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Intuitionistic Fuzzy Modal Topological Structure

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Abstract: The concept of an Intuitionistic Fuzzy Modal Topological Structure (IFMTS) or for brevity, Intuitionistic Fuzzy Modal Topology (IFMT), is introduced. It is proved that the two standard intuitionistic fuzzy topological operators $\mathcal{C}$ and $\mathcal{I}$, and the two standard intuitionistic fuzzy modal operators $\Box$ and $\Diamond$ generate two different IFMTs. Some basic properties of both IFMTs are discussed. Some important properties of the intuitionistic fuzzy modal and topological operators are discussed. These properties will be a basis of next research on the IFMTSs. Ideas for future development of the IFMT theory are formulated.

Keywords: Intuitionistic Fuzzy Modal Topology; intuitionistic fuzzy operator; Intuitionistic Fuzzy Set

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1. Introduction

In the present paper, we combine the ideas and definitions from the areas of (general) topology (see, e.g., [1–3]), of (standard) modal logic (see, e.g., [4–7]) and of intuitionistic fuzziness (see, e.g., [8–10]), and introduce the concept of an Intuitionistic Fuzzy Modal Topological Structure (IFMTS) or for brevity (by analogy with [1]) – Intuitionistic Fuzzy Modal Topology (IFMT).

Initially, short remarks over Intuitionistic Fuzzy Sets (IFSs) are given (in Section 2), after this, in Section 3, the new objects are introduced and some of their basic properties will be discussed. In the Conclusion, new directions of the development of the present ideas are discussed.

During the last years, the Intuitionistic Fuzzy Topology (IFT) has developed very actively. In [11–37], the first steps in this process were published. It will be interesting, in the future, to conduct a systematic research on IFT development. On the other hand, all research in the area of topology are related to set-theoretical operations "union" and/or "intersection", i.e., on the level of first order logic, but not to higher logical objects, e.g., modal logic operators. With the present research, we would like to introduce this direction of future development of the topology and, in the present case, intuitionistic fuzzy topology.

2. Short Remarks over IFSs

The IFSs are extensions of the standard fuzzy sets of Lotfi Zadeh [38]. All results that are valid for the fuzzy sets can be transformed here, too. Moreover, all studies, for which the apparatus of the fuzzy sets can be used, can be described in terms of the IFSs. On the other hand, not only operations similar to the ordinary fuzzy set operations are defined over the IFSs ones, but also operators that cannot be defined in the case of ordinary fuzzy sets.

Let a set E be fixed. An IFS A in E is an object of the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Let for every $x \in E$:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

Therefore, function π determines the degree of uncertainty.

Obviously, for every ordinary fuzzy set $\pi_A(x) = 0$ for each $x \in E$ and these sets have the form:

$$\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}.$$

Let everywhere below, the universe E be given. One of the geometrical interpretations of the IFSs uses figure F on Figure 1.

Following [9,39], it is important to mention that the functions μ, ν (and also π) can be continuous or discrete with respect of the concrete cases. If universe E and the three functions are constructive objects, then the operations over the IFSs with universe E preserve the constructiveness.

For every two IFSs A and B a lot of relations and operations are defined (see, e.g., [8,9,40]). The most important of them are the following:

$$\begin{array}{lll}
A \subset B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \ \& \ \nu_A(x) \geq \nu_B(x)); \\
A \supset B & \text{iff} & B \subset A; \\
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \ \& \ \nu_A(x) = \nu_B(x)); \\
\neg A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}; \\
A \cap B & = & \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
A \cup B & = & \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
A + B & = & \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \nu_A(x) \cdot \nu_B(x)\rangle \mid x \in E\}; \\
A \cdot B & = & \{\langle x, \mu_A(x) \cdot \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x)\rangle \mid x \in E\}.
\end{array}$$

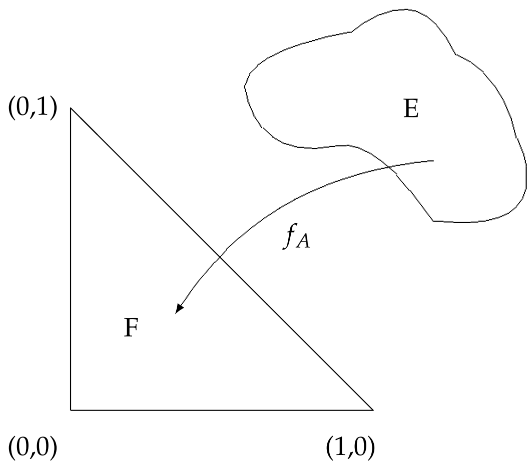

Figure 1. Geometrical interpretation of the elements of a given IFS.

