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The Fuzzy Complex Linear Systems Based on a New Representation of Fuzzy Complex Numbers

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Abstract: Since the product of complex numbers and rectangular fuzzy complex numbers (RFCN) is not necessarily a RFCN in the former fuzzy complex linear system (FCLS), the scalar multiplication and addition operations of complex numbers and fuzzy complex numbers (FCN) based on a new representation of FCN are proposed. We also introduce a new method for solving FCLS, which can convert FCLS into two distinct linear systems. One is an $n \times n$ complex linear system, and the other is an $(mn) \times (mn)$ real linear system, where n is the number of unknown variables, and m is the number of substitutional cyclic sets composed of coefficients of FCLS. In particular, using this method to solve one-dimensional fuzzy linear systems, a $(2n) \times (2n)$ RLS is obtained, which is consistent with Friedman's method. Finally, FCLS based on the RFCN as a special case are also investigated.

Keywords: fuzzy numbers; fuzzy complex number; fuzzy complex linear systems

MSC: 65F99

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1. Introduction

In 1989, the concept of FCN was first proposed by Buckley [1]. The basic arithmetic operations, algebraic and exponential forms of FCN, and a distance on the space of FCN were defined and studied. Subsequently, the derivative of fuzzy complex-valued functions (FCVF) was further generalized and developed [2] based on the concept of derivatives of real fuzzy mapping proposed by Dubois and Prade [3,4]. At the same time, the contour integral of FCVF on the complex plane was given [5]. In 1999, Wu and Qiu [6] improved the results in [1], which were different from those defined in [2,5], and introduced the derivatives and integrals of functions that map complex numbers to generalized FCN. In 2000, Qiu et al. studied the sequences and series of FCN and their convergence [7]. In 2001, Qiu et al. considered the continuity and differentiability of functions that map complex numbers to FCN or FCN to complex numbers [8].

In the process of mathematical modeling involving some practical problems, such as optimization problems, often due to measurement error and incomplete information, there will be some uncertain parameters and variables that can be expressed as a fuzzy number. Therefore, the linear system modeled by fuzzy numbers has many potential applications in actual production and life. In 1998, Friedman et al. proposed a method to solve the fuzzy linear system (FLS) $n \times n$ [9] by using the embedding method given in [10] and replacing the original FLS $n \times n$ with a crisp function linear system $(2n) \times (2n)$. Based on this method, some researchers have proposed various methods for solving FLS [11–33].

As far as we know, the research results of FCLS are relatively few. In 2009, Rahgooy et al. considered the solution of FCLS and applied it to the circuit analysis problem [34]. In 2010, Jahantigh et al. proposed a numerical procedure for FCLS [35]. In 2012, Behera and Chakraverty introduced a new and simple centre- and width-based method for solving FCLS [36]. Hladik and Djanybekov also investigated a solution of the complex interval linear systems (FILS) [37,38]. It is worth noting that a new and simple method for solving the general FCLS was proposed by Behera and Chakraverty [39,40].

Although Behera and Chakraverty have defined the scalar multiplication of complex number and FCN [39], just as Buckley pointed out in [1] that the product of two RFCN is not necessarily a RFCN, in fact, the scalar multiplication of complex numbers and RFCN is not necessarily a RFCN either. For example, let $\tilde{z} = \tilde{a} + \mathrm{i}\tilde{b}$ be a RFCN defined by the membership function

$$\tilde{z}(z) = \begin{cases} 1, & (x,y) \in [-1,1] \times [-1,1], \\ 0, & \text{otherwise}, \end{cases}$$

where $z = x + \mathrm{i}y$ and $\tilde{a}, \tilde{b}$ are fuzzy numbers defined by the membership functions

$$\tilde{a}(x) = \begin{cases} 1, & x \in [-1,1], \\ 0, & \text{otherwise}, \end{cases} \quad \tilde{b}(y) = \begin{cases} 1, & y \in [-1,1], \\ 0, & \text{otherwise}, \end{cases}$$

respectively. It is easy to verify that $w\tilde{z}$ is not an RFCN by the extension principle, where $w = \frac{\sqrt{2}}{2}(1+\mathrm{i})$. In fact,

$$w\tilde{z}(z) = \begin{cases} 1, & (x,y) \in \left([-1,0] \times [-(x+\sqrt{2}), x+\sqrt{2}]\right) \\ & \qquad \cup \left([0,1] \times [x-\sqrt{2}, \sqrt{2}-x]\right), \\ 0, & \text{otherwise}, \end{cases}$$

which does not write the form of $\tilde{a} + \mathrm{i}\tilde{b}$. Therefore, the left and right sides of FCLS may not be equal in [39]. For the above reason, based on the new representation and operation of FCN, we propose a new method to solve FCLS, which can convert the FCLS into two distinct linear systems. One is a complex linear system $n \times n$, and the other is an RLS $(mn) \times (mn)$. Finally, FCLS based on RFCN as a special case is also investigated.

The structure of this paper is as follows. Section 2 provides background notions related to FCN. In Section 3, a new representation and the arithmetic operations of FCN are provided. In Section 4, a new method for solving FCLS is proposed, and Section 5 studies a system of functional equations with period 2π. A especial case of FCLS is considered in Section 6. Section 7 presents our conclusions.

2. Background

A fuzzy set $\tilde{u}$ is a mapping from $\mathbb{R}$ to $[0,1]$ defined by the function $\tilde{u}(x)$. Let $[\tilde{u}]^r = \{x | \tilde{u}(x) \geqslant r\}$ for $r \in (0,1]$ and $[\tilde{u}]^0 = \overline{\{x | \tilde{u}(x) > 0\}}$. A fuzzy set $\tilde{u}$ is a fuzzy number if $\tilde{u}$ is a normal, convex fuzzy set that is upper semi-continuous, and supp $\tilde{u} = \overline{\{x | \tilde{u}(x) > 0\}}$ is compact.

We denote a complex number by $z = x + \mathrm{i}y$, and denote the complex plane by $\mathbb{C}$. A fuzzy complex set $\tilde{z}$ is a mapping from $\mathbb{C}$ to $[0,1]$ defined by the function $\tilde{z}(z)$. Let $[\tilde{z}]^r = \{z | \tilde{z}(z) \geqslant r\}$ for $r \in (0,1]$, $[\tilde{z}]^0 = \bigcup_{0<r\leqslant 1} [\tilde{z}]^r$ [1,7].

