x_1, \dots, x_n . $\text{col}(x_1, \dots, x_n) = [x_1^T, \dots, x_n^T]^T$, $\text{Sym}\{Z\} = Z + Z^T$. $*$ stands for the symmetric terms in a symmetric matrix. $\binom{n}{m}$ represents $\frac{n!}{m!(n-m)!}$.

2. Preliminaries

In this section, we consider the following NNs system:

$$\dot{x}(t) = -Ax(t) + W_0 f(Wx(t)) + W_1 f(Wx(t-d(t))), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ denotes the neuron state vector and $A \in \mathbb{R}^{n \times n}$ is a diagonal matrix. $W = \text{col}(\bar{W}_1, \bar{W}_2, \dots, \bar{W}_n)$, W_0 and W_1 are the interconnection weight matrices, and $f(Wx(t)) = \text{col}(f_1(\bar{W}_1 x_1(t)), f_2(\bar{W}_2 x_2(t)), \dots, f_n(\bar{W}_n x_n(t))) \in \mathbb{R}^n$ is the activation function with $f_i(0) = 0$. The time-varying delay $d(t)$ satisfies

$$0 \leq d(t) \leq h, -\mu \leq \dot{d}(t) \leq \mu, \quad (2)$$

where h and μ are known positive constants. The neuron activation function $f(\cdot)$ is bounded by

$$\delta_i^- \leq \frac{f_i(u_1) - f_i(u_2)}{u_1 - u_2} \leq \delta_i^+, i = 1, 2, \dots, n, \quad (3)$$

where $u_1, u_2 \in \mathbb{R}$, $u_1 \neq u_2$, δ_i^- and δ_i^+ are real scalars. For convenience, define $K_p = \text{diag}(\delta_1^+, \delta_2^+, \dots, \delta_n^+)$, $K_m = \text{diag}(\delta_1^-, \delta_2^-, \dots, \delta_n^-)$.

In order to derive the stability criteria for the system (1), the following lemmas were utilized.

Lemma 1 ([24]). For an integer $m \geq 0$, two scalars a and b with $a < b$, a real positive matrix R and a differentiable function $x : [a, b] \rightarrow \mathbb{R}^n$ such that the integration below is well defined, then

$$-(b-a) \int_a^b \dot{x}^T(s) R \dot{x}(s) ds \leq -(\Theta_m \Lambda_m \omega_m)^T \mathcal{R}_m \Theta_m \Lambda_m \omega_m \quad (4)$$

where $\mathcal{R}_m = \text{diag}(R, 3R, \dots, (2m+1)R)$,

$$\Theta_m = \begin{bmatrix} I & 0 & \dots & 0 \\ I & (-1)^1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I & (-1)^1 \begin{pmatrix} m \\ 1 \end{pmatrix} \begin{pmatrix} m+1 \\ 1 \end{pmatrix} I & \dots & (-1)^m \begin{pmatrix} m \\ m \end{pmatrix} \begin{pmatrix} 2m \\ m \end{pmatrix} I \end{bmatrix}, \quad (5)$$

$$\Lambda_m = \begin{bmatrix} I & -I & 0 & 0 & \dots & 0 \\ 0 & -I & I & 0 & \dots & 0 \\ 0 & -I & 0 & 2I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -I & 0 & 0 & \dots & mI \end{bmatrix}, \quad (6)$$

$\omega_m = \text{col}(x(b), x(a), \gamma_1(a, b), \dots, \gamma_m(a, b))$, $\gamma_k(a, b) = \frac{1}{(b-a)^k} \int_a^b (b-s)^{k-1} x(s) ds$, $(k = 1, 2, \dots, m)$.

Lemma 2 ([21]). For a given scalar $\alpha \in (0, 1)$, vectors $\omega_1, \omega_2 \in \mathbb{R}^n$ and matrices $R > 0$, $X_1 > 0$, $X_2 > 0$, $Y_1, Y_2 \in \mathbb{R}^{n \times n}$, such that

$$\begin{bmatrix} R - X_1 & Y_1 \\ * & R \end{bmatrix} \geq 0, \begin{bmatrix} R & Y_2 \\ * & R - X_2 \end{bmatrix} \geq 0,$$

the following inequality holds.

$$\begin{aligned} \frac{1}{\alpha} \omega_1^T R \omega_1 + \frac{1}{1-\alpha} \omega_2^T R \omega_2 &\geq \omega_1 [R + (1-\alpha)X_1] \omega_1 + \omega_2 [R + \alpha X_2] \omega_2 \\ &\quad + 2\omega_1 [\alpha Y_1 + (1-\alpha)Y_2] \omega_2. \end{aligned} \quad (7)$$

Lemma 3 ([25]). Consider an integrable function $x(u)$ in $[a, b]$, $R > 0$ and a matrix T with appropriate dimensions. The following inequality is true.

$$-\int_a^b x^T(s)Rx(s)ds \leq \text{Sym}\{\beta^T T \lambda\} + (b-a)\beta^T T R^{-1}T^T \beta, \quad (8)$$

where β is an arbitrary vector, and $\lambda = \int_a^b x(s)ds$.

Lemma 4. Consider the following polynomial described by $f(z) = a_{2n}z^{2n} + a_{2n-1}z^{2n-1} + \dots + a_1z + a_0$, where $n \in \mathbb{N}_+$, $z \in [0, h]$ with h is a constant; then,

(i) $f(z) < 0$ for all $z \in [0, h]$ if and only if there exists a scalar $m \geq 0$ such that

$$Y_2 = \begin{bmatrix} a_0 & q_1 & \cdots & 0 & 0 \\ * & q_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ * & * & \cdots & q_{2n-2} & q_{2n-1} \\ * & * & \cdots & * & q_{2n} \end{bmatrix} < 0, \quad (9)$$

where $q_{2n-1} = \frac{1}{2}a_{2n-1} + hm$, and $q_{2n} = a_{2n} - 2m$.