The above operations and relations are defined similarly to those from the fuzzy set theory. More interesting are the modal operators that can be defined over the IFSs. They do not have analogues in fuzzy set theory.

Here, we give definitions of only the first two modal operators (see, e.g., [8,9]) that are intuitionistic fuzzy interpretations of the classical modal logic operators (see, e.g., [4–7]):

$$\square A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\};$$
$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}.$$

The geometrical interpretation of both intuitionistic fuzzy modal operators is given on Figure 2.

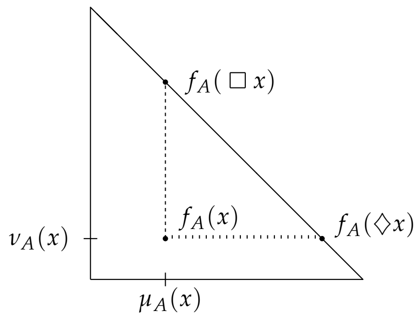

Figure 2. Geometrical interpretation of the two intuitionistic fuzzy modal operators.

If we have an ordinary fuzzy set A, then

$$\square A = A = \Diamond A,$$

while for a proper IFS A, i.e., an IFS with at least one element $x \in E$, for which $\pi_A(x) > 0$:

$$\square A \subset A \subset \Diamond A$$

and

$$\square A \neq A \neq \Diamond A.$$

Let

$$O^* = \{\langle x, 0, 1\rangle | x \in E\},$$
$$U^* = \{\langle x, 0, 0\rangle | x \in E\},$$
$$E^* = \{\langle x, 1, 0\rangle | x \in E\}.$$

Let for each set X

$$\mathcal{P}(X) = \{Y | Y \subseteq X\}.$$

Let for each set E, $FS(E)$ and $IFS(E)$ be the sets of all FSs and IFSs, respectively, with universe E. Then, we observe that

$$\mathcal{P}(E^*) = \{A | A \subseteq E\},$$

where

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \subseteq E^*.$$

Therefore, $\mathcal{P}(E^*)$ coincides with $IFS(E)$. On the other hand side, $FS(E)$ coincides with the set

$$\{A | A \subseteq E \;\&\; A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}\} = \{A | A \subseteq E \;\&\; A = \square A\}\}.$$

3. Definition of an Intuitionistic Fuzzy Modal Topological Structure

In [41], the idea about a feeble version of the IFMT was discussed, while the present research is the first one over proper IFMT.

By analogy with [2], and extending the definitions from there, we will call a $cl-$IFMT the object

$$\langle \mathcal{P}(E^*), cl, \Delta, \nabla, \circ \rangle,$$

where E is a fixed universe, $cl : IFS(E^*) \to IFS(E^*)$ is an operator over E, $\Delta, \nabla : IFS(E^*) \times IFS(E^*) \to IFS(E^*)$ are operations over E such that for every two $A, B \in \mathcal{P}(E^*)$:

$$A \nabla B = \neg(\neg A \Delta \neg B),$$

$\circ : IFS(E^*) \to FS(E^*)$ is a modal operator over E, and for every two IFSs $A, B \in \mathcal{P}(E^*)$:

C1 $cl(A \Delta B) = cl(A) \Delta cl(B)$,
C2 $A \subseteq cl(A)$,
C3 $cl(O^*) = O^*$,
C4 $cl(cl(A)) = cl(A)$,
C5 $\circ(A \nabla B) = \circ A \nabla \circ B$,
C6 $\circ A \subseteq A$,
C7 $\circ E^* = E^*$,
C8 $\circ \circ A = \circ A$,
C9 $\circ\, cl(A) = cl(\circ A)$.

The first two (simplest) analogues of the topological operators "closure" and "interior" (defined over IFSs) are introduced, e.g., in [8], by

$$\mathcal{C}(A) = \{\langle x, K, L \rangle | x \in E\},$$

$$\mathcal{I}(A) = \{\langle x, k, l \rangle | x \in E\},$$

where

$$K = \sup_{y \in E} \mu_A(y),$$

$$L = \inf_{y \in E} \nu_A(y),$$

$$k = \inf_{y \in E} \mu_A(y),$$

$$l = \sup_{y \in E} \nu_A(y).$$

The geometrical interpretations of both operators are given in Figures 3 and 4.