Definition 1 ([7])**.** *A fuzzy complex set $\tilde{z}$ is called an FCN if the following conditions are satisfied:*

(1) *$\tilde{z}$ is a upper semi-continuous function;*

(2) *$[\tilde{z}]^r$ is a compact set for $0 \leqslant r \leqslant 1$;*

(3) *$\tilde{z}$ is normal, i.e., there exists a z_0 such that $\tilde{z}(z_0) = 1$;*

(4) *$\tilde{z}$ is a fuzzy convex set, i.e., $\tilde{z}(\lambda z_1 + (1-\lambda) z_2) \geqslant \min\{\tilde{z}(z_1), \tilde{z}(z_2)\}$ for all z_1, $z_2 \in \mathbb{C}$, $\lambda \in [0,1]$.*

We use $\tilde{z}, \tilde{w}$ for FCN and $\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}$ for fuzzy numbers. Let $\tilde{\mathbb{C}}$ be the set of FCN. If $f(z_1, z_2) = w$ is a mapping from $\mathbb{C} \times \mathbb{C}$ into $\mathbb{C}$, we may define a mapping $f(\tilde{z}_1, \tilde{z}_2) = \tilde{w}$ from $\tilde{\mathbb{C}} \times \tilde{\mathbb{C}}$ into $\tilde{\mathbb{C}}$ by the extension principle as [1]

$$\tilde{w}(w) = \sup\{\min(\tilde{z}_1(z_1), \tilde{z}_2(z_2)) | f(z_1, z_2) = w\}.$$

Let $f(z_1, z_2) = z_1 + z_2$ or $f(z_1, z_2) = z_1 z_2$; then we may define the sum or multiplication of two FCN by $\tilde{w} = \tilde{z}_1 + \tilde{z}_2$ or $\tilde{w} = \tilde{z}_1 \tilde{z}_2$. We also define the subtraction of two FCN by

$$\tilde{z}_1 - \tilde{z}_2 = \tilde{z}_1 + (-\tilde{z}_2),$$

where

$$-\tilde{z}_2(z) = \tilde{z}_2(-z).$$

The sum and multiplication of FCN satisfy following properties:

$$[\tilde{z}_1 + \tilde{z}_2]^\alpha = [\tilde{z}_1]^\alpha + [\tilde{z}_2]^\alpha, \ \ [\tilde{z}_1 \tilde{z}_2]^\alpha = [\tilde{z}_1]^\alpha [\tilde{z}_2]^\alpha.$$

A FCN is called a trivial zero fuzzy number (TZFN) if $[\tilde{z}]^0 = \{0\}$. A FCN is called a nontrivial zero fuzzy number (NTZFN) if $0 \in [\tilde{z}]^1$ and $[\tilde{z}]^0 \neq \{0\}$.

3. A New Representation of FCN

In this section, a new representation of FCN is given, and the scalar multiplication and addition operations of complex numbers and FCN based on the representation of FCN are proposed.

Theorem 1 ([41]). *Let $\tilde{o}$ be a NTZFN of complex plane. Then*

$$\tilde{o} = \bigcup_{\varphi \in [0, 2\pi]} \left(\tilde{R}(\varphi) e^{\mathrm{i}\varphi} \right),$$

where $\tilde{R}(\varphi) = \bigcup_{r \in [0,1]} \left(r^* \cap [0, \rho(r, \varphi)] \right)$ is a fuzzy number for all $\varphi \in [0, 2\pi]$.

We give a definition of FCN based on the above representation theorem of NTZFN.

Definition 2 ([41]). *If $\tilde{z}$ is a FCN, then*

$$\tilde{z} = \hat{z} + \bigcup_{\varphi \in [0, 2\pi]} \left(\tilde{R}(\varphi) e^{\mathrm{i}\varphi} \right),$$

where $[\hat{z}]^r = \{z = x_0 + \mathrm{i}y_0\}$, for all $r \in [0, 1]$,

$$x_0 = \frac{\int \int_{[\tilde{z}]^1} x \mathrm{d}x \mathrm{d}y}{\int \int_{[\tilde{z}]^1} \mathrm{d}x \mathrm{d}y}, \ \ y_0 = \frac{\int \int_{[\tilde{z}]^1} y \mathrm{d}x \mathrm{d}y}{\int \int_{[\tilde{z}]^1} \mathrm{d}x \mathrm{d}y},$$

$\bigcup_{\varphi \in [0, 2\pi]} \left(\tilde{R}(\varphi) e^{\mathrm{i}\varphi} \right)$ is the new representation of NTZFN $\tilde{z} - \hat{z}$.

For simplicity, FCN $\tilde{z} = \hat{z} + \bigcup_{\varphi \in [0, 2\pi]} \left(\tilde{R}(\varphi) e^{\mathrm{i}\varphi} \right)$ is written as $\tilde{z} = z + \bigcup_{\varphi \in [0, 2\pi]} \left(\tilde{R}(\varphi) e^{\mathrm{i}\varphi} \right)$. FCN $\tilde{z} = z + \bigcup_{\varphi \in [0, 2\pi]} \left(\tilde{R}(\varphi) e^{\mathrm{i}\varphi} \right)$ can also be written as $\tilde{z} = z + \bigcup_{\varphi \in [0, 2\pi]} \left(\tilde{R}(\varphi + \frac{k\pi}{2}) e^{\mathrm{i}(\varphi + \frac{k\pi}{2})} \right)$, $k \in \mathbb{Z}^+$.

Theorem 2 ([41]). *Let $\tilde{z} = re^{\mathrm{i}\theta} + \bigcup_{\varphi \in [0, 2\pi]} \left(\tilde{R}(\varphi) e^{\mathrm{i}\varphi} \right), z = r' e^{\mathrm{i}\theta'} \in \mathbb{C}, k \in \mathbb{R}$. Then*

$$(1)\ z\tilde{z} = \left(r' e^{\mathrm{i}\theta'} \right) \left(re^{\mathrm{i}\theta} + \bigcup_{\varphi \in [0, 2\pi]} \left(\tilde{R}(\varphi) e^{\mathrm{i}\varphi} \right) \right) = r' r e^{\mathrm{i}(\theta' + \theta)} + \bigcup_{\varphi \in [0, 2\pi]} \left(r' \tilde{R}(\varphi) e^{\mathrm{i}(\theta' + \varphi)} \right).$$

$$(2)\ k\tilde{z} = \begin{cases} (kr)e^{\mathrm{i}\theta} + \bigcup_{\varphi \in [0, 2\pi]} \left(\left(k\tilde{R}(\varphi) \right) e^{\mathrm{i}\varphi} \right), k \geqslant 0, \\ \quad (-kr)e^{\mathrm{i}(\theta + \pi)} \\ + \bigcup_{\varphi \in [0, 2\pi]} \left(\left((-k)\tilde{R}(\varphi) \right) e^{\mathrm{i}(\varphi + \pi)} \right), k < 0. \end{cases}$$

Definition 3 ([41]). *Let*

$$\tilde{z}_1 = a_1 + \mathrm{i}b_1 + \bigcup_{\varphi \in [0, 2\pi]} \left(\tilde{R}_1(\varphi) e^{\mathrm{i}\varphi} \right), \quad \tilde{z}_2 = a_2 + \mathrm{i}b_2 + \bigcup_{\varphi \in [0, 2\pi]} \left(\tilde{R}_2(\varphi) e^{\mathrm{i}\varphi} \right).$$

If
$$\tilde{w} = (a_1 + a_2) + \mathrm{i}(b_1 + b_2) + \bigcup_{\varphi \in [0,2\pi]} \left((\tilde{R}_1(\varphi) + \tilde{R}_2(\varphi)) e^{\mathrm{i}\varphi} \right)$$
is a FCN, then $\tilde{w}$ is called the strong sum of two FCN $\tilde{z}_1$ and $\tilde{z}_2$, which is denoted by $\tilde{z}_1 \oplus \tilde{z}_2$.

