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Efficient Numerical Solutions to a SIR Epidemic Model

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Abstract: Two non-standard predictor-corrector type finite difference methods for a SIR epidemic model are proposed. The methods have useful and significant features, such as positivity, basic stability, boundedness and preservation of the conservation laws. The proposed schemes are compared with classical fourth order Runge–Kutta and non-standard difference methods (NSFD). The stability analysis is studied and numerical simulations are provided.

Keywords: epidemic models; numerical methods; elementary stability; positivity

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1. Introduction

Ordinary differential conditions are extensively used in demonstrating numerous natural and physical applications. Mathematical strategies dependent on finite differences [1,2], Taylor series [3], interpolation, such as Runge–Kutta, Euler, and multistep techniques [4,5], and some different strategies [6,7] are broadly utilized. A large number of problems in mathematical epidemiology are modeled by autonomous systems of nonlinear ordinary differential equations, which implies that the boundaries of the model are autonomous, regarding time. In these models, the factors address subpopulations of susceptibles, infected, recovered, etc. Consequently, the solutions of the ODE framework portray the advancement of the various classes of subpopulations in the model over the long haul. The use of different schemes brings up such issues as what the truncation error is or how large the region of steadiness is. Numerous standard strategies, such as forward Euler, Runge–Kutta and others, fail to demonstrate the non-actual motions, bifurcations and chaos (see, for example, [8]).

One method for forestalling these kinds of mathematical problems is the development of mathematical methods dependent on non-standard finite-difference techniques. This sort of technique was initially devised by Mickens [9,10]. Piyanwong et al. [11] and Jansen and Twizell [1] have planned positive and genuinely stable plans for the SIR and SEIR epidemic models, individually. In any case, in their created solutions, they have not applied the preservation law unequivocally, which can prompt impossible or unreasonable arrangements.

The best situation to accurately resolve an ODE based-model is the point at which a careful difference solution can be developed in [12]. Although the non-standard finite difference technique preserve the standard properties of the approximation solution, such as consistency and convergence, it can also preserve the qualitative properties of the solution, such as boundedness, monotonicity, positivity, and so on [13–22]. In this paper, we have developed two predictor–corrector types of NSFD techniques to obtain mathematical results for the SIR epidemic model. The new mathematical strategy has the heuristic properties of the ODE framework solution and is basically stable. These strategies are also very practical due to the huge time step used.

The sections of the paper are as follows: In Section 2, we introduce the SIR epidemic model and its steady-state point numerical model. In Section 3, we present the new methods. In Sections 4 and 5, we explore positivity and basic stability results. In Section 6, other NSFD techniques are constructed to be used as correctors in the event of larger stepsizes. In Section 7, we introduce the results of the new methods compared to different schemes used for inspection, such as the the classical Runge–Kutta (RK4) strategy, ode45, etc. We finish the paper with the conclusions and discussion.

2. The SIR Model

We consider the SIR epidemic model as given in [11]. This model incorporates three independent variables to represent the individuals:
$s(t)$: susceptible
$i(t)$: infected
$r(t)$: recovered

A variant of this model was utilized in [23] to depict the elements of beating hack epidemics in London involving occasional varieties in susceptibility. The model is depicted by a nonlinear ODEs framework as follows:

$$\begin{aligned}
&s'(t) = \mu N - \mu s(t) - \beta N i(t) s(t),\\
&i'(t) = -(\mu+\nu) i(t) + \beta N s(t) i(t),\\
&r'(t) = -\mu r(t) + \nu i(t),\\
&t > 0, \quad i(0) = i_0, \quad s(0) = s_0, \quad r(0) = r_0,
\end{aligned} \tag{1}$$

in which

- β is the coefficient of transmission;
- μ is the death rate or the birth rate, which are equal to each other;
- ν is the recovery rate;
- N is the total number of individuals, $N(t) = i(t) + s(t) + r(t)$.

All are assumed to be positive.

The equations of the continuous system, if added together, satisfy the conservation law
$$\frac{dN}{dt} = 0.$$

Since N is constant, system (1) can be written as follows by referring to [24] with the assumption $S(t) = \frac{s(t)}{N}$, $I = \frac{i(t)}{N}$ and $R(t) = \frac{r(t)}{N}$

$$\begin{aligned}
&S'(t) = \mu - \mu S(t) - \beta I(t) S(t),\\
&I'(t) = -(\mu+\nu) I(t) + \beta S(t) I(t),\\
&R'(t) = -\mu R(t) + \nu I(t),\\
&t > 0, \quad I(0) = I_0, \quad S(0) = S_0, \quad R(0) = R_0,
\end{aligned} \tag{2}$$

where
$s(t)$: susceptible
$i(t)$: infected
$r(t)$: recovered

The reproduction number of (2) is

$$\mathcal{R}_0 = \frac{\beta}{\mu+\nu}.$$

If $1 > \mathcal{R}_0$, model (2) has steady-state (disease-free) $E_0^* = (1, 0, 0)$, and it has a unique endemic steady-state, then

$$E^* = \Big(\frac{1}{\mathcal{R}_0}, \frac{\mu}{\mu+\nu}(1 - \frac{1}{\mathcal{R}_0}), \frac{\nu}{\mu+\nu}(1 - \frac{1}{\mathcal{R}_0})\Big),$$

if $\mathcal{R}_0 > 1$.

