mathematics

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Review

Convexity, Markov Operators, Approximation, and Related Optimization

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Abstract: The present review paper provides recent results on convexity and its applications to the constrained extension of linear operators, motivated by the existence of subgradients of continuous convex operators, the Markov moment problem and related Markov operators, approximation using the Krein–Milman theorem, related optimization, and polynomial approximation on unbounded subsets. In many cases, the Mazur–Orlicz theorem also leads to Markov operators as solutions. The common point of all these results is the Hahn–Banach theorem and its consequences, supplied by specific results in polynomial approximation. All these theorems or their proofs essentially involve the notion of convexity.

Keywords: convex operator; Hahn–Banach theorem; Markov moment problem; norm of the solution; Markov operator; Krein–Milman theorem; optimization; polynomial approximation; unbounded subsets; quadratic forms

MSC: 46A22; 46B42; 46N10; 41A10

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Citation: Olteanu, O. Convexity, Markov Operators, Approximation, and Related Optimization. *Mathematics* **2022**, *10*, 2775. https://doi.org/10.3390/math10152775

Academic Editor: Ioannis K. Argyros

Received: 10 July 2022
Accepted: 2 August 2022
Published: 4 August 2022

1. Introduction

This review paper provides information on recent applications of convexity to constrained extension of linear operators, Markov moment problems, Markov operators, approximation, and elements of convex optimization. These topics have further applications in analysis, functional analysis, and other research domains.

For notions, results, and further references related to this paper, see the monographs or books [1–11]. A very powerful tool in solving the existence problem for solutions of many classical moment problems is pointed out in [12]. Results on Hahn–Banach theorem and its applications, some of them in the operator setting, have been published [13–26]. On the other hand, knowing the explicit form of non-negative polynomials on multidimensional closed unbounded subsets is a very difficult task. It was solved in [27] for non-negative polynomials on a strip. Old results on characterizations of the existence of a linear extension with two constraints [16–18] found recent applications and completions in [19], where they appear among other interesting results.

In [28], new checkable sufficient conditions for the moment determinacy of measures on $\mathbb{R}$ and on $\mathbb{R}_+$ are proven. References [29–31] emphasize various aspects of the moment problem, some of them being related to optimization theory. Other results on optimization and related subjects are discussed in articles [32–34], while approximation on weighted spaces or on unbounded subsets can be found in articles/papers [35,36].

In the sequel, we review the classical moment problem and Markov moment problem, and their relationship or applications to other areas of research. The classical full moment problem is formulated as follows: given a sequence $(y_n)_{n\in\mathbb{N}^n}$ of real numbers, and a non-empty closed subset $F \subseteq \mathbb{R}^n$, find a positive regular Borel measure ν on F such that the interpolation moment conditions hold:

$$\int_F t^j d\nu = y_j,\ j \in \mathbb{N}^n.$$

Here we use the notations

$$\mathbb{N} = \{0,1,2,\ldots\},\ \varphi_j(t) = t^j = t_1^{j_1}\cdots t_n^{j_n},\ j = (j_1,\ldots,j_n) \in \mathbb{N}^n,$$

$$t = (t_1,\ldots,t_n) \in F \subseteq \mathbb{R}_n, \mathcal{P} = \mathbb{R}[t_1,\ldots,t_n], n \in \mathbb{N}, n \geq 1.$$

The moment problem is an inverse problem: we are looking for an unknown positive measure ν on F, starting from its moments $\int_F t^j d\nu = y_j,\ j \in \mathbb{N}^n$. Finding the measure ν means studying its existence and uniqueness and eventually constructing it. If $n = 1$, the corresponding moment problem is called one-dimensional. If $n \geq 2$, we have a multidimensional moment problem. In a Markov moment problem, an upper constraint on the solution ν is required. This constraint might be:

$$\int_F \psi d\nu \leq P(\psi), \text{ for all } \psi \in X_+,$$

where X is an ordered Banach space (see below) of functions defined on F, containing the polynomials and all real-valued continuous compactly supported functions on F. With X_+ we denote the positive cone of X, which is the convex cone of functions which take non-negative values at any point of F or for almost all $t \in F$, with respect to the measure ν. The vector-valued Markov moment problem variant is: being given an order-complete Banach lattice Y and a sequence $(y_n)_{n\in\mathbb{N}^n}$ of elements in Y, find a positive linear operator $T : X \to Y$ such that the interpolation moment conditions

$$T(\varphi_j) = y_j,\ j \in \mathbb{N}^n$$

are satisfied and $T \leq P$ on X, where $P:\ X \to Y$ is a given continuous convex operator. If P is sublinear (and continuous), then this last upper bondedness condition on T controls the continuity and sometimes the norm of the solution T. A variant of the full Markov moment problem is the following one: find T with the interpolation moment conditions, such that

$$T_1(\psi) \leq T(\psi) \leq T_2(\psi),\ \psi \in X_+,$$

where T_1, T_2 are given bounded linear operators applying X into Y. In the case of a truncated (reduced) moment problem, the interpolation moment conditions $T(\varphi_j) = y_j$ should hold only for $j_k \leq d,\ k = 1,\ldots,n$, where d is a positive integer, $j = (j_1,\ldots,j_n)$. If $K \subset \mathbb{R}^n$ is a compact subset, then any bounded linear solution of the full moment problem is unique due to the Weierstrass approximation theorem; any continuous real function $f \in C(K)$ is the limit of a sequence of polynomials, the convergence being uniform on K. Unlike the this case, the solution of a truncated classical moment problem cannot be unique (its values on the entire space $C(K)$ cannot be decided by its values on a finite dimensional subspace of $C(K)$). An interesting problem is that of solving moment problems on closed unbounded subsets F of $\mathbb{R}^n,\ n \geq 2$ in terms of quadratic forms. This problem is not easy since, unlike the one-dimensional case [1,4], the non-negative polynomials on unbounded subsets (such as $\mathbb{R}^n,\ \mathbb{R}^n_+$) are not expressible in terms of sums of squares [4]. In case of the Markov moment problem, the problem was discussed and solved by means of approximation of any continuous nonnegative compactly supported real-valued function defined on $\mathbb{R}^n$, or on $\mathbb{R}^n_+$, by a sequence of special nonnegative polynomials, which are expressible in terms of sums of squares ([21,36] and the references therein). The approximation holds in spaces $L^1_\nu(\mathbb{R}^n)$, or respectively, in $L^1_\nu(\mathbb{R}^n_+)$, where $\nu = \nu_1 \times \cdots \times \nu_n,\ \nu_j$, being a positive regular Borel moment determinate measure on $\mathbb{R}$ or $\mathbb{R}_+,\ j = 1,\ldots,n$, respectively, with finite moments of all orders. We recall that a measure μ on the closed subset $F \subseteq \mathbb{R}^n$ is called moment-determinate (M-determinate) if it is uniquely determined by its moments $\int_F t^j d\mu,\ j \in \mathbb{N}^n$ (all the involved measures appearing in this work are assumed to be regular Borel measures, with finite moments of all orders. Although the approximations on unbounded subsets were motivated first by solving Markov moment problems in terms of quadratic forms, such results found an application to characterizing the positivity of some bounded linear operators defined on $L^1_\nu(\mathbb{R}^n)$ only in terms of quadratic forms, where ν is as above. A similar result works for bounded linear operators defined on $L^1_\nu(\mathbb{R}^n_+)$, $(\nu = \nu_1 \times \cdots \times \nu_n$, each ν_j being M-determinate on $\mathbb{R}_+$, $j = 1, \ldots, n)$. Another application of the solution of an abstract moment problem was recently reviewed in [37], where the existence of at least one feasible solution for an optimization problem with infinitely many constraints is characterized ([37], Theorem 4). The existence of at least one optimal feasible solution is proven as well. The idea of solving such problems comes from the PhD thesis [38], in which a similar problem involving a finite number of moments, in an L^∞ space, is solved. Unlike the case of infinitely many moments interpolation conditions, if only a finite number of moments are involved, construction of the solutions comes under attention. This is also the case for references [39,40], in which construction of a solution for a truncated Markov moment problem is studied, and related algorithms are provided using deep results in matrix theory, measure theory, and other areas of analysis and algebra. Other optimization problems related to reduced moment problems are discussed in [30,31]. In [21], among other theorems, the positivity of some bounded linear operators on spaces of functions of several real variables is characterized only in terms of quadratic forms. In [41], an unusual sandwich-type result on finite simplicial sets is proven for the first time. The point is the fact that a finite simplicial set may be unbounded in any locally convex topology on the domain space. Finally, the article [42] provides a characterization of the finite dimensional bounded convex subsets in an arbitrary vector space in terms of the lower boundedness of convex operators defined on them. The codomain is assumed to be order-complete to make the Hahn–Banach theorem and its application to the existence of subgradients of convex operators effective. Other old results from [16,17] on constrained extension and decomposition as a difference of two positive linear operators found recent applications in [19]. Namely, in [19], among other results, the isotonicity (the property of being monotone increasing) for a continuous convex operator defined on the positive cone of the domain space is characterized in terms of its subgradients. Notably, we recall that Krein–Milman theorem plays an important role in the representation theory [2,8,21]. Almost all the abovementioned results have interesting geometric meaning, which is supported by clear rigorous proofs. For example, the sandwich-type result proven in reference [41] is relevant since it proves the existence of an affine function h, $f \leq h \leq g$ on F, in which the two functions f and $-g$ are given convex functions on a finite simplicial convex set F. This is an unusual sandwich result since it is a converse variant of sandwich results based on the separation of convex sets. Moreover, generally, a finite simplicial convex set may be unbounded. The proof of the first such result is essentially based on the solution of an abstract Markov moment problem.