The strong sum and the sum of two FCN are equivalent when the condition of Corollary 5.4 is satisfied in [41].

For simplicity, we also write FCN $\tilde{z} = \hat{z} + \bigcup_{\varphi \in [0,2\pi]} \left(\tilde{R}(\varphi) e^{\mathrm{i}\varphi} \right)$ as $\tilde{z} = z + \tilde{R}(\varphi) e^{\mathrm{i}\varphi}$, $\varphi \in [0, 2\pi]$, the scalar multiplication and the strong sum of FCN as

$$z'\tilde{z} = z'z + r'\tilde{R}(\varphi) e^{\mathrm{i}(\theta' + \varphi)}, \quad \varphi \in [0, 2\pi],$$

$$k\tilde{z} = \begin{cases} kz + k\tilde{R}(\varphi) e^{\mathrm{i}\varphi}, \ \varphi \in [0, 2\pi], & k \geqslant 0, \\ kz + (-k)\tilde{R}(\varphi) e^{\mathrm{i}(\varphi + \pi)}, \ \varphi \in [0, 2\pi], & k < 0, \end{cases}$$

$$\tilde{z}_1 + \tilde{z}_2 = z_1 + z_2 + (\tilde{R}_1(\varphi) + \tilde{R}_2(\varphi)) e^{\mathrm{i}\varphi}, \quad \varphi \in [0, 2\pi]$$

respectively, where $z' = r' e^{\mathrm{i}\theta'}$ is a complex number.

It is worth noting that since FCN can be regarded as a generalization of one-dimensional fuzzy numbers, we can similarly represent one-dimensional fuzzy numbers and define the operations of one-dimensional fuzzy numbers; see reference [42].

4. FCLS

The $n \times n$ FCLS is written as

$$\begin{cases} c_{11}\tilde{z}_1 + c_{12}\tilde{z}_2 + \cdots + c_{1n}\tilde{z}_n = \tilde{w}_1, \\ c_{21}\tilde{z}_1 + c_{22}\tilde{z}_2 + \cdots + c_{2n}\tilde{z}_n = \tilde{w}_2 \\ \quad \vdots \\ c_{n1}\tilde{z}_1 + c_{n2}\tilde{z}_2 + \cdots + c_{nn}\tilde{z}_n = \tilde{w}_n. \end{cases} \tag{1}$$

We may write the above as
$$C\tilde{Z} = \tilde{W}, \tag{2}$$
where $C = (c_{jk})_{n \times n}$ is a crisp $n \times n$ complex matrix, $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_n)^T$ is a column vector of known FCN, and $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_n)^T$ is the vector of unknown FCN.

System (1) may be also written as

$$\sum_{j=1}^{n} c_{kj}\tilde{z}_j = \tilde{w}_k, \tag{3}$$

where $k = 1, 2, \cdots, n$. Let $c_{kj} = r_{kj}^c e^{\mathrm{i}\theta_{kj}^c}$, $\tilde{w}_k = w_k + \tilde{R}_k^w e^{\mathrm{i}\varphi}$, $\tilde{z}_k = z_k + \tilde{R}_k^z e^{\mathrm{i}\varphi}$, $\varphi \in [0, 2\pi]$, where w_k, z_k are some complex numbers, $\tilde{R}_k^w = \tilde{R}_k^w(\varphi)$, $\tilde{R}_k^z = \tilde{R}_k^z(\varphi)$ are some fuzzy numbers, and i is an imaginary unit. By (3), we obtain the system

$$\sum_{j=1}^{n} (c_{kj}z_j + c_{kj}\tilde{R}_j^z e^{\mathrm{i}\varphi}) = w_k + \tilde{R}_k^w e^{\mathrm{i}\varphi},$$

i.e.,
$$\sum_{j=1}^{n} (c_{kj}z_j + r_{kj}^c \tilde{R}_j^z e^{\mathrm{i}(\varphi + \theta_{kj}^c)}) = w_k + \tilde{R}_k^w e^{\mathrm{i}\varphi},$$

where $k = 1, 2, \cdots, n$. The above system is written as

$$\sum_{j=1}^{n} c_{kj}z_j = w_k, \tag{4}$$

and

$$\sum_{j=1}^{n} r_{kj}^{c} \tilde{R}_{j}^{z} e^{\mathrm{i}(\varphi+\theta_{kj}^{c})} = \tilde{R}_{k}^{w} e^{\mathrm{i}\varphi}, \tag{5}$$

where $k = 1, 2, \cdots, n$, and (4) is a complex linear system, which may be denoted by

$$CZ = W, \tag{6}$$

where $Z = (z_1, z_2, \cdots, z_n)^T, W = (w_1, w_2, \cdots, w_n)^T$. System (5) is written as

$$\sum_{j=1}^{n} r_{kj}^{c} \tilde{R}_{j}^{z} (\varphi - \theta_{kj}^{c}) e^{\mathrm{i}\varphi} = \tilde{R}_{k}^{w} e^{\mathrm{i}\varphi},$$

i.e.,

$$\sum_{j=1}^{n} r_{kj}^{c} \tilde{R}_{j}^{z} (\varphi - \theta_{kj}^{c}) = \tilde{R}_{k}^{w} (\varphi), \tag{7}$$

where $k = 1, 2, \cdots, n$, and r_{kj}^{c} are some positive real numbers for every φ, $\tilde{R}_{k}^{w}(\varphi)$ are some known fuzzy numbers known and $\tilde{R}_{k}^{z}(\varphi)$ are some unknown fuzzy numbers.

Let

$$[\tilde{R}_{k}^{w}]^r = [\tilde{R}_{k}^{w}(\varphi)]^r = [0, b_k(r, \varphi)],$$
$$[\tilde{R}_{k}^{z}]^r = [\tilde{R}_{k}^{z}(\varphi)]^r = [0, \rho_k(r, \varphi)],$$
$$b_k(\varphi) = b_k(r, \varphi), \;\; \rho_k(\varphi) = \rho_k(r, \varphi).$$

Then, System (7) can be equivalently written as the following system

$$\sum_{j=1}^{n} r_{kj}^{c} \rho_j (\varphi - \theta_{kj}^{c}) = b_k(\varphi), \tag{8}$$

where $k = 1, 2, \cdots, n$, and $\rho_i(\varphi)$ are some unknown functions with period 2π, and $b_i(\varphi)$ are some known functions with period 2π.

We obtain the following theorem by the above discussion.

