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Riemann–Liouville Fractional Integral Inequalities for Generalized Pre-Invex Functions of Interval-Valued Settings Based upon Pseudo Order Relation

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Abstract: The concepts of convex and non-convex functions play a key role in the study of optimization. So, with the help of these ideas, some inequalities can also be established. Moreover, the principles of convexity and symmetry are inextricably linked. In the last two years, convexity and symmetry have emerged as a new field due to considerable association. In this paper, we study a new version of interval-valued functions (*I-V·Fs*), known as left and right χ-pre-invex interval-valued functions (LR-χ-pre-invex *I-V·Fs*). For this class of non-convex *I-V·Fs*, we derive numerous new dynamic inequalities interval Riemann–Liouville fractional integral operators. The applications of these repercussions are taken into account in a unique way. In addition, instructive instances are provided to aid our conclusions. Meanwhile, we'll discuss a few specific examples that may be extrapolated from our primary findings.

Keywords: LR-χ-pre-invex interval-valued function; interval Riemann–Liouville fractional integral operator; Hermite–Hadamard inequality; Hermite–Hadamard Fejér inequality

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1. Introduction

The Hermite–Hadamard inequality (see [1,2], p. 137) is a well-known inequality in convex function theory, with a geometrical explanation and a wide range of applications. Hermite–Hadamard inequality (*H-H* inequality) is a development of the concept of convexity, and it logically follows from Jensen's inequality. In recent years, the *H-H* inequality for convex functions has sparked a lot of attention, and several refinements and extensions have been investigated; see [3–14] and the references therein.

On the other hand, interval analysis is a subset of set-valued analysis and is concerned with the study of intervals in the context of mathematical analysis and topology. It was developed as a means of dealing with interval uncertainty, which is included in many mathematical or computer models of deterministic real-world systems. A historical example of an interval enclosure is Archimedes' method for calculating the circumference of a circle. In 1966, Moore [15] released the first book on interval analysis. Moore is recognized as being the first usage of intervals in computer mathematics. Following the release of his book, a lot of scientists began to study interval arithmetic's theory and applications. Because of its universality, interval analysis is currently a useful approach in a range of sectors that are interested in ambiguous data. Moreover, interval analysis has also applications in different fields like in error analysis, computer graphics, error analysis, experimental and computational physics, and many more.

Numerous significant inequalities for *I-V·Fs* (Hermite–Hadamard, Ostrowski, etc.) have been investigated in recent years. In [16,17], Chalco–Cano et al. constructed Ostrowski type inequalities for *I-V·Fs* using the Hukuhara derivative for *I-V·Fs*. Román-Flores et al. developed Minkowski and Beckenbach's inequality for *I-V·Fs* in [18]. For more information, see [18–22] and the references therein. Moreover, inequalities can be examined for the more general set-valued mappings for example, Sadowska [23] introduced the *H-H* inequality general set-valued mappings. Similarly, for generalized inequalities, we refer to the following articles, see [24,25] and the references therein. Recently, Khan et al. extended the interval *H-H* inequalities in terms of fuzzy interval *H-H* inequalities using fuzzy Riemannian and fuzzy Riemann–Liouville fractional integral operators such as in [26]. Khan et al. also presented the new class of convex fuzzy mappings known as (χ_1, χ_2)-convex fuzzy-interval-valued functions ((χ_1, χ_2)-convex *F-I-V·F*) and obtained the new version of *H-H* inequalities for (χ_1, χ_2)-convex *F-I-V·Fs*. Moreover, Khan et al. introduced new notions of generalized convex *F-I-V·Fs*, and derived new fractional *H-H* type and *H-H* type inequalities for convex *F-I-V·Fs* [27–32]. For more analysis and applications of *F-I-V·Fs*, see [33–50] and the references therein.

This study is organized as follows: Section 2 presents preliminary and new concepts and results in interval space, and convex analysis. Section 3 obtains interval *H-H* inequalities and *H-H* Fejér inequalities for LR-χ-pre-invex *I-V·Fs* via interval Riemann–Liouville fractional integral operators. In addition, some interesting examples are also given to verify our results. Section 4 gives conclusions and future plans.

2. Preliminaries

Let $\mathcal{K}_C$ stand for the collection of all closed and bounded intervals of $\mathbb{R}$. We use $\mathcal{K}_C^+$ to represent the set of all positive intervals. The collections of all Riemann integrable real valued functions and Riemann integrable *I-V-F* are denoted by $\mathcal{R}_{[\mu,\omega]}$ and $\mathcal{IR}_{[\mu,\omega]}$, respectively. For more conceptions on *I-V·Fs*, see [36]. Moreover, we have:

Remark 1. [35] *(i) The relation "* $\leq_p$ *" defined on* $\mathcal{K}_C$ *by:*

$$[\mathfrak{U}_*, \mathfrak{U}^*] \leq_p [\mathcal{Z}_*, \mathcal{Z}^*] \text{ if and only if } \mathfrak{U}_* \leq \mathcal{Z}_*, \ \mathfrak{U}^* \leq \mathcal{Z}^*, \tag{1}$$

for all $[\mathfrak{U}_*, \mathfrak{U}^*], [\mathcal{Z}_*, \mathcal{Z}^*] \in \mathcal{K}_C$, *it is a pseudo-order relation. For given* $[\mathfrak{U}_*, \mathfrak{U}^*], [\mathcal{Z}_*, \mathcal{Z}^*] \in \mathcal{K}_C$, *we say that* $[\mathfrak{U}_*, \mathfrak{U}^*] \leq_p [\mathcal{Z}_*, \mathcal{Z}^*]$ *if and only if* $\mathfrak{U}_* \leq \mathcal{Z}_*, \mathfrak{U}^* \leq \mathcal{Z}^*$ *or* $\mathfrak{U}_* \leq \mathcal{Z}_*, \mathfrak{U}^* < \mathcal{Z}^*$. *The relation* $[\mathfrak{U}_*, \mathfrak{U}^*] \leq_p [\mathcal{Z}_*, \mathcal{Z}^*]$ *coincident to* $[\mathfrak{U}_*, \mathfrak{U}^*] \leq [\mathcal{Z}_*, \mathcal{Z}^*]$ *on* $\mathcal{K}_C$.

(ii) It can be easily seen that " $\leq_p$ *" looks like "left and right" on the real line* $\mathbb{R}$, *so we call "* $\leq_p$ *" "left and right" (or "LR" order, in short).*

The concept of the Riemann integral for *I-V-F* first introduced by Moore [15] is defined as follows:

Theorem 1. [15] *If* $\mathfrak{S} : [\mu, \omega] \subset \mathbb{R} \to \mathcal{K}_C$ *is an I-V·F on* $[\mu, \omega]$ *such that* $\mathfrak{S}(x) = [\mathfrak{S}_*(x), \mathfrak{S}^*(x)]$, *then* $\mathfrak{S}$ *is Riemann integrable over* $[\mu, \omega]$ *if and only if,* $\mathfrak{S}_*(x)$ *and* $\mathfrak{S}^*(x)$ *both are Riemann integrable over* $[\mu, \omega]$ *such that:*

$$(IR)\int_\mu^\omega \mathfrak{S}(x)dx = \left[(R)\int_\mu^\omega \mathfrak{S}_*(x)dx, \ (R)\int_\mu^\omega \mathfrak{S}^*(x)dx\right] \tag{2}$$

Lupulescu and Budak et al. [36,37] introduced the following interval Riemann–Liouville fractional integral operators:

Let $\alpha > 0$ and $L\left([\mu,\ \omega], \mathcal{K}_C^+\right)$ be the collection of all Lebesgue measurable I-V-Fs on $[\mu, \omega]$. Then the interval left and right Riemann–Liouville fractional integrals of $\mathfrak{S} \in L\left([\mu,\ \omega], \mathcal{K}_C^+\right)$ with order $\alpha > 0$ are defined by:

$$\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}(x) = \frac{1}{\Gamma(\alpha)} \int_{\mu}^{x} (x-\varsigma)^{\alpha-1} \mathfrak{S}(\varsigma) d\varsigma, \quad (x > \mu), \tag{3}$$

and:

$$\mathcal{I}_{\omega^-}^{\alpha}\ \mathfrak{S}(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\omega} (\varsigma - x)^{\alpha-1} \mathfrak{S}(\varsigma) d\varsigma, \quad (x < \omega) \tag{4}$$

respectively, where $\Gamma(x) = \int_0^\infty \varsigma^{x-1} e^{-\varsigma} d\varsigma$ is the Euler gamma function.

Definition 1. [34] *A real-valued function $\mathfrak{S} : [\mu,\ \omega] \to \mathbb{R}^+$ is named as convex function if:*

$$\mathfrak{S}(\varsigma x + (1-\varsigma) z) \le \varsigma \mathfrak{S}(x) + (1-\varsigma)\mathfrak{S}(z), \tag{5}$$

for all $x,\ z \in [\mu,\ \omega]$, $\varsigma \in [0,\ 1]$. If (5) is reversed, then $\mathfrak{S}$ is named as concave.

Definition 2. [40] *A real valued function $\mathfrak{S} : [\mu,\ \omega] \to \mathbb{R}^+$ is named as pre-invex function if:*

$$\mathfrak{S}(x + (1-\varsigma)\varphi(z, x)) \le \varsigma \mathfrak{S}(x) + (1-\varsigma)\mathfrak{S}(z), \tag{6}$$

for all $x,\ z \in [\mu,\ \omega]$, $\varsigma \in [0,\ 1]$, where $\varphi : [\mu, \omega] \times [\mu, \omega] \to \mathbb{R}$. If (6) is reversed, then $\mathfrak{S}$ is named as pre-incave.

Definition 3. [35] *The I-V·F $\mathfrak{S} : [\mu,\ \omega] \to \mathcal{K}_C^+$ is named as LR-convex I-V·F on $[\mu,\ \omega]$ if:*

$$\mathfrak{S}(\varsigma x + (1-\varsigma) z) \le_p \varsigma \mathfrak{S}(x) + (1-\varsigma)\mathfrak{S}(z), \tag{7}$$

for all $x,\ z \in [\mu,\ \omega]$, $\varsigma \in [0,\ 1]$. If (7) is reversed, then $\mathfrak{S}$ is named as LR-concave I-V·F on $[\mu,\ \omega]$. $\mathfrak{S}$ is affine, if and only if it is both LR-convex and LR-concave I-V·F.