The notion of a Markov operator is emphasized in [19,20]. These special bounded linear operators have interesting properties and appear naturally as subgradients of continuous sublinear operators, or as solutions of Markov moment problems or Mazur–Orlicz type problems. We hope this work will be useful for readers interested in analysis and its applications because it joins various directions of applications under the general thematic of convexity. In the sequel, we recall some basic definitions. If X, Y are ordered vector spaces with positive cones X_+ and Y_+, respectively, then a linear operator $T : X \to Y$ is called positive if

$$x \in X_+ \Rightarrow T(x) \in Y_+.$$

In other words, $x \geq \mathbf{0}$ in $X \Rightarrow T(x) \geq \mathbf{0}$ in Y. From the linearity of T, we infer that T is positive if and only if $x_1 \leq x_2 \in X \Rightarrow T(x_1) \leq T(x_2)$, which means that T is isotone (increasingly monotone) on X. An ordered vector space X which is also a Banach space is called an ordered Banach space if the positive cone X_+ is topologically closed and the norm is increasingly monotone on X_+ :

$$\mathbf{0} \leq x_1 \leq x_2 \Rightarrow \|x_1\| \leq \|x_2\|.$$

Thus, in an ordered Banach space, the norm is compatible with the (linear) order relation. Banach lattice X is a Banach space which is also a vector lattice such that

$$|x_1| \leq |x_2| \Rightarrow \|x_1\| \leq \|x_2\|.$$

We recall that in a vector lattice X, for any $x \in X$, one defines

$$|x| := sup\{x, -x\} := x \vee (-x).$$

Thus, $|x|$ is the least upper bound of the set $\{x, -x\}$; it exists according to the definition of a vector lattice (see [6] or [7]). From these definitions, it follows that any Banach lattice is an ordered Banach space, but the converse is not true. Most of the function spaces (and sequence spaces) are Banach lattices. An example of an ordered Banach space which is not a lattice is the space $\mathcal{SM}(n)$ of all symmetric $n \times n$ matrixes with real entries. The order relation on this space is defined by: $A \leq B \Longleftrightarrow \langle Ah, h \rangle \leq \langle Bh, h \rangle$ for all $h \in \mathbb{R}^n$. The norm of a matrix $A \in \mathcal{SM}(n)$ is defined by $\|A\| = \sup\limits_{\|h\| \leq 1} |\langle Ah, h \rangle|$, where $\|h\| = \langle h, h \rangle^{1/2}$ is the Euclidean norm on $\mathbb{R}^n$. As is well known, the symmetric matrix A naturally defines a symmetric linear operator (denoted also by A), acting on $\mathbb{R}^n$. The symmetry condition for this linear operator is written as follows: $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in \mathbb{R}^n$. Conversely, being given a linear operator A which maps $\mathbb{R}^n$ into itself for which the above symmetry condition is verified, the associated matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$, where $a_{i,j} := \langle Ae_i, e_j \rangle$, $1 \leq i, j \leq n$, is symmetric. Here, $\{e_1, \ldots, e_n\}$ is an orthonormal base of $\mathbb{R}^n$. Referencing the infinite dimensional Hilbert space case, let H be a real or complex infinite dimensional Hilbert space. A self-adjoint operator A acting on H is a bounded linear symmetric operator applying H into H. The symmetry condition is written as follows, from the matrix setting mentioned above:

$$\langle Ax, y \rangle = \langle x, Ay \rangle \text{ for all } x, y \in H.$$

The order relation and the norm on the real vector space $\mathcal{A}(H)$ of all self-adjoint operators acting on H can be written as above for symmetric matrixes. With these structures, $\mathcal{A}(H)$ is an ordered real Banach space, which is not a lattice. Finally, we recall that any positive linear operator acting on ordered Banach spaces is continuous [9,19]. This result is relevant because it avoids the effort of the reader in proving the continuity of such operators. For example, if X, Y are Banach lattices and T_1, T_2 are two linear operators mapping X into Y such that $\mathbf{0} \leq T_1 \leq T_2$ on X_+, then, according to the invoked result, the two operators are continuous. Moreover, it is easy to prove that our conditions imply $\|T_1\| \leq \|T_2\|$.

The rest of the paper is organized as follows. Section 2 summarizes the basic methods applied along the proofs of the results. In Section 3, the results are stated, and some of them are proven. Section 4 concludes the paper.

2. Methods

The basic methods applied in the sequel are:

1. General notions and results on positive linear operators, convex operators and their subdifferentials, ordered Banach spaces, Banach lattices.
2. General Hahn–Banach-type theorems and some of their applications.
3. The classical full and/or truncated moment problem. Markov operators as solutions of moment problems in concrete spaces or as solutions of Mazur–Orlicz theorems.

4. Extending inequalities from a small set to an entire convex cone via approximation provided by the Krein–Milman theorem. Elements of convex optimization theory are also applied.
5. Results on polynomial approximation on unbounded subsets are applied in Section 3.3. The role played by regular moment determinate positive Borel measures on closed unbounded finite dimensional sets is a key point in this topic [21,36].

