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An Axiomatization of the Value α for Games Restricted by Augmenting Systems

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Abstract: The value α for augmenting structures was introduced and axiomatically characterized by Algaba, Bilbao and Slikker. In this paper, we provide a new axiomatization of the value α for augmenting structures by using marginality for augmenting structures and the standard axioms of component efficiency, equal treatment of necessary players and loop-null.

Keywords: TU-game; augmenting system; Shapley value; marginality; Myerson value

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1. Introduction

Communication plays a significant role in many social and economic situations. Cooperation under communication restrictions and surplus distributions can be described and analyzed through cooperative games restricted by a combinatorial structure. The first model in which the restrictions are defined by the connected subgraphs of a graph was introduced by Myerson [1–13]. Since then, many other situations in which the communication restrictions are described by graphs or hypergraphs have been studied in cooperative game theory.

Bilbao [4] introduced the restricted cooperation model derived from a combinatorial structure called augmenting system. The combinatorial structure is a generalization of the antimatroid structure [2,3] and the system of connected subgraphs of a graph [13,14]. For games under augmenting systems, Algaba et al. [1] proposed an allocation rule called the value α, which generalizes the Myerson value for games restricted by graphs and the Shapley value for games restricted by permission structures. This value α for augmenting structures has been characterized axiomatically by using either component efficiency, loop-null and balanced contributions, or standardness for two-person restricted games and the consistency of this value. The variants of balanced contributions have been suggested and applied to axiomatic characterizations of the other values for cooperative games in [6,15–19]. More studies for games on augmenting systems can be found in [5,12].

The marginality principle has a long tradition in economic theory [17]. The reason is that the outcome of a player in cooperative games is largely based on the player's marginal contributions to all coalitions. Marginality requires that a player's payoff only depends on her own productivity. Many allocation rules in cooperative games have been characterized axiomatically by using the property of marginality. For example, the Shapley value [20] and the Owen value [9,10]. It is known that the Myerson value for games restricted by graphs does not satisfy the axiom of marginality. However, Manuel et al. [11] introduced a PL-marginality associated with the set of links for games restricted by graphs and gave an axiomatization of the Myerson value by using PL-marginality and the other standard axioms in the literature.

The purpose of this paper is to provide a new axiomatization of the value α for augmenting structures by introducing a new property of marginality, namely marginality for augmenting structures. We show that the value α for augmenting structures is uniquely determined by the marginality for augmenting structures as well as the standard axioms: component efficiency, the loop-null property used by Algaba et al. in [1] and equal treatment of necessary players used by van den Brink and Gilles in [7]. The property of marginality for augmenting structures requires that equal marginal contributions in games restricted by augmenting systems imply equal allocation.

In Section 2, we give preliminaries. In Section 3, we propose an axiomatic characterization of the value α for augmenting structures by marginality for augmenting structures. Section 4 gives some concluding remarks.

2. Preliminaries

2.1. TU-Games

A *cooperative game with transferable utility*, in short, a TU-game, is composed of a nonempty set N and a *characteristic function* defined on the collection of all subsets of N and having the property $v(\emptyset) = 0$. We shall denote the TU-game given through N and v by (N, v), or simply v, and the collection of all TU-games with a player set N by $\mathcal{G}^N$. Each subset S of N is called a *coalition* and $v(S)$ is the *worth* of coalition S, i.e., the members of N can obtain total payoff $v(S)$ by agreeing to cooperate. For the simplicity of notation, we write $v(i, \ldots, k)$ and $S \setminus i$ for $v(\{i, \ldots, k\})$ and $S \setminus \{i\}$, respectively. The cardinality of set A is denoted by $|A|$ or the corresponding lower case letter $a = |A|$.