(ii) $f(z) > 0$ for all $z \in [0, h]$ if and only if there exists a scalar $m \geq 0$ such that

$$Y_1 = \begin{bmatrix} a_0 & p_1 & \cdots & 0 & 0 \\ * & p_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ * & * & \cdots & p_{2n-2} & p_{2n-1} \\ * & * & \cdots & * & p_{2n} \end{bmatrix} > 0, \quad (10)$$

where $p_{2n-1} = \frac{1}{2}a_{2n-1} - hm$, and $p_{2n} = a_{2n} + 2m$.

Proof. (i) First, for $z \in [0, h]$, we define $g(z) = z(z^{2n-1} - 2hz^{2n-2} + \dots + z^3 - 2hz^2 + z - 2h) \leq 0$. $f(z)$ and $g(z)$ can be represented as $\hat{f}(\hat{z})$ and $\hat{g}(\hat{z})$, respectively.

$$\hat{f}(\hat{z}) = \hat{z}^T \Lambda_1 \hat{z}, \quad \hat{g}(\hat{z}) = \hat{z}^T \Lambda_2 \hat{z}, \quad \hat{z} = \text{col}(1, z, \dots, z^{N-1}, z^N),$$

$$\Lambda_1 = \begin{bmatrix} a_0 & \frac{1}{2}a_1 & \cdots & 0 & 0 \\ \frac{1}{2}a_1 & a_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{2n-2} & \frac{1}{2}a_{2n-1} \\ 0 & 0 & \cdots & \frac{1}{2}a_{2n-1} & a_{2n} \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 0 & -h & \cdots & 0 & 0 \\ -h & 2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 & -h \\ 0 & 0 & \cdots & -h & 2 \end{bmatrix}.$$

(Sufficiency) If there exists a scalar $m \geq 0$ satisfying $f(z) - mg(z) = \hat{z}^T Y_1 \hat{z} < 0$, then $f(z) < 0$ for all $z \in [0, h]$.

(Necessity) $\mathcal{H} = \{(\hat{f}(\hat{z}), \hat{g}(\hat{z})) : \hat{z} \in \mathbb{R}^N\}$ is a convex set. Then, we set an open convex cone as $\mathcal{T} = \{(\alpha_1, \alpha_2) : \alpha_1 > 0, \alpha_2 < 0\}$. If there exists a nonzero two-tuple $(\beta_1, -\beta_2)$, we have

$$\alpha_1 \beta_1 - \alpha_2 \beta_2 > 0, \quad \forall (\alpha_1, \alpha_2) \in \mathcal{T} \quad (11)$$

$$\beta_1 \hat{f}(\hat{z}) - \beta_2 \hat{g}(\hat{z}) \leq 0, \quad \forall (\hat{f}(\hat{z}), \hat{g}(\hat{z})) \in \mathcal{H} \quad (12)$$

For any given $\varepsilon > 0$, we have $(\varepsilon, \alpha_2) \in \mathcal{T}$, for all $\alpha_1 > 0$. Thus, (11) implies that $\beta_2 > 0$. Since $\hat{g}(\hat{z}) < 0$ is regular, (12) shows that $\beta_1 > 0$. Hence, the following inequality can be obtained by using $\beta_1 > 0$:

$$\hat{f}(\hat{z}) - \frac{\beta_2}{\beta_1} \hat{g}(\hat{z}) \leq 0 \quad (13)$$

for $m = \frac{\beta_2}{\beta_1} \geq 0$. This means that there exists $m \geq 0$ such that $\hat{f}(z) - m\hat{g}(z) = \hat{z}^T Y_1 \hat{z} < 0$. This completes the proof of (i).

(ii) Set $\tilde{f}(z) = -f(z) > 0$; then, $g(z)$ can be rewritten as $\tilde{g}(\hat{z}) = \hat{z}^T \Lambda_3 \hat{z}$, where

$$\Lambda_3 = \begin{bmatrix} 0 & h & \cdots & 0 & 0 \\ h & -2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -2 & h \\ 0 & 0 & \cdots & h & -2 \end{bmatrix},$$

Then, the proof is straightforward from the proof of (i). \square

Remark 1. Lemma 4 provides necessary and sufficient conditions of high-order polynomial inequality $f(s) < 0$ and $f(s) > 0$ for $\forall z \in [0, h]$. When $a_{2n} = 0$, $f(z)$ represents an odd polynomial on z if $a_{2n-1} \neq 0$. Thus, Lemma 4 covers both even and odd polynomials. It is worth mentioning that Lemma 2 in [26] is a special case of Lemma 4 with $n = 1$. A quadratic function $f(z) = a_2 z^2 + a_1 z + a_0 < 0$ for all $z \in [0, h]$ if and only if there exists a scalar $m \geq 0$ such that

$$\begin{bmatrix} a_0 & \frac{1}{2}a_1 + hm \\ \frac{1}{2}a_1 + hm & a_2 - 2m \end{bmatrix} < 0.$$

3. Main Results

In this section, two improved stability criteria are developed. For simplification, the following notations are defined.