3. Results

3.1. Hahn–Banach Theorem, Markov Moment Problem, and Markov Operators

First variants of Hahn–Banach theorem on extension of linear operators dominated by convex continuous operators have been applied to the subject of subdifferentiability of convex continuous operators [13–15]. Another domain of applications of Hahn–Banach-type results—and, conversely, a motivation for stating and proving such results—was that of solving moment problems. The full classical moment problem in the operator setting is an interpolation problem, with the positivity constraint on the linear solution T on the positive cone of the domain function Banach space X, which must contain the polynomials $\mathcal{P}$ and the space $C_c(F)$ of real-valued continuous compactly supported functions defined on an arbitrary closed subset F of $\mathbb{R}^n$. The values of the solution T on the basic polynomials

$$\varphi_j(t) = t^j = t_1^{j_1} \cdots t_n^{j_n},\ t = (t_1, \ldots, t_n) \in F,\ j = (j_1, \ldots, j_n) \in \mathbb{N}^n, \mathbb{N} = \{0, 1, 2, \ldots\},$$

are prescribed: we must have

$$T(\varphi_j) = y_j,\ j \in \mathbb{N}^n, \tag{1}$$

and the positivity condition

$$\psi \in X_+ \Rightarrow T(\psi) \in Y_+. \tag{2}$$

Here, the elements $y_j,\ j \in \mathbb{N}^n$ are contained in the order-complete Banach lattice Y. The order completeness requirement on Y allows applying Hahn–Banach-type results to the extension of linear operators having Y as their codomain. The existence, the uniqueness, and the construction of the linear solution T are studied. The Markov moment problem requires one more condition on the solution T. This requirement can be written as

$$T \leq P \text{ on } X \text{ or } T \leq P \text{ on } X_+, \tag{3}$$

where $P : X \to Y$ is a continuous sublinear operator (or, more generally, P is a continuous convex operator). Sometimes, P is defined only on X_+, with range contained in Y. The above stated moment problem is the full moment problem since it involves the moments of all orders. If condition (1) is required only for $j_l \leq m, l = 1, \ldots, n$, for some fixed positive integer m, then we have a truncated (reduced) moment problem.

Notably, the existence problem is an extension problem of the linear operator defined on the space $\mathcal{P}$ of all polynomial functions to the entire Banach space X such that the moment conditions (1) and the constraints (2) (and, for a Markov moment problem also (3)) are satisfied. Hence, we can apply a Hahn–Banach-type theorem. We start by recalling a general result and its variant related to the Mazur–Orlicz theorem. Then, we apply these theoretical results to concrete spaces, pointing out the notion of a Markov operator.

Theorem 1 ([18]). *Let X be a preordered vector space, Y an order complete vector lattice, $P : X \to Y$ a convex operator, $\{x_j\}_{j\in J} \subset X$, $\{y_j\}_{j\in J} \subset Y$ given families. The following statements are equivalent:*

(a) *there exists a linear positive operator $T : X \to Y$ such that*

$$T(x_j) = y_j,\ j \in J,\ T(x) \leq P(x),\ x \in X;$$

(b) *for any finite subset $J_0 \subseteq J$, and any $\{\lambda_j; j \in J_0\} \subseteq \mathbb{R}$, we have*

$$\sum_{j \in J_0} \lambda_j x_j \leq x \in X \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq P(x).$$

If we additionally we assume that P is isotone $(u \leq v \Rightarrow P(u) \leq P(v))$, the assertions (a) and (b) are equivalent to (c), where:

(c) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}$, the following inequality holds:*

$$\sum_{j \in J_0} \lambda_j y_j \leq P\left(\sum_{j \in J_0} \lambda_j x_j\right).$$

Theorem 1 is the reformulation of the following extension result, in which the interpolation moment conditions do not appear explicitly. Theorem 2 below was initially stated in [16] and proven in [17], where it was deduced from a more general result (Theorem 1 of [16]).

Theorem 2. *Let X be an ordered vector space, Y an order complete vector space, $M \subset X$ a vector subspace, $T_1 : M \to Y$ a linear operator, $P : X \to Y$ a convex operator. The following statements are equivalent:*

(a) *there exists a positive linear extension $T : X \to Y$ of T_1 such that $T \leq P$ on X;*

(b) *we have $T_1(h) \leq P(x)$ for all $(h, x) \in M \times X$ such that $h \leq x$.*

One observes that in the very particular case $X_+ = \{\mathbf{0}\}$, when the order relation on X is the equality, from Theorem 2, one obtains the Hahn–Banach extension theorem for linear operators dominated by convex operators.

Another version of the same result is found in [21].

Theorem 3. *Let X be an ordered vector space, Y an order complete vector space, $M \subset X$ a vector subspace, $T_1 : M \to Y$ a linear operator, $P : X_+ \to Y$ a convex operator. The following statements are equivalent:*

(a) *there exists a positive linear extension $T : X \to Y$ of T_1 such that $T|_{X_+} \leq P$;*

(b) *we have $T_1(h) \leq P(x)$ for all $(h, x) \in M \times X_+$ such that $h \leq x$.*

Theorem 4 (Mazur-Orlicz [18])**.** *Let X be a preordered vector space, Y an order complete vector space, $\{x_j\}_{j \in J}, \{y_j\}_{j \in J}$ families of elements in X, respectively in Y, $P : X \to Y$ a sublinear operator. The following statements are equivalent:*

(a) *there exists a linear positive operator $T : X \to Y$ such that*

$$T(x_j) \geq y_j,\ j \in J,\ T(x) \leq P(x),\ x \in X;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}_+ = [0, \infty)$, the following implication holds true*

$$\sum_{j \in J_0} \lambda_j x_j \leq x \in X \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq P(x).$$

If we additionally assume that P is isotone, the assertions (a) and (b) are equivalent to the following condition (c):

(c) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j\in J_0} \subset \mathbb{R}_+$, the following inequality holds*

$$\sum_{j\in J_0} \lambda_j y_j \le P\left(\sum_{j\in J_0} \lambda_j x_j\right).$$

The next results refer to the Markov moment problem on the space $C(K)$ as the domain space of the solution, where K is an arbitrary compact Hausdorff topological space.

A version of the abstract Markov moment problem has been already stated in Theorem 1 above. The classical moment problem and Markov moment problem have been reviewed in the Introduction. In the next theorem, $\mathbf{1} \in C(K)$ denotes the constant function which equals the number 1 at any point of K and the class of the same function, regarded as an element of $L^\infty_\mu(K)$.

Theorem 5. *Let K be a compact Hausdorff topological space, μ a positive regular Borel measure on K, $C(K)$ the Banach lattice of all real valued continuous functions on K, $\{\varphi_j\}_{j\in J}$ a family of linearly independent elements in $C(K)$, $\{y_j\}_{j\in J}$ a given family of elements in $L^\infty_\mu(K)$. The following two statements are equivalent*

(a) *there exists a linear (positive) bounded operator $T : C(K) \to L^\infty_\mu(K)$ such that*

$$T(\varphi_j) = y_j,\ j \in J,\ T(\varphi) \le \left(\underset{t\in K}{sup}\varphi(t)\right)\mathbf{1},\ \forall\varphi \in C(K)$$

In particular, the following equalities hold

$$T(\mathbf{1}) = \mathbf{1}, \|T\| = 1;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\alpha_j; j \in J_0\} \subset \mathbb{R}$, the following relation holds true*

$$\sum_{j\in J_0} \alpha_j y_j \le \underset{t\in K}{sup}\left(\sum_{j\in J_0} \alpha_j \varphi_j(t)\right)\mathbf{1}. \tag{4}$$

Proof. The implication (a) $\Rightarrow$ (b) is obvious, thanks to the properties of T. To prove (b) $\Rightarrow$ (a), one applies Theorem 1, implying (c) $\Rightarrow$ (a), for

$$X = C(K),\ Y = L^\infty_\mu(K),\ P(\psi) := \left(\underset{t\in K}{sup}\psi(t)\right)\mathbf{1}, \psi \in X. \tag{5}$$

Observe that P defined by (5) is a scalar-valued sublinear nondecreasing functional multiplied by the class of the constant function $\mathbf{1}$ in $L^\infty_\mu(K)$ and therefore is an isotone sublinear operator. The inequality (4) is equivalent to the fact that the condition written at point (c) of Theorem 1 is accomplished. Since $L^\infty_\mu(K)$ is an order-complete vector lattice, according to Theorem 1, there exists a positive linear operator $T : X \to L^\infty_\mu(K)$ with the properties mentioned at point (a) of the latter theorem. It results $T(\varphi_j) = T_0(\varphi_j) := y_j$, $j \in J$. Moreover, the following implications hold:

$$\varphi \in X \Rightarrow T(\varphi) \le \left(\underset{t\in K}{sup}\varphi(t)\right)\mathbf{1}, -T(\varphi) \le \left(\underset{t\in K}{sup} - \varphi(t)\right)\mathbf{1}$$

$$= -\left(\underset{t\in K}{inf}\varphi(t)\right)\mathbf{1} \Rightarrow \left(\underset{t\in K}{inf}\varphi(t)\right)\mathbf{1} \le T(\varphi) \le \left(\underset{t\in K}{sup}\varphi(t)\right)\mathbf{1}.$$