Definition 4. [41] *The I-V·F $\mathfrak{S} : [\mu,\ \omega] \to \mathcal{K}_C^+$ is named as LR-pre-invex I-V·F on invex interval $[\mu,\ \omega]$ if:*

$$\mathfrak{S}(x + (1-\varsigma)\varphi(z, x)) \le_p \varsigma \mathfrak{S}(x) + (1-\varsigma)\mathfrak{S}(z), \tag{8}$$

for all $x,\ z \in [\mu,\ \omega]$, $\varsigma \in [0,\ 1]$, where $\varphi : [\mu, \omega] \times [\mu, \omega] \to \mathbb{R}$. If (8) is reversed then, $\mathfrak{S}$ is named as LR-pre-incave I-V·F on $[\mu,\ \omega]$. $\mathfrak{S}$ is a LR-affine if and only if, it is both LR-pre-invex and LR-pre-incave I-V·Fs.

Definition 5. *Let $\chi\ : [0,\ 1] \subseteq [\mu, \omega] \to \mathbb{R}^+$ such that $\chi 0$. Then, I-V·F $\mathfrak{S} : [\mu, \omega] \to \mathcal{K}_C^+$ is said to be LR-χ-pre-invex I-V·F on $[\mu, \omega]$ if:*

$$\mathfrak{S}(x + (1-\varsigma)\varphi(x, z)) \le_p \chi(\varsigma)\mathfrak{S}(x) + \chi(1-\varsigma)\mathfrak{S}(z), \tag{9}$$

for all $x,\ z \in [\mu, \omega]$, $\varsigma \in [0,\ 1]$, where $\varphi : [\mu, \omega] \times [\mu, \omega] \to \mathbb{R}$. If $\mathfrak{S}$ is LR-χ-pre-incave on $[\mu, \omega]$, then inequality (9) is reversed.

Remark 2. *If $\chi(\varsigma) = \varsigma$, then LR-$\chi$-pre-invex I-V·F becomes LR-pre-invex I-V·F. If $\chi(\varsigma) \equiv 1$, then LR-χ-pre-invex I-V·F becomes LR-P I-V·F, that is:*

$$\mathfrak{S}(x + (1-\varsigma)\varphi(x, z)) \le_p \mathfrak{S}(x) + \mathfrak{S}(z),\ \forall\ \ x,\ z \in [\mu, \omega],\ \varsigma \in [0,\ 1]. \tag{10}$$

Theorem 2. *Let $\chi : [0,\ 1] \subseteq [\mu, \omega] \to \mathbb{R}$ be a non-negative real valued function such that $\chi 0$ and let $\mathfrak{S} : [\mu, \omega] \to \mathcal{K}_C^+$ be a I-V·F such that:*

$$\mathfrak{S}(z) = [\mathfrak{S}_*(z),\ \mathfrak{S}^*(z)], \tag{11}$$

for all $z \in [\mu, \omega]$. Then, $\mathfrak{S}(z)$ is LR-χ-pre-invex I-V·F on $[\mu, \omega]$, if and only if, $\mathfrak{S}_*(z)$ and $\mathfrak{S}^*(z)$ both are χ-pre-invex.

Proof. Assume that, $\mathfrak{S}_*(x)$ and $\mathfrak{S}^*(x)$ are χ-pre-invex on $[\mu, \omega]$. Then from (6), we have:

$$\mathfrak{S}_*(x + (1-\varsigma)\varphi(x,z)) \leq \chi(\varsigma)\mathfrak{S}_*(x) + \chi(1-\varsigma)\mathfrak{S}_*(z),\ \forall\ x, z \in [\mu, \omega],\ \varsigma \in [0,\ 1],$$

and:

$$\mathfrak{S}^*(x + (1-\varsigma)\varphi(x,z)) \leq \chi(\varsigma)\mathfrak{S}^*(x) + \chi(1-\varsigma)\mathfrak{S}^*(z), \forall\ x, z \in [\mu, \omega],\ \varsigma \in [0,\ 1].$$

Then by (11), we obtain:

$$\begin{aligned} \mathfrak{S}(x + (1-\varsigma)\varphi(x,z)) &= [\mathfrak{S}_*(x + (1-\varsigma)\varphi(x,z)),\ \mathfrak{S}^*(x + (1-\varsigma)\varphi(x,z))], \\ &\leq [\chi(\varsigma)\mathfrak{S}_*(x),\ \chi(\varsigma)\mathfrak{S}^*(x)] + [\chi(1-\varsigma)\mathfrak{S}_*(z),\ \chi(1-\varsigma)\mathfrak{S}^*(z)], \end{aligned}$$

that is:

$$\mathfrak{S}(x + (1-\varsigma)\varphi(x,z)) \leq_p \chi(\varsigma)\mathfrak{S}(x) + \chi(1-\varsigma)\mathfrak{S}(z), \forall\ x, z \in [\mu, \omega],\ \varsigma \in [0,\ 1].$$

Hence, $\mathfrak{S}$ is LR-χ-pre-invex IVF on $[\mu, \omega]$.

Conversely, let $\mathfrak{S}$ be a LR-χ-pre-invex IVF on $[\mu, \omega]$. Then for all $x, z \in [\mu, \omega]$ and $\varsigma \in [0,\ 1]$, we have:

$$\mathfrak{S}(x + (1-\varsigma)\varphi(x,z)) \leq_p \chi(\varsigma)\mathfrak{S}(x) + \chi(1-\varsigma)\mathfrak{S}(z).$$

Therefore, from (11), we have:

$$\mathfrak{S}(x + (1-\varsigma)\varphi(x,z)) = [\mathfrak{S}_*(x + (1-\varsigma)\varphi(x,z)),\ \mathfrak{S}^*(x + (1-\varsigma)\varphi(x,z))].$$

Again, from (11), we obtain:

$$\chi(\varsigma)\mathfrak{S}(x) + \chi(1-\varsigma)\mathfrak{S}(x) = [\chi(\varsigma)\mathfrak{S}_*(x),\ \chi(\varsigma)\mathfrak{S}^*(x)] + [\chi(1-\varsigma)\mathfrak{S}_*(z),\ \chi(1-\varsigma)\mathfrak{S}^*(z)],$$

for all $x, z \in [\mu, \omega]$ and $\varsigma \in [0,\ 1]$. Then by LR-χ-pre-invexity of $\mathfrak{S}$, we have for all $x, z \in [\mu, \omega]$ and $\varsigma \in [0,\ 1]$ such that:

$$\mathfrak{S}_*(x + (1-\varsigma)\varphi(x,z)) \leq \chi(\varsigma)\mathfrak{S}_*(x) + \chi(1-\varsigma)\mathfrak{S}_*(z),$$

and:

$$\mathfrak{S}^*(x + (1-\varsigma)\varphi(x,z)) \leq \chi(\varsigma)\mathfrak{S}^*(x) + \chi(1-\varsigma)\mathfrak{S}^*(z),$$

hence, the result follows. □

Example 1. *We consider $\chi(\varsigma) = \varsigma,$ for $\varsigma \in [0,\ 1]$ and the I-V·F $\mathfrak{S} : [0,\ 4] \to \mathcal{K}_C^+$ defined by $\mathfrak{S}(z) = \left[z, 2e^{z^2}\right]$. Since end point functions $\mathfrak{S}_*(z)$, $\mathfrak{S}^*(z)$ are χ-pre-invex functions with respect to $\varphi(x,z) = x - z$. Hence $\mathfrak{S}(z)$ is LR-χ-pre-invex I-V·F.*

Remark 3. *If $\chi(\varsigma) \equiv \varsigma$ and $\mathfrak{S}_*(z) = \mathfrak{S}^*(z)$, then from (8), we obtain the inequality (6).*

If $\chi(\varsigma) \equiv \varsigma$ and $\mathfrak{S}_*(z) = \mathfrak{S}^*(z)$ and $\varphi(x,z) = x - z$, then from (8), we obtain the inequality (5).

We'll need to make the following assumption about the function $\varphi : [\mu, \omega] \times [\mu, \omega] \to \mathbb{R}$, which will be crucial in the major findings.

Condition C. [40]

$$\varphi(x, z + \varsigma\varphi(x, z)) = (1 - \varsigma)\varphi(x, z),$$
$$\varphi(z, z + \varsigma\varphi(x, z)) = -\varsigma\varphi(x, z).$$

Note that $\forall\, z,\ x \in [\mu, \omega]$ and $\varsigma \in [0,\ 1]$, then from condition C we have

$$\varphi(z + \varsigma_2\varphi(x, z), z + \varsigma_1\varphi(x, z)) = (\varsigma_2 - \varsigma_1)\varphi(x, z)$$

Clearly for $\varsigma = 0$, we have $\varphi(x, z) = 0$ if and only if $x = z$, for all $z,\ x \in [\mu, \omega]$. For the application of condition C, see [40,41].

3. Interval Fractional Hermite–Hadamard Inequalities

In this section, we will prove some new *H-H* type inequalities for LR-χ-pre-invex *I-V·Fs* via Riemann–Liouville fractional integral operators. In the next results, we will denote $L\big([\mu,\ \mu + \varphi(\omega, \mu)], \mathcal{K}_C^+\big)$ as the family of Lebesgue measurable *I-V·Fs*.