$$\begin{aligned} v_1(t) &= \text{col} \left(\int_{t-d(t)}^t \frac{x(s)}{d(t)} ds, \int_{t-d(t)}^t \frac{(t-s)x(s)}{d^2(t)} ds \right), \\ v_2(t) &= \text{col} \left(\int_{t-h}^{t-d(t)} \frac{x(s)}{h-d(t)} ds, \int_{t-h}^{t-d(t)} \frac{(t-d(t)-s)x(s)}{(h-d(t))^2} ds \right), \\ v_3(t) &= \int_{t-d(t)}^t f(Wx(s)) ds, \quad v_4(t) = \int_{t-h}^{t-d(t)} f(Wx(s)) ds, \\ \eta_a(t) &= \text{col}(x(t), x(t-d(t)), x(t-h)), \quad \eta_b(t) = \text{col}(v_3(t), v_4(t)), \\ \eta_1(t) &= \text{col}(\eta_a(t), \eta_b(t), d(t)v_1(t), (h-d(t))v_2(t)), \\ \eta_2(s, t) &= \text{col} \left(\dot{x}(s), x(s), x(t), \int_{t-d(t)}^s x(\theta) d\theta, \int_s^t x(\theta) d\theta \right), \\ \eta_3(s, t) &= \text{col} \left(x(s), x(s), x(t), \int_{t-h}^s x(\theta) d\theta, \int_s^{t-d(t)} x(\theta) d\theta \right), \quad \eta_4(t) = \text{col}(\dot{x}(t), x(t), f(Wx(t))), \\ f(t) &= f(Wx(t)), \quad f_d(t) = f(W(t-d(t))), \quad f_h(t) = f(Wx(t-h)), \\ \xi(t) &= \text{col}(\eta_a(t), \dot{x}(t-d(t)), \dot{x}(t-h), v_1(t), v_2(t), f(t), f_d(t), f_h(t), \eta_b(t)), \\ e_i &= \begin{pmatrix} 0_{n \times (i-1)n} & I_n & 0_{n \times (14-i)n} \end{pmatrix} (i = 1, 2, \dots, 14), \quad e_s = -Ae_1 + W_0 e_{10} + W_1 e_{11}. \end{aligned}$$

Now, we present the following stability conditions.

Theorem 1. For given scalars $h > 0$ and $u > 0$, the system (1) is asymptotically stable if there exist symmetric matrices $P_i > 0$, $Q_i > 0$, $R_i > 0$ ($i = 1, 2$), $Z > 0$, $T > 0$, diagonal matrices $U_j > 0$, $V_j > 0$ ($j = 1, 2, 3$), $\Delta_k > 0$ ($k = 1, \dots, 6$) and any matrices X_l , Y_l , M_l , N_l ($l = 1, 2$), such that

$$\begin{bmatrix} \tilde{Z} - X_1 & Y_1 \\ * & \tilde{Z} \end{bmatrix} \geq 0, \begin{bmatrix} \tilde{Z} & Y_2 \\ * & \tilde{Z} - X_2 \end{bmatrix} \geq 0, \quad (14)$$

$$\begin{bmatrix} M_i + M_i^T & N_1^T & N_2^T \\ * & hT & 0 \\ * & * & hT \end{bmatrix} > 0, \quad (15)$$

$$\begin{bmatrix} \Phi_0(-\mu) & 0.5\Phi_1(-\mu) + hM_1 & hN_2^T & 0 \\ * & \Phi_2(-\mu) - M_1 - M_1^T & -N_2^T & N_1^T \\ * & * & -hT & 0 \\ * & * & * & -hT \end{bmatrix} < 0, \quad (16)$$

$$\begin{bmatrix} \Phi_0(\mu) & 0.5\Phi_1(\mu) + hM_2 & hN_2^T & 0 \\ * & \Phi_2(\mu) - M_2 - M_2^T & -N_2^T & N_1^T \\ * & * & -hT & 0 \\ * & * & * & -hT \end{bmatrix} < 0, \quad (17)$$

where $\tilde{Z} = \text{diag}(Z, 3Z, 5Z)$, $\Phi(d(t), \dot{d}(t)) = \Phi_2(\dot{d}(t))d^2(t) + \Phi_1(\dot{d}(t))d(t) + \Phi_0(\dot{d}(t))$ with

$$\begin{aligned} \Phi_0(\dot{d}(t)) = & \text{Sym}\left\{hr_{11}^T P_2 r_2\right\} + \dot{d}(t)r_{11}^T(P_1 - P_2)r_{11} + r_{31}^T Q_1 r_{31} - r_{41}^T Q_2 r_{41} \\ & - (1 - \dot{d}(t))\left(r_{51}^T Q_1 r_{51} - r_{61}^T Q_2 r_{61}\right) + \text{Sym}\left\{r_9^T Q_1 r_{71} + r_{10}^T Q_2 r_{81}\right\} \\ & + \text{Sym}\left\{\left((K_p W e_1 - e_{10})^T \Delta_1 + (e_{10} - K_m W e_1)^T \Delta_2\right) W e_s\right\} \\ & + \text{Sym}\left\{\left((K_p W e_2 - e_{11})^T \Delta_3 + (e_{11} - K_m W e_2)^T \Delta_4\right) W e_4(1 - \dot{d}(t))\right\} \\ & + \text{Sym}\left\{\left((K_p W e_3 - e_{12})^T \Delta_5 + (e_{12} - K_m W e_3)^T \Delta_6\right) W e_5\right\} \\ & + \text{Sym}\left\{\sum_{c=1}^3 (K_p W e_c - e_{c+9})^T U_c(e_{c+9} - K_m W e_c)\right\} \\ & + \text{Sym}\left\{\sum_{c=1}^2 (K_p W(e_c - e_{c+1}) - (e_{c+9} - e_{c+10}))^T V_c((e_{c+9} - e_{c+10}) - K_m W(e_c - e_{c+1}))\right\} \\ & + \text{Sym}\left\{(K_p W(e_1 - e_3) - (e_{10} - e_{12}))^T V_3((e_{10} - e_{12}) - K_m W(e_1 - e_3))\right\} \\ & + \rho_1^T R_1 \rho_1 + (1 - \dot{d}(t))\rho_2^T(R_2 - R_1)\rho_2 - \rho_3^T R_2 \rho_3 + h^2 e_s^T Z e_s \\ & - \rho_4^T(\tilde{Z} + X_1)\rho_4 - \rho_5^T \tilde{Z} \rho_5 - \text{Sym}\left\{\rho_4^T Y_2 \rho_5\right\} + h e_{10}^T T e_{10} + \text{Sym}\left\{e_{13}^T N_1 + e_{14}^T N_2\right\}, \\ \Phi_1(\dot{d}(t)) = & \text{Sym}\left\{r_{11}^T(P_1 - P_2)r_2 + hr_{12}^T P_2 r_2 + \dot{d}(t)r_{11}^T(P_1 - P_2)r_{12} + r_{31}^T Q_1 r_{32} - r_{41}^T Q_2 r_{42}\right. \\ & \left. - (1 - \dot{d}(t))\left(r_{51}^T Q_1 r_{52} - r_{61}^T Q_2 r_{62}\right) + r_9^T Q_1 r_{72} + r_{10}^T Q_2 r_{82}\right\} - \frac{1}{h} \text{Sym}\left\{\rho_4^T(Y_1 - Y_2)\rho_5\right\} \\ & + \frac{1}{h}\left(\rho_4^T X_1 \rho_4 - \rho_5^T X_2 \rho_5\right), \\ \Phi_2(\dot{d}(t)) = & \text{Sym}\left\{r_{12}^T(P_1 - P_2)r_2\right\} + \dot{d}(t)r_{12}^T(P_1 - P_2)r_{12} + r_{32}^T Q_1 r_{32} - r_{42}^T Q_2 r_{42} \\ & - (1 - \dot{d}(t))\left(r_{52}^T Q_1 r_{52} - r_{62}^T Q_2 r_{62}\right) + \text{Sym}\left\{r_9^T Q_1 r_{73} + r_{10}^T Q_2 r_{83}\right\}, \end{aligned}$$