For $\varphi = \mathbf{1}$, it results in $T(\mathbf{1}) = \mathbf{1}$; we say that T is unital. Since any $\varphi \in X$ with $\|\varphi\|_X \leq 1$ is situated in the order interval $[-\mathbf{1}, \mathbf{1}]$, the positivity of T leads to $T(\varphi) \in [T(-\mathbf{1}),\ T(\mathbf{1})] = [-\mathbf{1}, \mathbf{1}] \Rightarrow \|T(\varphi)\|_Y \leq 1 \Rightarrow \|T\| \leq 1$. However, we have already seen that $\|T(\mathbf{1})\|_Y = \|\mathbf{1}\|_Y = 1$. Hence, $\|T\| = 1$ and the proof is complete. □

In the next theorem, K will be a compact subset of $\mathbb{R}^n$ ($n \geq 1$ is a natural number), $X := C(K)$, and Y is an order-complete Banach lattice with a strong order unit u_Y such that the order interval $[-u_Y, u_Y]$ is equal to the closed unit ball of Y. As usual, we denote:

$$j = (j_1, \ldots, j_n) \in \mathbb{N}^n.\ t = (t_1, \ldots, t_n) \in K,\ |j| = \sum_{k=1}^n j_k,\ \varphi_j(t) = t^j = t_1^{j_1} \cdots t_n^{j_n}.$$

Theorem 6. *Let $\{y_j; |j| \leq m\} \subset Y$, for some fixed $m \in \mathbb{N},\ m \geq 1$. The following statements are equivalent:*

(a) *there exists a positive linear operator $T : C(K) \to Y$ such that*

$$T(\varphi_j) = y_j, |j| \leq m,\ T(\varphi) \leq \left(\sup_{t \in K} \varphi(t)\right) u_Y,\ \forall \varphi \in C(K),\ \|T\| = 1, \\ T(\mathbf{1}) = u_Y; \tag{6}$$

(b) *for any $\{\beta_j; |j| \leq m\} \subset \mathbb{R}$, the following relation holds*

$$\sum_{\substack{j \in \mathbb{N}^n \\ |j| \leq m}} \beta_j y_j \leq \left(\sup_{t \in K} \left(\sum_{|j| \leq m} \beta_j t^j\right)\right) u_Y. \tag{7}$$

Proof. One repeats the proof of Theorem 5, in which we replace $L_\mu^\infty(K)$ with Y to obtain

$$\varphi_j(t) = t^j,\ t \in K,\ j \in J = \mathbb{N}^n,\ |j| \leq m,\ P(\psi) = \left(\sup_{t \in K} \psi(t)\right) u_Y, \psi \in X := C(K).$$

Some of the notations have been defined before the statement. Clearly, from (6) with positive and unital T, relation (7) follows. For the converse, repeating the arguments from the proof of Theorem 5, the existence of a positive linear operator verifying (6) follows (via Theorem 1, (c)⇒(a)). Here, the subspace $M = Sp\{\varphi_j; j \in \mathbb{N}^n,\ |j| \leq m\}$ is the linear subspace of all polynomial functions on K of degree $\leq m$. From (6), in particular, $T(\mathbf{1}) = u_Y$ follows as well, and the proof is complete.□

When applying the Mazur–Orlicz theorem (Theorem 4), one can work with the subspace of all polynomial functions on $K \subset \mathbb{R}^n$, without any restriction on the degree (one proves a full Mazur–Orlicz theorem). Such a result is not a direct consequence of the density of polynomials in $C(K)$, unlike the case of the full moment problem for $C(K),\ K \subset \mathbb{R}^n$. In Theorem 6, a solution for a truncated moment problem was proposed. A linear operator T mapping $C(K)$ into Y is called a Markov operator if T is positive and $T(\mathbf{1}) = u_Y$ (the definition is valid for any Hausdorff compact topological space K). It is easy to observe that a linear operator $T \in L(C(K), Y)$ is a Markov operator if and only if $T(\varphi) \leq \left(\sup_{t \in K} \varphi(t)\right) u_Y\ \forall \varphi \in C(K)$. In particular, solutions T from Theorems 5, 6, and 7 (the last one being proven below) are Markov operators. Let Y be and order complete Banach lattice with a strong order unit u_Y, $(y_j)_{j \in \mathbb{N}^n}$ a sequence in Y. We prove the following theorem.

Theorem 7. *Let* $K = K_1 \times \cdots \times K_n \subset \mathbb{R}_+^n$ *be such that:* $K_l \subset \mathbb{R}_+$ *is compact and denote* $r_l = supK_l,\ l = 1, \ldots, n,\ \ r^j = r_1^{j_1} \cdots r_n^{j_n},\ j = (j_1, \ldots, j_n) \in \mathbb{N}^n$. *The following statements are equivalent:*

(a) *there exists a (positive) linear operator* $T : C(K) \to Y$ *such that*

$$T(\varphi_j) \geq y_j,\ j \in \mathbb{N}^n,\ T(\varphi) \leq \left(\underset{t \in K}{sup} \varphi(t) \right) u_Y\ \forall \varphi \in C(K),\ \|T\| = 1;$$

(b) $y_j \leq r^j u_Y\ \forall j \in \mathbb{N}^n$.

Proof. The implication (a)⇒(b) is obvious thanks to the properties of T. Namely, the following relations hold true:

$$y_j \leq T(\varphi_j) \leq \left(\underset{t \in K}{sup} \varphi_j(t) \right) u_Y = \left(\underset{t \in K}{sup} (t_1^{j_1} \cdots t_n^{j_n}) \right) u_Y = r^j u_Y,\ j \in \mathbb{N}^n$$

To prove (b)⇒(a), we use the implication (c)⇒(a) of Theorem 4. The conditions mentioned at (c) of Theorem 4 are satisfied, since for any finite subset $J_0 \subset \mathbb{N}^n$, the following inequalities hold:

$$y_j \leq r^j u_Y,\ \lambda_j \geq 0,\ \forall j \in J_0 \Rightarrow$$

$$\sum_{j \in J_0} \lambda_j y_j \leq \left(\sum_{j \in J_0} \lambda_j r^j \right) u_Y = \left((\sum_{j \in J_0} \lambda_j t^j)|_{t=(r_1, \ldots, r_n)} \right) u_Y = \underset{t \in K}{sup} \left(\sum_{j \in J_0} \lambda_j t^j \right) u_Y = P\left(\sum_{j \in J_0} \lambda_j \varphi_j \right),$$

$$P(\psi) := \left(\underset{t \in K}{sup} \psi(t) \right) u_Y,\ \psi \in C(K).$$

According to Theorem 4, (c)⇒(a), there exists a positive linear operator $T : C(K) \to Y$ with the properties mentioned at point (a) of the present theorem. This ends the proof. □

We next recall an example of an order complete Banach lattice of self-adjoint operators acting on an arbitrary Hilbert space, which is also a commutative real algebra [7]. This example works for symmetric $n \times n$ matrices with real entries.

Example 1. *Let* H *be a Hilbert space and* $A \in \mathcal{A}(H)$ *be a self-adjoint operator acting on* H. *We define:*

$$Y_1 = Y_1(A) := \{V \in \mathcal{A}(H); AV = VA\},\ Y = Y(A) := \{U \in \mathcal{A}(H); UV = VU\ \forall V \in Y_1(A)\}.$$

Then, $Y(A)$ is an order complete Banach lattice with strong order unit for identity operator I. By its definition, it is also a commutative real algebra.