Theorem 3. *Let $\mathfrak{S} : [\mu,\ \mu + \varphi(\omega, \mu)] \to \mathcal{K}_C^+$ be a LR--pre-invex I-V·F on $[\mu,\ \mu + \varphi(\omega, \mu)]$ such that $\mathfrak{S}(z) = [\mathfrak{S}_*(z),\ \mathfrak{S}^*(z)]$ for all $z \in [\mu,\ \mu + \varphi(\omega, \mu)]$. If φ satisfies condition C and $\mathfrak{S} \in L\big([\mu,\ \mu + \varphi(\omega, \mu)], \mathcal{K}_C^+\big)$, then:*

$$\begin{aligned}\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) &\leq_p \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}^\alpha_{\mu^+}\ \mathfrak{S}(\mu+\varphi(\omega,\mu)) + \mathcal{I}^\alpha_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}(\mu)\right] \\ &\leq_p (\mathfrak{S}(\mu) + \mathfrak{S}(\mu+\varphi(\omega,\mu)))\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma) - \chi(1-\varsigma)]d\varsigma \\ &\leq_p (\mathfrak{S}(\mu) + \mathfrak{S}(\omega))\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma) - \chi(1-\varsigma)]d\varsigma\end{aligned} \tag{12}$$

If $\mathfrak{S}(z)$ is pre-incave I-V·F, then:

$$\begin{aligned}\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) &\geq_p \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}^\alpha_{\mu^+}\ \mathfrak{S}(\mu+\varphi(\omega,\mu)) + \mathcal{I}^\alpha_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}(\mu)\right] \\ &\geq_p (\mathfrak{S}(\mu) + \mathfrak{S}(\mu+\varphi(\omega,\mu)))\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma) - \chi(1-\varsigma)]d\varsigma \\ &\geq_p (\mathfrak{S}(\mu) + \mathfrak{S}(\omega))\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma) - \chi(1-\varsigma)]d\varsigma\end{aligned} \tag{13}$$

Proof. Let $\mathfrak{S} : [\mu,\ \mu + \varphi(\omega, \mu)] \to \mathcal{K}_C^+$ be a LR--pre-invex *I-V·F*. If condition C holds then, by hypothesis, we have:

$$\frac{1}{\chi\left(\frac{1}{2}\right)}\mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) \leq_p \mathfrak{S}(\mu + (1-\varsigma)\varphi(\omega,\mu)) + \mathfrak{S}(\mu + \varsigma\varphi(\omega,\mu)).$$

Therefore, we have:

$$\begin{aligned}\frac{1}{\chi\left(\frac{1}{2}\right)}\mathfrak{S}_*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) &\leq \mathfrak{S}_*(\mu + (1-\varsigma)\varphi(\omega,\mu)) + \mathfrak{S}_*(\mu + \varsigma\varphi(\omega,\mu)), \\ \frac{1}{\chi\left(\frac{1}{2}\right)}\mathfrak{S}^*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) &\leq \mathfrak{S}^*(\mu + (1-\varsigma)\varphi(\omega,\mu)) + \mathfrak{S}^*(\mu + \varsigma\varphi(\omega,\mu)).\end{aligned}$$

Multiplying both sides by $\varsigma^{\alpha-1}$ and integrating the obtained result with respect to ς over $(0, 1)$, we have

$$\begin{aligned}\frac{1}{\chi\left(\frac{1}{2}\right)}\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)d\varsigma &\leq \int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*(\mu + (1-\varsigma)\varphi(\omega,\mu))d\varsigma + \int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*(\mu + \varsigma\varphi(\omega,\mu))d\varsigma, \\ \frac{1}{\chi\left(\frac{1}{2}\right)}\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)d\varsigma &\leq \int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*(\mu + (1-\varsigma)\varphi(\omega,\mu))d\varsigma + \int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*(\mu + \varsigma\varphi(\omega,\mu))d\varsigma.\end{aligned}$$

Let $x = \mu + (1-\varsigma)\varphi(\omega,\mu)$ and $x = \mu + \varsigma\varphi(\omega,\mu)$. Then, we have:

$$\begin{aligned}
\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{S}_*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) &\leq \frac{1}{(\varphi(\omega,\mu))^\alpha}\int_\mu^{\mu+\varphi(\omega,\mu)}(\mu+\varphi(\omega,\mu)-x)^{\alpha-1}\mathfrak{S}_*(x)dx \\
&+\frac{1}{(\varphi(\omega,\mu))^\alpha}\int_\mu^{\mu+\varphi(\omega,\mu)}(z-\mu)^{\alpha-1}\mathfrak{S}_*(z)dz \\
\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{S}^*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) &\leq \frac{1}{(\varphi(\omega,\mu))^\alpha}\int_\mu^{\mu+\varphi(\omega,\mu)}(\mu+\varphi(\omega,\mu)-x)^{\alpha-1}\mathfrak{S}^*(x)dx \\
&+\frac{1}{(\varphi(\omega,\mu))^\alpha}\int_\mu^{\mu+\varphi(\omega,\mu)}(z-\mu)^{\alpha-1}\mathfrak{S}^*(z)dz, \\
&\leq \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}_*(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}_*(\mu)\right] \\
&\leq \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}^*(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}^*(\mu)\right],
\end{aligned}$$

that is:

$$\begin{aligned}
&\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\left[\mathfrak{S}_*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right),\ \mathfrak{S}^*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\right] \\
&\leq_p \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}_*(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}_*(\mu),\ \mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}^*(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}^*(\mu)\right]
\end{aligned}$$

thus,

$$\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\ \mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) \leq_p \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}(\mu)\right]. \tag{14}$$

In a similar way as above, we have:

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}(\mu)\right] \\
&\leq_p [\mathfrak{S}(\mu)+\mathfrak{S}(\mu+\varphi(\omega,\mu))]\int_0^1\varsigma^{\alpha-1}[\chi(\varsigma)-\chi(1-\varsigma)]d\varsigma.
\end{aligned} \tag{15}$$

Combining (14) and (15), we have:

$$\begin{aligned}
\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) &\leq_p \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}(\mu)\right] \\
&\leq_p [\mathfrak{S}(\mu)+\mathfrak{S}(\mu+\varphi(\omega,\mu))]\int_0^1\varsigma^{\alpha-1}[\chi(\varsigma)-\chi(1-\varsigma)]d\varsigma \\
&\leq_p [\mathfrak{S}(\mu)+\mathfrak{S}(\omega)]\int_0^1\varsigma^{\alpha-1}[\chi(\varsigma)-\chi(1-\varsigma)]d\varsigma
\end{aligned}$$

hence, the required result. □

Remark 4. *From Theorem 3 we clearly see that:*

If $\varphi(\omega,\mu) = \omega-\mu$, then from Theorem 3, we get the following new result in fractional calculus, see [42].

$$\mathfrak{Q}\left(\frac{\mu+\omega}{2}\right) \leq_p \frac{\Gamma(\alpha+1)}{2(\omega-\mu)^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{Q}(\omega)+\mathcal{I}_{\omega^-}^\alpha\ \mathfrak{Q}(\mu)\right] \leq_p \frac{\mathfrak{Q}(\mu)+\mathfrak{Q}(\omega)}{2} \tag{16}$$

If $\alpha = 1$, then from Theorem 3, we obtain the following results for LR--pre-invex I-V·F, which are also new ones:

$$\begin{aligned}
&\frac{1}{2\chi\left(\frac{1}{2}\right)}\ \mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) \leq_p \frac{1}{\varphi(\omega,\mu)}\ (FR)\int_\mu^{\mu+\varphi(\omega,\mu)}\mathfrak{S}(z)dz \\
&\leq_p [\mathfrak{S}(\mu)+\mathfrak{S}(\mu+\varphi(\omega,\mu))]\int_0^1\chi(\varsigma)d\varsigma.
\end{aligned} \tag{17}$$

If $\chi(\varsigma) = \varsigma$, then Theorem 3 reduces to the result for LR-pre-invex I-V·F, see [41]*:*

$$\mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) \leq_p \frac{\Gamma(\alpha+1)}{2(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}(\mu)\right] \leq_p \frac{\mathfrak{S}(\mu)+\mathfrak{S}(\mu+\varphi(\omega,\mu))}{2} \tag{18}$$

Let $\alpha = 1$ and $\chi(\varsigma) = \varsigma$. Then, Theorem 3 reduces to the result for LR-pre-invex-I-V·F, see [41]*:*

$$\mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) \leq_p \frac{1}{\varphi(\omega,\mu)}\,(FR)\int_{\mu}^{\mu+\varphi(\omega,\mu)} \mathfrak{S}(z)dz \leq_p \frac{\mathfrak{S}(\mu)+\mathfrak{S}(\omega)}{2} \tag{19}$$

Example 2. *$\alpha = \frac{1}{2}$, $\chi(\varsigma) = \varsigma$, for all $\varsigma \in [0,\ 1]$ and the I-V·F $\mathfrak{S} : [\mu,\ \mu+\varphi(\omega,\mu)] = [2,\ 2+\varphi(3,2)] \to \mathcal{K}_C^+$, defined by $\mathfrak{S}(z) = [1,2]\left(2 - z^{\frac{1}{2}}\right)$. Since left and right end-point functions $\mathfrak{S}_*(z) = \left(2 - z^{\frac{1}{2}}\right)$, $\mathfrak{S}^*(z) = 2\left(2 - z^{\frac{1}{2}}\right)$, are LR-$\chi$-pre-invex functions, then $\mathfrak{S}(z)$ is LR-χ-pre-invex I-V·F. We clearly see that $\mathfrak{S} \in L([\mu,\ \mu+\varphi(\omega,\mu)], \mathcal{K}_C^+)$ and:*

$$\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{S}_*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) = \mathfrak{S}_*\left(\frac{5}{2}\right) = \frac{4-\sqrt{10}}{8}$$

$$\frac{1}{\alpha\chi\left(\frac{1}{2}\right)}\mathfrak{S}^*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right) = \mathfrak{S}^*\left(\frac{5}{2}\right) = \frac{4-\sqrt{10}}{4},$$

$$\frac{\mathfrak{S}_*(\mu)+\mathfrak{S}_*(\mu+\varphi(\omega,\mu))}{2}\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma)-\chi(1-\varsigma)]d\varsigma = \left(4-\sqrt{2}-\sqrt{3}\right)$$

$$\frac{\mathfrak{S}^*(\mu)+\mathfrak{S}^*(\mu+\varphi(\omega,\mu))}{2}\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma)-\chi(1-\varsigma)]d\varsigma = 2\left(4-\sqrt{2}-\sqrt{3}\right).$$