$$\begin{aligned}
r_1 &= r_{11} + d(t)r_{12}, \quad r_{11} = \text{col}(e_1, e_2, e_3, e_{13}, e_{14}, 0, 0, he_8, he_9), \quad r_{12} = \text{col}(0, 0, 0, 0, 0, e_6, e_2, -e_8, -e_9), \\
r_2 &= \text{col}(e_s, (1 - \dot{d}(t))e_4, e_5, e_{10} - (1 - \dot{d}(t))e_{11}, (1 - \dot{d}(t))e_{11} - e_{12}, e_1 - (1 - \dot{d}(t))e_2, \\
&\quad - (1 - \dot{d}(t))e_2 + e_6 - \dot{d}(t)e_7, (1 - \dot{d}(t))e_2 - e_3, -e_3 + (1 - \dot{d}(t))e_8 + \dot{d}(t)e_9), \\
r_i &= r_{i1} + d(t)r_{i2} \quad (i = 3, 4, 5, 6), \quad r_j = r_{j1} + d(t)r_{j2} + d^2(t)r_{j3} \quad (j = 7, 8), \\
r_{31} &= \text{col}(e_s, e_1, e_1, 0, 0), \quad r_{32} = \text{col}(0, 0, 0, e_6, 0), \quad r_{41} = \text{col}(e_5, e_3, e_1, 0, he_8), \\
r_{42} &= \text{col}(0, 0, 0, 0, -e_8), \quad r_{51} = \text{col}(e_4, e_2, e_1, 0, 0), \quad r_{52} = \text{col}(0, 0, 0, 0, e_6), \\
r_{61} &= \text{col}(e_4, e_2, e_1, he_8, 0), \quad r_{62} = \text{col}(0, 0, 0, -e_8, 0), \quad r_{71} = \text{col}(e_1 - e_2, 0, 0, 0, 0), \\
r_{72} &= \text{col}(0, e_6, e_1, 0, 0), \quad r_{73} = \text{col}(0, 0, 0, e_7, e_6 - e_7), \quad r_{83} = \text{col}(0, 0, 0, e_9, e_8 - e_9), \\
r_{81} &= \text{col}(e_2 - e_3, he_8, he_1, h^2e_9, h^2(e_8 - e_9)), \quad r_{82} = \text{col}(0, -e_8, -e_1, -2he_9, -2h(e_8 - e_9)), \\
r_9 &= \text{col}(0, 0, e_s, -(1 - \dot{d}(t))e_2, e_1), \quad r_{10} = \text{col}(0, 0, e_s, -e_3, (1 - \dot{d}(t))e_2), \\
\rho_1 &= \text{col}(e_s, e_1, e_{10}), \quad \rho_2 = \text{col}(e_4, e_2, e_{11}), \quad \rho_3 = \text{col}(e_5, e_3, e_{12}), \\
\rho_4 &= \text{col}(e_1 - e_2, e_1 + e_2 - 2e_6, e_1 - e_2 - 6e_6 + 12e_7), \\
\rho_5 &= \text{col}(e_2 - e_3, e_2 + e_3 - 2e_8, e_2 - e_3 - 6e_8 + 12e_9).
\end{aligned}$$

Proof. To prove the stability of the system (1), we constructed the following LKF:

$$V(t) = \sum_{i=1}^6 V_i(t), \quad (18)$$

where

$$\begin{aligned}
V_1(t) &= \eta_1^T(t)P(d(t))\eta_1(t), \\
V_2(t) &= \int_{t-d(t)}^t \eta_2^T(s, t)Q_1\eta_2(s, t)ds + \int_{t-h}^{t-d(t)} \eta_3^T(s, t)Q_2\eta_3(s, t)ds, \\
V_3(t) &= 2 \sum_{i=1}^n \int_0^{W_i x(t)} [\lambda_{1i}(\delta_i^+ s - f_i(s))] + [\lambda_{2i}(f_i(s) - \delta_i^- s)]ds, \\
&\quad + 2 \sum_{i=1}^n \int_0^{W_i x(t-d(t))} [\lambda_{3i}(\delta_i^+ s - f_i(s))] + [\lambda_{4i}(f_i(s) - \delta_i^- s)]ds, \\
&\quad + 2 \sum_{i=1}^n \int_0^{W_i x(t-h)} [\lambda_{5i}(\delta_i^+ s - f_i(s))] + [\lambda_{6i}(f_i(s) - \delta_i^- s)]ds, \\
V_4(t) &= \int_{t-d(t)}^t \eta_4^T(s)R_1\eta_4(s)ds + \int_{t-h}^{t-d(t)} \eta_4^T(s)R_2\eta_4(s)ds, \\
V_5(t) &= h \int_{t-h}^t (h - t + s)\dot{x}^T(s)Z\dot{x}(s)ds, \\
V_6(t) &= \int_{t-h}^t \int_{\theta}^t f^T(Wx(s))Tf(Wx(s))dsd\theta.
\end{aligned}$$