For nonempty $S \subseteq N$, the *subgame* of v with respect to S is $v_S(T) = v(T)$, for all $T \subseteq S$. The *unanimity game* with respect to S is defined by $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. Every game (N, v) is a unique linear combination of unanimity games,

$$v = \sum_{\emptyset \neq S \subseteq N} \lambda_S(v) u_S,$$

where $\lambda_S(v) = \sum_{\emptyset \neq T \subseteq S} (-1)^{s-t} v(T)$ is called the *unanimity coefficient* of S in (N, v). Hence, the worth of every coalition S can be written in terms of them as

$$v(S) = \sum_{\emptyset \neq T \subseteq S} \lambda_T(v). \tag{1}$$

An allocation rule (also called a *value*) on $\mathcal{G}^N$ is a function φ that assigns to every game $(N, v) \in \mathcal{G}^N$ a payoff vector $\varphi(N, v) \in \mathbb{R}^n$, $\varphi_i(N, v)$ representing the outcome of player i in the game (N, v).

The Shapley value is a well-known allocation rule that is defined in [18] by

$$Sh_i(N, v) = \sum_{i \in S, S \subseteq N} \frac{\lambda_S(v)}{|S|} \quad \text{for all } i \in N.$$

Shapley [18] introduced the first axiomatization of this value, which is founded on the axioms of efficiency, the null player property, symmetry and additivity. Let us state the axioms as follows.

- *Efficiency*. $\sum_{i \in N} \varphi_i(N, v) = v(N)$.
- *Null player property*. For any *null player* $i \in N$, i.e., $v(S \cup i) = v(S)$ for any $S \subseteq N \setminus i$, $\varphi_i(N, v) = 0$.
- *Symmetry*. For any *symmetric players* $i, j \in N$, i.e., $v(S \cup i) = v(S \cup j)$ for any $S \subseteq N \setminus \{i, j\}$, $\varphi_i(N, v) = \varphi_j(N, v)$.
- *Additivity*. For any $(N, v), (N, w) \in \mathcal{G}^N$, $\varphi(N, v + w) = \varphi(N, v) + \varphi(N, w)$.

2.2. Augmenting Systems

A *set system* on N is a pair $(N, \mathcal{F})$ where $\mathcal{F} \subseteq 2^N$ is a family of subsets of N. The sets belonging to $\mathcal{F}$ are called *feasible*. For a coalition $R \subseteq N$, the set system $(R, \mathcal{F}_R)$ induced by R is defined by $\mathcal{F}_R = \{S \in \mathcal{F} : S \subseteq R\}$. For $i \in N$, we define $\mathcal{F} \setminus i = \{S \in \mathcal{F} : i \notin S\}$. The set system $(N \setminus i, \mathcal{F} \setminus i)$ is the deletion of i in $(N, \mathcal{F})$.

An augmenting system is a set system $(N, \mathcal{F})$ with the following properties:

(i) $\emptyset \in \mathcal{F}$;
(ii) If $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, then $S \cup T \in \mathcal{F}$;
(iii) If $S, T \in \mathcal{F}$ with $S \subset T$, then there exists $i \in T \setminus S$ such that $S \cup i \in \mathcal{F}$.

By definition of an augmenting system, if $(N, \mathcal{F})$ is an augmenting system, then $(N \setminus i, \mathcal{F} \setminus i)$ is an augmenting system. Player $i \in N$ is called an *isolated player* (also called *loop player*) in an augmenting system $(N, \mathcal{F})$ if $i \in N \setminus \bigcup_{S \in \mathcal{F}} S$. Obviously, $(N, \mathcal{F} \setminus i)$ is also an augmenting system when i is an isolated player in $(N, \mathcal{F})$.

Let $(N, \mathcal{F})$ be a set system and let $S \subseteq N$ be a subset. The maximal nonempty feasible subsets of S are called *components* of S. We denote by $C_{\mathcal{F}}(S)$ the set of components of a subset $S \subseteq N$. Observe that the set $C_{\mathcal{F}}(S)$ may be the empty set. Clearly, i is isolated if and only if $i \notin C$ for all $C \in C_{\mathcal{F}}(N)$.