Note that:

$$\begin{aligned}
&\tfrac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}_*(\mu+\varphi(\omega,\mu)) + \mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}_*(\mu)\right] \\
&= \tfrac{\Gamma\left(\frac{1}{2}\right)}{2}\tfrac{1}{\sqrt{\pi}}\int_2^3 (3-z)^{\frac{-1}{2}}\cdot\left(2-z^{\frac{1}{2}}\right)dz \\
&+ \tfrac{\Gamma\left(\frac{1}{2}\right)}{2}\tfrac{1}{\sqrt{\pi}}\int_2^3 (z-2)^{\frac{-1}{2}}\cdot\left(2-z^{\frac{1}{2}}\right)dz \\
&= \tfrac{1}{2}\left[\tfrac{7393}{10{,}000} + \tfrac{9501}{10{,}000}\right] \\
&= \tfrac{8447}{20{,}000}.
\end{aligned}$$

$$\begin{aligned}
&\tfrac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}^*(\mu+\varphi(\omega,\mu)) + \mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}^*(\mu)\right] \\
&\tfrac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}^*(\mu+\varphi(\omega,\mu)) + \mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}^*(\mu)\right] \\
&+ \tfrac{\Gamma\left(\frac{1}{2}\right)}{2}\tfrac{1}{\sqrt{\pi}}\int_2^3 (z-2)^{\frac{-1}{2}}\cdot 2\left(2-z^{\frac{1}{2}}\right)dz \\
&= \left[\tfrac{7393}{10{,}000} + \tfrac{9501}{10{,}000}\right] \\
&= \tfrac{8447}{10{,}000}.
\end{aligned}$$

Therefore:

$$\left[\frac{4-\sqrt{10}}{8}, \frac{4-\sqrt{10}}{4}\right] \leq_p \left[\frac{8447}{20{,}000}, \frac{8447}{10{,}000}\right] \leq_p \left[\left(4-\sqrt{2}-\sqrt{3}\right), 2\left(4-\sqrt{2}-\sqrt{3}\right)\right]$$

and Theorem 3 is verified.

From Theorems 4 and 5, we obtain some interval fractional integral inequalities related to interval fractional *H-H* inequalities.

Theorem 4. *Let* $\mathfrak{S}, \mathcal{H} : [\mu,\ \mu+\varphi(\omega,\mu)] \to \mathcal{K}_C^+$ *be LR-*χ_1*-pre-invex and LR-*χ_2*-pre-invex I-V·Fs on* $[\mu,\ \mu+\varphi(\omega,\mu)]$*, respectively, such that* $\mathfrak{S}(z) = [\mathfrak{S}_*(z),\ \mathfrak{S}^*(z)]$ *and* $\mathcal{H}(z) = [\mathcal{H}_*(z),\ \mathcal{H}^*(z)]$ *for all* $z \in [\mu,\ \mu+\varphi(\omega,\mu)]$*. If* φ *satisfies condition C and* $\mathfrak{S} \times \mathcal{H} \in L([\mu,\ \mu+\varphi(\omega,\mu)], \mathcal{K}_C^+)$*, then:*

$$\begin{aligned}\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}&\left[\mathcal{I}^\alpha_{\mu^+}\ \mathfrak{S}(\mu+\varphi(\omega,\mu)) \times \mathcal{H}(\mu+\varphi(\omega,\mu)) + \mathcal{I}^\alpha_{\mu+\varphi(\omega,\mu)^-}\mathfrak{S}(\mu) \times \mathcal{H}(\mu)\right] \\ &\leq_p \xi(\mu,\mu+\varphi(\omega,\mu)) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(\varsigma) + \chi_1(1-\varsigma)\chi_2(1-\varsigma)]d\varsigma \\ &+\partial(\mu,\mu+\varphi(\omega,\mu)) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(1-\varsigma) + \chi_1(1-\varsigma)\chi_2(\varsigma)]d\varsigma.\end{aligned} \tag{20}$$

where $\xi(\mu,\mu+\varphi(\omega,\mu)) = \mathfrak{S}(\mu) \times \mathcal{H}(\mu) + \mathfrak{S}(\mu+\varphi(\omega,\mu)) \times \mathcal{H}(\mu+\varphi(\omega,\mu))$, $\partial(\mu,\mu+\varphi(\omega,\mu)) = \mathfrak{S}(\mu) \times \mathcal{H}(\mu+\varphi(\omega,\mu)) + \mathfrak{S}(\mu+\varphi(\omega,\mu)) \times \mathcal{H}(\mu)$, *and* $\xi(\mu,\mu+\varphi(\omega,\mu)) = [\xi_*((\mu,\mu+\varphi(\omega,\mu))),\ \xi^*((\mu,\mu+\varphi(\omega,\mu)))]$ *and* $\partial(\mu,\mu+\varphi(\omega,\mu)) = [\partial_*(\mu,\mu+\varphi(\omega,\mu)),\ \partial^*(\mu,\mu+\varphi(\omega,\mu))]$.

Proof. Since $\mathfrak{S},\ \mathcal{H}$ both are LR-χ_1-pre-invex and LR-χ_2-pre-invex *I-V·F* then, we have:

$$\begin{aligned}\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu)) &= \mathfrak{S}_*(\mu+\varphi(\omega,\mu)+\varsigma\varphi(\mu,\mu+\varphi(\omega,\mu))) \\ &\leq \chi_1(\varsigma)\mathfrak{S}_*(\mu) + \chi_1(1-\varsigma)\mathfrak{S}_*(\mu+\varphi(\omega,\mu)) \\ \mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu)) &= \mathfrak{S}^*(\mu+\varphi(\omega,\mu)+\varsigma\varphi(\mu,\mu+\varphi(\omega,\mu))) \\ &\leq \chi_1(\varsigma)\mathfrak{S}^*(\mu) + \chi_1(1-\varsigma)\mathfrak{S}^*(\mu+\varphi(\omega,\mu)).\end{aligned}$$

and:

$$\begin{aligned}\mathcal{H}_*(\mu+(1-\varsigma)\varphi(\omega,\mu)) = &\ \mathcal{H}_*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \\ &\leq \chi_2(\varsigma)\mathcal{H}_*(\mu) + \chi_2(1-\varsigma)\mathcal{H}_*(\mu+\varphi(\omega,\mu)) \\ \mathcal{H}^*(\mu+(1-\varsigma)\varphi(\omega,\mu)) = &\ \mathcal{H}^*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \\ &\leq \chi_2(\varsigma)\mathcal{H}^*(\mu) + \chi_2(1-\varsigma)\mathcal{H}^*(\mu+\varphi(\omega,\mu)).\end{aligned}$$

From the definition of LR--pre-invex *I-V·F*, we have:

$$\begin{aligned}&\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \\ &\quad\leq \chi_1(\varsigma)\chi_2(\varsigma)\mathfrak{S}_*(\mu) \times \mathcal{H}_*(\mu) + \chi_1(1-\varsigma)\chi_2(1-\varsigma)\mathfrak{S}_*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu+\varphi(\omega,\mu)) \\ &\quad+\chi_1(\varsigma)\chi_2(1-\varsigma)\mathfrak{S}_*(\mu) \times \mathcal{H}_*(\mu+\varphi(\omega,\mu)) + \chi_1(1-\varsigma)\chi_2(\varsigma)\mathfrak{S}_*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu) \\ &\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \\ &\quad\leq \chi_1(\varsigma)\chi_2(\varsigma)\mathfrak{S}^*(\mu) \times \mathcal{H}^*(\mu) + \chi_1(1-\varsigma)\chi_2(1-\varsigma)\mathfrak{S}^*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu+\varphi(\omega,\mu)) \\ &\quad+\chi_1(\varsigma)\chi_2(1-\varsigma)\mathfrak{S}^*(\mu) \times \mathcal{H}^*(\mu+\varphi(\omega,\mu)) + \chi_1(1-\varsigma)\chi_2(\varsigma)\mathfrak{S}^*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu).\end{aligned} \tag{21}$$

Analogously, we have:

$$\begin{aligned}&\mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu))\mathcal{H}_*(\mu+\varsigma\varphi(\omega,\mu)) \\ &\quad\leq \chi_1(1-\varsigma)\chi_2(1-\varsigma)\mathfrak{S}_*(\mu) \times \mathcal{H}_*(\mu) + \chi_1(\varsigma)\chi_2(\varsigma)\mathfrak{S}_*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu+\varphi(\omega,\mu)) \\ &\quad+\chi_1(1-\varsigma)\chi_2(\varsigma)\mathfrak{S}_*(\mu) \times \mathcal{H}_*(\mu+\varphi(\omega,\mu)) + \chi_1(\varsigma)\chi_2(1-\varsigma)\mathfrak{S}_*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu) \\ &\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu+\varsigma\varphi(\omega,\mu)) \\ &\quad\leq \chi_1(1-\varsigma)\chi_2(1-\varsigma)\mathfrak{S}^*(\mu) \times \mathcal{H}^*(\mu) + \chi_1(\varsigma)\chi_2(\varsigma)\mathfrak{S}^*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu+\varphi(\omega,\mu)) \\ &\quad+\chi_1(1-\varsigma)\chi_2(\varsigma)\mathfrak{S}^*(\mu) \times \mathcal{H}^*(\mu+\varphi(\omega,\mu)) + \chi_1(\varsigma)\chi_2(1-\varsigma)\mathfrak{S}^*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu).\end{aligned} \tag{22}$$

Adding (21) and (22), we have:

$$
\begin{aligned}
&\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu+(1-\varsigma)\varphi(\omega,\mu)) + \mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu+\varsigma\varphi(\omega,\mu)) \\
&\quad \leq \begin{bmatrix} \chi_1(\varsigma)\chi_2(\varsigma) \\ +\chi_1(1-\varsigma)\chi_2(1-\varsigma) \end{bmatrix} [\mathfrak{S}_*(\mu) \times \mathcal{H}_*(\mu) + \mathfrak{S}_*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu+\varphi(\omega,\mu))] \\
&\qquad + \begin{bmatrix} \chi_1(\varsigma)\chi_2(1-\varsigma) \\ +\chi_1(1-\varsigma)\chi_2(\varsigma) \end{bmatrix} [\mathfrak{S}_*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu) + \mathfrak{S}_*(\mu) \times \mathcal{H}_*(\mu+\varphi(\omega,\mu))] \\
&\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu+(1-\varsigma)\varphi(\omega,\mu)) + \mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu+\varsigma\varphi(\omega,\mu)) \\
&\quad \leq \begin{bmatrix} \chi_1(\varsigma)\chi_2(\varsigma) \\ +\chi_1(1-\varsigma)\chi_2(1-\varsigma) \end{bmatrix} [\mathfrak{S}^*(\mu) \times \mathcal{H}^*(\mu) + \mathfrak{S}^*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu+\varphi(\omega,\mu))] \\
&\qquad + \begin{bmatrix} \chi_1(\varsigma)\chi_2(1-\varsigma) \\ +\chi_1(1-\varsigma)\chi_2(\varsigma) \end{bmatrix} [\mathfrak{S}^*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu) + \mathfrak{S}^*(\mu) \times \mathcal{H}^*(\mu+\varphi(\omega,\mu))].
\end{aligned} \tag{23}
$$