where $P(d(t)) = d(t)P_1 + (h - d(t))P_2$.

Then, taking the derivatives of $V(t)$ along the trajectory of system (1) yields

$$\dot{V}_1(t) = 2\eta_1^T(t)P(d(t))\dot{\eta}_1(t) + \eta_1^T(t)\dot{P}(d(t))\eta_1(t), \quad (19)$$

$$\begin{aligned} \dot{V}_2(t) = & \eta_2^T(t,t)Q_1\eta_2(t,t) - (1-\dot{d}(t))\eta_2^T(t-d(t),t)Q_1\eta_2(t-d(t),t) \\ & - \eta_3^T(t-h,t)Q_2\eta_3(t-h,t) + (1-\dot{d}(t))\eta_3^T(t-d(t),t)Q_2\eta_3(t-d(t),t) \\ & + 2\frac{\partial}{\partial t}\eta_2^T(s,t)Q_1\int_{t-d(t)}^t\eta_2(s,t)ds + 2\frac{\partial}{\partial t}\eta_3^T(s,t)Q_2\int_{t-h}^{t-d(t)}\eta_3(s,t)ds, \end{aligned} \quad (20)$$

$$\begin{aligned} \dot{V}_3(t) = & \text{Sym}\left\{\left((K_p Wx(t) - f(t))^T\Delta_1 + (f(t) - K_m Wx(t))^T\Delta_2\right)W\dot{x}(t)\right. \\ & + \left((K_p Wx(t-d(t)) - f_d(t))^T\Delta_3 + (f_d(t) - K_m Wx(t-d(t)))^T\Delta_4\right)W\dot{x}(t-d(t))(1-\dot{d}(t)) \\ & \left.+ \left((K_p Wx(t-h) - f_h(t))^T\Delta_5 + (f_h(t) - K_m Wx(t-h))^T\Delta_6\right)W\dot{x}(t-h)\right\}, \end{aligned} \quad (21)$$

$$\begin{aligned} \dot{V}_4(t) = & \eta_4^T(t)R_1\eta_4(t) + (1-\dot{d}(t))\eta_4^T(t-d(t))(R_2 - R_1)\eta_4(t-d(t)) \\ & - \eta_4^T(t-h)R_2\eta_4(t-h), \end{aligned} \quad (22)$$

$$\dot{V}_5(t) = h^2\dot{x}^T(t)Z\dot{x}(t) - h\int_{t-h}^t\dot{x}^T(s)Z\dot{x}(s)ds, \quad (23)$$

$$\dot{V}_6(t) = hf^T(Wx(t))Tf(Wx(t)) - \int_{t-h}^tf^T(Wx(t))Tf(Wx(t))ds. \quad (24)$$

Based on Lemmas 1 and 2, it follows from (14) that

$$\begin{aligned} -h\int_{t-h}^t\dot{x}^T(s)Z\dot{x}(s)ds &= -h\int_{t-h}^{t-d(t)}\dot{x}^T(s)Z\dot{x}(s)ds - h\int_{t-d(t)}^t\dot{x}^T(s)Z\dot{x}(s)ds \\ &\leq -\frac{h}{d(t)}\xi^T(t)\rho_4^TZ\rho_4\xi(t) - \frac{h}{h-d(t)}\xi^T(t)\rho_5^TZ\rho_5\xi(t) \\ &\leq \xi^T(t)\left(-\rho_4^T(\tilde{Z} + X_1)\rho_4 - \rho_5^T\tilde{Z}\rho_5 - \text{Sym}\left\{\rho_4^TY_2\rho_5\right\}\right. \\ &\quad \left.+ \frac{d(t)}{h}\left(\rho_4^TX_1\rho_4 - \rho_5^TX_2\rho_5 - \text{Sym}\left\{\rho_4^T(Y_1 - Y_2)\rho_5\right\}\right)\right)\xi(t), \end{aligned} \quad (25)$$

Using Lemma 3, one has the following inequality:

$$\begin{aligned} & -\int_{t-h}^tf^T(Wx(t))Tf(Wx(t))ds \\ & \leq \xi^T(t)(\text{Sym}\{e_{13}^TN_1 + e_{14}^TN_2\} + d(t)N_1^TT^{-1}N_1 + (h-d(t))N_2^TT^{-1}N_2)\xi(t). \end{aligned} \quad (26)$$