Definition 1. *A discrete version of* (2) *is called the nonstandard method provided that one of the conditions below is satisfied:*

(i) *In the discrete derivatives of $\frac{dS}{dt}$, $\frac{dI}{dt}$ and $\frac{dR}{dt}$, a non-negative function $\varphi(h)$ substitutes the step size h, such that*

$$\varphi(h) = h + O(h^2) \quad as \quad 0 < h \to 0; \tag{3}$$

(ii) *Nonlinear terms in the right hand side of* (2) *are approximated in a nonlocal way, that is to say, by an appropriate function of some points in the mesh.*

For instance,

$$\begin{aligned} S &\approx 2S_n - S_{n+1}, \\ S^2 &\approx S_n * S_{n+1}, \\ S^3 &\approx \alpha S_n^2 * S_{n+1} + (1-\alpha) S_n * S_{n+1} * S_{n-1}. \end{aligned}$$

3. Construction of New Schemes

We express system (2) as

$$\begin{aligned} \frac{S^{n+1} - S^n}{\varphi(h)} &= \mu - \mu(\delta S^n + \gamma S^{n+1}) - \beta(\eta S^{n+1} + \theta S^n) I^n, \\ \frac{I^{n+1} - I^n}{\varphi(h)} &= \beta(\eta S^{n+1} + \theta S^n) I^n - \mu(\gamma I^{n+1} + \delta I^n) - \nu(\theta I^n + \eta I^{n+1}), \\ \frac{R^{n+1} - R^n}{\varphi(h)} &= \nu(\theta I^n + \eta I^{n+1}) - \mu(\gamma R^{n+1} + \delta R^n), \end{aligned} \tag{4}$$

with $1 = \eta + \theta = \gamma + \delta$,

$$\varphi(h) = exp(h) - 1.$$

We are free to choose the parameters keeping the condition $1 = \eta + \theta = \gamma + \delta$, and here we chose theta to achieve a set of stable explicit methods.

Remark 1. *The condition $0 < \varphi(h) < 1$ for $h > 0$ is essential, and there exist several functions satisfying this condition, e.g., $\varphi(h) = exp(h) - 1$, $\varphi(h) = 1 - exp(-h)$ or $\varphi(h) = sin(h)$.*

It should be noted that by writing the Taylor expansion of S^{n+1} and $\varphi(h) = h + O(h^2)$ in the first relation of Equation (4), we have

$$\begin{aligned} \frac{S^n + hS'^n + \frac{h^2}{2!}S''^n + \cdots - S^n}{h + O(h^2)} &= \mu - \mu(\delta S^n + \gamma(S^n + hS'^n + \frac{h^2}{2!}S''^n + \cdots)) \\ &\quad - \beta(\eta(S^n + hS'^n + \frac{h^2}{2!}S''^n + \cdots) + \theta S^n) I^n \end{aligned}$$

with $h \to 0$, from which we have

$$S'^n = \mu - \mu(\delta S^n + \gamma S^n) - \beta(\eta(S^n + \theta S^n)) I^n = \mu - \mu(\delta + \gamma) S^n - \beta((\eta + \theta) S^n) I^n$$

On the other hand, $\delta + \gamma = 1$ and $\eta + \theta = 1$, then

$$S'^n = \mu - \mu S^n - \beta S^n I^n$$

The second and third relations of the equation can be written in the same way. Therefore, we can move from the discrete form to the continuous form with $h \to 0$.

Lemma 1. *The new NSFD family in* (4) *preserves the conservation law.*

Proof. We use induction to prove it. Since $S + I + R = 1$, for the initial values, it is $S_0 + I_0 + R_0 = 1$. Hence, for $n = 0$, $S_1 + I_1 + R_1 - 1 = \mu\varphi(h)\gamma(1 - (S_1 + I_1 + R_1))$, and thus $S_1 + I_1 + R_1 = 1$, which implies $S_{n+1} + I_{n+1} + R_{n+1} = 1$; as a result, the new family preserves the conservation law. □

After investigating the characteristics of the new family, we present the following methods:

3.1. Scheme 1

Letting $\gamma = \frac{3}{2}, \theta = 0, \delta = -\frac{1}{2}, \eta = 1$ leads to

$$S^{n+1} = \frac{\mu\varphi(h) + S^n(1 + \frac{1}{2}\varphi(h)\mu)}{\frac{3}{2}\mu\varphi(h) + \varphi(h)\beta I^n + 1} = F(I^n, S^n), \tag{5a}$$

$$I^{n+1} = \frac{I^n\left(\frac{1}{2}\mu\varphi(h) + 1 + \beta\varphi(h)F(S^n, I^n)\right)}{\frac{3}{2}\mu\varphi(h) + \nu\varphi(h) + 1} = G(S^n, I^n) \tag{5b}$$

$$R^{n+1} = 1 - I^{n+1} - S^{n+1}. \tag{5c}$$

3.2. Scheme 2

Letting $\theta = 0, \gamma = 2, \eta = 1, \delta = -1$ gives

$$S^{n+1} = \frac{\varphi(h)\mu + (1 + \varphi(h)\mu)S^n}{1 + 2\varphi(h)\mu + \varphi(h)I^n\beta} = \bar{F}(S^n, I^n), \tag{6a}$$

$$I^{n+1} = \frac{I^n\left(\mu\varphi(h) + 1 + \beta\varphi(h)\bar{F}(I^n, S^n)\right)}{1 + 2\varphi(h)\mu + \nu\varphi(h)} = \bar{G}(I^n, S^n) \tag{6b}$$

$$R^{n+1} = 1 - I^{n+1} - S^{n+1}. \tag{6c}$$

4. Positivity

In this part, we analyze the positivity features of the proposed strategies. With positivity, we mean that the component-wise non-negativity of the initial vector is preserved in time for the numerical solution. It must be mentioned that the positivity property of a mathematical technique is significant when it is employed to address differential models emerging in population science because these state factors address subpopulations, which never take negative qualities. Many papers have been written about the positivity property (see for instance [25]).