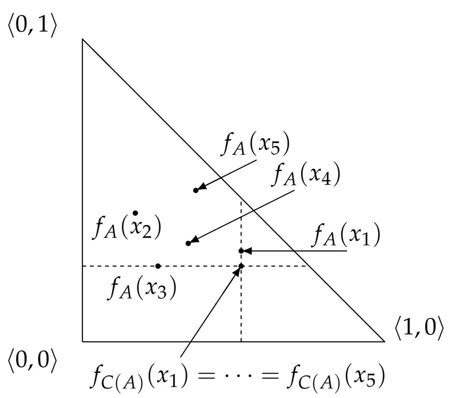

Figure 3. Geometrical interpretation of operator $\mathcal{C}$.

Having in mind that for every two IFSs $A, B \in \mathcal{P}(E^*)$ the De Morgan's laws

$$A \cup B = \neg(\neg A \cap \neg B),$$

$$A \cap B = \neg(\neg A \cup \neg B)$$

hold, we formulate and prove the following theorem.

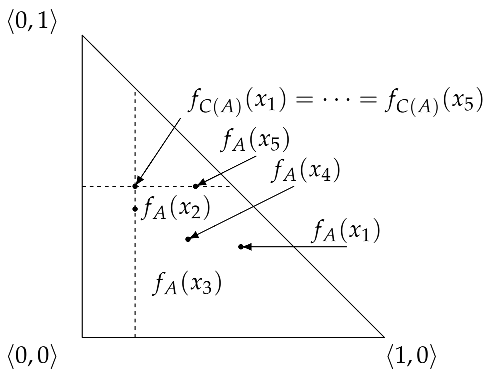

Figure 4. Geometrical interpretation of operator $\mathcal{I}$.

Theorem 1. *$\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \cap, \Box \rangle$ is a cl-IFMT.*

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. Then, we check sequentially that

$$\begin{aligned}
\mathcal{C}(A \cup B) &= \mathcal{C}(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \cup \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}) \\
&= \mathcal{C}(\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}) \\
&= \{\langle x, \sup_{y \in E} \max(\mu_A(y), \mu_B(y)), \inf_{y \in E} \min(\nu_A(y), \nu_B(y))\rangle | x \in E\} \\
&= \{\langle x, \max(\sup_{y \in E} \mu_A(y), \sup_{y \in E} \mu_B(y)), \min(\inf_{y \in E} \nu_A(y), \inf_{y \in E} \nu_B(y))\rangle | x \in E\} \\
&= \mathcal{C}(A) \cup \mathcal{C}(B);
\end{aligned}$$

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \subseteq \{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y)\rangle | x \in E\} = \mathcal{C}(A);$$

$$\begin{aligned}
\mathcal{C}(O^*) &= \mathcal{C}(\{\langle x, 0, 1\rangle | x \in E\}) \\
&= \{\langle x, \sup_{y \in E} 0, \inf_{y \in E} 1\rangle | x \in E\} = \{\langle x, 0, 1\rangle | x \in E\} = O^*;
\end{aligned}$$

$$\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(\{\langle x, K, L\rangle | x \in E\}) = \{\langle x, K, L\rangle | x \in E\} = \mathcal{C}(A);$$

$$\begin{aligned}
\Box(A \cap B) &= \Box(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \cap \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}) \\
&= \Box(\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}) \\
&= \{\langle x, \min(\mu_A(x), \mu_B(x)), 1 - \min(\mu_A(x), \mu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(1 - \mu_A(x), 1 - \mu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\} \cap \{\langle x, \mu_B(x), 1 - \mu_B(x)\rangle | x \in E\} = A \cap B;
\end{aligned}$$

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\} \subseteq \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} = A;$$

$$\square E^* = \square \{\langle x, 1, 0\rangle | x \in E\} = E^*;$$

$$\square \square A = \square \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\} = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\} = \square A;$$

$$\begin{aligned}
\square \mathcal{C}(A) &= \square \{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y)\rangle | x \in E\} \\
&= \{\langle x, \sup_{y \in E} \mu_A(y), 1 - \sup_{y \in E} \mu_A(y)\rangle | x \in E\} \\
&= \{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E}(1 - \mu_A(y))\rangle | x \in E\} = \mathcal{C} \square A.
\end{aligned}$$