3.2. On Some Applications of the Krein–Milman Theorem

The next results follow via Krein–Milman Theorem. If B is a compact convex subset of a locally convex space, we denote with $Extr(B)$ the set of all extreme points of B.

Theorem 8. *Let* X *be a reflexive Banach space endowed with a linear order relation defined by a closed positive cone* X_+, *and* $B \subset X_+$ *a convex bounded closed subset such that any* $x \in X_+ \backslash \{\mathbf{0}_X\}$ *can be represented uniquely as* $x = \rho b$ *for some* $\rho \in (0, \infty)$ *and* $b \in B$ *(*B *is a base for* X_+*). Assume that* Y *is a topological vector space endowed with an order relation defined by a closed positive cone*

Y_+, $\Phi : X \to Y$ is a bounded sublinear operator, and $T \in B(X, Y)$ is a bounded linear operator such that

$$\Phi(e) \le T(e)\ \forall e \in Extr(B)$$

Then $\Phi(x) \le T(x)\ \forall x \in X_+$.

Proof. Clearly, B is weakly compact and convex, so $B = cl(co(Extr(B)))$. The topological closure of the convex set $co(Extr(B))$ in the weak topology equals its topological closure in the topology defined by the norm of X. Let

$$x_n = \sum_{j=1}^{n} \alpha_j e_j \in co(Extr(B)),\ \alpha_j \in [0, \infty),\ \sum_{j=1}^{n} \alpha_j = 1, e_j \in Extr(B),\ j = 1, \ldots, n$$

Then, from the assumptions of the statement, we derive

$$\Phi(x_n) \le \sum_{j=1}^{n} \alpha_j \Phi(e_j) \le \sum_{j=1}^{n} \alpha_j T(e_j) = T\left(\sum_{j=1}^{n} \alpha_j e_j\right) = T(x_n),\ n \in \mathbb{N},\ n \ge 1.$$

For $b \in cl(co(Extr(B))) = B,\ b = \lim_n x_n,\ x_n \in co(Extr(B))$. For all $n \ge 1$, the continuity of Φ, T, as well as the hypothesis that the positive cone of Y is closed, leads to

$$\Phi(b) = \lim_n \Phi(x_n) \le \lim_n T(x_n) = T(b).$$

Now let $x \in X_+ \setminus \{\mathbf{0}_X\}$, $x = \rho b,\ \rho > 0,\ b \in B$. Then,

$$\Phi(x) = \Phi(\rho b) = \rho \Phi(b) \le \rho T(b) = T(\rho b) = T(x).$$

This ends the proof. □

Corollary 1. *Under the hypothesis of Theorem 8, additionally assume that X, Y are normed vector lattices (their norms are solid), and Φ is isotone. Then $\|\Phi\| \le \|T\|$.*

Proof. According to Theorem 8, we have already seen that $\Phi(w) \le T(w)$ for all $w \in X_+$. Using this and the monotonicity of Φ, we derive

$$(\Phi(x) \le \Phi(|x|) \le T(|x|), -\Phi(x) \le \Phi(-x) \le \Phi(|x|) \le T(|x|)) \Rightarrow$$

$$|\Phi(x)| \le T(|x|) \Rightarrow \|\Phi(x)\| \le \|T\| \|x\|\ \ \forall x \in X \Rightarrow \|\Phi\| \le \|T\|.$$

This concludes the proof. □

The next results extend an inequality occurring on a small set to a much larger subset.

Theorem 9. *Let X be a reflexive Banach lattice, Y an order complete Banach lattice in which every topological bounded subset is order-bounded and $y_n \uparrow y$ implies $y_n \to y$, $\Phi : X_+ \to Y$ a quasiconvex continuous positively homogeneous operator, and $T \in B_+(X, Y)$ a positive linear operator such that $\Phi(e) \le T(e)$ for all extreme points e of the set $K := X_+ \cap B_{1,X}$. Then*

$$\Phi(x) \le \|x\| \cdot \sup_{e \in Ex(K)} \Phi(e) \in Y_+\ \forall x \in X_+.$$

Proof. Recall that operator Φ from a convex subset C of vector space X to vector lattice Y is called quasiconvex if

$$\Phi((1-t)x_1 + t x_2) \le sup\{\varphi(x_1), \varphi(x_2)\},\ \forall t \in [0, 1],\ \forall x_1, x_2 \in C.$$

Following the proof (by induction) of Jensen's inequality for real convex functions, for any convex combination $\sum_{j=1}^{n} \alpha_j x_j,\ x_j \in C,\ j = 1, \ldots, n$, quasiconvex operator Φ verifies

$$\Phi\left(\sum_{j=1}^{n} \alpha_j x_j\right) \leq sup\{\Phi(x_1), \ldots, \Phi(x_n)\}$$

See [8] for details, examples, and exercises related to this important notion. The set $K := X_+ \cap\ B_{1,X}$ is convex, weakly compact, and

$$K = cl(co(Extr(K))) \tag{8}$$

holds due to the Krein–Milman theorem. Let

$$x_n = \sum_{j=1}^{n} \alpha_j e_j \in co(Extr(K)),\ e_j \in Ex(K),\ \alpha_j \in [0, \infty),\ j = 1, \ldots, n,\ \sum_{j=1}^{n} \alpha_j = 1.$$

From the above remarks, and using the hypothesis, we infer that

$$\Phi(x_n) \leq sup\{\Phi(e_1), \ldots, \Phi(e_n)\} \leq sup\{T(e_1), \ldots, T(e_n)\} \tag{9}$$

On the other side, any positive linear operator from X to Y is continuous, so that the image of the bounded set $K \subset X_+$ through the positive (bounded) linear operator T is topologically bounded. Therefore, it is $o-$ bounded in Y_+. Thus it results

$$T(K) \subset [\mathbf{0}_Y, y_0] \text{ for some } y_0 \in Y_+.$$

From this and using (9), we find that

$$\Phi(x_n) \leq y_0,\ x_n \in co(Extr(K)),\ n \in \mathbb{N},\ n \geq 1.$$

If $x = \lim_{n \to \infty} x_n \in cl(co(Extr(K))) = K$ where $x_n \in co(Extr(K))$ for all $n \geq 1$, thanks to the continuity of Φ, we are leaded to

$$\Phi(x) = \lim_{n \to \infty} \Phi(x_n) \leq \lim_{n \to \infty} \left(sup\{\Phi(e_j); j = 1, \ldots, n\}\right) =$$

$$\sup_{n \geq 1} \Phi(e_n) \leq \sup_{e \in Extr(K)} \Phi(e) \leq \sup_{e \in Extr(K)} T(e) \leq y_0,\ x \in K.$$

Since Φ is positively homogeneous, application of this evaluation to $x/\|x\| \in K,$ for all $x \in X_+,\ x \neq 0_X$, yields

$$\Phi(x) \leq \|x\| \cdot \sup_{e \in Ex(K)} \Phi(e) \leq \|x\| y_0\ , \quad x \in X_+.$$

This ends the proof. □

Theorem 10. *Let X be an order complete normed vector lattice, $K \subset X$ a finite dimensional compact subset, and $(\Phi_n)_{n \geq 0}$ a sequence of continuous sublinear operators from X into X such that for each $x \in X$, there exists $\widetilde{\Phi}(x) := \lim_{n \to \infty} \Phi_n(x) \in X$. Assume that for each $n \in \mathbb{N}$, there exists an affine operator T_n from X to X, such that $T_n(x) \leq \Phi_n(x)\ \forall x \in X$ and there exists*

$$\widetilde{T}(e) := \lim_{n \to \infty} T_n(e) = e\ \forall e \in Extr(K).$$

Then, $x \leq \widetilde{\Phi}(x)\ \forall x \in Cone(K)$, where $Cone(K)$ is the convex cone generated by $(co(K)) \bigcup \{\mathbf{0}_X\}$.