Taking multiplication of (23) with $\varsigma^{\alpha-1}$ and integrating the obtained result with respect to ς over (0,1), we have:

$$
\begin{aligned}
&\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \\
&\qquad + \varsigma^{\alpha-1}\mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu+\varsigma\varphi(\omega,\mu))d\varsigma \\
&\quad \leq \xi_*((\mu,\mu+\varphi(\omega,\mu))) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(\varsigma)+\chi_1(1-\varsigma)\chi_2(1-\varsigma)]d\varsigma \\
&\quad +\partial_*((\mu,\mu+\varphi(\omega,\mu))) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(1-\varsigma)+\chi_1(1-\varsigma)\chi_2(\varsigma)]d\varsigma \\
&\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu+(1-\varsigma)\varphi(\omega,\mu)) \\
&\qquad + \varsigma^{\alpha-1}\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu+\varsigma\varphi(\omega,\mu))d\varsigma \\
&\quad \leq \xi^*((\mu,\mu+\varphi(\omega,\mu))) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(\varsigma)+\chi_1(1-\varsigma)\chi_2(1-\varsigma)]d\varsigma \\
&\quad +\partial^*((\mu,\mu+\varphi(\omega,\mu))) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(1-\varsigma)+\chi_1(1-\varsigma)\chi_2(\varsigma)]d\varsigma.
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
&\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}_*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu+\varphi(\omega,\mu)) + \mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}_*(\mu) \times \mathcal{H}_*(\mu)\right] \\
&\quad \leq \xi_*((\mu,\mu+\varphi(\omega,\mu))) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(\varsigma)+\chi_1(1-\varsigma)\chi_2(1-\varsigma)]d\varsigma \\
&\quad +\partial_*((\mu,\mu+\varphi(\omega,\mu))) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(1-\varsigma)+\chi_1(1-\varsigma)\chi_2(\varsigma)]d\varsigma \\
&\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}^*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu+\varphi(\omega,\mu)) + \mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}^*(\mu) \times \mathcal{H}^*(\mu)\right] \\
&\quad \leq\ \xi^*((\mu,\mu+\varphi(\omega,\mu))) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(\varsigma)+\chi_1(1-\varsigma)\chi_2(1-\varsigma)]d\varsigma \\
&\quad +\partial^*((\mu,\mu+]\varphi(\omega,\mu))) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(1-\varsigma)+\chi_1(1-\varsigma)\chi_2(\varsigma)]d\varsigma.
\end{aligned}
$$

It results that:

$$
\begin{aligned}
&\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}_*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}_*(\mu+\varphi(\omega,\mu)) + \mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}_*(\mu) \times \mathcal{H}_*(\mu),\ \ \mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}^*(\mu+\varphi(\omega,\mu)) \times \mathcal{H}^*(\mu+\varphi(\omega,\mu)) + \mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha\ \mathfrak{S}^*(\mu) \times \mathcal{H}^*(\mu)\right] \\
&\leq_p [\xi_*((\mu,\mu+\varphi(\omega,\mu))),\ \xi^*((\mu,\mu+\varphi(\omega,\mu)))] \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(\varsigma)+\chi_1(1-\varsigma)\chi_2(1-\varsigma)]d\varsigma \\
&+[\partial_*((\mu,\mu+\varphi(\omega,\mu))),\ \partial^*((\mu,\mu+\varphi(\omega,\mu)))] \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(1-\varsigma)+\chi_1(1-\varsigma)\chi_2(\varsigma)]d\varsigma
\end{aligned}
$$

that is:

$$
\begin{aligned}
&\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^\alpha\ \mathfrak{S}(\mu+\varphi(\omega,\mu)) \times \mathcal{H}(\mu+\varphi(\omega,\mu)) + \mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^\alpha \mathfrak{S}(\mu) \times \mathcal{H}(\mu)\right] \\
&\quad \leq_p \xi(\mu,\mu+\varphi(\omega,\mu)) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(\varsigma)+\chi_1(1-\varsigma)\chi_2(1-\varsigma)]d\varsigma \\
&\quad +\partial(\mu,\mu+\varphi(\omega,\mu)) \int_0^1 \varsigma^{\alpha-1}[\chi_1(\varsigma)\chi_2(1-\varsigma)+\chi_1(1-\varsigma)\chi_2(\varsigma)]d\varsigma
\end{aligned}
$$

and the theorem has been established. □

Theorem 5. *Let $\mathfrak{S},\ \mathcal{H}\ : [\mu,\ \mu+\varphi(\omega,\mu)] \to \mathcal{K}_C^+$ be two LR-χ_1-pre-invex and LR-χ_2-pre-invex I-V·Fs, respectively, such that $\mathfrak{S}(z) = [\mathfrak{S}_*(z),\ \mathfrak{S}^*(z)]$ and $\mathcal{H}(z) = [\mathcal{H}_*(z),\ \mathcal{H}^*(z)]$ for all $z \in [\mu,\ \mu+\varphi(\omega,\mu)]$. If φ satisfies condition C and $\mathfrak{S} \times \mathcal{H} \in L\big([\mu,\ \mu+\varphi(\omega,\mu)], \mathcal{K}_C^+\big)$, then:*

$$\begin{array}{c}\frac{1}{\alpha\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)}\,\mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\times\mathcal{H}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\\ \leq_p \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}^{\alpha}_{\mu^+}\,\mathfrak{S}(\mu+\varphi(\omega,\mu))\times\mathcal{H}(\omega)+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\,\mathfrak{S}(\mu)\times\mathcal{H}(\mu)\right]\\ +\partial(\mu,\mu+\varphi(\omega,\mu))\int_0^1\left[\varsigma^{\alpha-1}+(1-\varsigma)^{\alpha-1}\right]\chi_1(\varsigma)\chi_2(1-\varsigma)d\varsigma\\ +\xi(\mu,\mu+\varphi(\omega,\mu))\int_0^1\left[\varsigma^{\alpha-1}+(1-\varsigma)^{\alpha-1}\right]\chi_1(1-\varsigma)\chi_2(1-\varsigma)d\varsigma,\end{array} \tag{24}$$

where $\xi(u,u+\varphi(\nu,u)) = \mathfrak{S}(u)\times\mathcal{H}(u)+\mathfrak{S}(\mu+\varphi(\omega,\mu))\times\mathcal{H}(\mu+\varphi(\omega,\mu))$, $\partial(\mu,\mu+\varphi(\omega,\mu)) = \mathfrak{S}(\mu)\times\mathcal{H}(\mu+\varphi(\omega,\mu))+\mathfrak{S}(\mu+\varphi(\omega,\mu))\times\mathcal{H}(\mu)$, *and* $\xi(\mu,\mu+\varphi(\omega,\mu)) = [\xi_*(\mu,\mu+\varphi(\omega,\mu)),\ \xi^*(\mu,\mu+\varphi(\omega,\mu))]$ *and* $\partial(\mu,\mu+\varphi(\omega,\mu)) = [\partial_*((\mu,\mu+\varphi(\omega,\mu))),\ \partial^*(\mu,\mu+\varphi(\omega,\mu))]$.

Proof. Consider $\mathfrak{S},\mathcal{H}\ :[\mu,\ \mu+\varphi(\omega,\mu)]\to\mathcal{K}_C^+$**.** are LR-$\chi_1$-pre-invex and LR-$\chi_2$-pre-invex *I-V·Fs*. Then, by hypothesis, we have:

$$\begin{array}{c}\mathfrak{S}_*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\times\mathcal{H}_*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\\ \mathfrak{S}^*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\times\mathcal{H}^*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\\ \leq\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\times\mathcal{H}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\\ +\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\times\mathcal{H}_*(\mu+\varsigma\varphi(\omega,\mu))\end{array}\right]\\ +\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu))\times\mathcal{H}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\\ +\mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu))\times\mathcal{H}_*(\mu+\varsigma\varphi(\omega,\mu))\end{array}\right]\\ \leq\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\times\mathcal{H}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\\ +\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\times\mathcal{H}^*(\mu+\varsigma\varphi(\omega,\mu))\end{array}\right]\\ +\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu))\times\mathcal{H}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\\ +\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu))\times\mathcal{H}^*(\mu+\varsigma\varphi(\omega,\mu))\end{array}\right],\\ \leq\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\times\mathcal{H}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\\ +\mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu))\times\mathcal{H}_*(\mu+\varsigma\varphi(\omega,\mu))\end{array}\right]\\ +\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}(\chi_1(\varsigma)\mathfrak{S}_*(\mu)+\chi_1(1-\varsigma)\mathfrak{S}_*(\mu+\varphi(\omega,\mu),))\\ \times(\chi_2(1-\varsigma)\mathcal{H}_*(\mu)+\chi_2(\varsigma)\mathcal{H}_*(\mu+\varphi(\omega,\mu),))\\ +(\chi_1(1-\varsigma)\mathfrak{S}_*(\mu)+\chi_1(\varsigma)\mathfrak{S}_*(\mu+\varphi(\omega,\mu),))\\ \times(\chi_2(\varsigma)\mathcal{H}_*(\mu)+\chi_2(1-\varsigma)\mathcal{H}_*(\mu+\varphi(\omega,\mu),))\end{array}\right]\\ \leq\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\times\mathcal{H}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\\ +\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu))\times\mathcal{H}^*(\mu+\varsigma\varphi(\omega,\mu))\end{array}\right]\\ +\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}(\chi_1(\varsigma)\mathfrak{S}^*(\mu)+\chi_1(1-\varsigma)\mathfrak{S}^*(\mu+\varphi(\omega,\mu),))\\ \times(\chi_2(1-\varsigma)\mathcal{H}^*(\mu)+\chi_2(\varsigma)\mathcal{H}^*(\mu+\varphi(\omega,\mu),))\\ +(\chi_1(1-\varsigma)\mathfrak{S}^*(\mu)+\chi_1(\varsigma)\mathfrak{S}^*(\mu+\varphi(\omega,\mu),))\\ \times(\chi_2(\varsigma)\mathcal{H}^*(\mu)+\chi_2(1-\varsigma)\mathcal{H}^*(\mu+\varphi(\omega,\mu),))\end{array}\right],\\ =\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\times\mathcal{H}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\\ +\mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu))\times\mathcal{H}_*(\mu+\varsigma\varphi(\omega,\mu))\end{array}\right]\\ +\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\{\chi_1(\varsigma)\chi_2(1-\varsigma)+\chi_1(1-\varsigma)\chi_2(\varsigma)\}\partial_*((\mu,\mu+\varphi(\omega,\mu)))\\ +\{\chi_1(\varsigma)\chi_2(\varsigma)+\chi_1(1-\varsigma)\chi_2(1-\varsigma)\}\xi_*((\mu,\mu+\varphi(\omega,\mu)))\end{array}\right]\\ =\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\times\mathcal{H}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\\ +\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu))\times\mathcal{H}^*(\mu+\varsigma\varphi(\omega,\mu))\end{array}\right]\\ +\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\{\chi_1(\varsigma)\chi_2(1-\varsigma)+\chi_1(1-\varsigma)\chi_2(\varsigma)\}\partial^*(\mu,\mu+\varphi(\omega,\mu))\\ +\{\chi_1(\varsigma)\chi_2(\varsigma)+\chi_1(1-\varsigma)\chi_2(1-\varsigma)\}\xi^*(\mu,\mu+\varphi(\omega,\mu))\end{array}\right].\end{array} \tag{25}$$