This completes the proof. □

Now, we can define for each IFS $A \in \mathcal{P}(E^*)$:

$$\mathcal{I}(A) = \neg \mathcal{C}(\neg A),$$
$$\Diamond A = \neg \square \neg A.$$

Therefore,

$$\begin{aligned}
\mathcal{I}(A) &= \neg \mathcal{C}(\neg A) \\
&= \neg \mathcal{C}(\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}) \\
&= \neg \{\langle x, \sup_{y \in E} \nu_A(y), \inf_{y \in E} \mu_A(y)\rangle | x \in E\} \\
&= \{\langle x, \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle | x \in E\};
\end{aligned}$$

$$\begin{aligned}
\Diamond A &= \neg \square \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\} \\
&= \neg \{\langle x, \nu_A(x), 1 - \nu_A(x)\rangle | x \in E\} \\
&= \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}.
\end{aligned}$$

By analogy with the above, we call an *in*-IFMT the object

$$\langle \mathcal{P}(E^*), in, \nabla, \Delta, *\rangle,$$

where E is a fixed universe, $in : IFS(E^*) \to IFS(E^*)$ is an operator over E, as above $\nabla, \Delta : IFS(E^*) \times IFS(E^*) \to IFS(E^*)$ are operations over E satisfying the De Morgan's laws, and $\circ : IFS(E^*) \to FS(E^*)$ is a modal operator over E, and for every two IFSs $A, B \in \mathcal{P}(E^*)$:

D1 $in(A \nabla B) = in(A) \nabla in(B)$,
D2 $in(A) \subseteq A$,
D3 $in(E^*) = E^*$,
D4 $in(in(A)) = in(A)$,
D5 $*(A \Delta B) = *A \Delta * B$,
D6 $A \subseteq *A$,
D7 $*O^* = O^*$,
D8 $* * A = *A$,
D9 $* \, in(A) = in(*A)$.

Theorem 2. *$\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \cup, \Diamond\rangle$ is an in-IFMT.*

The proof is similar to the proof of Theorem 1.

Theorem 3. *For every two IFSs A and B:*

(a) $\mathcal{I}(A) \subset A \subset \mathcal{C}(A)$;
(b) $\mathcal{C}(\mathcal{I}(A)) = \mathcal{I}(A)$;
(c) $\mathcal{I}(\mathcal{C}(A)) = \mathcal{C}(A)$;
(d) $\mathcal{C}(A \cap B) \subset \mathcal{C}(A) \cap \mathcal{C}(B)$;
(e) $\mathcal{I}(A \cup B) \supset \mathcal{I}(A) \cup \mathcal{I}(B)$;
(f) $\mathcal{C}(E^*) = E^*$;
(g) $\mathcal{C}(U^*) = U^*$;
(h) $\mathcal{I}(O^*) = O^*$;
(i) $\mathcal{I}(U^*) = U^*$,
(j) $\Box(\mathcal{C}(A)) = \mathcal{C}(\Box(A))$,
(k) $\Box(\mathcal{I}(A)) = \mathcal{I}(\Box(A))$,
(l) $\Diamond(\mathcal{C}(A)) = \mathcal{C}(\Diamond(A))$,
(m) $\Diamond(\mathcal{I}(A)) = \mathcal{I}(\Diamond(A))$.

Proof. We check the validity of (d)

$$\begin{aligned}
\mathcal{C}(A \cap B) &= \mathcal{C}(\{\langle x, \min(\mu_A(y), \mu_B(y)), \max(\nu_A(y), \nu_B(y))\rangle | x \in E\}) \\
&= \{\langle x, \sup_{y \in E}(\min(\mu_A(y), \mu_B(y))), \inf_{y \in E}(\max(\nu_A(y), \nu_B(y)))\rangle | x \in E\} \\
&\subseteq \{\langle x, \min(\sup_{y \in E} \mu_A(y), \sup_{y \in E} \mu_B(y)), \max(\inf_{y \in E} \nu_A(y), \inf_{y \in E} \nu_B(y))\rangle | x \in E\} \\
&= \{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y)\rangle | x \in E\} \cap \{\langle x, \sup_{y \in E} \mu_B(y), \inf_{y \in E} \nu_B(y)\rangle | x \in E\} \\
&= C(A) \cap C(B).
\end{aligned}$$