Definition 2. *A finite difference technique is said to preserve the positivity property, if, for h and* $y_0 \in \mathbb{R}^n_+$, $y_k \in \mathbb{R}^n_+$ *for every* $k \in \mathbf{N}$.

Theorem 1. *NSFD schemes (5) and (6) are positivity preserving.*

Proof. Because $\nu, \beta, \mu > 0$, the (5) and (6) positive NSFD schemes for any $\varphi(h) > 0$ if $0 < S_n < 1, 0 < I_n < 1$ and $0 < R_n < 1$ for all $n \geq 0$. □

5. Elementary Stability

We give sufficient conditions for schemes (5) and (6) to maintain the stability properties of the steady points of the model (2). A difference scheme with this stability property is called an elementary stable scheme [26].

Definition 3. *If the linear stability features of the discrete and differential models are the same, the finite-difference scheme is said to be elementary stable.*

The following result can be readily obtained by using the well-known Jury conditions [27].

Lemma 2. *Consider $\lambda^2 - A\lambda + B = 0$. Both roots satisfy $|\lambda_i| < 1$, if*

- $B < 1$,
- $1 + A + B > 0$,
- $1 - A + B > 0$.

Theorem 2. *Schemes in (5) and (6) are elementary stable.*

Proof. Consider only Equations (5a), (5b), (6a) and (6b): Steady-state points of (5) are E_0^* and E^* of (2).

The Jacobian of (5) is $J(S^n, I^n) = \begin{bmatrix} \frac{\partial F}{\partial S} & \frac{\partial F}{\partial I} \\ \frac{\partial G}{\partial S} & \frac{\partial G}{\partial I} \end{bmatrix}$, where

$$\frac{\partial F}{\partial S} = \frac{1 + \frac{1}{2}\mu\varphi(h)}{\frac{3}{2}\varphi(h)\mu + \varphi(h)\beta I^n + 1},$$

$$\frac{\partial F}{\partial I} = -\frac{\beta\varphi(h)\Big(\mu\varphi(h) + S^n(1 + \frac{1}{2}\mu\varphi(h))\Big)}{(1 + \frac{3}{2}\mu\varphi(h) + \beta\varphi(h)I^n)^2},$$

$$\frac{\partial G}{\partial S} = \frac{\beta\varphi(h)I^n\dfrac{\partial F}{\partial S}}{1 + \frac{3}{2}\mu\varphi(h) + \nu\varphi(h)} = \frac{\beta\varphi(h)I^n(1 + \frac{1}{2}\mu\varphi(h))}{(1 + \frac{3}{2}\mu\varphi(h) + \varphi(h)\beta I^n)(1 + \frac{3}{2}\mu\varphi(h) + \nu\varphi(h))},$$

$$\frac{\partial G}{\partial I} = \frac{1 + \frac{1}{2}\mu\varphi(h) + \beta\varphi(h)F(I^n, S^n) + \beta I^n\varphi(h)\dfrac{\partial F}{\partial I}}{\frac{3}{2}\mu\varphi(h) + 1 + \varphi(h)}\nu$$

$$= \frac{1 + \frac{1}{2}\mu\varphi(h) + \beta\varphi(h)\left(\dfrac{\mu\varphi(h) + S^n(1 + \frac{1}{2}\varphi(h))}{1 + \frac{3}{2}\mu\varphi(h) + \varphi(h)\beta I^n}\right) - \beta^2\varphi^2(h)I^n\left(\dfrac{\Big(\mu\varphi(h) + S^n(1 + \frac{1}{2}\mu\varphi(h))\Big)}{(1 + \frac{3}{2}\mu\varphi(h) + \beta\varphi(h)I^n)^2}\right)}{1 + \frac{3}{2}\mu\varphi(h) + \nu\varphi(h)}.$$

Writing E_0^* into $J(S^n, I^n)$,

$$J(E_0^*) = \begin{pmatrix} \dfrac{2 + \varphi(h)\mu}{2 + 3\mu\varphi(h)} & \dfrac{-2\beta\varphi(h)}{2 + 3\mu\varphi(h)} \\ 0 & \dfrac{2 + \mu\varphi(h) + 2\beta\varphi(h)}{2 + 3\mu\varphi(h) + 2\varphi(h)\nu} \end{pmatrix},$$

from which we obtain the two eigenvalues

$$\lambda_1 = \frac{2 + \mu\varphi(h)}{2 + 3\mu\varphi(h)}, \quad \lambda_2 = \frac{2 + \mu\varphi(h) + 2\beta\varphi(h)}{2 + 3\mu\varphi(h) + 2\nu\varphi(h)}.$$

Hence, we have $|\lambda_1| < 1$. On the other hand, if $\mathcal{R}_0 < 1$, i.e., $\beta < \mu + \nu$, then $|\lambda_2| < 1$. However, if $\mathcal{R}_0 < 1$, the disease-free steady-state point is asymptotically stable; otherwise, it is unstable.