To make use of the information of the activation function, we can obtain

$$\phi_j(u) = 2(KpWx(u) - f(Wx(u)))^TU_j(f(Wx(u)) - KmWx(u)) \geq 0, \quad (27)$$

$$\begin{aligned} \phi_j(u_1, u_2) = & 2(KpW(x(u_1) - x(u_2)) - (f(Wx(u_1)) - f(Wx(u_2))))^T \\ & \times V_j((f(Wx(u_1)) - f(Wx(u_2))) - KmW(x(u_1) - x(u_2))) \geq 0, \end{aligned} \quad (28)$$

where $u_1, u_2 \in \mathbb{R}$, $U_j = \text{diag}\{u_{j1}, u_{j2}, \dots, u_{jn}\} \in S_+^n$, $V_j = \text{diag}\{v_{j1}, v_{j2}, \dots, v_{jn}\} \in S_+^n$ ($j = 1, 2, 3$); then, it is clear that

$$\phi_1(t) + \phi_2(t-d(t)) + \phi_3(t-h) \geq 0, \quad (29)$$

$$\phi_1(t, t-d(t)) + \phi_2(t-d(t), t-h) + \phi_3(t, t-h) \geq 0. \quad (30)$$

From (19) to (30), the following inequality can be obtained:

$$\dot{V}(t) \leq \xi^T(t)\Omega\xi(t), \quad (31)$$

where

$$\begin{aligned}\Omega &= \Phi_2(\dot{d}(t))d^2(t) + \left(\Phi_1(\dot{d}(t)) + N_1^T T^{-1} N_1 - N_2^T T^{-1} N_2\right)d(t) \\ &\quad + \Phi_0(\dot{d}(t)) + hN_2^T T^{-1} N_2.\end{aligned}\quad (32)$$

To guarantee that $\dot{V}(t) < 0$, applying Lemma 2 and $\bar{M}_i + \bar{M}_i^T > 0$ ($i = 1, 2$), $\Omega < 0$ is equivalent to

$$\begin{bmatrix} \Phi_0(\dot{d}(t)) & 0.5\Phi_1(\dot{d}(t)) + hM_i \\ * & \Phi_2(\dot{d}(t)) - M_i - M_i^T \end{bmatrix} + \begin{bmatrix} hN_1^T T^{-1} N_1 & -N_1^T T^{-1} N_1 \\ * & \frac{1}{h}N_1^T T^{-1} N_1 + \frac{1}{h}N_2^T T^{-1} N_2 \end{bmatrix} < 0, \quad (33)$$

where $\bar{M}_i = -\frac{1}{2h}N_1^T T^{-1} N_1 - \frac{1}{2h}N_2^T T^{-1} N_2 + M_i$ ($i = 1, 2$). Finally, if (16) and (17) are satisfied, then $\dot{V}(t) < 0$ for M_i satisfying (15) by a Schur complement [27], which means that the NNs (1) are asymptotically stable. The proof is completed. \square

Remark 2. To derive the stability criterion in terms of linear matrix inequality, the matrix \bar{M}_i ($i = 1, 2$) should be appropriately designed. In Theorem 1, \bar{M}_i ($i = 1, 2$) are set as $\bar{M}_i = -\frac{1}{2h}N_1^T T^{-1} N_1 - \frac{1}{2h}N_2^T T^{-1} N_2 + M_i$ ($i = 1, 2$) to handle $\frac{1}{2}N_1^T T^{-1} N_1$ and $\frac{1}{2}N_2^T T^{-1} N_2$ of Ω . The slack variable \bar{M}_i ($i = 1, 2$) helps in reducing the conservativeness by giving the freedom for checking the feasibility of the linear matrix inequality conditions.

Remark 3. It should be pointed out that most literature [11,28] only propose the information of activation functions, such as $f(x(t))$, $f(x(t-d(t)))$ and $f(x(t-d))$. In this paper, the information of single integrations of activation functions, i.e., $\int_{t-d(t)}^t f(Wx(s))ds$ and $\int_{t-h}^{t-d(t)} f(Wx(s))ds$, are considered in $V_1(t)$. According to this construction, more activation function information on NNs can be adequately utilized, which leads to less conservative results. $\int_{t-h}^t \int_{\theta}^t f^T(Wx(s))Tf(Wx(s))dsd\theta$ is firstly proposed in [11]; different from its derivative handled method, this paper estimates the integral term by Lemma 3, which makes full use of cross-terms from the single integrations of activation function to others, and leads to a tighter bound.

Corollary 1. For given scalars $h > 0$ and $u > 0$, the system (1) is asymptotically stable if there exist symmetric matrices $P > 0$, $Z > 0$, $T > 0$, $Q_i > 0$, $R_i > 0$ ($i = 1, 2$), diagonal matrices $U_j > 0$, $V_j > 0$ ($j = 1, 2, 3$), $\Delta_k > 0$ ($k = 1, \dots, 6$) and any matrices X_l , Y_l , M_l , N_l ($l = 1, 2$), such that

$$\begin{bmatrix} \tilde{Z} - X_1 & Y_1 \\ * & \tilde{Z} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \tilde{Z} & Y_2 \\ * & \tilde{Z} - X_2 \end{bmatrix} \geq 0, \quad (34)$$

$$\begin{bmatrix} M_i + M_i^T & N_1^T & N_2^T \\ * & hT & 0 \\ * & * & hT \end{bmatrix} > 0, \quad (35)$$

$$\begin{bmatrix} \Phi_0(-\mu) & 0.5\Phi_1(-\mu) + hM_1 & hN_2^T & 0 \\ * & \Phi_2(-\mu) - M_1 - M_1^T & -N_2^T & N_1^T \\ * & * & -hT & 0 \\ * & * & * & -hT \end{bmatrix} < 0, \quad (36)$$

$$\begin{bmatrix} \Phi_0(\mu) & 0.5\Phi_1(\mu) + hM_2 & hN_2^T & 0 \\ * & \Phi_2(\mu) - M_2 - M_2^T & -N_2^T & N_1^T \\ * & * & -hT & 0 \\ * & * & * & -hT \end{bmatrix} < 0, \quad (37)$$