This completes the proof. $\square$

Theorem 4. *For every IFS A:*

(a) $\Box\mathcal{C}(\Box A) = \Diamond\mathcal{C}(\Box A) = \neg\Box\mathcal{I}(\Diamond\neg A) = \neg\Diamond\mathcal{I}(\Diamond\neg A) = \{\langle x, K, 1-K\rangle | x \in E\}$,
(b) $\Box\mathcal{C}(\Diamond A) = \Diamond\mathcal{C}(\Diamond A) = \neg\Box\mathcal{I}(\Box\neg A) = \neg\Diamond\mathcal{I}(\Box\neg A) = \{\langle x, 1-L, L\rangle | x \in E\}$,
(c) $\Box\mathcal{I}(\Box A) = \Diamond\mathcal{I}(\Box A) = \neg\Box\mathcal{C}(\Diamond\neg A) = \neg\Diamond\mathcal{C}(\Diamond\neg A) = \{\langle x, k, 1-k\rangle | x \in E\}$,
(d) $\Box\mathcal{I}(\Diamond A) = \Diamond\mathcal{I}(\Diamond A) = \neg\Box\mathcal{C}(\Box\neg A) = \neg\Diamond\mathcal{C}(\Box\neg A) = \{\langle x, 1-l, l\rangle | x \in E\}$,
(e) $\Box\mathcal{C}(\Box\neg A) = \Diamond\mathcal{C}(\Box\neg A) = \neg\Box\mathcal{I}(\Diamond A) = \neg\Diamond\mathcal{I}(\Diamond A) = \{\langle x, l, 1-l\rangle | x \in E\}$,
(f) $\Box\mathcal{C}(\Diamond\neg A) = \Diamond\mathcal{C}(\Diamond\neg A) = \neg\Box\mathcal{I}(\Box A) = \neg\Diamond\mathcal{I}(\Box A) = \{\langle x, 1-k, k\rangle | x \in E\}$,
(g) $\Box\mathcal{I}(\Box\neg A) = \Diamond\mathcal{I}(\Box\neg A) = \neg\Box\mathcal{C}(\Diamond A) = \neg\Diamond\mathcal{C}(\Diamond A) = \{\langle x, L, 1-L\rangle | x \in E\}$,
(h) $\Box\mathcal{I}(\Diamond\neg A) = \Diamond\mathcal{I}(\Diamond\neg A) = \neg\Box\mathcal{C}(\Box A) = \neg\Diamond\mathcal{C}(\Box A) = \{\langle x, 1-K, K\rangle | x \in E\}$.

Proof. For (a) we obtain

$$\begin{aligned}
\Box\mathcal{C}(\Box A) &= \Box\mathcal{C}(\Box\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}) \\
&= \Box\mathcal{C}(\{\langle x, \mu_A(x), 1-\mu_A(x)\rangle | x \in E\}) \\
&= \Box\{\langle x, K, \min_{y \in E}(1-\mu_A(y))\rangle | x \in E\} \\
&= \{\langle x, K, 1-K\rangle | x \in E\}.
\end{aligned}$$

All other equalities are checked in the same manner. $\square$

Let for a fixed IFS A:

$$\begin{aligned}
s(A) &= \{\Box\mathcal{C}(\Box A), \Diamond\mathcal{C}(\Box A), \neg\Box\mathcal{I}(\Diamond\neg A), \neg\Diamond\mathcal{I}(\Diamond\neg A)\},\\
t(A) &= \{\Box\mathcal{C}(\Diamond A), \Diamond\mathcal{C}(\Diamond A), \neg\Box\mathcal{I}(\Box\neg A), \neg\Diamond\mathcal{I}(\Box\neg A)\},\\
u(A) &= \{\Box\mathcal{I}(\Box A), \Diamond\mathcal{I}(\Box A), \neg\Box\mathcal{C}(\Diamond\neg A), \neg\Diamond\mathcal{C}(\Diamond\neg A)\},\\
v(A) &= \{\Box\mathcal{I}(\Diamond A), \Diamond\mathcal{I}(\Diamond A), \neg\Box\mathcal{C}(\Box\neg A), \neg\Diamond\mathcal{C}(\Box\neg A)\},\\
w(A) &= \{\Box\mathcal{C}(\Box\neg A), \Diamond\mathcal{C}(\Box\neg A), \neg\Box\mathcal{I}(\Diamond A), \neg\Diamond\mathcal{I}(\Diamond A)\},\\
x(A) &= \{\Box\mathcal{C}(\Diamond\neg A), \Diamond\mathcal{C}(\Diamond\neg A), \neg\Box\mathcal{I}(\Box A), \neg\Diamond\mathcal{I}(\Box A)\},\\
y(A) &= \{\Box\mathcal{I}(\Box\neg A), \Diamond\mathcal{I}(\Box\neg A), \neg\Box\mathcal{C}(\Diamond A), \neg\Diamond\mathcal{C}(\Diamond A)\},\\
z(A) &= \{\Box\mathcal{I}(\Diamond\neg A), \Diamond\mathcal{I}(\Diamond\neg A), \neg\Box\mathcal{C}(\Box A), \neg\Diamond\mathcal{C}(\Box A)\}.
\end{aligned}$$

