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On Groups in Which Many Automorphisms Are Cyclic

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Abstract: Let G be a group. An automorphism α of G is said to be a cyclic automorphism if the subgroup $\langle x, x^\alpha \rangle$ is cyclic for every element x of G. In [F. de Giovanni, M.L. Newell, A. Russo: On a class of normal endomorphisms of groups, J. Algebra and its Applications 13, (2014), 6pp] the authors proved that every cyclic automorphism is central, namely, that every cyclic automorphism acts trivially on the factor group $G/Z(G)$. In this paper, the class FW of groups in which every element induces by conjugation a cyclic automorphism on a (normal) subgroup of finite index will be investigated.

Keywords: FC-groups; FW-groups; cyclic automorphisms; cyclicizer

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1. Introduction

Let G be a group. Following the work in [1], an automorphism α of G is called a *cyclic automorphism* if the subgroup $\langle x, x^\alpha \rangle$ is cyclic for every element x of G. Clearly, any *power automorphism* of G (i.e., an automorphism which maps every subgroup onto itself) is cyclic; however, the multiplication by a rational number greater than 1 is a cyclic automorphism of the additive group of rational numbers which is not a power automorphism. Finally, it is easy to show that any cyclic automorphism of a periodic group is a power automorphism.

In [1], it was proved that any cyclic automorphism of a group G is *central*, i.e., it acts trivially on the factor group $G/Z(G)$. Notice that this result is an extension to cyclic automorphisms of a renowned theorem by Cooper [2] for power automorphisms. It is not difficult to prove that the set $CAut(G)$ of all cyclic automorphisms of G forms a normal abelian subgroup of the automorphism group $Aut(G)$ of G. In [3], the structure of $CAut(G)$ has been investigated in detail and some well-known properties of power automorphisms (see in [2]) has been extended to cyclic automorphisms. Moreover, the groups in which every automorphism is cyclic have been characterized there.

In the following, we will say that an element g of a group G induces by conjugation a *weakly cyclic automorphism* of G if there exists a normal subgroup $W(g)$ of G such that the index $|G : W(g)|$ is finite and the subgroup $\langle x, x^g \rangle$ is cyclic for each element x of $W(g)$. Let g_1 and g_2 be elements of G inducing weakly cyclic automorphisms and put $W = W(g_1) \cap W(g_2)$. If x is an element of W, then $\langle x, x^{g_1} \rangle = \langle y \rangle$ for some $y \in W$, and so $\langle x, x^{g_1} \rangle$ is contained in the cyclic subgroup $\langle y, y^{g_2} \rangle$. It follows that $g_1 g_2$ induces a weakly cyclic automorphism of G and hence the set $FW(G)$ of all elements of G inducing by conjugation weakly cyclic automorphisms of G is a subgroup of G. Moreover, if g is an element of FW, x is an element of $W(g)$ and y is an element of G, we have that $\langle x^{y^{-1}}, x^{y^{-1}g} \rangle^y$ is again a cyclic subgroup of $W(g)$, so that $FW(G)$ is a normal subgroup of G. We name this subgroup the FW-*centre* of G. A group which coincides with its FW-center will be called an FW-*group*.

Recall that the *cyclic norm* $C(G)$ of a group G is defined as the intersection of the normalizers of every maximal locally cyclic subgroup of G. By [3], Lemma 2.1, any cyclic automorphism of G fixes all maximal locally cyclic subgroups of G. It follows that $C(G)$ coincides with the set of all elements of G inducing cyclic automorphisms of G. In particular, $C(G)$ is a subgroup of $FW(G)$.

In the first part of the article, the class $\mathcal{FW}$ of groups in which every element induces by conjugation a weakly cyclic automorphism will be investigated. In particular, it will be proved that the class $\mathcal{FW}$ coincides with the class $\mathcal{FP}$ recently studied by De Falco et al. [4]. Recall here that a group G is said to be an FP-*group* if every element of G induces by conjugation a power automorphism on some subgroup of finite index of G. Clearly, the groups with finitely many conjugacy classes (the so-called FC-*groups*) are FP-groups, while every FP-group is an FW-group. The consideration of the infinite dihedral group D_∞ shows that there are FP-groups which are not FC-groups.

Let G be a group and denote by $Cyc(G)$ the set of all elements x of G such that $\langle x, y\rangle$ is cyclic for every y in G. It is easy to show that $Cyc(G)$ is a central, characteristic subgroup of G called the *cyclicizer* of G (see [5,6]). Clearly, $Cyc(G)$ is locally cyclic and hence every automorphism of G induces a cyclic automorphism on $Cyc(G)$. In the last part of the article, groups with non-trivial cyclicizer will be investigated extending to the infinite case some results in [6–8]. In particular, it is shown that any torsion-free or primary generalized soluble group with non-trivial cyclicizer is an FW-group. Moreover, the well-known characterization of finite p-groups with only one subgroup of order p (see, for instance, [9], 5.3.6) will be extended to locally finite groups. Finally, it is proved that the factor group $G/Cyc(G)$ is finite if and only if G has a finite covering of locally cyclic subgroups.