Theorem 3. *A vector $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_n)^T$ is a solution (a unique solution) of Equation (2) if and only if the vectors $Z = (z_1, z_2, \cdots, z_n)^T$, $U_\rho = (\rho_1(\varphi), \rho_2(\varphi), \cdots, \cdots, \rho_n(\varphi))^T$ are solutions (two unique solutions) of Equation (6) and System (8), respectively, and*

$$[\tilde{z}_k]^r = z_k + \bigcup_{\varphi \in [0, 2\pi]} \left([0, \rho_k(\varphi)] e^{\mathrm{i}\varphi} \right).$$

Example 1. *Consider the 3×3 FCLS*

$$\begin{cases} (1+\mathrm{i})\tilde{z}_1 + \tilde{z}_2 + \mathrm{i}\tilde{z}_3 = \tilde{w}_1, \\ 3\tilde{z}_1 + (1-\mathrm{i})\tilde{z}_2 + \tilde{z}_3 = \tilde{w}_2, \\ \tilde{z}_1 - \mathrm{i}\tilde{z}_2 - (1+\mathrm{i})\tilde{z}_3 = \tilde{w}_3, \end{cases} \tag{9}$$

where

$$\tilde{w}_1 = 3 + 2\mathrm{i} + \tilde{R}_{1}^{w}(\varphi) e^{\mathrm{i}\varphi}, \;\; [\tilde{R}_{1}^{w}(\varphi)]^r = [0, (2 + 2\sqrt{2} + |\sin\varphi|)(1-r)],$$

$$\tilde{w}_2 = 5 - \mathrm{i} + \tilde{R}_{2}^{w}(\varphi) e^{\mathrm{i}\varphi}, \;\; [\tilde{R}_{2}^{w}(\varphi)]^r = [0, (7 + \sqrt{2} + \sqrt{2}|\sin(\varphi + \frac{\pi}{4})|)(1-r)],$$

$$\tilde{w}_3 = 1 + \tilde{R}_{3}^{w}(\varphi) e^{\mathrm{i}\varphi}, \;\; [\tilde{R}_{3}^{w}(\varphi)]^r = [0, (3 + 2\sqrt{2} + |\cos\varphi|)(1-r)].$$

Let

$$\tilde{z}_k = z_k + \tilde{R}_k^z e^{\mathrm{i}\varphi},\ \ [\tilde{R}_k^z]^r = [\tilde{R}_k^z(\varphi)]^r = [0, \rho_k(r,\varphi)],\ \ \rho_k(\varphi) = \rho_k(r,\varphi), k = 1,2,3.$$

Equation (9) is written as

$$\begin{cases} (1+\mathrm{i})(z_1 + \tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi}) + (z_2 + \tilde{R}_2^z(\varphi)e^{\mathrm{i}\varphi}) + \mathrm{i}(z_3 + \tilde{R}_3^z(\varphi)e^{\mathrm{i}\varphi}) = 3 + 2\mathrm{i} + \tilde{R}_1^w(\varphi)e^{\mathrm{i}\varphi}, \\ 3(z_1 + \tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi}) + (1-\mathrm{i})(z_2 + \tilde{R}_2^z(\varphi)e^{\mathrm{i}\varphi}) + (z_3 + \tilde{R}_3^z(\varphi)e^{\mathrm{i}\varphi}) = 5 - \mathrm{i} + \tilde{R}_2^w(\varphi)e^{\mathrm{i}\varphi}, \\ (z_1 + \tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi}) - \mathrm{i}(z_2 + \tilde{R}_2^z(\varphi)e^{\mathrm{i}\varphi}) - (1+\mathrm{i})(z_3 + \tilde{R}_3^z(\varphi)e^{\mathrm{i}\varphi}) = 1 + \tilde{R}_3^w(\varphi)e^{\mathrm{i}\varphi}. \end{cases}$$

Equation above is equivalent to

$$\begin{cases} (1+\mathrm{i})z_1 + z_2 + \mathrm{i}z_3 = 3 + 2\mathrm{i}, \\ 3z_1 + (1-\mathrm{i})z_2 + z_3 = 5 - \mathrm{i} \\ z_1 - \mathrm{i}z_2 - (1+\mathrm{i})z_3 = 1, \end{cases} \tag{10}$$

and

$$\begin{cases} (1+\mathrm{i})(\tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi}) + \tilde{R}_2^z(\varphi)e^{\mathrm{i}\varphi} + \mathrm{i}(\tilde{R}_3^z(\varphi)e^{\mathrm{i}\varphi}) = \tilde{R}_1^w(\varphi)e^{\mathrm{i}\varphi}, \\ 3(\tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi}) + (1-\mathrm{i})(\tilde{R}_2^z(\varphi)e^{\mathrm{i}\varphi}) + (\tilde{R}_3^z(\varphi)e^{\mathrm{i}\varphi}) = \tilde{R}_2^w(\varphi)e^{\mathrm{i}\varphi}, \\ \tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi} - \mathrm{i}(\tilde{R}_2^z(\varphi)e^{\mathrm{i}\varphi}) - (1+\mathrm{i})(\tilde{R}_3^z(\varphi)e^{\mathrm{i}\varphi}) = \tilde{R}_3^w(\varphi)e^{\mathrm{i}\varphi}. \end{cases} \tag{11}$$

The solution of Equation (10) is

$$z_1 = 1,\ \ z_2 = 1 + \mathrm{i},\ \ z_3 = -\mathrm{i}. \tag{12}$$

Equation (11) is written as

$$\begin{cases} (\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}})(\tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi}) + \tilde{R}_2^z(\varphi)e^{\mathrm{i}\varphi} + e^{\mathrm{i}\frac{\pi}{2}}(\tilde{R}_3^z(\varphi)e^{\mathrm{i}\varphi}) = \tilde{R}_1^w(\varphi)e^{\mathrm{i}\varphi}, \\ 3(\tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi}) + (\sqrt{2}e^{\mathrm{i}(-\frac{\pi}{4})})(\tilde{R}_2^z(\varphi)e^{\mathrm{i}\varphi}) + \tilde{R}_3^z(\varphi)e^{\mathrm{i}\varphi} = \tilde{R}_2^w(\varphi)e^{\mathrm{i}\varphi}, \\ \tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi} + e^{\mathrm{i}(-\frac{\pi}{2})}(\tilde{R}_2^z(\varphi)e^{\mathrm{i}\varphi}) + (\sqrt{2}e^{\mathrm{i}\frac{5\pi}{4}})(\tilde{R}_3^z(\varphi)e^{\mathrm{i}\varphi}) = \tilde{R}_3^w(\varphi)e^{\mathrm{i}\varphi}. \end{cases}$$

i.e.,

$$\begin{cases} \sqrt{2}\tilde{R}_1^z(\varphi)e^{\mathrm{i}(\varphi+\frac{\pi}{4})} + \tilde{R}_2^z(\varphi)e^{\mathrm{i}\varphi} + \tilde{R}_3^z(\varphi)e^{\mathrm{i}(\varphi+\frac{\pi}{2})} = \tilde{R}_1^w(\varphi)e^{\mathrm{i}\varphi}, \\ 3\tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi} + \sqrt{2}\tilde{R}_2^z(\varphi)e^{\mathrm{i}(\varphi-\frac{\pi}{4})} + \tilde{R}_3^z(\varphi)e^{\mathrm{i}\varphi} = \tilde{R}_2^w(\varphi)e^{\mathrm{i}\varphi}, \\ \tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi} + \tilde{R}_2^z(\varphi)e^{\mathrm{i}(\varphi-\frac{\pi}{2})} + \sqrt{2}\tilde{R}_3^z(\varphi)e^{\mathrm{i}(\varphi+\frac{5\pi}{4})} = \tilde{R}_3^w(\varphi)e^{\mathrm{i}\varphi}. \end{cases}$$