Proof. It is known that for any finite dimensional compact K, its convex hull $co(K)$ is compact as well (the proof of this assertion is based on Carathéodory's theorem, which leads to a way of expressing $co(K)$ as the image of a compact (finite dimensional) subset through a continuous mapping). Let m be the dimension of the linear variety generated by K (and by $co(K)$) and $x \in co(K)$. Assume that $m \geq 2$. Due to Carathéodory's theorem, there exist at most $m+1$ extreme points in the compact (convex) subset $co(K)$—for example, $e_1, \ldots, e_{m+1}$ and $\{\alpha_1, \ldots, \alpha_{m+1}\} \subset [0, \infty)$, $\sum_{j=1}^{m+1} \alpha_j = 1$, such that $x = \sum_{j=1}^{m+1} \alpha_j e_j$. Additionally, it is known that any extreme point of $co(K)$ is an extreme point of K. From this hypothesis, we derive

$$T_n(x) = \sum_{j=1}^{m+1} \alpha_j T_n(e_j) \to \sum_{j=1}^{m+1} \alpha_j \widetilde{T}(e_j) = \sum_{j=1}^{m+1} \alpha_j e_j = x, \;\; n \to \infty.$$

Thus, there exists $\widetilde{T}(x) = x \; \forall x \in co(K)$. On the other side, the positive cone X_+ of the space X is closed, and we have assumed that $\Phi_n(x) - T_n(x) \in X_+$ for all $x \in X$ and all $n \in \mathbb{N}$. Passing to the limit, one obtains

$$\widetilde{\Phi}(x) - \widetilde{T}(x) = \widetilde{\Phi}(x) - x \in X_+ \quad \forall x \in co(K) \Leftrightarrow \widetilde{\Phi}(x) \geq x \;\; \forall x \in co(K). \tag{10}$$

Since $\widetilde{T}$ is the pointwise limit of affine operators, it is affine on $co(K)$; a Hahn–Banach argument shows that it has an affine extension defined on the whole space X into X. We denote this extension with $\widetilde{T}$ as well. Recall that if $\mathbf{0}_X$ is not an element of $co(K)$, then we denote

$$C := Cone(K) = \{\alpha x; \; \alpha \in [0, \infty), \; x \in co(K)\}. \tag{11}$$

It is easy to see that in this case, $C \cap (-C) = \{\mathbf{0}_X\}$. Now (10) and (11) yield

$$\widetilde{\Phi}(\alpha x) = \alpha \widetilde{\Phi}(x) \geq \alpha x \;\; \forall \alpha \in [0, \infty), \, \forall x \in co(K) \Leftrightarrow \widetilde{\Phi}(w) \geq w \;\; \forall w \in C.$$

If $\mathbf{0}_X \in \partial(co(K)) \backslash ri(co(K))$, then C could satisfy the condition $C \cap (-C) = \{\mathbf{0}_X\}$, or $C \cap (-C)$ might be a nonzero vector subspace (here $ri(co(K))$ is the relative interior of $co(K)$). We denote with $co(K)$ the convex hull of the set K. When $\mathbf{0}_X \in ri(co(K))$, $C - C$ is an m-dimensional vector subspace of X. In both of these last two cases, the conclusion of the theorem still holds true, following the same proof as in the first case. This concludes the proof. □

In the sequel, we recall a consequence of the Carathéodory's theorem [8,21] and give a possible formulation for the infinite dimensional convex compact subsets. Namely, the following useful maximum principle for convex continuous real functions on finite dimensional convex compact subsets holds. If f is a continuous convex real function on a convex compact subset, $\mathrm{K} \subset \mathbb{R}^{\mathrm{n}}$ $(\mathrm{n} \in \{1, 2, \ldots\})$, then f attains a global maximum at an extreme point of K. For infinite dimensional convex compact subsets we derive from the Krein-Milman theorem the next result.

Theorem 11. *Let K be a compact convex subset in the locally convex space X, and $f : K \to \mathbb{R}$ a convex continuous function. Then $\sup\limits_{x \in K} f(x) = \sup\limits_{e e Extr(K)} f(e)$.*

Proof. Let $x \in K$. The continuity of f implies the compactness of $f(K)$, so $\sup\limits_{x \in K} f(x) < \infty$.

On the other hand, from Krein–Milman theorem, we know there exists a net $(x_\delta)_{\delta \in \Delta}$ with terms in the convex hull $co(Extr(K))$ of the set $Extr(K)$ of all extreme points of K such that

$$x = \lim_{\delta} x_\delta.$$

If $x_\delta = \sum_{i=1}^{k} \alpha_i e_i, \alpha_i \in [0, \infty),\ i = 1, \ldots, k,\ \sum_{i=1}^{k} \alpha_i = 1, e_i \in Extr(K),\ i = 1, \ldots, k$, then the convexity and the continuity of the function f yields

$$f(x_\delta) \leq \sum_{i=1}^{k} \alpha_i f(e_i) \leq \max_{1 \leq i \leq k} f(e_i) \leq \sup_{e \in Extr(K)} f(e) \leq \sup_{x \in K} f(x) < \infty,\ \delta \in \Delta,$$

$$f(x) = \lim_{\delta} f(x_\delta) \leq \sup_{e \in Extr(K)} f(e),$$

for all $x \in K$. This results in $\sup_{x \in K} f(x) \leq \sup_{e e Extr(K)} f(e)$. The inequality in the reversed sense is obvious. This ends the proof. □

Here follows a vector valued version of Theorem 11.

Corollary. 2 *Let K be a compact convex subset in the locally convex space X, and let Y be an order complete Banach lattice endowed with a strong order unit u_Y such that the closed unit ball in Y equals the order interval $[-u_Y, u_Y]$. Let $P : K \to Y$ be a continuous convex operator. Then,*

$$\sup_{x \in K} P(x) = \sup_{e e Extr(K)} P(e) \in Y.$$

Proof. The subset $P(K)$ is compact in Y thanks to the continuity of the operator P. In particular, $P(K)$ is bounded, so there exists $R > 0$ great enough such that the closed ball $B_R(\mathbf{0}_Y)$ centered in $\mathbf{0}_Y$ of radius R contains $P(K)$. We infer that:

$$x \in K \Rightarrow P(x) \in R[-u_Y, u_Y] = [-Ru_Y, Ru_Y] \Rightarrow P(x) \leq Ru_Y.$$

A first conclusion is that there exists $\sup_{x \in K} P(x) \leq Ru_Y$ because of the order completeness of Y. The rest of the proof follows the method and the inequalities from the proof of Theorem 11. This ends the proof. □

In what follows, we add related remarks. To do this, we require the following lemma [11].