If $1 < \mathcal{R}_0$, the system (2) has an endemic steady state. We have

$$J(E^*) = \begin{pmatrix} \frac{a}{b} & -\frac{c}{b^2} \\ \frac{af}{bd} & \frac{1}{d}(a + \frac{c}{b} - \frac{cf}{b^2}) \end{pmatrix},$$

where

$$a = 1 + \frac{1}{2}\mu\varphi(h) > 1,$$

$$b = 1 + \frac{3}{2}\mu\varphi(h) + \beta\varphi(h)I^* = 1 + \frac{3}{2}\mu\varphi(h) + \mu\varphi(h)(\mathcal{R}_0 - 1) = 1 + \frac{1}{2}\mu\varphi(h) + \mu\varphi(h)\mathcal{R}_0 > 1,$$

$$c = \beta\varphi(h)\Big(\varphi(h)\mu + S^*(1 + \frac{1}{2}\mu\varphi(h))\Big) = \beta\varphi(h)\Big(\mu\varphi(h) + \frac{1}{\mathcal{R}_0}(1 + \frac{1}{2}\mu\varphi(h))\Big)$$
$$= (\mu + \nu)\mu\varphi^2(h)\mathcal{R}_0 + (\mu + \nu)\mu\varphi(h)(1 + \frac{1}{2}\mu\varphi(h)) > 0,$$

$$f = \beta\varphi(h)I^* = \mu\varphi(h)(\mathcal{R}_0 - 1) > 0,$$

$$d = 1 + \frac{3}{2}\mu\varphi(h) + \nu\varphi(h) > 0.$$

The characteristic equation obtained from $J(E^*)$ is $\lambda^2 - A\lambda + B = 0$ in which

$$A = Trace J(E^*) = \frac{ab^2 + bda + c(b - f)}{b^2 d},$$

$$B = Det J(E^*) = \frac{a^2b^2 + acb - caf}{b^3 d} + \frac{caf}{b^3 d} = \frac{a^2b^2 + acb}{b^3 d} = \frac{a^2 b + ac}{b^2 d}.$$

Thus, E^* is stable if Lemma 2 holds. Clearly $B > 0$ since $b > f$, it is also $A > 0$. Since A and B are positive, we have

$$1 + A + B > 0. \tag{7}$$

Additionally, since $\frac{a}{b} < 1$ and

$$\frac{1}{d}(a + \frac{c}{b} - \frac{cf}{b^2}) < \frac{1}{d}(a + \frac{c}{b}) = \frac{ab + c}{bd} = 1,$$

we have

$$1 - A + B = (1 - \frac{a}{b})\Big(1 - \frac{1}{d}(a + \frac{c}{b} - \frac{cf}{b^2})\Big) + \frac{acf}{b^3 d} > 0. \tag{8}$$

Finally, we have

$$B = \frac{a^2}{bd} + \frac{ac}{b^2 d} < \frac{a^2}{bd} + \frac{c}{bd} = \frac{a^2+c}{bd}$$

$$= \frac{(1+\frac{1}{2}\mu\varphi(h))^2 + \mu\beta\varphi(h)\Big(\varphi(h) + \frac{1}{\mathcal{R}_0}(1+\frac{1}{2}\mu\varphi(h))\Big)}{(1+\frac{1}{2}\varphi(h)\mu + \mu\varphi(h)\mathcal{R}_0)(1+\frac{3}{2}\mu\varphi(h) + \nu\varphi(h))} < 1, \tag{9}$$

From (7), (8) and (9), we observe that Lemma 2 holds. Hence, $J(E^*)$ has eigenvalues which are < 1 in modulus, regardless of the size of h, as long as $\mathcal{R}_0 > 1$. Therefore, we have verified the dynamical consistency between system (2) and scheme (5) around every steady state that results in the elementary stability of (5).

Similarly, we have $J(S^n, I^n) = \begin{bmatrix} \frac{\partial \bar{F}}{\partial S} & \frac{\partial \bar{F}}{\partial I} \\ \frac{\partial \bar{G}}{\partial S} & \frac{\partial \bar{G}}{\partial I} \end{bmatrix}$, in which

$$\frac{\partial \bar{F}}{\partial S} = \frac{1+\mu\varphi(h)}{1+2\mu\varphi(h)+\varphi(h)\beta I^n},$$

$$\frac{\partial \bar{F}}{\partial I} = -\frac{\beta\varphi(h)\Big(\mu\varphi(h) + S^n(1+\mu\varphi(h))\Big)}{(1+2\mu\varphi(h)+\beta\varphi(h)I^n)^2},$$

$$\frac{\partial \bar{G}}{\partial S} = \frac{\beta\varphi(h)I^n\dfrac{\partial \bar{F}}{\partial S}}{1+2\mu\varphi(h)+\nu\varphi(h)} = \frac{\beta\varphi(h)I^n(1+\mu\varphi(h))}{(1+2\mu\varphi(h)+\varphi(h)\beta I^n)(1+2\mu\varphi(h)+\nu\varphi(h))},$$

$$\frac{\partial \bar{G}}{\partial I} = \frac{1+\varphi(h)\mu + \beta\varphi(h)\bar{F}(S^n, I^n) + \beta\varphi(h)I^n\dfrac{\partial \bar{F}}{\partial I}}{1+2\mu\varphi(h)+\nu\varphi(h)},$$

$$= \frac{1+\varphi(h)\mu + \beta\varphi(h)\Big(\dfrac{\mu\varphi(h)+S^n(1+\varphi(h)\mu)}{1+2\mu\varphi(h)+\varphi(h)\beta I^n}\Big) - \beta^2\varphi^2(h)I^n\left(\dfrac{\Big(\mu\varphi(h)+S^n(1+\varphi(h)\mu)\Big)}{(1+2\mu\varphi(h)+\beta\varphi(h)I^n)^2}\right)}{1+2\mu\varphi(h)+\nu\varphi(h)}.$$

Writing E_0^* in $J(S^n, I^n)$, we obtain

$$J(E_0^*) = \begin{pmatrix} \dfrac{1+\mu\varphi(h)}{1+2\mu\varphi(h)} & \dfrac{-\beta\varphi(h)}{1+2\mu\varphi(h)} \\ 0 & \dfrac{1+\mu\varphi(h)+\beta\varphi(h)}{1+2\mu\varphi(h)+\nu\varphi(h)} \end{pmatrix},$$

hence, we obtain the two eigenvalues

$$\lambda_1 = \frac{1+\mu\varphi(h)}{1+2\mu\varphi(h)}, \quad \lambda_2 = \frac{1+\mu\varphi(h)+\beta\varphi(h)}{1+2\mu\varphi(h)+\nu\varphi(h)}.$$

It is clear that $|\lambda_1| < 1$ always holds. On the other hand, if $\mathcal{R}_0 < 1$, i.e., $\beta < \mu + \nu$, then $|\lambda_2| < 1$. Therefore, if $\mathcal{R}_0 < 1$, the disease-free equilibrium is asymptotically stable; otherwise, it is unstable. If $\mathcal{R}_0 > 1$, (2) has an endemic steady state.