where

$$\begin{aligned}
\Phi_0(\dot{d}(t)) = & \text{Sym}\left\{r_{11}^T P r_2\right\} + r_{31}^T Q_1 r_{31} - r_{41}^T Q_2 r_{41} - (1 - \dot{d}(t))\left(r_{51}^T Q_1 r_{51} - r_{61}^T Q_2 r_{61}\right) \\
& + \text{Sym}\left\{r_9^T Q_1 r_{71} + r_{10}^T Q_2 r_{81}\right\} + \text{Sym}\left\{\left((K_p W e_1 - e_{10})^T \Delta_1 + (e_{10} - K_m W e_1)^T \Delta_2\right) W e_s\right\} \\
& + \text{Sym}\left\{\left((K_p W e_2 - e_{11})^T \Delta_3 + (e_{11} - K_m W e_2)^T \Delta_4\right) W e_4(1 - \dot{d}(t))\right\} \\
& + \text{Sym}\left\{\left((K_p W e_3 - e_{12})^T \Delta_5 + (e_{12} - K_m W e_3)^T \Delta_6\right) W e_5\right\} \\
& + \text{Sym}\left\{\sum_{c=1}^3\left(K_p W e_c - e_{c+9}\right)^T U_c\left(e_{c+9} - K_m W e_c\right)\right\} \\
& + \text{Sym}\left\{\sum_{c=1}^2\left(K_p W\left(e_c - e_{c+1}\right) - \left(e_{c+9} - e_{c+10}\right)\right)^T V_c\left(\left(e_{c+9} - e_{c+10}\right) - K_m W\left(e_c - e_{c+1}\right)\right)\right\} \\
& + \text{Sym}\left\{\left(K_p W\left(e_1 - e_3\right) - \left(e_{10} - e_{12}\right)\right)^T V_3\left(\left(e_{10} - e_{12}\right) - K_m W\left(e_1 - e_3\right)\right)\right\} + \rho_1^T R_1 \rho_1 \\
& + (1 - \dot{d}(t)) \rho_2^T\left(R_2 - R_1\right) \rho_2 - \rho_3^T R_2 \rho_3 + h^2 e_s^T Z e_s - \rho_4^T\left(\tilde{Z} + X_1\right) \rho_4 - \rho_5^T \tilde{Z} \rho_5 \\
& - \text{Sym}\left\{\rho_4^T Y_2 \rho_5\right\} + h e_{10}^T T e_{10} + \text{Sym}\left\{e_{13}^T N_1 + e_{14}^T N_2\right\} \\
\Phi_1(\dot{d}(t)) = & \text{Sym}\left\{r_{12}^T p r_2 + r_{31}^T Q_1 r_{32} - r_{41}^T Q_2 r_{42} - (1 - \dot{d}(t))\left(r_{51}^T Q_1 r_{52} - r_{61}^T Q_2 r_{62}\right) + r_9^T Q_1 r_{72}\right. \\
& \left.+ r_{10}^T Q_2 r_{82}\right\} - \frac{1}{h} \text{Sym}\left\{\rho_4^T\left(Y_1 - Y_2\right) \rho_5\right\} + \frac{1}{h}\left(\rho_4^T X_1 \rho_4 - \rho_5^T X_2 \rho_5\right), \\
\Phi_2(\dot{d}(t)) = & r_{32}^T Q_1 r_{32} - r_{42}^T Q_2 r_{42} - (1 - \dot{d}(t))\left(r_{52}^T Q_1 r_{52} - r_{62}^T Q_2 r_{62}\right) + \text{Sym}\left\{r_9^T Q_1 r_{73} + r_{10}^T Q_2 r_{83}\right\},
\end{aligned}$$

where the other notions are defined well in Theorem 1.

Proof. From the proof of Theorem 1, we choose the LKF (19) with $P(d(t)) = P$ in $V_1(t)$. The rest of the process is similar to that in the proof of Theorem 1; hence, we omit it here. \square

Remark 4. Theorem 1 and Corollary 1 are related to the stability problem of NNs with time-varying delays. It is possible to apply the proposed methods to other control problems, e.g., H_∞ control [29] and sampled-data control [30,31].

4. Illustrative Examples

In this section, two numerical examples are presented to demonstrate the superiority and the effectivity of the derived criteria.

Example 1. Consider the delayed NNs (1) with the following parameters:

$$\begin{aligned}
A &= \text{diag}(1.5, 0.7), \quad W = \text{diag}(1, 1), \quad K_m = \text{diag}(0, 0), \quad K_p = \text{diag}(0.3, 0.8), \\
W_0 &= \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix}.
\end{aligned}$$

For various μ , the allowable maximum delay bounds by Theorem 1 and Corollary 1 in this paper and the related methods in [17,32–34] are listed in Table 1. The results show that Theorem 1 provides a bigger allowable maximum upper bound of delays than the related methods of [17,32–34] do, which verifies the lower conservatism of the proposed criteria. Theorem 1 and Corollary 1 are derived via a similar approach; it can be seen that Theorem 1 provides larger allowable upper delay bounds than Corollary 1, which means that the LKF that contains delay-dependent matrices can lead to a less conservative result. In addition, when $h = 42.2617$, $\mu = 0.4$, $f(x) = \text{col}(0.3 \tanh(x_1), 0.8 \tanh(x_2))$, the initial state $x(t) = \text{col}(1, -1)$, $t \in [-42.2617, 0]$ and $d(t) = \frac{42.2617}{2} + \frac{42.2617}{2} \sin(\frac{0.8}{42.2617} t)$. The state

trajectories of the $x(t)$ given in Figure 1 show that the NNs (1) are stable at their equilibrium points, which verifies the effectiveness of our methods.