Let $\mathbb{P}$ be the set of all positive integers and let $N = \{1, \ldots, n\} \subseteq \mathbb{P}$. An *augmenting structure* on N is a triple $(N, v, \mathcal{F})$, where (N, v) is a TU-game on $\mathcal{G}^N$ and $(N, \mathcal{F})$ is an augmenting system. The set of all augmenting structures with player set N is denoted by AS^N, and the set of all augmenting structures is given by $AS = \bigcup_{N \subseteq \mathbb{P}} AS^N$.

An allocation rule φ on AS^N is a map φ: $AS^N \to \mathbb{R}^n$, $\varphi_i(N, v, \mathcal{F})$ representing the outcome for player i in the augmenting structure $(N, v, \mathcal{F})$.

To introduce the allocation rule α, Bilbao [4] defined the restricted game under augmenting systems. Let $(N, v, \mathcal{F}) \in AS^N$. The *restricted game* $v^{\mathcal{F}}$ with respect to augmenting system $\mathcal{F}$ is defined by

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T) \text{ for all } S \subseteq N.$$

For any $T \in \mathcal{F}$, let $T^+ = T \cup T^*$ where $T^* = \{i \in N \setminus T : T \cup i \in \mathcal{F}\}$. Bilbao [4] obtained the following properties of the restricted game $v^{\mathcal{F}}$ inspired by the result of Owen [16] for graph-restricted games.

Lemma 1. *Let $(N, v, \mathcal{F})$ be an augmenting structure. Then the restricted game $v^{\mathcal{F}}$ satisfies*

$$v^{\mathcal{F}} = \sum_{S \in \mathcal{F}, S \neq \emptyset} \lambda_S(v^{\mathcal{F}}) u_S,$$

where

$$\lambda_S(v^{\mathcal{F}}) = \sum_{T \in \mathcal{F}, T \subseteq S \subseteq T^+} (-1)^{s-t} v(T)$$

and $\lambda_S(v^{\mathcal{F}}) = 0$ for all $S \notin \mathcal{F}$.

The allocation rule α on AS^N is defined in [1] as the Shapley value of the restricted game $v^{\mathcal{F}}$, i.e.,

$$\alpha(N, v, \mathcal{F}) = Sh(N, v^{\mathcal{F}}).$$

By Lemma 1,

$$\alpha_i(N, v, L) = Sh_i(N, v^{\mathcal{F}}) = \sum_{i \in S \in \mathcal{F}} \frac{\lambda_S(v^{\mathcal{F}})}{|S|} \text{ for all } i \in N. \tag{2}$$

Note that the value $\alpha(N, v, \mathcal{F})$ coincides with the Myerson value for games restricted by graphs when $(N, \mathcal{F})$ is an augmenting system such that $\{i\} \in \mathcal{F}$ for all $i \in N$.

Algaba et al. [1] showed that value α on AS^N is uniquely determined by the axioms of either component efficiency, loop-null and balanced contributions or consistency and standardness for two-person restricted games below.

Let φ be an allocation rule on AS^N.

Component efficiency. For all $(N, v, \mathcal{F}) \in AS^N$ and $C \in C_{\mathcal{F}}(N)$,

$$\sum_{i \in C} \varphi_i(N, v, \mathcal{F}) = v(C).$$

Loop-null. For all $(N, v, \mathcal{F}) \in AS^N$ and for any isolated player i in $(N, \mathcal{F})$, $\varphi_i(N, v, \mathcal{F}) = 0$.

Loop-null states that every player who is not in any admissible (or, feasible) coalition obtains zero. The property is also called an *isolated property* in the literature.

Balanced contributions. For all $(N, v, \mathcal{F}) \in AS^N$ and any two players $i, j \in N$ with $i \neq j$,

$$\varphi_j(N, v, \mathcal{F}) - \varphi_j(N, v, \mathcal{F} \setminus i) = \varphi_i(N, v, \mathcal{F}) - \varphi_i(N, v, \mathcal{F} \setminus j).$$

Algaba et al. [1] established an axiomatic characterization of the value α for augmenting structures in terms of component efficiency, loop-null and balanced contributions.