Taking multiplication of (25) with $\varsigma^{\alpha-1}$ and integrating over $(0,\ 1)$, we get:

$$\begin{aligned}
&\frac{1}{\alpha\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)}\mathfrak{S}_*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\times\mathcal{H}_*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\\
&\quad\le\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}_*(\mu+\varphi(\omega,\mu))\times\mathcal{H}_*(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}_*(\mu)\times\mathcal{H}_*(\mu)\right]\\
&\qquad+\partial_*(\mu,\mu+\varphi(\omega,\mu))\int_0^1\left[\varsigma^{\alpha-1}+(1-\varsigma)^{\alpha-1}\right]\chi_1(\varsigma)\chi_2(1-\varsigma)d\varsigma b\\
&\qquad+\xi_*(\mu,\mu+\varphi(\omega,\mu))\int_0^1\left[\varsigma^{\alpha-1}+(1-\varsigma)^{\alpha-1}\right]\chi_1(1-\varsigma)\chi_2(1-\varsigma)d\varsigma\\
&\frac{1}{\alpha\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)}\mathfrak{S}^*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\times\mathcal{H}^*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\\
&\quad\le\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}^*(\mu+\varphi(\omega,\mu))\times\mathcal{H}^*(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}^*(\mu)\times\mathcal{H}^*(\mu)\right]\\
&\qquad+\partial^*(\mu,\mu+\varphi(\omega,\mu))\int_0^1\left[\varsigma^{\alpha-1}+(1-\varsigma)^{\alpha-1}\right]\chi_1(\varsigma)\chi_2(1-\varsigma)d\varsigma\\
&\qquad+\xi^*(\mu,\mu+\varphi(\omega,\mu))\int_0^1\left[\varsigma^{\alpha-1}+(1-\varsigma)^{\alpha-1}\right]\chi_1(1-\varsigma)\chi_2(1-\varsigma)d\varsigma.
\end{aligned}$$

It follows that:

$$\begin{aligned}
&\frac{1}{\alpha\chi_1\left(\frac{1}{2}\right)\chi_2\left(\frac{1}{2}\right)}\ \mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\times\mathcal{H}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\\
&\le_p\ \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\Big[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}(\mu+\varphi(\omega,\mu))\times\mathcal{H}(\mu+\varphi(\omega,\mu))\\
&\qquad+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}(\mu)\times\mathcal{H}(\mu)\Big]\\
&+\partial(\mu,\mu+\varphi(\omega,\mu))\int_0^1\left[\varsigma^{\alpha-1}+(1-\varsigma)^{\alpha-1}\right]\chi_1(\varsigma)\chi_2(1-\varsigma)d\varsigma\\
&+\xi(\mu,\mu+\varphi(\omega,\mu))\int_0^1\left[\varsigma^{\alpha-1}+(1-\varsigma)^{\alpha-1}\right]\chi_1(1-\varsigma)\chi_2(1-\varsigma)d\varsigma
\end{aligned}$$

Hence, the required result. □

Now, we present the successful reformative version of the generalized version of interval *H-H* inequality on invex set for LR-χ-pre-invex *I-V·F* via interval Riemann–Liouville fractional integral.

Theorem 6. *(Second fractional H-H Fejér inequality) Let* $\mathfrak{S}:[\mu,\ \mu+\varphi(\omega,\mu)]\to\mathcal{K}_C^+$ *be a LR-*χ*-pre-invex I-V·F with* $\mu<\mu+\varphi(\omega,\mu)$*, such that* $\mathfrak{S}(z)=[\mathfrak{S}_*(z),\ \mathfrak{S}^*(z)]$ *for all* $z\in[\mu,\ \mu+\varphi(\omega,\mu)]$*. If* $\mathfrak{S}\in L\left([\mu,\ \mu+\varphi(\omega,\mu)],\mathcal{K}_C^+\right)$ *and* $\mathcal{D}:[\mu,\ \mu+\varphi(\omega,\mu)]\to\mathbb{R}$, $\mathcal{D}(z)\ge 0$, *symmetric with respect to* $\frac{2\mu+\varphi(\omega,\mu)}{2}$*, then:*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}\mathcal{D}(\mu)\right]\\
&\le_p\ (\mathfrak{S}(\mu)+\mathfrak{S}(\mu+\varphi(\omega,\mu)))\int_0^1\ \varsigma^{\alpha-1}[\chi(\varsigma)+\chi(1-\varsigma)]\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\ .
\end{aligned}\tag{26}$$

If $\mathfrak{S}$ *is pre-incave I-V·F, then inequality (26) is reversed.*

Proof. Let $\mathfrak{S}$ be a LR-χ-pre-invex *I-V·F* and $\varsigma^{\alpha-1}\mathcal{D}(\mu+(1-\varsigma)\varphi(\omega,\mu))\ge 0$. Then, we have:

$$\begin{aligned}
&\varsigma^{\alpha-1}\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\mathcal{D}(\mu+(1-\varsigma)\varphi(\omega,\mu))\\
&\quad\le\varsigma^{\alpha-1}(\chi(\varsigma)\mathfrak{S}_*(\mu)+\chi(1-\varsigma)\mathfrak{S}_*(\mu+\varphi(\omega,\mu)))\mathcal{D}(\mu+(1-\varsigma)\varphi(\omega,\mu))\\
&\varsigma^{\alpha-1}\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\mathcal{D}(\mu+(1-\varsigma)\varphi(\omega,\mu))\\
&\quad\le\varsigma^{\alpha-1}(\chi(\varsigma)\mathfrak{S}^*(\mu)+\chi(1-\varsigma)\mathfrak{S}^*(\mu+\varphi(\omega,\mu)))\mathcal{D}(\mu+(1-\varsigma)\varphi(\omega,\mu)),
\end{aligned}\tag{27}$$

and:

$$\begin{aligned}
&\varsigma^{\alpha-1}\mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))\\
&\quad\le\varsigma^{\alpha-1}(\chi(1-\varsigma)\mathfrak{S}_*(\mu)+\chi(\varsigma)\mathfrak{S}_*(\mu+\varphi(\omega,\mu)))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))\\
&\varsigma^{\alpha-1}\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))\\
&\quad\le\varsigma^{\alpha-1}(\chi(1-\varsigma)\mathfrak{S}^*(\mu)+\chi(\varsigma)\mathfrak{S}^*(\mu+\varphi(\omega,\mu)))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu)).
\end{aligned}\tag{28}$$

After adding (27) and (28), and integrating over $[0,\ 1]$, we get:

$$\begin{array}{l}\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\mathcal{D}(\mu+(1-\varsigma)\varphi(\omega,\mu))d\varsigma \\ \quad +\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*(\mu+\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma \\ \leq \int_0^1 \left[\begin{array}{c} \varsigma^{\alpha-1}\mathfrak{S}_*(\mu)\{\chi(\varsigma)\mathcal{D}(\mu+(1-\varsigma)\varphi(\omega,\mu))+\chi(1-\varsigma)\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))\} \\ +\varsigma^{\alpha-1}\mathfrak{S}_*(\mu+\varphi(\omega,\mu))\{\chi(1-\varsigma)\mathcal{D}(\mu+(1-\varsigma)\varphi(\omega,\mu))+\chi(\varsigma)\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))\}\end{array}\right]d\varsigma, \\ \quad = \mathfrak{S}_*(\mu)\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma)+\chi(1-\varsigma)]D(\mu+(1-\varsigma)\varphi(\omega,\mu))\, d\varsigma \\ \quad +\mathfrak{S}_*(\mu+\varphi(\omega,\mu))\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma)+\chi(1-\varsigma)]D(\mu+\varsigma\varphi(\omega,\mu))\, d\varsigma, \\ \int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma \\ \quad +\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\mathcal{D}(\mu+(1-\varsigma)\varphi(\omega,\mu))d\varsigma \\ \leq \int_0^1 \left[\begin{array}{c} \varsigma^{\alpha-1}\mathfrak{S}^*(\mu)\{\chi(\varsigma)\mathcal{D}(\mu+(1-\varsigma)\varphi(\omega,\mu))+\chi(1-\varsigma)\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))\} \\ +\varsigma^{\alpha-1}\mathfrak{S}^*(\mu+\varphi(\omega,\mu))\{\chi(1-\varsigma)\mathcal{D}(\mu+(1-\varsigma)\varphi(\omega,\mu))+\chi(\varsigma)\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))\}\end{array}\right]d\varsigma, \\ \quad = \mathfrak{S}^*(\mu)\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma)+\chi(1-\varsigma)]D(\mu+(1-\varsigma)\varphi(\omega,\mu))\, d\varsigma \\ \quad +\mathfrak{S}^*(\mu+\varphi(\omega,\mu))\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma)+\chi(1-\varsigma)]D(\mu+\varsigma\varphi(\omega,\mu))\, d\varsigma.\end{array} \tag{29}$$