Lemma 1. *Let X be a real normed linear space, $L \in X^*$ a continuous linear functional on X, and $M := \{x \in X; L(x) = \alpha\}$ a closed hyperplane defined by L, $x_0 \in X$. Then, the distance $d(x_0, M)$ from x_0 to M is provided by the formula*:

$$d(x_0, M) = \frac{|L(x_0) - \alpha|}{\|L\|}\ .$$

Remark 1. *Assume now that $\mathbf{0}_X$ is not an element of $co(K)$.Then, there exists a strictly positive linear continuous form T on X endowed with the order relation defined by C such that $\|T\| = 1$ and that there is a constant $\beta > 0$ with*

$$\inf_{x \in co(K)} T(x) = \beta = T(e) \text{ for some } e \in Extr(co(K)).$$

Indeed, denote by $d_0 := d(\mathbf{0}_X, co(K)) > 0,\ V := B_{d_0}(\mathbf{0}_X) = \{x \in X; \|x\| < d_0\}$. Then, V is a convex open neighborhood of the origin which does not intersect $co(K)$. Using the geometric form of the Hahn–Banach theorem (namely a separation theorem), there exists a closed hyperplane separating V and $co(K)$ and not intersecting V, i.e., there exists a linear continuous functional T on X such that $0 < supT(V) \leq \beta \leq infT(co(K))$. In particular,

$$T(x) \geq \beta > 0 \ \ \forall x \in co(K) \Rightarrow T(w) > 0 \, \forall w \in C \backslash \{\mathbf{0}_X\}.$$

Hence T is strictly positive and for any $\gamma > 0$, the set $B = \{x \in C; T(x) = \gamma\}$ is a compact base for C. Now, scaling T by a positive scalar, we obtain a new strictly continuous positive form (which we also denote by T) with the special property

$$0 < supT(V) = d_0 = infT(co(K)), \ T(x) < d_0 \, \forall x \in V.$$

This results in

$$T\left(d_0 \frac{x}{\|x\| + \varepsilon}\right) \langle d_0 \ \forall \varepsilon \rangle 0, \ \forall x \in X \Rightarrow |T(x)| \leq \|x\|, \ \forall x \in X \Rightarrow \|T\| \leq 1.$$

Let $x_\star \in co(K) \cap \ (\partial V)$ be such that $T(x_\star) = d_0$. Then, $\|x_\star\| = d_0$ and $T\left(\frac{x_\star}{\|x_\star\|}\right) = 1$. Thus, $\|T\| = 1$. Consider the hyperplane $H := \{x; T(x) = d_0\}$ and the base $B = H \cap \ C$. Then, the distance

$$d(\mathbf{0}_X, \ H) = \frac{|T(\mathbf{0}) - d_0|}{\|T\|} = d_0 = d(\mathbf{0}_X, co(K)),$$

as expected, (H and B are separating $cl(V)$ and $co(K)$, but they are "tangent" to both these closed convex subsets). If X is a real Hilbert space and T possesses the properties from above, then $x_\star$ is the orthogonal (or metric) projection of $\mathbf{0}_X$ to $co(K)$. Consequently,

$$\|x_\star\| = d(\mathbf{0}_X, co(K)) = d_0, \ \ x_\star \perp H$$

Having in mind the idea of the proof of the Riesz representation theorem for linear continuous forms on a Hilbert space, this results in T being represented by a vector which is collinear to $x_\star$. Since $\|T\| = 1$, we must normalize $x_\star$. This gives

$$T(x) = \left\langle \frac{x_\star}{\|x_\star\|}, x \right\rangle = \left\langle \frac{x_\star}{d_0}, x \right\rangle \quad \forall x \in X.$$

3.3. Applying Polynomial Approximation on Unbounded Subsets

Applying polynomial approximation [21,36] on $\mathbb{R}_+ := [0, \infty)$ of any nonnegative continuous compactly supported function by dominating polynomials on the entire semiaxes $\mathbb{R}_+$, as well as the explicit form of such a polynomial, namely

$$p \in \mathcal{P} = \mathbb{R}[t], p(t) \geq 0 \forall t \in \mathbb{R}_+ \Longleftrightarrow p(t) = q^2 + tr^2(t) \forall t \in \mathbb{R}_+$$

for some $q, r \in \mathcal{P}$, one can prove the following result. As is well known, any nonnegative polynomial p on the entire real axes is the sum of two squares [1]: $p = q^2 + r^2$, q, $r \in P = \mathbb{R}[t]$.

Theorem 12. *Let $X = L^1_\nu(\mathbb{R}_+)$, where ν is a moment-determinate measure on $\mathbb{R}_+$. Assume that Y is an arbitrary order complete Banach lattice and that $(y_n)_{n \geq 0}$ is a given sequence with its terms in Y. Let T_1, T_2 be two linear operators from X to Y , such that $0 \leq T_1 \leq T_2$ on X_+. As usual, we denote $\varphi_j(t) = t^j$, $j \in \mathbb{N}$, $t \in \mathbb{R}_+$. The following statements are equivalent:*

(a) *There exists a unique bounded linear operator T from X into Y, $T_1 \leq T \leq T_2$ on X_+, $\|T_1\| \leq \|T\| \leq \|T_2\|$, such that $T(\varphi_n) = y_n$ for all $n \in \mathbb{N}$;*

(b) *If $J_0 \subset \mathbb{N}$ is a finite subset, and $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, then*

$$\sum_{i,j\in J_0} \lambda_i\lambda_j T_1\left(\varphi_{i+j+k}\right) \le \sum_{i,j\in J_0} \lambda_i\lambda_j y_{i+j+k} \le \sum_{i,j\in J_0} \lambda_i\lambda_j T_2\left(\varphi_{i+j+k}\right),\ k \in \{0,1\}.$$

Corollary 3. *Let ν be a moment-determinate measure on $\mathbb{R}_+$. Assume that h_1, h_2 are two functions in $L^\infty_\nu(\mathbb{R}_+)$ such that $0 \le h_1 \le h_2$ almost everywhere. Let $(y_n)_{n\ge 0}$ be a given sequence of real numbers. The following statements are equivalent:*

(a) *there exists $h \in L^\infty_\nu(\mathbb{R}_+)$ such that $h_1 \le h \le h_2$ $\nu-$ almost everywhere; $\int_{\mathbb{R}_+} t^j h(t)d\nu = y_j$ for all $j \in \mathbb{N}$;*

(b) *if $J_0 \subset \mathbb{N}$ is a finite subset and $\left\{\lambda_j; j \in J_0\right\} \subset \mathbb{R}$, then:*

$$\sum_{i,j\in J_0} \lambda_i\lambda_j \int_{\mathbb{R}_+} t^{i+j+k} h_1(t)d\nu \le \sum_{i,j\in J_0} \lambda_i\lambda_j y_{i+j+k} \le \sum_{i,j\in J_0} \lambda_i\lambda_j \int_{\mathbb{R}_+} t^{i+j+k} h_2(t)d\nu,\ k \in \{0,1\}.$$

4. Discussion

First, results reviewed in this paper refer to the Hahn–Banach- and Mazur–Orlicz-type theorems involving linear operators as extensions, and convex or sublinear operators as dominating constraint on the extension. The positivity of the solutions is also a result of these general theorems. Next, applications to concrete spaces are discussed. Here, Markov operators appear as linear solutions. Second, approximation via the Krein–Milman theorem is applied to extending properties from a small set to the entire positive cone of the domain space. Elements of optimization are deduced as well. Third, polynomial approximation on closed unbounded subsets is applied to the characterization of existence and uniqueness of the solution for a Markov moment problem in terms of quadratic forms. Here, the Hahn–Banach theorem does not appear explicitly, but it is essentially used in proving previous results on which Section 3.3 is based. The common point of these three subjects is the notion of convexity (especially related to the Hahn–Banach theorem) completed by polynomial approximation in $L^1_\nu(F)$ spaces, where $F \subseteq \mathbb{R}^n$, $n \ge 1$, $n \in \mathbb{N}$ is a closed unbounded subset and ν is a moment determinate positive regular Borel measure on F. The reader could find other aspects of the subjects discussed here in the references below. As a direction for continuing this work, one can study more deeply the multidimensional case for results as those of Section 3.3. In the Introduction, a few applications of Hahn–Banach theorem are briefly discussed by means of the reference citations. Connections or applications of the moment problem and of the Hahn–Banach theorem to optimization theory, polynomial approximation on unbounded subsets, characterizing positivity of some bounded linear operators, nonstandard sandwich type results, and elements of representation theory, are pointed out. A common point of all subsections of Section 3 is the fact that all these results are directly or indirectly related to Hahn–Banach-type theorems and/or the moment problem.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Akhiezer, N.I. *The Classical Moment Problem and Some Related Questions in Analysis*; Oliver and Boyd: Edinburgh, UK, 1965.
2. Berg, C.; Christensen, J.P.R.; Ressel, P. *Harmonic Analysis on Semigroups*; Theory of Positive Definite and Related Functions; Springer: New York, NY, USA, 1984.
3. Krein, M.G.; Nudelman, A.A. *Markov Moment Problem and Extremal Problems*; American Mathematical Society: Providence, RI, USA, 1977.