Theorem 1. *The value α is the unique allocation rule on AS^N that satisfies component efficiency, loop-null and balanced contributions.*

In [1] Algaba et al. also provided another characterization of value α by means of consistency and standardness for two-person restricted games. The consistency axiom was introduced by Hart and Mas-Colell [8] and applied to the characterization of the Shapley value.

3. The Axiomatization

In this section, we shall give an alternative axiomatization of the value $\alpha(N, v, L)$. For this purpose, let us introduce more terminology and properties.

For any $(N, v) \in \mathcal{G}^N$, we define

$$v_i = \sum_{i \in T, T \subseteq N} \lambda_T(v) u_T. \tag{3}$$

The following formulation directly holds by (1).

$$v_i(S) = v(S) - v(S \setminus i) \text{ for all } S \subseteq N. \tag{4}$$

A player $i \in N$ is a *necessary player* in $(N, v) \in \mathcal{G}^N$ if $v(S) = 0$ for all $S \subseteq N \setminus i$ (see [7]). The *marginal contribution* of a player $i \in N$ to a coalition $S \subseteq N \setminus i$ is measured as $v(S \cup i) - v(S)$.

Let φ be an allocation rule on AS^N and any $(N, v, \mathcal{F}) \in AS^N$.

Equal treatment of necessary players. For any $i, j \in N$, if i, j are necessary players in (N, v), then $\varphi_i(N, v, \mathcal{F}) = \varphi_j(N, v, \mathcal{F})$.

This axiom requires that all players necessary to produce worth shall receive the same payoff. It has been applied to the characterizations of values in [11,15]. By definition of the restricted game $v^{\mathcal{F}}$, if i, j are necessary players in (N, v) for $i, j \in N$, then i, j are both necessary and symmetric players in $(N, v^{\mathcal{F}})$.

Marginality for augmenting structures. For any $(N, v, \mathcal{F}), (N, w, \mathcal{F}) \in AS^N$ and $i \in N$, if $v^{\mathcal{F}}(S \cup i) - v^{\mathcal{F}}(S) = w^{\mathcal{F}}(S \cup i) - w^{\mathcal{F}}(S)$ for all $S \subseteq N \setminus i$, then $\varphi_i(N, v, \mathcal{F}) = \varphi_i(N, w, \mathcal{F})$.

Marginality for augmenting structures states that a player's payoff should depend only on his own productivity in the restricted game $v^{\mathcal{F}}$.

Lemma 2. *For any $(N, v, \mathcal{F}) \in AS^N$ and $S \in \mathcal{F}$, we have*

$$\lambda_S\left[(v^{\mathcal{F}})_i\right] = \begin{cases} \lambda_S(v^{\mathcal{F}}), & \text{if } i \in S \\ 0, & \text{if } i \notin S \end{cases}$$

Proof. Using (4), we have

$$\begin{aligned} \lambda_S\left[(v^{\mathcal{F}})_i\right] &= \sum_{T \subseteq S} (-1)^{s-t} (v^{\mathcal{F}})_i(T) \\ &= \sum_{T \subseteq S} (-1)^{s-t} \left[v^{\mathcal{F}}(T) - v^{\mathcal{F}}(T \setminus i)\right]. \end{aligned}$$

If $i \notin S$, then $\lambda_S\left[(v^{\mathcal{F}})_i\right] = 0$ as $v^{\mathcal{F}}(T) = v^{\mathcal{F}}(T \setminus i)$. If $i \in S$, then

$$\begin{aligned} \lambda_S\left[(v^{\mathcal{F}})_i\right] &= \sum_{i \in T \subseteq S} (-1)^{s-t} \left[v^{\mathcal{F}}(T) - v^{\mathcal{F}}(T \setminus i)\right] \\ &= \sum_{T \subseteq S} (-1)^{s-t} v^{\mathcal{F}}(T) = \lambda_S(v^{\mathcal{F}}), \end{aligned}$$