The Jacobian at the endemic steady state is

$$J(E^*) = \begin{pmatrix} \frac{a}{b} & -\frac{c}{b^2} \\ \\ \frac{af}{bd} & \frac{1}{d}(a + \frac{c}{b} - \frac{cf}{b^2}) \end{pmatrix},$$

where

$$a = 1 + \varphi(h)\mu > 1,$$

$$b = 1 + 2\mu\varphi(h) + \beta\varphi(h)I^* = 1 + 2\mu\varphi(h) + \mu\varphi(h)(\mathcal{R}_0 - 1) = 1 + \mu\varphi(h) + \mu\varphi(h)\mathcal{R}_0 > 1,$$

$$c = \beta\varphi(h)\Big(\mu\varphi(h) + S^*(1 + \mu\varphi(h))\Big) = \beta\varphi(h)\Big(\mu\varphi(h) + \frac{1}{\mathcal{R}_0}(1 + \mu\varphi(h))\Big) > 0,$$

$$f = 1\beta\varphi(h)I^* = \varphi(h)\mu(\mathcal{R}_0 - 1) > 0,$$

$$d = 1 + 2\mu\varphi(h) + \varphi(h)\nu > 0.$$

Hence, from $\lambda^2 - A\lambda + B = 0$, we obtain the eigenvalues as

$$A = TraceJ(E^*) = \frac{ab^2 + bda + c(b - f)}{b^2 d},$$

$$B = DetJ(E^*) = \frac{a^2b^2 + acb - caf}{b^3 d} + \frac{caf}{b^3 d} = \frac{a^2b^2 + acb}{b^3 d} = \frac{a^2 b + ac}{b^2 d}.$$

Thus, E^* is stable if Lemma 2 holds. Clearly $B > 0$, and, since $b > f$, it is also $A > 0$. Since A and B are positive, we have

$$1 + A + B > 0. \tag{10}$$

Again, since $\frac{a}{b} < 1$ and

$$\frac{1}{d}(a + \frac{c}{b} - \frac{cf}{b^2}) < \frac{1}{d}(a + \frac{c}{b}) = \frac{ab + c}{bd} = 1$$

we have

$$1 - A + B = (1 - \frac{a}{b})\Big(1 - \frac{1}{d}(a + \frac{c}{b} - \frac{cf}{b^2})\Big) + \frac{acf}{b^3 d} > 0. \tag{11}$$

Finally, we have

$$B = \frac{a^2}{bd} + \frac{ac}{b^2 d} < \frac{a^2}{bd} + \frac{c}{bd} = \frac{a^2 + c}{bd}$$

$$= \frac{(1 + \mu\varphi(h))^2 + \beta\varphi(h)\Big(\mu\varphi(h) + \frac{1}{\mathcal{R}_0}(1 + \mu\frac{1}{2}\varphi(h))\Big)}{(1 + \mu\varphi(h) + \mu\varphi(h)\mathcal{R}_0)(1 + 2\mu\varphi(h) + \nu\varphi(h))} < 1. \tag{12}$$

From (10), (11) and (12) we observe that Lemma 2 holds. Accordingly, $J(E^*)$ has eigenvalues which are less than 1 in modulus, regardless of h, provided that $\mathcal{R}_0 > 1$. We have therefore verified the dynamical consistence between system (2) and scheme (6) around each steady state that gives us the elementary stability of (6). □

6. The New Nonstandard Discretizations of SIR Model

To improve the performance of the NSFD schemes (5) and (6), we formulate them by a predictor–corrector type method. In the NSFD methods, if step size h is relatively large, some of their features, including convergence, conservation law and positivity, may be lost. Therefore, it is useful to build robust computational methods that can cope with these drawbacks. In general, explicit methods produce larger errors than implicit methods, but implicit methods require solving nonlinear models. Next we will introduce two NSFD schemes of predictor-corrector type schemes (5) and (6), where, for larger step sizes, h maintains the aforementioned significant features. To develop scheme (5), we employ (5) as a predictor scheme, that is

$$S_p^{n+1} = \frac{\mu\varphi(h) + S^n(1 + \frac{1}{2}\mu\varphi(h))}{1 + \frac{3}{2}\mu\varphi(h) + \varphi(h)\beta I^n}, \tag{13a}$$

$$I_p^{n+1} = \frac{\left(1 + \frac{1}{2}\mu\varphi(h) + \beta\varphi(h)S_p^{n+1}\right)I^n}{1 + \frac{3}{2}\mu\varphi(h) + \nu\varphi(h)}, \tag{13b}$$

Now, we introduce an implicit NSFD method for solving the system in (2)

$$\begin{aligned}
\frac{S^{n+1} - S^n}{\varphi(h)} &= \mu - \mu(\frac{3S^{n+1} - S^n}{2}) - \beta S^{n+1}I^{n+1} - \frac{S^{n+1}}{\varphi(h)} + \frac{S^{n+1}}{\varphi(h)}, \\
\frac{I^{n+1} - I^n}{\varphi(h)} &= \beta S^{n+1}I^{n+1} - \mu(\frac{3I^{n+1} - I^n}{2}) - \nu I^{n+1}, \\
\frac{R^{n+1} - R^n}{\varphi(h)} &= \nu I^{n+1} - \mu(\frac{3R^{n+1} - R^n}{2}).
\end{aligned} \tag{14}$$