That is,

$$\begin{cases} \sqrt{2}\tilde{R}_1^z(\varphi-\frac{\pi}{4})e^{\mathrm{i}\varphi} + \tilde{R}_2^z(\varphi)e^{\mathrm{i}\varphi} + \tilde{R}_3^z(\varphi-\frac{\pi}{2})e^{\mathrm{i}\varphi} = \tilde{R}_1^w(\varphi)e^{\mathrm{i}\varphi}, \\ 3\tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi} + \sqrt{2}\tilde{R}_2^z(\varphi+\frac{\pi}{4})e^{\mathrm{i}\varphi} + \tilde{R}_3^z(\varphi)e^{\mathrm{i}\varphi} = \tilde{R}_2^w(\varphi)e^{\mathrm{i}\varphi}, \\ \tilde{R}_1^z(\varphi)e^{\mathrm{i}\varphi} + \tilde{R}_2^z(\varphi+\frac{\pi}{2})e^{\mathrm{i}\varphi} + \sqrt{2}\tilde{R}_3^z(\varphi-\frac{5\pi}{4})e^{\mathrm{i}\varphi} = \tilde{R}_3^w(\varphi)e^{\mathrm{i}\varphi}. \end{cases}$$

It follows that

$$\begin{cases} \sqrt{2}\tilde{R}_1^z(\varphi-\frac{\pi}{4}) + \tilde{R}_2^z(\varphi) + \tilde{R}_3^z(\varphi-\frac{\pi}{2}) = \tilde{R}_1^w(\varphi), \\ 3\tilde{R}_1^z(\varphi) + \sqrt{2}\tilde{R}_2^z(\varphi+\frac{\pi}{4}) + \tilde{R}_3^z(\varphi) = \tilde{R}_2^w(\varphi), \\ \tilde{R}_1^z(\varphi) + \tilde{R}_2^z(\varphi+\frac{\pi}{2}) + \sqrt{2}\tilde{R}_3^z(\varphi-\frac{5\pi}{4}) = \tilde{R}_3^w(\varphi). \end{cases}$$

i.e.,

$$\begin{cases} \sqrt{2}\rho_1(\varphi-\frac{\pi}{4}) + \rho_2(\varphi) + \rho_3(\varphi-\frac{\pi}{2}) = b_1(\varphi), \\ 3\rho_1(\varphi) + \sqrt{2}\rho_2(\varphi+\frac{\pi}{4}) + \rho_3(\varphi) = b_2(\varphi), \\ \rho_1(\varphi) + \rho_2(\varphi+\frac{\pi}{2}) + \sqrt{2}\rho_3(\varphi-\frac{5\pi}{4}) = b_3(\varphi), \end{cases} \tag{13}$$

where $\rho_i(\varphi)$ are three functions unknown with period 2π, $b_i(\varphi)$ are three functions known with period 2π, and

$$b_1(\varphi) = (2 + 2\sqrt{2} + |\sin\varphi|)(1-r),$$

$$b_2(\varphi) = (7 + \sqrt{2} + \sqrt{2}|\sin(\varphi + \frac{\pi}{4})|)(1 - r),$$
$$b_3(\varphi) = (3 + 2\sqrt{2} + |\cos\varphi|)(1 - r).$$

It is to easy verify that

$$\rho_1(\varphi) = 2(1 - r),\ \ \rho_2(\varphi) = (1 + |\sin\varphi|)(1 - r),\ \ \rho_3(\varphi) = 1 - r$$

are solutions of Equation (13). From Theorem 3 and Equation (12), we obtain

$$[\tilde{z}_1]^r = 1 + \bigcup_{\varphi\in[0,2\pi]} \big([0, 2(1 - r)]e^{\mathrm{i}\varphi}\big),$$

$$[\tilde{z}_2]^r = (1 + i) + \bigcup_{\varphi\in[0,2\pi]} \big([0, (1 + |\sin\varphi|)(1 - r)]e^{\mathrm{i}\varphi}\big),$$

$$[\tilde{z}_3]^r = (-i) + \bigcup_{\varphi\in[0,2\pi]} \big([0, (1 - r)]e^{\mathrm{i}\varphi}\big).$$

5. System of Functional Equations with Period 2π

Since System (8) is a system of functional equations with period 2π that cannot be solved directly, we need to convert System (8) into a $(mn) \times (mn)$ RLS. First, we give the following three definitions.

Definition 4. *A set $C = \{\varphi - \theta_1, \varphi - \theta_2, \cdots, \varphi - \theta_m\}$ is said to be m-substitutional cyclic if for any $\varphi - \theta_i, \varphi - \theta_j \in C$, we have $\varphi - \theta_i - \theta_j \in C$.*

Definition 5. *A m-substitutional cyclic set C is said to be generated by the set $\{\varphi - \theta_1, \varphi - \theta_2, \cdots, \varphi - \theta_k\}$ if $\{\varphi - \theta_1, \varphi - \theta_2, \cdots, \varphi - \theta_k\} \subseteq C$ and for any m-substitution cyclic set D including the set $\{\varphi - \theta_1, \varphi - \theta_2, \cdots, \varphi - \theta_k\}$ satisfies $C \subseteq D$, denoted by*

$$C =< \varphi - \theta_1, \varphi - \theta_2, \cdots, \varphi - \theta_k > .$$

Definition 6. *The system (8) is said to be m-substitutional cyclic if there exists a m-substitutional cyclic set $C = \{\varphi - \theta_1, \varphi - \theta_2, \cdots, \varphi - \theta_m\}$ such that*

$$C =< \varphi - \theta^c_{11}, \cdots, \varphi - \theta^c_{1n}, \varphi - \theta^c_{21}, \cdots, \varphi - \theta^c_{2n}, \cdots \varphi - \theta^c_{n1}, \cdots, \varphi - \theta^c_{nn} > .$$

Obviously, we can get the following theorem by Definition 6.

Theorem 4. *If System (8) is m-substitutional cyclic, then the vector*

$$\big(\rho_1(\varphi), \rho_2(\varphi), \cdots, \rho_n(\varphi)\big)^T$$

is a solution of System (8) if only if

$$\big(\rho_1(\varphi - \theta_1), \cdots, \rho_n(\varphi - \theta_1), \rho_1(\varphi - \theta_2), \cdots, \rho_n(\varphi - \theta_2), \cdots, \rho_1(\varphi - \theta_m), \cdots, \rho_n(\varphi - \theta_m)\big)^T$$

is a solution of the following $(mn) \times (mn)$ system

$$\begin{cases} \sum_{j=1}^n r^c_{kj}\rho_j(\varphi - \theta_1 - \theta^c_{kj}) = b_k(\varphi - \theta_1), \\ \sum_{j=1}^n r^c_{kj}\rho_j(\varphi - \theta_2 - \theta^c_{kj}) = b_k(\varphi - \theta_2), \\ \vdots \\ \sum_{j=1}^n r^c_{kj}\rho_j(\varphi - \theta_m - \theta^c_{kj}) = b_k(\varphi - \theta_m), \end{cases} \tag{14}$$

where $k = 1, 2, \cdots, n$, $\rho_i(\varphi)$ *are some unknown functions with period* 2π, *and* $b_i(\varphi)$ *are some functions known with period* 2π.