Showing the assertion. □

Theorem 2. *The value $\alpha(N, v, L)$ is the unique allocation rule on AS^N satisfying component efficiency, equal treatment of necessary players, loop-null and marginality for augmenting structures.*

Proof. It is easy to check that the value $\alpha(N, v, L)$ satisfies the four properties in Theorem 2. It has been shown that $\alpha(N, v, L)$ satisfies component efficiency and loop-null by Algaba et al. [1]. $\alpha(N, v, L)$ satisfies the properties of equal treatment of necessary and marginality for augmenting structures follows directly from the fact that the Shapley value satisfies symmetry and marginality in $v^{\mathcal{F}}$, respectively.

Let φ be an allocation rule satisfying the four properties in Theorem 2, we have to show that $\varphi = \alpha$. If $|N| = 1$, then clearly $\alpha(N, v, \mathcal{F}) = \varphi(N, v, \mathcal{F})$ by component efficiency and loop-null. Therefore, we may assume that $|N| \geq 2$.

We establish this by contradiction. Let $(N, v, \mathcal{F}) \in AS^N$ be a game with a minimum number of terms $\lambda_S(v^{\mathcal{F}}) \neq 0$ under the summation below such that $\varphi \neq \alpha$.

$$v^{\mathcal{F}} = \sum_{\emptyset \neq S \in \mathcal{F}} \lambda_S(v^{\mathcal{F}}) u_S. \tag{5}$$

Let $D(N, v^{\mathcal{F}}) = \{S \in \mathcal{F} : \lambda_S(v^{\mathcal{F}}) \neq 0\}$. Note that $(v^{\mathcal{F}})^{\mathcal{F}} = v^{\mathcal{F}}$ by the definition of $v^{\mathcal{F}}$. Thus, by marginality for augmenting structures of φ,

$$\varphi(N, v, \mathcal{F}) = \varphi(N, v^{\mathcal{F}}, \mathcal{F}). \tag{6}$$

If $|D(N, v^{\mathcal{F}})| = 0$, then $v^{\mathcal{F}} = \mathbf{0}$, and so each $i \in N$ is necessary in $v^{\mathcal{F}}$. This implies that each pair $i, j \in N$ is symmetric in $v^{\mathcal{F}}$. By symmetry of the Shapley value in $v^{\mathcal{F}}$ and component efficiency of the value $\alpha(N, v, L)$,

$$\begin{aligned} \sum_{i \in N} \alpha_i(N, v, \mathcal{F}) &= \sum_{i \in N} Sh_i(N, v^{\mathcal{F}}) = |N| Sh_i(N, v^{\mathcal{F}}) \\ \sum_{i \in N} \alpha_i(N, v, \mathcal{F}) &= v^{\mathcal{F}}(N) = 0. \end{aligned}$$

Therefore, $\alpha_i(N, v, \mathcal{F}) = Sh_i(N, v^{\mathcal{F}}) = 0$ for all $i \in N$. On the other hand, by component efficiency of φ in $v^{\mathcal{F}}$,

$$\sum_{i \in N} \varphi_i(N, v^{\mathcal{F}}, \mathcal{F}) = (v^{\mathcal{F}})^{\mathcal{F}}(N) = v^{\mathcal{F}}(N) = 0.$$

By equal treatment of necessary players of φ in $v^{\mathcal{F}}$, we obtain $\varphi_i(N, v^{\mathcal{F}}, \mathcal{F}) = 0$ for all $i \in N$. By (6), $\varphi_i(N, v, \mathcal{F}) = 0$ for all $i \in N$. Hence $\alpha(N, v, \mathcal{F}) = \varphi(N, v, \mathcal{F})$, a contradiction. Thus $|D(N, v^{\mathcal{F}})| \geq 1$.