Thus,

$$S_c^{n+1} = \frac{\mu\varphi(h) + S^n(1 + \frac{1}{2}\mu\varphi(h)) + S_p^{n+1}}{2 + \frac{3}{2}\mu\varphi(h) + \varphi(h)\beta I_p^{n+1}}, \tag{15a}$$

$$I_c^{n+1} = \frac{\left(1 + \frac{1}{2}\mu\varphi(h)\right)I^n + \beta\varphi(h)S_p^{n+1}I_p^{n+1}}{1 + \frac{3}{2}\mu\varphi(h) + \nu\varphi(h)}, \tag{15b}$$

$$R^{n+1} = 1 - S_c^{n+1} - I_c^{n+1}. \tag{15c}$$

Similarly, to develop method (6),

$$S_p^{n+1} = \frac{\mu\varphi(h) + S^n(1 + \varphi(h)\mu)}{1 + 2\mu\varphi(h) + \varphi(h)\beta I^n}, \tag{16a}$$

$$I_p^{n+1} = \frac{\left(1 + \mu\varphi(h) + \beta\varphi(h)S_p^{n+1}\right)I^n}{1 + 2\mu\varphi(h) + \nu\varphi(h)} \tag{16b}$$

Applying the implicit NSFD method to solve (2), we obtain

$$\begin{aligned}
\frac{S^{n+1}-S^n}{\varphi(h)} &= \mu-\mu(2S^{n+1}-S^n)-\beta S^{n+1}I^{n+1}-\frac{S^{n+1}}{\varphi(h)}+\frac{S^{n+1}}{\varphi(h)},\\
\frac{I^{n+1}-I^n}{\varphi(h)} &= \beta S^{n+1}I^{n+1}-\mu(2I^{n+1}-I^n)-\nu I^{n+1},\\
\frac{R^{n+1}-R^n}{\varphi(h)} &= \nu I^{n+1}-\mu(2R^{n+1}-R^n).
\end{aligned} \tag{17}$$

By the conservation law (since the population is constant), we have

$$S_c^{n+1} = \frac{\mu\varphi(h)+S^n(1+\mu\varphi(h))+S_p^{n+1}}{2+2\mu\varphi(h)+\varphi(h)\beta I_p^{n+1}}, \tag{18a}$$

$$I_c^{n+1} = \frac{\Big(1+\mu\varphi(h)\Big)I^n+\beta\varphi(h)S_p^{n+1}I_p^{n+1}}{1+2\mu\varphi(h)+\nu\varphi(h)} \tag{18b}$$

$$R^{n+1} = 1-I_c^{n+1}-S_c^{n+1}. \tag{18c}$$

The algorithm to get the computational solution may be written as follows:

- Step 1 Choose $0<\epsilon<<1$, and I^0, S^0, R^0 such that $S^0+I^0+R^0=1$.
- Step 2 For $n=0,1,\ldots$ do
- Step 3 Evaluate S_p^{n+1}.
- Step 4 Via S_p^{n+1} and I^n, evaluate I_p^{n+1}.
- Step 5 Correct the value S_c^{n+1}, using S^n, S_p^{n+1}, I_p^{n+1}.
- Step 6 Correct the value I_c^{n+1}, using I^n, S_p^{n+1}, I_p^{n+1}.
- Step 7 If $\|S_c^{n+1}-S^n\|<\epsilon$ and $\|I_c^{n+1}-I^n\|<\epsilon$ then
- Step 8 Calculate R^{n+1}, else $S^n=S_c^{n+1}$, $I^n=I_c^{n+1}$ and go to step 5.

7. Numerical Results

In this section, we address some useful simulations to affirm the hypothetical outcomes and illustrate the upsides of the developed NSFD methods. Consider the model in (2) by constants

$$\beta=123, \quad \nu=24, \quad \mu=0.04.$$

If $\mathcal{R}_0<1$, (2) has an asymptotically stable disease-free point and an endemic equilibrium point if $\mathcal{R}_0>1$. Figure 1 illustrates that for $\mathcal{R}_0>1$, the present method will approach the correct epidemic point and generate positive quantities for all time t. At the same time, the conventional Matlab programming procedures do not meet or sometimes produce unreasonable negative qualities for the contaminated population. It should be noted that the fourth request graph of Runge–Kutta with a period step of $h=0.005$ and the new plan with a cycle step of $h=0.01$ are combined with the correct popularity points. The comparisons of the absolute errors provided by different methods for different values of h show that the proposed schemes are more precise than the other techniques. The absolute value of the errors with the new methods, the method presented in [28], and Runge-Kutta method as a reference method are illustrated in Figure 2. Figure 3 shows that the behavior of the P-C schemes with $h=0.01$ is similar to the Runge–Kutta method with $h=0.005$, while the NSFD schemes do not work like them. This means that P-C methods perform better than NSFD methods. We can also see in Figure 4 that P-C 1 has lower dissipation than P-C 2.