Example 2. *Since the m-substitutional cyclic set*

$$\begin{aligned} C &= \{\varphi - \frac{5\pi}{4}, \varphi - \pi, \varphi - \frac{3\pi}{4}, \varphi - \frac{\pi}{2}, \varphi - \frac{\pi}{4}, \varphi, \varphi + \frac{\pi}{4}, \varphi + \frac{\pi}{2}\} \\ &= < \varphi - \frac{5\pi}{4}, \varphi - \frac{\pi}{2}, \varphi - \frac{\pi}{4}, \varphi, \varphi + \frac{\pi}{4}, \varphi + \frac{\pi}{2} > \end{aligned}$$

is in System (13), *we know that System* (13) *is m-substitutional cyclic, which is equivalent to the following system:*

$$\begin{cases} \sqrt{2}\rho_1(\varphi + \frac{\pi}{2}) + \rho_2(\varphi - \frac{5\pi}{4}) + \rho_3(\varphi + \frac{\pi}{4}) = b_1(\varphi - \frac{5\pi}{4}), \\ 3\rho_1(\varphi - \frac{5\pi}{4}) + \sqrt{2}\rho_2(\varphi - \pi) + \rho_3(\varphi - \frac{5\pi}{4}) = b_2(\varphi - \frac{5\pi}{4}), \\ \rho_1(\varphi - \frac{5\pi}{4}) + \rho_2(\varphi - \frac{3\pi}{4}) + \sqrt{2}\rho_3(\varphi - \frac{\pi}{2}) = b_3(\varphi - \frac{5\pi}{4}), \\ \sqrt{2}\rho_1(\varphi - \frac{5\pi}{4}) + \rho_2(\varphi - \pi) + \rho_3(\varphi + \frac{\pi}{2}) = b_1(\varphi - \pi), \\ 3\rho_1(\varphi - \pi) + \sqrt{2}\rho_2(\varphi - \frac{3\pi}{4}) + \rho_3(\varphi - \pi) = b_2(\varphi - \pi), \\ \rho_1(\varphi - \pi) + \rho_2(\varphi - \frac{\pi}{2}) + \sqrt{2}\rho_3(\varphi - \frac{\pi}{4}) = b_3(\varphi - \pi), \\ \sqrt{2}\rho_1(\varphi - \pi) + \rho_2(\varphi - \frac{3\pi}{4}) + \rho_3(\varphi - \frac{5\pi}{4}) = b_1(\varphi - \frac{3\pi}{4}), \\ 3\rho_1(\varphi - \frac{3\pi}{4}) + \sqrt{2}\rho_2(\varphi - \frac{\pi}{2}) + \rho_3(\varphi - \frac{3\pi}{4}) = b_3(\varphi - \frac{3\pi}{4}), \\ \rho_1(\varphi - \frac{3\pi}{4}) + \rho_2(\varphi - \frac{\pi}{4}) + \sqrt{2}\rho_3(\varphi) = b_3(\varphi - \frac{3\pi}{4}), \\ \sqrt{2}\rho_1(\varphi - \frac{3\pi}{4}) + \rho_2(\varphi - \frac{\pi}{2}) + \rho_3(\varphi - \pi) = b_1(\varphi - \frac{\pi}{2}), \\ 3\rho_1(\varphi - \frac{\pi}{2}) + \sqrt{2}\rho_2(\varphi - \frac{\pi}{4}) + \rho_3(\varphi - \frac{\pi}{2}) = b_2(\varphi - \frac{\pi}{2}), \\ \rho_1(\varphi - \frac{\pi}{2}) + \rho_2(\varphi) + \sqrt{2}\rho_3(\varphi + \frac{\pi}{4}) = b_3(\varphi - \frac{\pi}{2}), \\ \sqrt{2}\rho_1(\varphi - \frac{\pi}{2}) + \rho_2(\varphi - \frac{\pi}{4}) + \rho_3(\varphi - \frac{3\pi}{4}) = b_1(\varphi - \frac{\pi}{4}), \\ 3\rho_1(\varphi - \frac{\pi}{4}) + \sqrt{2}\rho_2(\varphi) + \rho_3(\varphi - \frac{\pi}{4}) = b_2(\varphi - \frac{\pi}{4}), \\ \rho_1(\varphi - \frac{\pi}{4}) + \rho_2(\varphi + \frac{\pi}{4}) + \sqrt{2}\rho_3(\varphi + \frac{\pi}{2}) = b_3(\varphi - \frac{\pi}{4}), \\ \sqrt{2}\rho_1(\varphi - \frac{\pi}{4}) + \rho_2(\varphi) + \rho_3(\varphi - \frac{\pi}{2}) = b_1(\varphi), \\ 3\rho_1(\varphi) + \sqrt{2}\rho_2(\varphi + \frac{\pi}{4}) + \rho_3(\varphi) = b_2(\varphi), \\ \rho_1(\varphi) + \rho_2(\varphi + \frac{\pi}{2}) + \sqrt{2}\rho_3(\varphi - \frac{5\pi}{4}) = b_3(\varphi), \end{cases}$$

$$\begin{cases} \sqrt{2}\rho_1(\varphi) + \rho_2(\varphi + \frac{\pi}{4}) + \rho_3(\varphi - \frac{\pi}{4}) = b_1(\varphi + \frac{\pi}{4}), \\ 3\rho_1(\varphi + \frac{\pi}{4}) + \sqrt{2}\rho_2(\varphi + \frac{\pi}{2}) + \rho_3(\varphi + \frac{\pi}{4}) = b_2(\varphi + \frac{\pi}{4}), \\ \rho_1(\varphi + \frac{\pi}{4}) + \rho_2(\varphi - \frac{5\pi}{4}) + \sqrt{2}\rho_3(\varphi - \pi) = b_3(\varphi + \frac{\pi}{4}), \\ \sqrt{2}\rho_1(\varphi + \frac{\pi}{4}) + \rho_2(\varphi + \frac{\pi}{2}) + \rho_3(\varphi) = b_1(\varphi + \frac{\pi}{2}), \\ 3\rho_1(\varphi + \frac{\pi}{2}) + \sqrt{2}\rho_2(\varphi - \frac{5\pi}{4}) + \rho_3(\varphi + \frac{\pi}{2}) = b_2(\varphi + \frac{\pi}{2}), \\ \rho_1(\varphi + \frac{\pi}{2}) + \rho_2(\varphi + \pi) + \sqrt{2}\rho_3(\varphi - \frac{3\pi}{4}) = b_3(\varphi + \frac{\pi}{2}). \end{cases}$$