Taking the right hand side of inequality (29), we have:

$$\begin{array}{c}\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma \\ +\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma \\ = \frac{1}{(\varphi(\omega,\mu))^{\alpha}}\int_{\mu}^{\mu+\varphi(\omega,\mu)}(z-\mu)^{\alpha-1}\mathfrak{S}_*(2\mu+\varphi(\omega,\mu)-z)\mathcal{D}(z)dz \\ +\frac{1}{(\varphi(\omega,\mu))^{\alpha}}\int_{\mu}^{\mu+\varphi(\omega,\mu)}(z-\mu)^{\alpha-1}\mathfrak{S}_*(z)\mathcal{D}(z)dz \\ = \frac{1}{(\varphi(\omega,\mu))^{\alpha}}\int_{\mu}^{\mu+\varphi(\omega,\mu)}(\mu+\varphi(\omega,\mu)-z)^{\alpha-1}\mathfrak{S}_*(z)\mathcal{D}(2\mu+\varphi(\omega,\mu)-z)dz \\ +\frac{1}{(\varphi(\omega,\mu))^{\alpha}}\int_{\mu}^{\mu+\varphi(\omega,\mu)}(z-\mu)^{\alpha-1}\mathfrak{S}_*(z)\mathcal{D}(z)dz \\ = \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}_*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}_*\mathcal{D}(\mu)\right], \\ \int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma \\ +\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma \\ = \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}^*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}^*\mathcal{D}(\mu)\right]\end{array} \tag{30}$$

From (30), we have:

$$\begin{array}{l}\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}_*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}_*\mathcal{D}(\mu)\right] \\ \quad \leq (\mathfrak{S}_*(\mu)+\mathfrak{S}_*(\mu+\varphi(\omega,\mu)))\int_0^1 \varsigma^{\alpha-1}\left[\begin{array}{c}\chi(\varsigma) \\ +\chi(1-\varsigma)\end{array}\right]D(\mu+\varsigma\varphi(\omega,\mu)) \\ \frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}^*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}^*\mathcal{D}(\mu)\right] \\ \quad \leq (\mathfrak{S}^*(\mu)+\mathfrak{S}^*(\mu+\varphi(\omega,\mu)))\int_0^1 \varsigma^{\alpha-1}\left[\begin{array}{c}\chi(\varsigma) \\ +\chi(1-\varsigma)\end{array}\right]D(\mu+\varsigma\varphi(\omega,\mu)),\end{array}$$

that is:

$$\begin{array}{l}\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}_*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}_*\mathcal{D}(\mu),\ \mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}^*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}^*\mathcal{D}(\mu)\right] \\ \leq_p [\mathfrak{S}_*(\mu)+\mathfrak{S}_*(\mu+\varphi(\omega,\mu)),\ \mathfrak{S}^*(\mu)+\mathfrak{S}^*(\mu+\varphi(\omega,\mu))]\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma)+\chi(1-\varsigma)]D(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\end{array}$$

Hence,

$$\begin{array}{c}\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}\mathcal{D}(\mu)\right] \\ \leq_p (\mathfrak{S}(\mu)+\mathfrak{S}(\omega))\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma)+\chi(1-\varsigma)]D(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\end{array}$$

□

Now, we propose the first *H-H* Fejér inequality for LR--pre-invex *I-V·F* using the interval Riemann–Liouville fractional integral. Then we will prove the validity of Theorem 6 and Theorem 7 with a nontrivial Example 3.

Theorem 7. *(First fractional H-H Fejér inequality) Let $\mathfrak{S} : [\mu,\ \mu + \varphi(\omega, \mu)] \to \mathcal{K}_C^+$ be a LR--pre-invex I-V·F such that $\mathfrak{S}(z) = [\mathfrak{S}_*(z),\ \mathfrak{S}^*(z)]$ for all $z \in [\mu,\ \mu + \varphi(\omega, \mu)]$. Let $\mathfrak{S} \in L([\mu,\ \mu + \varphi(\omega, \mu)], \mathcal{K}_C^+)$ and $\mathcal{D} : [\mu,\ \mu + \varphi(\omega, \mu)] \to \mathbb{R}$, $\mathcal{D}(z) \geq 0$, symmetric with respect to $\frac{2\mu+\varphi(\omega,\mu)}{2}$. If φ satisfies condition C, then:*

$$\begin{aligned}\frac{1}{2\chi\left(\frac{1}{2}\right)}\mathfrak{S}\left(\tfrac{2\mu+\varphi(\omega,\mu)}{2}\right)&\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathcal{D}(\mu)\right]\\ &\leq_p \left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}\mathcal{D}(\mu)\right]\end{aligned} \tag{31}$$

If $\mathfrak{S}$ is pre-incave I-V·F, then inequality (31) is reversed.

Proof. Since $\mathfrak{S}$ is a LR-χ-pre-invex *I-V·F* then, we have:

$$\begin{aligned}\mathfrak{S}_*\left(\tfrac{2\mu+\varphi(\omega,\mu)}{2}\right)&\leq \chi\left(\tfrac{1}{2}\right)(\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))+\mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu)))\\ \mathfrak{S}^*\left(\tfrac{2\mu+\varphi(\omega,\mu)}{2}\right)&\leq \chi\left(\tfrac{1}{2}\right)(\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))+\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu))).\end{aligned} \tag{32}$$

Since $\mathcal{D}(\mu + (1-\varsigma)\varphi(\omega, \mu)) = \mathcal{D}(\mu + \varsigma\varphi(\omega, \mu))$, then by multiplying (32) by $\varsigma^{\alpha-1}\mathcal{D}(\mu + \varsigma\varphi(\omega, \mu))$ and integrate it with respect to ς over $[0,\ 1]$, we obtain:

$$\begin{aligned}&\mathfrak{S}_*\left(\tfrac{2\mu+\varphi(\omega,\mu)}{2}\right)\int_0^1 \varsigma^{\alpha-1}\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\\ &\quad\leq \chi\left(\tfrac{1}{2}\right)\left(\begin{array}{c}\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\\ +\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\end{array}\right),\\ &\mathfrak{S}^*\left(\tfrac{2\mu+\varphi(\omega,\mu)}{2}\right)\int_0^1 \varsigma^{\alpha-1}\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\\ &\quad\leq \chi\left(\tfrac{1}{2}\right)\left(\begin{array}{c}\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\\ +\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\end{array}\right).\end{aligned} \tag{33}$$

Let $x = \mu + \varsigma\varphi(\omega, \mu)$. Then, on the right hand side of inequality (33), we have:

$$\begin{aligned}&\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*(\mu+(1-\varsigma)\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\\ &\quad+\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}_*(\mu+\varsigma\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\\ &\quad=\frac{1}{(\varphi(\omega,\mu))^{\alpha}}\int_{\mu}^{\mu+\varphi(\omega,\mu)}(z-\mu)^{\alpha-1}\mathfrak{S}_*(2\mu+\varphi(\omega,\mu)-z)\mathcal{D}(z)dz\\ &\quad+\frac{1}{(\varphi(\omega,\mu))^{\alpha}}\int_{\mu}^{\mu+\varphi(\omega,\mu)}(z-\mu)^{\alpha-1}\mathfrak{S}_*(z)\mathcal{D}(z)dz\\ &\quad=\frac{1}{(\varphi(\omega,\mu))^{\alpha}}\int_{\mu}^{\mu+\varphi(\omega,\mu)}(z-\mu)^{\alpha-1}\mathfrak{S}_*(z)\mathcal{D}(\mu-\omega-z)dz\\ &\quad+\frac{1}{(\varphi(\omega,\mu))^{\alpha}}\int_{\mu}^{\mu+\varphi(\omega,\mu)}(z-\mu)^{\alpha-1}\mathfrak{S}_*(z)\mathcal{D}(z)dz\\ &\quad=\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}_*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}_*\mathcal{D}(\mu)\right],\\ &\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*(\mu+(1-\varsigma)\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\\ &\quad+\int_0^1 \varsigma^{\alpha-1}\mathfrak{S}^*(\mu+\varsigma\varphi(\omega,\mu))\mathcal{D}(\mu+\varsigma\varphi(\omega,\mu))d\varsigma\\ &\quad=\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^{\alpha}}\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}^*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}^*\mathcal{D}(\mu)\right].\end{aligned} \tag{34}$$

Then from (34), we have:

$$\begin{aligned}&\frac{1}{2\chi\left(\frac{1}{2}\right)}\mathfrak{S}_*\left(\tfrac{2\mu+\varphi(\omega,\mu)}{2}\right)\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathcal{D}(\mu)\right]\\ &\qquad\leq\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}_*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}_*\mathcal{D}(\mu)\right]\\ &\frac{1}{2\chi\left(\frac{1}{2}\right)}\mathfrak{S}^*\left(\tfrac{2\mu+\varphi(\omega,\mu)}{2}\right)\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathcal{D}(\mu)\right]\\ &\qquad\leq\left[\mathcal{I}_{\mu^+}^{\alpha}\ \mathfrak{S}^*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}_{\mu+\varphi(\omega,\mu)^-}^{\alpha}\ \mathfrak{S}^*\mathcal{D}(\mu)\right],\end{aligned}$$

from which, we have:

$$\frac{1}{2\chi\left(\frac{1}{2}\right)}\left[\mathfrak{S}_*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right),\ \mathfrak{S}^*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\right]\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathcal{D}(\mu)\right]$$
$$\leq_p\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathfrak{S}_*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}_*\mathcal{D}(\mu),\ \mathcal{I}^{\alpha}_{\mu^+}\ \mathfrak{S}^*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}^*\mathcal{D}(\mu)\right],$$

and it follows that:

$$\frac{1}{2\chi\left(\frac{1}{2}\right)}\mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathcal{D}(\mu)\right]$$
$$\leq_p\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathfrak{S}\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}\mathcal{D}(\mu)\right]$$