Let $A = \bigcap_{S \in D(N, v^{\mathcal{F}})} S$. We now consider each $i \in N$.

Suppose $i \in A$. Then i is a necessary player in $v^{\mathcal{F}}$ since

$$v^{\mathcal{F}}(S) = \sum_{T \subseteq S} \lambda_T(v^{\mathcal{F}}) = \sum_{T \in \mathcal{F}, T \subseteq S} \sum_{R \in \mathcal{F}, R \subseteq T \subseteq R^+} (-1)^{(t-r)} v(R) = 0 \text{ for any } S \subseteq N \setminus i$$

By Lemma 1. The property of equal treatment of necessary players of φ in $v^{\mathcal{F}}$ requires the allocation of exactly the same payoff to either of these players in A, i.e., $\varphi_i(N, v^{\mathcal{F}}, \mathcal{F}) = \varphi_j(N, v^{\mathcal{F}}, \mathcal{F})$ for all $i, j \in A$. By (6),

$$\varphi_i(N, v, \mathcal{F}) = \varphi_j(N, v, \mathcal{F}) \text{ for all } i, j \in A.$$

For the value α, since α satisfies equal treatment of necessary players in v,

$$\alpha_i(N, v, \mathcal{F}) = \alpha_j(N, v, \mathcal{F}) \text{ for all } i, j \in A.$$

Suppose $i \notin A$. We first claim that $\left[\left(v^{\mathcal{F}}\right)_i\right]^{\mathcal{F}} = \left(v^{\mathcal{F}}\right)_i$. Indeed, by (4)

$$\left(v^{\mathcal{F}}\right)_i(S) = v^{\mathcal{F}}(S) - v^{\mathcal{F}}(S \setminus i) = \sum_{C \in C_{\mathcal{F}}(S)} v(C) - \sum_{C \in C_{\mathcal{F}}(S \setminus i)} v(C) = v(C_i) - v^{\mathcal{F}}(C_i \setminus i)$$

For any $S \subseteq N$ and $i \in S$, where $C_i \in C_{\mathcal{F}}(S)$ is the component of S containing player i. On the other hand, by (4),

$$\left[\left(v^{\mathcal{F}}\right)_i\right]^{\mathcal{F}}(S) = \sum_{C \in C_{\mathcal{F}}(S)} \left(v^{\mathcal{F}}\right)_i(C) = \left(v^{\mathcal{F}}\right)_i(C_i) = v(C_i) - v^{\mathcal{F}}(C_i \setminus i)$$

For any $S \subseteq N$ and $i \in S$. Hence $\left[\left(v^{\mathcal{F}}\right)_i\right]^{\mathcal{F}} = \left(v^{\mathcal{F}}\right)_i$.

Again by (4), $\left[\left(v^{\mathcal{F}}\right)_i\right]^{\mathcal{F}}(S \setminus i) = \left(v^{\mathcal{F}}\right)_i(S \setminus i) = 0$. Hence

$$\begin{aligned} \left[\left(v^{\mathcal{F}}\right)_i\right]^{\mathcal{F}}(S) - \left[\left(v^{\mathcal{F}}\right)_i\right]^{\mathcal{F}}(S \setminus i) &= \left(v^{\mathcal{F}}\right)_i(S) - \left(v^{\mathcal{F}}\right)_i(S \setminus i) = \left(v^{\mathcal{F}}\right)_i(S) \\ &= v^{\mathcal{F}}(S) - v^{\mathcal{F}}(S \setminus i), \end{aligned}$$

For all $S \subseteq N$. The property of marginality for augmenting structures implies that $\varphi_i(N, v, \mathcal{F}) = \varphi_i(N, \left(v^{\mathcal{F}}\right)_i, \mathcal{F})$. Note that $|D(N, \left(v^{\mathcal{F}}\right)_i)| < |D(N, v^{\mathcal{F}})|$. By the minimality of $|D(N, v^{\mathcal{F}})|$, we have $\varphi(N, \left(v^{\mathcal{F}}\right)_i, \mathcal{F}) = \alpha(N, \left(v^{\mathcal{F}}\right)_i, \mathcal{F})$. Then $\varphi_i(N, v, \mathcal{F}) = \alpha_i(N, \left(v^{\mathcal{F}}\right)_i, \mathcal{F})$.