We may write the above as

$$KX = B,$$

where

$$\begin{aligned} X = \big(&\rho_1(\varphi - \tfrac{5\pi}{4}), \rho_2(\varphi - \tfrac{5\pi}{4}), \rho_3(\varphi - \tfrac{5\pi}{4}), \rho_1(\varphi - \pi),\ \rho_2(\varphi - \pi),\ \rho_3(\varphi - \pi), \\ &\rho_1(\varphi - \tfrac{3\pi}{4}), \rho_2(\varphi - \tfrac{3\pi}{4}), \rho_3(\varphi - \tfrac{3\pi}{4}), \rho_1(\varphi - \tfrac{\pi}{2}),\ \rho_2(\varphi - \tfrac{\pi}{2}),\ \rho_3(\varphi - \tfrac{\pi}{2}), \\ &\rho_1(\varphi - \tfrac{\pi}{4}),\ \rho_2(\varphi - \tfrac{\pi}{4}),\ \rho_3(\varphi - \tfrac{\pi}{4}),\ \rho_1(\varphi),\ \rho_2(\varphi),\ \rho_3(\varphi), \\ &\rho_1(\varphi + \tfrac{\pi}{4}),\ \rho_2(\varphi + \tfrac{\pi}{4}),\ \rho_3(\varphi + \tfrac{\pi}{4}),\ \rho_1(\varphi + \tfrac{\pi}{2}),\ \rho_2(\varphi + \tfrac{\pi}{2}),\ \rho_3(\varphi + \tfrac{\pi}{2})\big)^T, \\ B = \big(&b_1(\varphi - \tfrac{5\pi}{4}), b_2(\varphi - \tfrac{5\pi}{4}), b_3(\varphi - \tfrac{5\pi}{4}), b_1(\varphi - \pi),\ b_2(\varphi - \pi),\ b_3(\varphi - \pi), \\ &b_1(\varphi - \tfrac{3\pi}{4}), b_2(\varphi - \tfrac{3\pi}{4}), b_3(\varphi - \tfrac{3\pi}{4}), b_1(\varphi - \tfrac{\pi}{2}),\ b_2(\varphi - \tfrac{\pi}{2}),\ b_3(\varphi - \tfrac{\pi}{2}), \\ &b_1(\varphi - \tfrac{\pi}{4}),\ b_2(\varphi - \tfrac{\pi}{4}),\ b_3(\varphi - \tfrac{\pi}{4}),\ b_1(\varphi),\ b_2(\varphi),\ b_3(\varphi), \\ &b_1(\varphi + \tfrac{\pi}{4}),\ b_2(\varphi + \tfrac{\pi}{4}),\ b_3(\varphi + \tfrac{\pi}{4}),\ b_1(\varphi + \tfrac{\pi}{2}),\ b_2(\varphi + \tfrac{\pi}{2}),\ b_3(\varphi + \tfrac{\pi}{2})\big)^T, \end{aligned}$$

$$K = \begin{pmatrix} U \\ LU \\ L^2U \\ L^3U \\ L^4U \\ L^5U \\ L^6U \\ L^7U \end{pmatrix},$$

$$U = (U_1, U_2, \cdots, U_8),$$

$$U_1 = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\ U_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ U_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix},$$

$$U_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ U_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ U_8 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$LU = L(U_1, U_2, \cdots, U_8) = (U_8, U_1, \cdots, U_7).$$

Similarly, we also may denote System (14) in the form of a matrix as

$$KX = B, \tag{15}$$

where

$$\begin{aligned} X = \big(&\rho_1(\varphi-\theta_1),\cdots,\rho_n(\varphi-\theta_1),\rho_1(\varphi-\theta_2),\cdots,\rho_n(\varphi-\theta_2),\cdots,\rho_1(\varphi-\theta_m),\\ &\cdots,\rho_n(\varphi-\theta_m)\big)^T,\\ B = \big(&b_1(\varphi-\theta_1),\cdots,b_n(\varphi-\theta_1),b_1(\varphi-\theta_2),\cdots,b_n(\varphi-\theta_2),\cdots,b_1(\varphi-\theta_m),\\ &\cdots,b_n(\varphi-\theta_m)\big)^T, \end{aligned}$$

$$K = \begin{pmatrix} U \\ LU \\ L^2U \\ \vdots \\ L^{m-1}U \end{pmatrix},$$

$$U = (U_1, U_2, \cdots, U_m)$$

U_i $(i = 1, 2, \cdots, m)$ denotes an $n \times m$ coefficient matrix of the first n equations of System (14) determined by the variable $\varphi - \theta_i$ in order $\rho_1, \rho_2, \cdots, \rho_n$. L denotes putting the previous column of the block matrix into the latter column, and the last column of the block matrix into the first column, i.e.,

$$LU = L(U_1, U_2, \cdots, U_m) = (U_m, U_1, \cdots, U_{m-1}).$$

We obtain the following theorem by the above discussion.

Theorem 5. *A vector $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_n)^T$ is a solution (a unique solution) of Equation* (2) *if and only if the vectors $Z = (z_1, z_2, \cdots, z_n)^T$ and*

$$\begin{aligned} X = \big(&\rho_1(\varphi-\theta_1),\cdots,\rho_n(\varphi-\theta_1),\rho_1(\varphi-\theta_2),\cdots,\rho_n(\varphi-\theta_2),\cdots,\rho_1(\varphi-\theta_m),\\ &\cdots,\rho_n(\varphi-\theta_m)\big)^T, \end{aligned}$$

are solutions (two unique solutions) of Equation (6) *and* (15), *respectively, and*

$$[\tilde{z}_k]^r = z_k + \bigcup_{\varphi\in[0,2\pi]} \left([0, \rho_k(\varphi)]e^{\mathrm{i}\varphi}\right).$$

FCLS can be transformed into two distinct linear systems: one is an $n \times n$ complex linear system, and the other is an $(mn) \times (mn)$ RLS. Since FCN can be regarded as an extension of one-dimensional fuzzy numbers, we discuss the one-dimensional FLS by using the method of studying FCLS. Our method can also transform the one-dimensional FLS into two distinct linear systems: one is an $n \times n$ RLS, and the other is a $(2n) \times (2n)$ RLS. The method of solving the one-dimensional FLS is similar to the method of solving FCLS [42].

It should be noted that the method introduced by Friedman et al. for solving the one-dimensional FLS is to transform the original $n \times n$ FLS into a $(2n) \times (2n)$ RLS [9]. The coefficient matrix of the $(2n) \times (2n)$ RLS is a block matrix, the positive part of the original FLS coefficient matrix is the main diagonal of the block matrix, and the negative part of the original FLS coefficient matrix is the negative diagonal of the block matrix. Our method is to transform the one-dimensional FLS into a $n \times n$ RLS and a $(2n) \times (2n)$ RLS, where the coefficients of the $n \times n$ RLS are the coefficients of the original FLS, and the coefficient matrix of the $(2n) \times (2n)$ RLS is also a block matrix. The positive part of the original FLS coefficient matrix is the main diagonal of the block matrix, and the negative part of the original FLS coefficient matrix is the negative diagonal of the block matrix. This is identical to the coefficient matrix of the $(2n) \times (2n)$ RLS in the approach developed by Friedman et al. Therefore, our method can be regarded as an extension of the method introduced by Friedman et al. in the case of FCLS.