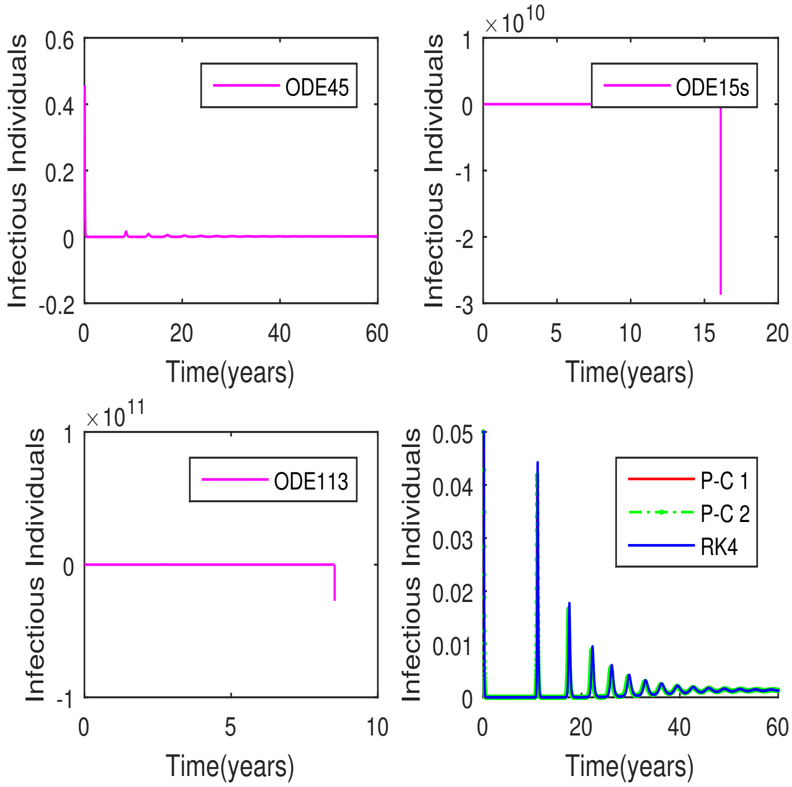

Figure 1. The numerical results of system (2) using P-C schemes with $h = 0.01$, ode45 and RK4 scheme with $h = 0.005$ taking parameter values ($\mathcal{R}_0 > 1$) and I.C. $S_0 = 0.9$, $I_0 = 0.05$, $\nu = 24$, $\mu = 0.04$, $\beta = 123$, $R_0 = 0.05$.

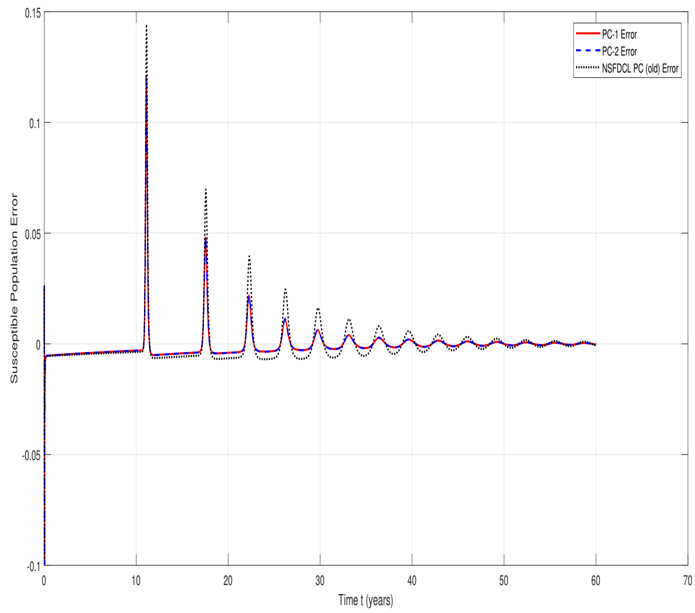

Figure 2. *Cont.*

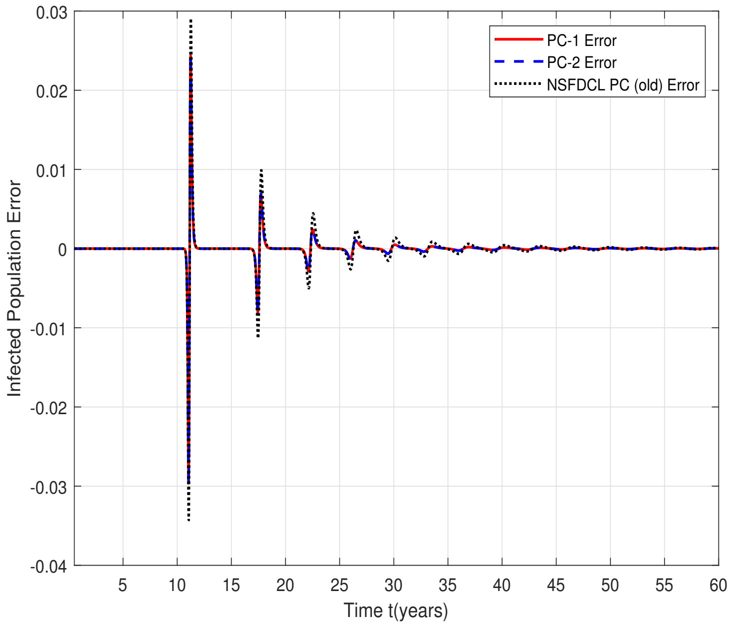

Figure 2. Absolute errors for the system (2) with h= 0.01 by the new schemes, the proposed method in [28] and I.C. $(S_0, I_0, R_0) = (0.9, 0.05, 0.05)$ with $\beta = 123$, $\mu = 0.04$, $\nu = 24$, $(\mathcal{R}_0 > 1)$ using Runge–Kutta as a reference solution.