By the definition of value α and the above equality $\left[\left(v^{\mathcal{F}}\right)_i\right]^{\mathcal{F}} = \left(v^{\mathcal{F}}\right)_i$,

$$\alpha_i\left(N, \left(v^{\mathcal{F}}\right)_i, \mathcal{F}\right) = Sh_i\left(N, \left[\left(v^{\mathcal{F}}\right)_i\right]^{\mathcal{F}}\right) = Sh_i\left(N, \left(v^{\mathcal{F}}\right)_i\right).$$

By (2) and Lemma 2, we have $Sh_i(N, \left(v^{\mathcal{F}}\right)_i) = Sh_i(N, v^{\mathcal{F}})$. Hence

$$\varphi_i(N, v, \mathcal{F}) = Sh_i(N, v^{\mathcal{F}}) = \alpha_i(N, v, \mathcal{F}) \text{ for all } i \in N \setminus A.$$

If $|A| = 0$, then $\varphi_i(N, v, \mathcal{F}) = \alpha_i(N, v, \mathcal{F})$ for all $i \in N$, contradicting the assumption that $\varphi \neq \alpha$. Thus $|A| > 0$. Furthermore, by component efficiency and (i),

$$\sum_{i \in N} \alpha_i(N, v, \mathcal{F}) = v^{\mathcal{F}}(N) = \sum_{i \in N} \varphi_i(N, v, \mathcal{F}) = |A| \varphi_i(N, v, \mathcal{F}) + \sum_{i \in N \setminus A} \alpha_i(N, v, \mathcal{F}).$$

Or, equivalently,

$$|A|\alpha_i(N,v,\mathcal{F}) = \sum_{i\in A} \alpha_i(N,v,\mathcal{F}) = |A|\varphi_i(N,v,\mathcal{F}).$$

Therefore, $\alpha_i(N,v,\mathcal{F}) = \varphi_i(N,v,\mathcal{F})$ for all $i \in A$. This implies that $\alpha_i(N,v,\mathcal{F}) = \varphi_i(N,v,\mathcal{F})$ for all $i \in N$, a contradiction. $\square$

Remark 1. *The property of equal treatment of necessary players in Theorem 2 can not be replaced by equal treatment of symmetric players. This is because the symmetry in (N,v) does not guarantee the symmetry in $(N,v^{\mathcal{F}})$, which implies that the value α does not satisfy the equal treatment of symmetric players. However, the property of marginality for augmenting structures in Theorem 2 can be replaced by the strong marginality for augmenting structures below.*

Strong marginality for augmenting structures. For any $(N,v,\mathcal{F}),(N,w,\mathcal{F}) \in AS^N$ and $i \in N$, if $v^{\mathcal{F}}(S\cup i) - v^{\mathcal{F}}(S) \geq w^{\mathcal{F}}(S\cup i) - w^{\mathcal{F}}(S)$ for all $S \subseteq N\setminus i$, then $\varphi_i(N,v,\mathcal{F}) \geq \varphi_i(N,w,\mathcal{F})$.

The reason is that strong marginality for augmenting structures clearly implies marginality for augmenting structures.

The independence of the four stated properties in Theorem 2 can be shown by the following examples.

For any $(N,v,\mathcal{F}) \in AS^N$, let $IP = \{i \in N : i \text{ is an isolated player in } (N,v,\mathcal{F})\}$.

(1) Let $f^1 : AS^N \to \mathbb{R}^n$ be defined by $f^1(N,v,\mathcal{F}) = \frac{1}{2}\alpha(N,v,\mathcal{F})$. Then the value f^1 satisfies all axioms in Theorem 2 except component efficiency.