6. FCLS Based on RFCN

From Theorem 5 in the reference [1] and Lemma 2, as well as Theorem 2.3 in the reference [43], we know that a RFCN $\tilde{z}$ may be written as a form of $\tilde{a} + \mathrm{i}\tilde{b}$. It follows from (2) of Theorem 2 and Definition 3 that

$$k\tilde{z} = k\tilde{a} + \mathrm{i}(k\tilde{b}), \quad \tilde{z}_1 + \tilde{z}_2 = (\tilde{a}_1 + \tilde{a}_2) + \mathrm{i}(\tilde{b}_1 + \tilde{b}_2),$$

where $\tilde{z} = \tilde{a} + \mathrm{i}\tilde{b}, \tilde{z}_1 = \tilde{a}_1 + \mathrm{i}\tilde{b}_1, \tilde{z}_2 = \tilde{a}_2 + \mathrm{i}\tilde{b}_2, k \in \mathbb{R}$.

However, by (1) of Theorem 2, we note that the scalar multiplication of an RFCN is not always an RFCN. Therefore, We consider only a FCLS based on an RFCN, and its coefficients are all real numbers.

The $n \times n$ FCLS based on RFCN is written as

$$\begin{cases} a_{11}\tilde{z}_1 + a_{12}\tilde{z}_2 + \cdots + a_{1n}\tilde{z}_n = \tilde{w}_1, \\ a_{21}\tilde{z}_1 + a_{22}\tilde{z}_2 + \cdots + a_{2n}\tilde{z}_n = \tilde{w}_2 \\ \qquad \vdots \\ a_{n1}\tilde{z}_1 + a_{n2}\tilde{z}_2 + \cdots + a_{nn}\tilde{z}_n = \tilde{w}_n, \end{cases} \tag{16}$$

where $\tilde{z}_j = \tilde{x}_j + \mathrm{i}\tilde{y}_j (j = 1, 2, \cdots, n)$ are unknown RFCN, $\tilde{w}_k = \tilde{c}_k + \mathrm{i}\tilde{d}_k (k = 1, 2, \cdots, n)$ are known RFCN, $a_{jk} (i, j = 1, 2, \cdots, n)$ are real numbers, and i is an imaginary unit.

We can write the above as

$$A\tilde{Z} = \tilde{W}, \tag{17}$$

where $A = (a_{jk})_{n\times n}$ is a real $n \times n$ matrix, $\tilde{W} = (\tilde{c}_k + \mathrm{i}\tilde{d}_k)_{n\times 1}$ is a column vector of the known RFCN and $\tilde{Z} = (\tilde{x}_j + \mathrm{i}\tilde{y}_j)_{n\times 1}$ is a column vector of the unknown RFCN.

Then, Equation (17) is equivalent to the equations

$$A\tilde{X} = \tilde{C}, \tag{18}$$

and

$$A\tilde{Y} = \tilde{D}, \tag{19}$$

where $\tilde{C} = (\tilde{c}_k)_{n\times 1}, \tilde{D} = (\tilde{d}_k)_{n\times 1}$ are two column vectors of known fuzzy numbers, and $\tilde{X} = (\tilde{x}_j)_{n\times 1}, \tilde{Y} = (\tilde{y}_j)_{n\times 1}$ are two column vectors of unknown fuzzy numbers.

We obtain the following theorem by the above discussion.

Theorem 6. *A column vector of RFCN $\tilde{Z} = \tilde{X} + \mathrm{i}\tilde{Y}$ is a solution of Equation (17) if and only if $\tilde{X}$ and $\tilde{Y}$ are the solutions of Equations (18) and (19), respectively.*

Example 3. *Let*

$$\begin{cases} \tilde{z}_1 - \tilde{z}_2 = \tilde{w}_1, \\ \tilde{z}_1 + 3\tilde{z}_2 = \tilde{w}_2, \end{cases} \tag{20}$$

be a 2×2 FCLS based on RFCN, where

$$[\tilde{w}_1]^\alpha = [\alpha, 2-\alpha] + \mathrm{i}[1+\alpha, 3-\alpha], \quad [\tilde{w}_2]^\alpha = [4+\alpha, 7-2\alpha] + \mathrm{i}[\alpha-4, -1-2\alpha].$$

Let

$$[\tilde{z}_1]^\alpha = [x^\alpha_{1-}, x^\alpha_{1+}] + \mathrm{i}[y^\alpha_{1-}, y^\alpha_{1+}], \quad [\tilde{z}_2]^\alpha = [x^\alpha_{2-}, x^\alpha_{2+}] + \mathrm{i}[y^\alpha_{2-}, y^\alpha_{2+}].$$

Then, Equation (20) is equivalent to the systems

$$\begin{cases} [x^\alpha_{1-}, x^\alpha_{1+}] - [x^\alpha_{2-}, x^\alpha_{2+}] = [\alpha, 2-\alpha], \\ [x^\alpha_{1-}, x^\alpha_{1+}] + 3[x^\alpha_{2-}, x^\alpha_{2+}] = [4+\alpha, 7-2\alpha], \end{cases} \tag{21}$$

and

$$\begin{cases} [y_{1-}^{\alpha}, y_{1+}^{\alpha}] - [y_{2-}^{\alpha}, y_{2+}^{\alpha}] = [1+\alpha, 3-\alpha], \\ [y_{1-}^{\alpha}, y_{1+}^{\alpha}] + 3[y_{2-}^{\alpha}, y_{2+}^{\alpha}] = [\alpha-4, -1-2\alpha]. \end{cases} \tag{22}$$

By System (21), *we get*

$$x_{1-}^{\alpha} = \frac{11}{8} + \frac{5\alpha}{8}, \; x_{1+}^{\alpha} = \frac{23}{8} - \frac{7\alpha}{8},$$

$$x_{2-}^{\alpha} = \frac{7}{8} + \frac{\alpha}{8}, \quad x_{2+}^{\alpha} = \frac{11}{8} - \frac{3\alpha}{8}.$$

By System (22), *we get*

$$y_{1-}^{\alpha} = \frac{1}{8} + \frac{5\alpha}{8}, \quad y_{1+}^{\alpha} = \frac{13}{8} - \frac{7\alpha}{8},$$

$$y_{2-}^{\alpha} = -\frac{11}{8} + \frac{\alpha}{8}, \; y_{2+}^{\alpha} = -\frac{7}{8} - \frac{3\alpha}{8}$$

Hece, we obtain that

$$[\tilde{z}_1]^{\alpha} = [\frac{11}{8} + \frac{5\alpha}{8}, \frac{23}{8} - \frac{7\alpha}{8}] + \mathrm{i}[\frac{1}{8} + \frac{5\alpha}{8}, \frac{13}{8} - \frac{7\alpha}{8}],$$

$$[\tilde{z}_2]^{\alpha} = [\frac{7}{8} + \frac{\alpha}{8}, \frac{11}{8} - \frac{3\alpha}{8}] + \mathrm{i}[-\frac{11}{8} + \frac{\alpha}{8}, -\frac{7}{8} - \frac{3\alpha}{8}].$$

Our calculation results are consistent with those in the literature [39].

7. Conclusions

We have discussed the scalar multiplication and addition operations of complex numbers and FCN based on the representation of FCN and also introduced a method for solving FCLS, which can convert FCLS into two distinct linear systems: one is a $n \times n$ complex linear system, and the other is an $(mn) \times (mn)$ RLS. Finally, FCLS based on the RFCN as a special case are also investigated. Our calculation results show the efficiency and effectiveness of the methodology by several examples. In the future, the research in this field will focus on fuzzy complex analysis and applications of FCLS.

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