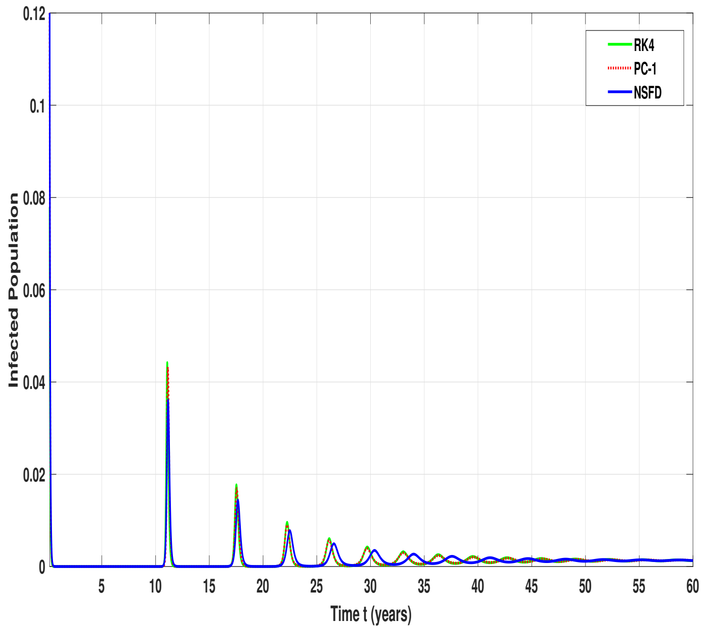

Figure 3. *Cont.*

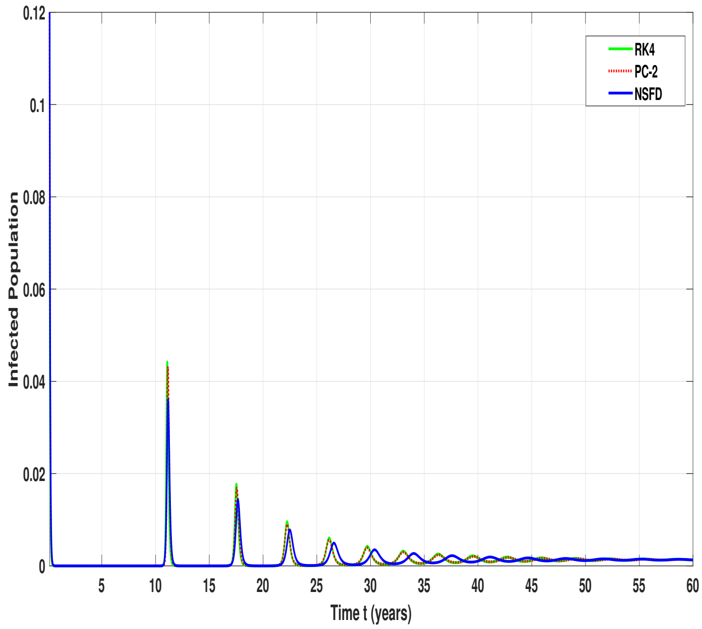

Figure 3. The numerical results of system (2) using the P-C schemes, NSFD schemes with $h = 0.01$ and RK4 with $h = 0.005$ following parameter values ($\mathcal{R}_0 > 1$) and I.C. $S_0 = 0.24$, $I_0 = 0.007$, $R_0 = 0.753$, $\beta = 123$, $\mu = 0.04$, $\nu = 24$.

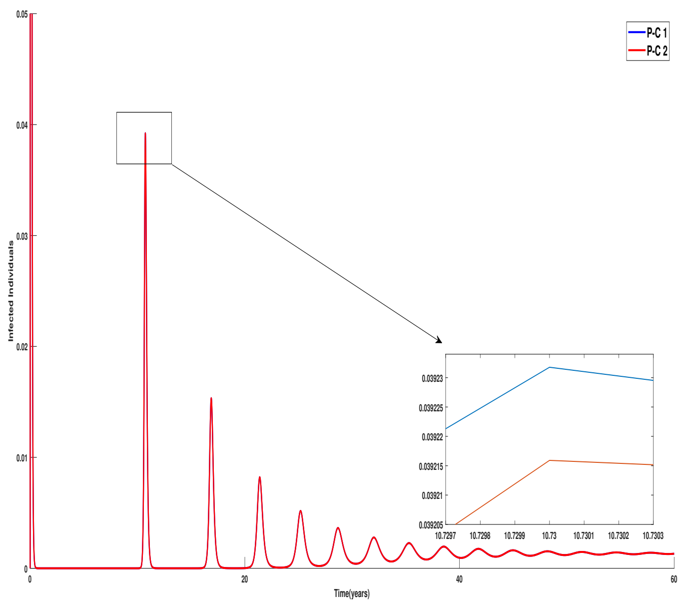

Figure 4. The numerical results of system (2) using the P-C 1 schemes and P-C 2 schemes with $h = 0.01$ following parameter values ($\mathcal{R}_0 > 1$) and I.C. $S_0 = 0.9$, $I_0 = 0.05$, $R_0 = 0.05$, $\beta = 123$, $\mu = 0.04$, $\nu = 24$.

8. Conclusions

In this article, we introduced two competing non-standard finite differences (NSFD) of the of predictor-correctdor type for the classic SIR epidemic model. The model has two biological equilibrium points: one is disease-free equilibrium point E_0^*, if and only if $\mathcal{R}_0 < 1$ is an asymptotically stable node, and the other is endemic equilibrium point E^*. The positivity, boundedness and stability of the proposed scheme are shown. The numerical comparison between the NSFD numerical scheme proposed in this paper and the Runge-

Kutta type scheme shows that the NSFD numerical scheme satisfying the conservation law is unconditionally stable for $\mathcal{R}_0 < 1$, and converges to the time step of the long disease-free equilibrium point. In addition, the same behavior is obtained for $\mathcal{R}_0 > 1$. Furthermore, we showed that the well-known methods in the Matlab software package did not converge to the popular equilibrium point. We conclude that the developed non-standard scheme retains the basic characteristics of the continuous SIR model. In the new scheme, large time steps can be used, making them more economical to reach a steady state over a long period of time.

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