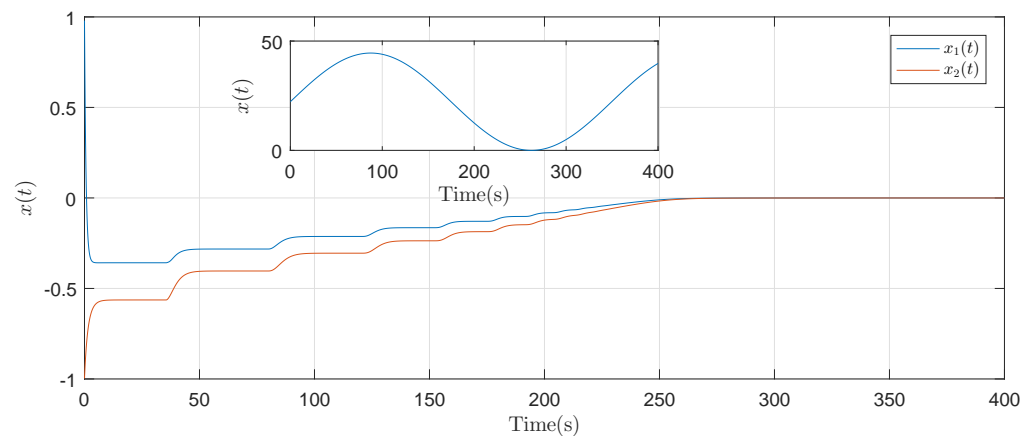


Figure 1. State trajectories for $\mu = 0.4$ and $h = 42.2617$.

Table 1. The allowable upper bound of delays for different μ (Example 1).

Methods	$\mu = 0.4$	$\mu = 0.5$
Theorem 1 (Case II) [32]	8.5669	7.2809
Theorem 3 [17]	13.8671	10.0050
Proposition 3 ($N = 2$) [33]	23.8409	14.8593
Corollary 1[34]	24.2322	15.0214
Theorem 1 [34]	33.5828	18.4200
Corollary 1	27.4594	14.9889
Theorem 1	42.2617	22.4641
Improvement	25.8433%	21.9549%

Example 2. Consider the delayed NNs (1) with the following parameters:

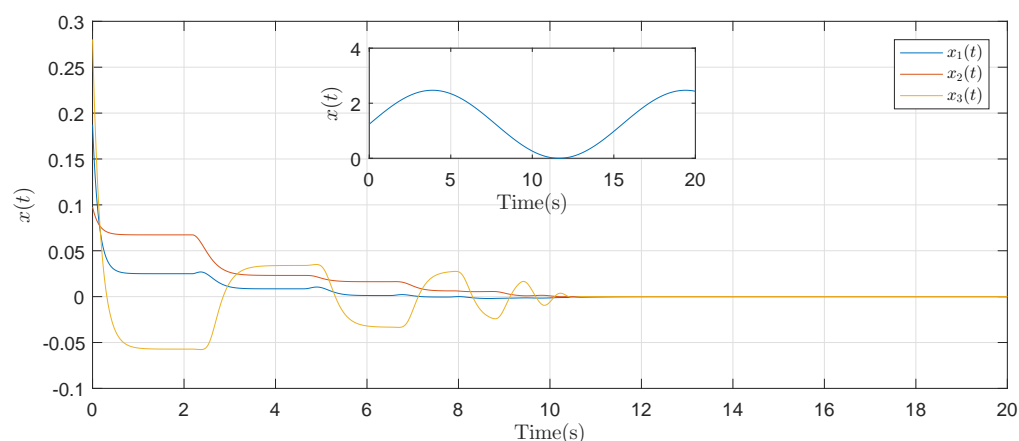
$$A = \text{diag}(7.3458, 6.9987, 5.5949), K_m = \text{diag}(0, 0, 0), K_p = \text{diag}(0.3680, 0.1795, 0.2876),$$

$$W_0 = \text{diag}(0, 0, 0), W_1 = \text{diag}(1, 1, 1), W = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7290 & -2.6334 & -20.1300 \end{bmatrix}.$$

It can be easily found that the proposed stability criteria can produce a larger allowable maximum upper bound for all cases than those given in the existing literature. This shows that the proposed criteria are indeed less conservative than the ones in the literature. Table 2 shows the allowable maximum upper bound of delays for various μ . When $h = 2.4705$, $\mu = 0.5$, $d(t) = \frac{2.4705}{2} + \frac{2.4705}{2} \sin(\frac{1}{2.4705}t)$, the initial state $x(t) = \text{col}(0.2, 0.1, 0.3)$, $t \in [-2.4705, 0]$ and the activation function $f(x) = \text{col}(0.3680 \tanh(x_1), 0.1795 \tanh(x_2), 0.2876 \tanh(x_3))$; the state trajectories of NNs in Example 2 are shown in Figure 2. The results given reveal that the LKF with a more integral term of activation function (i.e., $\int_{t-d(t)}^t f(Wx(s))ds$ and $\int_{t-h}^{t-d(t)} f(Wx(s))ds$) has improved the results.

Table 2. The allowable upper bound of delays for different μ (Example 2).

Methods	$\mu = 0.1$	$\mu = 0.5$
Theorem 1 ($N = 2$) [11]	1.1545	0.6010
Proposition 3 ($N = 3$) [33]	1.1554	0.5967
Corollary 1 [34]	1.1612	0.6162
Theorem 1 [34]	1.1845	0.6572
Theorem 1 [35]	2.0497	0.9860
Theorem 1 [1]	2.3679	1.0187
Corollary 1	7.9145	1.5412
Theorem 1	10.9334	2.4705
Improvement	361.7340%	142.5149%

**Figure 2.** State trajectories for $\mu = 0.5$ and $h = 2.4705$.

5. Conclusions

This paper has investigated the stability problem of NNs with time-varying delays. A positivity and negativity definiteness condition for a general polynomial with any chosen degree in time-varying delay has been established and a relaxed LKF that includes more previous information of the neuron activation function has been proposed. Benefiting from the relaxed LKF and the negative definiteness determination method, two stability criteria with less conservatism have been derived, which has been demonstrated by two numerical examples. Nevertheless, the definiteness condition is still proposed with conservatism, and future work on reducing its conservatism is meaningful.

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