It can be directly observed that for every two IFSs P and Q:

(a) If $P \in s(A)$ and $Q \in t(A)$, then $P \subseteq \mathcal{C}(A) \subseteq Q$;
(b) If $P \in u(A)$ and $Q \in v(A)$, then $P \subseteq \mathcal{I}(A) \subseteq Q$;
(c) If $P \in w(A)$ and $Q \in x(A)$, then $P \subseteq \mathcal{I}(A) \subseteq Q$;
(d) If $P \in y(A)$ and $Q \in z(A)$, then $P \subseteq \mathcal{C}(A) \subseteq Q$.

4. Conclusions or Ideas for the Future

The described above idea opens some directions for future research. Below, we discuss two of them.

First, as it was discussed in [10], operations "union" and "intersection" over IFSs can have different forms. For example, instead of operations "$\cup$" and $\cap$" we can use operations "+" and ".", respectively. Therefore, the topological operators "$\mathcal{C}$" and "$\mathcal{I}$", that are based on operations "$\cup$" and $\cap$" can obtain new forms, based on operations "+" and ".", respectively. Similarly, we can proceed with all other forms of operations "$\cup$" and "$\cap$".

The new operations from "$\cup$"- and "$\cap$"-types, will generate new topological operators from "closure" and "interior" types, respectively, but in some cases, they will satisfy feeble C- and D-conditions. Thus, in next research we will introduce new, already feeble topologies, generated by the new operations and operators. When the standard modal operators are used in these topologies, we will have modal topologies. In the sense of the above definitions, all these topological structures are from the intuitionistic fuzzy type. In the intuitionistic fuzziness, the standard modal operators are extended in some directions. Therefore, in the future, we will introduce extended modal topologies, changing the two modal standard ones with extended modal operators.

If we, using the terminology from [42,43], call *maps* the two types of IFMTs, discussed in the previous section, and also, all other IFMTs, generated from the other forms of operations "$\cup$" and "$\cap$", then all these maps, based on a fixed universe E will generate an *atlas*. Now, following the idea of Saul Kripke's worlds (see. e.g., [44]), we can interpret each atlas as a world in a *universe* (of universes) and study the properties from one hand side: of the maps in an atlas, and from another – of the atlases (worlds) in the universe.

Another direction of our research is related to extension of the forms of both topological operators discussed above, as follows. If $\mathcal{A}(E) \subseteq \mathcal{P}(E)$, then:

$$\overline{\mathcal{C}}(\mathcal{A}(E)) = \{\langle x, \sup_{A\in\mathcal{A}(E)} \mu_A(x), \inf_{A\in\mathcal{A}(E)} \nu_A(x)\rangle | x \in E\},$$

$$\overline{\mathcal{I}}(\mathcal{A}(E)) = \{\langle x, \inf_{A\in\mathcal{A}(E)} \mu_A(x), \sup_{A\in\mathcal{A}(E)} \nu_A(x)\rangle | x \in E\}.$$

Therefore, we can study the properties of the new topological operators (that of course, can have each one of the above discussed forms). For example, for them, we can prove the validity of the following

Theorem 5. *For each $\mathcal{A}(E) \subseteq \mathcal{P}(E)$, it holds that:*

$$\mathcal{C}(\overline{\mathcal{C}}(\mathcal{A}(E))) = \overline{\mathcal{C}}(\bigcup_{A \in \mathcal{A}(E)} \mathcal{C}(A)).$$

The third possible direction is related to the use of the extended topological operators, discussed in [9]. In practice, all of the above research can be re-written (in a more detailed form) for these extended topological operators, and new results, specific for them, will arise.

All these directions for development of IFMT will be the object of a future author's research.

In the near future, the possibility to apply the above mentioned object in different areas of Data mining, InterCriteria Analysis and others will be investigated.

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