(2) We define $f^2 : AS^N \to \mathbb{R}^n$ as

$$f_i^2(N,v,\mathcal{F}) = \begin{cases} 0 & \text{if } i \in IP, \\ \alpha_i(N,v,\mathcal{F}) + (|C_i|-1)\varepsilon, & \text{if } i = i^*, \\ \alpha_i(N,v,\mathcal{F}) - \varepsilon, & \text{if } i \in C_i \setminus i^*, \end{cases}$$

where $\varepsilon > 0$, $i^* = \max_j\{j \in C_i\}$ and $i \in C_i \in C_{\mathcal{F}}(N)$. It is easy to see that f^2 satisfies component efficiency, loop-null and marginality for augmenting structures, but not equal treatment of necessary players.

(3) Let $f^3 : AS^N \to \mathbb{R}^n$ be defined by

$$f_i^3(N,v,\mathcal{F}) = \begin{cases} 0 & \text{if } i \in IP, \\ \frac{v(C_i)}{|C_i|}, & \text{if } i \in N\setminus IP, \end{cases}$$

where IP is defined as above and $i \in C_i \in C_{\mathcal{F}}(N)$. It is easily verified that f^3 satisfies component efficiency, equal treatment of necessary players and loop-null, but not marginality for augmenting structures. Indeed, let $(N,v),(N,w) \in \mathcal{G}^N$ where $v(S) = |S|$, $w(S) = |S|-1$ for any $S \subseteq N$ with $|N| \geq 3$. For $\emptyset \neq N_0 \subset N$ with $|N_0| \geq 2$, $i \in N_0$, let

$$\mathcal{F} = \{F : F \subseteq N\setminus N_0)\} \cup \{F \cup \{i\} : \emptyset \neq F \subseteq N \setminus N_0\}.$$

Note that $IP = N_0 \setminus i$ and $\mathcal{C}_{\mathcal{F}}(N) = \{(N\setminus N_0)\cup\{i\}\}$. It can readily be checked that $v^{\mathcal{F}}(S\cup\{i\}) - v^{\mathcal{F}}(S) = w^{\mathcal{F}}(S\cup\{i\}) - w^{\mathcal{F}}(S)$ for every $S \subseteq N\setminus i$, but $f_i^3(N,v,\mathcal{F}) = 1$, $f_i^3(N,w,\mathcal{F}) = \frac{|N\setminus N_0|}{|N\setminus N_0|+1}$.

(4) Let $f^4 : AS^N \to \mathbb{R}^n$ be defined by

$$f_i^4(N,v,\mathcal{F}) = \begin{cases} v(i) & \text{if } i \in IP, \\ \alpha_i(N,v,\mathcal{F}), & \text{if } i \in N\setminus IP. \end{cases}$$

Then the value satisfies all axioms in Theorem 2 except loop-null.

4. Concluding Remarks

In this paper, we introduce the axiom of marginality for augmenting structures. We obtain an alternative axiomatic characterization of the value α for augmenting structures by replacing balanced contributions with marginality for augmenting structures and equal treatment of necessary players. However, this does not mean that the balanced contributions axiom is equivalent to the two axioms of marginality for augmenting structures and equal treatment of necessary players. For example, we consider the values $f_i^5(N,v) = Sh_i(N,v) + a_i$ and $f_i^6(N,v) = |N|Sh_i(N,v)$ for all $(N,v) \in \mathcal{G}^N$ and all $i \in N$, where $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Obviously, $f_i^5(N,v)$ satisfies balanced contributions but not symmetry (also called equal treatment of equals), while $f_i^6(N,v)$ satisfies symmetry and marginality but not balanced contributions. This implies that a value for augmenting structures satisfying balanced contributions does not necessarily satisfy marginality for augmenting structures and equal treatment of necessary players, and vice versa.

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