4. Schmüdgen, K. The Moment Problem. In *Graduate Texts in Mathematics*; Springer International Publishing AG: Cham, Switzerland, 2017.
5. Rudin, W. *Real and Complex Analysis*, 3rd ed.; McGraw-Hill Book Company: Singapore, 1987.
6. Schaefer, H.H.; Wolff, M.P. *Topological Vector Spaces*, 2nd ed.; Springer: New York, NY, USA, 1999.
7. Cristescu, R. *Ordered Vector Spaces and Linear Operators*; Academiei: Bucharest, Romania; Abacus Press: Tunbridge Wells, UK, 1976.
8. Niculescu, C.P.; Persson, L.-E. *Convex Functions and Their Applications. A Contemporary Approach*, 2nd ed.; CMS Books in Mathematics; Springer: New York, NY, USA, 2018; Volume 23.
9. Niculescu, C.; Popa, N. *Elements of Theory of Banach Spaces*; Academiei: Bucharest, Romania, 1981. (In Romanian)
10. Holmes, R.B. *Geometric Functional Analysis and its Applications*; Springer: New York, NY, USA, 1975.
11. Singer, I. *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*; Springer: Berlin, Germany; New York, NY, USA, 1970.
12. Haviland, E.K. On the momentum problem for distributions in more than one dimension. *Am. J. Math.* **1936**, *58*, 164–168. [CrossRef]
13. Valadier, M. Sous-Différentiabilité de fonctions convexes à valeurs dans un espace vectoriel ordonné (Subdifferentiability of convex functions having as codomain an ordered vector space). *Math. Scand.* **1972**, *30*, 65–74. [CrossRef]
14. Zowe, J. Sandwich Theorems for Convex Operators with Values in an Ordered Vector Space. *J. Math. Anal. Appl.* **1978**, *66*, 282–296. [CrossRef]
15. Kutateladze, S.S. Convex operators. *Russ. Math. Surv.* **1979**, *34*, 181–214. [CrossRef]
16. Olteanu, O. Convexité et prolongement d'opérateurs linéaires (Convexity and extension of linear operators). *C. R. Acad. Sci. Paris* **1978**, *286*, 511–514.
17. Olteanu, O. Théorèmes de prolongement d'opérateurs linéaires (Theorems of extension of linear operators). *Rev. Roum. Math. Pures Appl.* **1983**, *28*, 953–983.
18. Olteanu, O. Application de théorèmes de prolongement d'opérateurs linéaires au problème des moments e à une generalization d'un théorème de Mazur-Orlicz, (Applications of theorems on extension of linear operators to the moment problem and to a generalization of Mazur-Orlicz theorem). *C. R. Acad. Sci. Paris* **1991**, *313*, 739–742.
19. Niculescu, C.P.; Olteanu, O. From the Hahn-Banach extension theorem to the isotonicity of convex functions and the majorization theory. *Rev. R. Acad. Cienc. Exactas Fis. Nat.* **2020**, *114*, 171. [CrossRef]
20. Olteanu, O. New results on extension of linear operators and Markov moment problem. *MathLAB J.* **2020**, *5*, 143–154.
21. Olteanu, O. On Hahn-Banach theorem and some of its applications. *Open Math.* **2022**, *20*, 366–390. [CrossRef]
22. Montiel López, P.; Ruiz Galán, M. Revisiting the Hahn-Banach theorem and nonlinear infinite programming. *J. Math. Anal. Appl.* **2017**, *455*, 1037–1050. [CrossRef]
23. Ruiz Galán, M. A version of the Lax–Milgram theorem for locally convex spaces. *J. Convex Anal.* **2009**, *16*, 993–1002.
24. Simons, S. The asymmetric sandwich theorem. *J. Convex Anal.* **2013**, *20*, 107–124.
25. Simons, S. Bootstrapping the Mazur–Orlicz–König theorem and the Hahn–Banach–Lagrange theorem. *J. Convex Anal.* **2018**, *25*, 691–699.
26. Boccuto, A. Hahn-Banach and sandwich theorems for equivariant vector lattice-valued operators and applications, Tatra Mt. *Math. Publ.* **2020**, *76*, 11–34. [CrossRef]
27. Marshall, M. Polynomials non-negative on a strip. *Proc. Am. Math. Soc.* **2010**, *138*, 1559–1567. [CrossRef]
28. Stoyanov, J.M.; Lin, G.D.; Kopanov, P. New checkable conditions for moment determinacy of probability distributions. *SIAM Theory Probab. Appl.* **2020**, *65*, 497–509. [CrossRef]
29. Stochel, J. Solving the truncated moment problem solves the full moment problem. *Glasg. Math. J.* **2001**, *43*, 335–341. [CrossRef]
30. Tagliani, A. Maximum entropy solutions and moment problem in unbounded domains. *Appl. Math. Lett.* **2003**, *16*, 519–524. [CrossRef]
31. Inverardi, P.L.N.; Tagliani, A. Stieltjies and Hamburger reduced moment problem when MaxEnt solution does not exist. *Mathematics* **2021**, *9*, 309. [CrossRef]
32. Cobzaş, Ş. Geometric properties of Banach spaces and the existence of nearest and farthest points. *Abstr. Appl. Anal.* **2005**, *2005*, 259–285. [CrossRef]
33. Cobzaş, Ş. Ekeland variational principle and its equivalents in T_1 quasi-uniform spaces. *Optimization* **2022**, 1–32. [CrossRef]
34. Pakkaranang, N.; Kumam, P.; Cho, Y.J. Proximal point algorithms for solving convex minimization problem and common fixed points of asymptotically quasi-nonexpansive mappings in in CAT(0) spaces with convergenece analysis. *Numer. Algorithms* **2018**, *78*, 827–845. [CrossRef]
35. Bucur, I.; Păltineanu, G. De Branges type lemma and approximation in weighted spaces. *Mediterr. J. Math.* **2021**, *18*, 120. [CrossRef]
36. Olteanu, O. On Markov moment problem and related results. *Symmetry* **2021**, *13*, 986. [CrossRef]
37. Olteanu, O. On special properties for continuous convex operators and related linear operators. *Symmetry* **2022**, *14*, 1390. [CrossRef]
38. Norris, D.T. Optimal Solutions to the L_∞ Moment Problem with Lattice Bounds. Ph.D. Thesis, Department of Mathematics, College of Arts and Sciences, University Colorado Boulder, Boulder, CO, USA, 2002.
39. Gosse, L.; Runborg, O. Resolution of the finite Markov moment problem. *C. R. Acad. Sci. Paris* **2005**, *341*, 775–780. [CrossRef]

40. Gosse, L.; Runborg, O. Existence, uniqueness, and a constructive solution algorithm for a class of finite Markov moment problems. *SIAM J. Appl. Math.* **2008**, *68*, 16181640. [CrossRef]
41. Ambrozie, C.; Olteanu, O. A sandwich theorem, the moment problem, finite-simplicial sets and some inequalities. *Rev. Roum. Math. Pures Appl.* **2004**, *49*, 189–210.
42. Olteanu, O. Sur les fonctions convexes définies sur les ensembles convexes bornés de $\mathbb{R}^n$ (On convex functions defined on bounded convex subsets of $\mathbb{R}^n$). *C. R. Acad. Sci. Paris* **1980**, *290*, 837–838.