This completes the proof. □

Remark 5. *If $\mathcal{D}(z)=1$, then from (26) and (31), we get Theorem 3.*

If $\chi(\varsigma)=\varsigma$, then from (26) and (31), we achieve the following coming inequality, see [42]*:*

$$\begin{aligned}\mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)&\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathcal{D}(\mu)\right]\\ &\leq_p\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathfrak{S}\mathcal{D}(\mu+\{\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}\mathcal{D}(\mu)\right]\\ &\leq_p\frac{\mathfrak{S}(\mu)+\mathfrak{S}(\mu+\varphi(\omega,\mu))}{2}\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathcal{D}(\mu)\right]\\ &\leq_p\frac{\mathfrak{S}(\mu)+\mathfrak{S}(\omega)}{2}\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathcal{D}(\mu)\right]\end{aligned}\tag{35}$$

Let $\chi(\varsigma)=\varsigma$ and $\alpha=1$. Then, from (26) and (31), we achieve the following coming inequality:

$$\mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\leq_p\frac{1}{\int_{\mu}^{\mu+\varphi(\omega,\mu)}\mathcal{D}(z)dz}\ (FR)\int_{\mu}^{\mu+\varphi(\omega,\mu)}\mathfrak{S}(z)\mathcal{D}(z)dz\leq_p\frac{\mathfrak{S}(\mu)+\mathfrak{S}(\omega)}{2}\tag{36}$$

If $\mathfrak{S}_*(z)=\mathfrak{S}^*(z)$ and $\chi(\varsigma)=\varsigma$, then from (26) and (31), we achieve the following coming inequality:

$$\begin{aligned}\mathfrak{S}\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathcal{D}(\omega)+\mathcal{I}^{\alpha}_{\omega^-}\ \mathcal{D}(\mu)\right]&\leq\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathfrak{S}\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}\mathcal{D}(\mu)\right]\\ &\leq\frac{\mathfrak{S}(\mu)+\mathfrak{S}(\mu+\varphi(\omega,\mu))}{2}\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathcal{D}(\mu)\right]\\ &\leq\frac{\mathfrak{S}(\mu)+\mathfrak{S}(\omega)}{2}\left[\mathcal{I}^{\alpha}_{\mu^+}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^{\alpha}_{\mu+\varphi(\omega,\mu)^-}\ \mathcal{D}(\mu)\right]\end{aligned}\tag{37}$$

If $\mathfrak{S}_*(z)=\mathfrak{S}^*(z)$ and $\alpha=1$ and $\chi(\varsigma)=\varsigma$, then from (26) and (31), we acquire the classical H-H Fejér inequality.

If $\mathfrak{S}_*(z)=\mathfrak{S}^*(z)$ and $\mathcal{D}(z)=\alpha=1$ and $\chi(\varsigma)=\varsigma$, then from (26) and (31), we acquire the classical H-H inequality.

Example 3. *We consider the I-V·F $\mathfrak{S}:[0,\ \varphi(2,0)]\to\mathcal{K}^+_C$ defined by, $\mathfrak{S}(z)=[1,2]\left(2-\sqrt{z}\right)$. Since end-point functions $\mathfrak{S}_*(z)$, $\mathfrak{S}^*(z)$ are LR-χ-pre-invex functions, then $\mathfrak{S}(z)$ is LR-χ-pre-invex I-V·F.*

If:

$$\mathcal{D}(z)=\begin{cases}\sqrt{z}, & \sigma\in[0,1],\\ \sqrt{2-z}, & \sigma\in(1,\ 2],\end{cases}$$

then $\mathcal{D}(2-z)=\mathcal{D}(z)\geq 0$, for all $z\in[0,\ 2]$. Since $\mathfrak{S}_*(z)=\left(2-\sqrt{z}\right)$ and $\mathfrak{S}^*(z)=2\left(2-\sqrt{z}\right)$. If $\chi(\varsigma)=\varsigma$ and $\alpha=\frac{1}{2}$, then we compute the following:

$$\begin{aligned}\frac{\mathfrak{S}_*(\mu)+\mathfrak{S}_*(\mu+\varphi(\omega,\mu))}{2}\int_0^1\ \varsigma^{\alpha-1}[\chi(\varsigma)+\chi(1-\varsigma)]D(\mu+\varsigma\varphi(\omega,\mu))&=\frac{\pi}{\sqrt{2}}\left(\frac{4-\sqrt{2}}{2}\right),\\ \frac{\mathfrak{S}^*(\mu)+\mathfrak{S}^*(\mu+\varphi(\omega,\mu))}{2}\int_0^1\ \varsigma^{\alpha-1}[\chi(\varsigma)+\chi(1-\varsigma)]D(\mu+\varsigma\varphi(\omega,\mu))&=\frac{\pi}{\sqrt{2}}\left(4-\sqrt{2}\right),\end{aligned}\tag{38}$$

$$
\begin{array}{r}
\frac{\mathfrak{S}_*(\mu)+\mathfrak{S}_*(\mu+\varphi(\mu+\varphi(\omega,\mu),\mu))}{2}\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma)+\chi(1-\varsigma)]D(\mu+\varsigma\varphi(\omega,\mu)) = \frac{\pi}{\sqrt{2}}\left(\frac{4-\sqrt{2}}{2}\right), \\
\frac{\mathfrak{S}^*(\mu)+\mathfrak{S}^*(\mu+\varphi(\omega,\mu))}{2}\int_0^1 \varsigma^{\alpha-1}[\chi(\varsigma)+\chi(1-\varsigma)]D(\mu+\varsigma\varphi(\omega,\mu)) = \frac{\pi}{\sqrt{2}}\left(4-\sqrt{2}\right), \\
\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}^\alpha_{\mu^+}\ \mathfrak{S}_*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^\alpha_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}_*\mathcal{D}(\mu)\right] = \frac{1}{\sqrt{\pi}}\left(2\pi+\frac{4-8\sqrt{2}}{3}\right), \\
\frac{\Gamma(\alpha)}{(\varphi(\omega,\mu))^\alpha}\left[\mathcal{I}^\alpha_{\mu^+}\ \mathfrak{S}^*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^\alpha_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}^*\mathcal{D}(\mu)\right] = \frac{2}{\sqrt{\pi}}\left(2\pi+\frac{4-8\sqrt{2}}{3}\right).
\end{array} \tag{39}
$$

From (38) and (39), we have:

$$
\frac{1}{\sqrt{\pi}}\left[\left(2\pi+\frac{4-8\sqrt{2}}{3}\right),\ 2\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)\right] \quad \leq_p \quad \frac{\pi}{\sqrt{2}}\left[\frac{4-\sqrt{2}}{2},\ 4-\sqrt{2}\right].
$$

Hence, Theorem 6 is verified.

For Theorem 7, we have:

$$
\begin{array}{l}
\mathcal{I}^\alpha_{\mu^+}\ \mathfrak{S}_*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^\alpha_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}_*\mathcal{D}(\mu) \\
\quad = \frac{1}{\sqrt{\pi}}\int_0^2 (2-z)^{\frac{-1}{2}}\ \mathcal{D}(z)\left(2-\sqrt{z}\right)dz + \frac{1}{\sqrt{\pi}}\int_0^2 (z)^{\frac{-1}{2}}\ \mathcal{D}(z)\left(2-\sqrt{z}\right)dz \\
\quad = \frac{1}{\sqrt{\pi}}\left(\pi+\frac{8-8\sqrt{2}}{3}\right)+\frac{1}{\sqrt{\pi}}\left(\pi-\frac{4}{3}\right) = \frac{1}{\sqrt{\pi}}\left(2\pi+\frac{4-8\sqrt{2}}{3}\right) \\
\mathcal{I}^\alpha_{\mu^+}\ \mathfrak{S}^*\mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^\alpha_{\mu+\varphi(\omega,\mu)^-}\ \mathfrak{S}^*\mathcal{D}(\mu) \\
\quad = \frac{2}{\sqrt{\pi}}\int_0^2 (2-z)^{\frac{-1}{2}}\ \mathcal{D}(z)\left(2-\sqrt{z}\right)dz + \frac{2}{\sqrt{\pi}}\int_0^2 (z)^{\frac{-1}{2}}\ \mathcal{D}(z)\left(2-\sqrt{z}\right)dz \\
\quad = \frac{2}{\sqrt{\pi}}\left(\pi+\frac{8-8\sqrt{2}}{3}\right)+\frac{2}{\sqrt{\pi}}\left(\pi-\frac{4}{3}\right) = \frac{2}{\sqrt{\pi}}\left(2\pi+\frac{4-8\sqrt{2}}{3}\right).
\end{array} \tag{40}
$$

$$
\begin{array}{r}
\frac{1}{2\chi\left(\frac{1}{2}\right)}\mathfrak{S}_*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\left[\mathcal{I}^\alpha_{\mu^+}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^\alpha_{\mu+\varphi(\omega,\mu)^-}\ \mathcal{D}(\mu)\right] = \sqrt{\pi}, \\
\frac{1}{2\chi\left(\frac{1}{2}\right)}\mathfrak{S}^*\left(\frac{2\mu+\varphi(\omega,\mu)}{2}\right)\left[\mathcal{I}^\alpha_{\mu^+}\ \mathcal{D}(\mu+\varphi(\omega,\mu))+\mathcal{I}^\alpha_{\mu+\varphi(\omega,\mu)^-}\ \mathcal{D}(\mu)\right] = 2\sqrt{\pi}.
\end{array} \tag{41}
$$

4. Conclusions

We have proposed the class of LR-χ-pre-invexity for *I-V·Fs*. By using this class, we have presented several interval *H-H* inequalities and interval *H-H* Fejér inequalities using interval Riemann–Liouville fractional integral operators. Useful examples that illustrate the applicability of theory developed in this study are also presented. In future, we intend to discuss generalized LR-χ-pre-invex *I-V·Fs*. We hope that this concept will be helpful for other authors to play their roles in different fields of sciences.

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