Most of our notation is standard and can be found in [10].

2. *FW*-Groups

Our first result is an easy remark concerning cyclic automorphisms of finite order.

Lemma 1. *Let G be a group. Every periodic cyclic automorphism of G is a power automorphism.*

Proof. Let α be a cyclic automorphism of G, let g be an element of G, and consider a maximal locally cyclic subgroup M of G such that $g \in M$. As one can easily see that $M^\alpha = M$ (see, for instance, in [3], Lemma 2.1), then the normal closure $\langle x\rangle^{\langle\alpha\rangle}$ is locally cyclic and hence there exists an element x of G such that $\langle g\rangle^{\langle\alpha\rangle} = \langle x\rangle$. Clearly, $\langle x\rangle^{\langle\alpha\rangle} = \langle x\rangle$ and we may suppose that g has infinite order. Therefore, $x^\alpha = x^{-1}$ and $g^\alpha = g^{-1}$. Thus, α induces a power automorphism on G. □

Let G be a group. A normal subgroup W of G is said to be *weakly central* if every element of G induces by conjugation a cyclic automorphism of W. Clearly, if G contains a weakly central subgroup of finite index, then G is an FW-group.

Proposition 1. *Let G be a group. If W is a weakly central subgroup of finite index of G, then every subgroup of W is normal in G. In particular, G is an FP-group.*

Proof. First, assume that every inner automorphism of G is cyclic. Then, G coincides with its cyclic norm and hence every maximal locally cyclic subgroup of G is normal. Let g be an element of G and consider a maximal locally cyclic subgroup M containing g. As G is an FC-group (see [3], Theorem 4.2), then the normal closure $\langle g\rangle^G$ of g in G is a finitely generated subgroup of M. Therefore, $\langle g\rangle$ is normal in G and thus G is a Dedekind group.

The above argument shows that W is a Dedekind group. Since a cyclic automorphism of a periodic group is a power automorphism (see in [3], Lemma 2.3), we may suppose that W is abelian. It follows that the factor group $G/C_G(W)$ is finite and hence every element g of G induces on W a cyclic automorphism of finite order. The statement now follows from Lemma 1. □

Corollary 1. *Let G be a group all of whose inner automorphisms are cyclic automorphisms. Then G is a Dedekind group.*

Let G be a group. We denote here with $FP(G)$ the *FP-centre* of G, namely the subgroup of all elements of G inducing by conjugation power automorphisms on some subgroup of finite index of G. Clearly, $FP(G)$ is a subgroup of $FW(G)$.

Recall that a non-periodic group is said to be *weak* if it can be generated by its elements of infinite order, while it is said to be *strong* otherwise. In particular, all non-periodic abelian groups are weak.

Theorem 1. *Let G be a group. Then, FW-centre and FP-centre of G coincide.*

Proof. As the FP-centre of G is a subgroup of $FW(G)$, we just have to show that every element of G inducing a weakly cyclic automorphism of G induces a weakly power automorphism of G. Therefore, let g be an element of $FW(G)$ and let $W(g)$ be a normal subgroup of finite index of G such that g induces on $W(g)$ a cyclic automorphism. By Lemma 1, we may assume that g induces an aperiodic automorphism on $W(g)$. Clearly, $g^n \in W(g)$ for some positive integer n and $g^n \neq 1$. If $W(g)$ is weak, then g acts universally on $W(g)$ (see [3], Theorem 3.5) and then $[W(g), g] = \{1\}$ as g^n belongs to $W(g)$, so we may further assume that $W(g)$ is strong. If we let W be the subgroup of G generated by every element of infinite order of G, by Theorem 3.5 in [3], g fixes W and G/W elementwise. Let now x be an element of finite order of $W(g)$ and let m be the order of x. As $\langle x \rangle$ and $\langle x^g \rangle$ are both subgroups of order m of the cyclic group $\langle x, x^g \rangle$, they coincide and this shows that g acts as a power automorphism on every finite cyclic subgroup of $W(g)$. As g centralizes every element of infinite order of G, it follows that g induces a power automorphism on $W(g)$ and our thesis is proved. $\square$

Corollary 2. *Let G be a group. Then, G is an FW-group if and only if G is an FP-group.*

Recall that a subgroup X of a group G is said to be *pronormal* if the subgroups X and X^g are conjugate in the subgroup $\langle X, X^g \rangle$ for all elements g of G. As any subnormal and pronormal subgroup of a group is normal, it follows that a group all of whose subgroups are pronormal is a $\overline{T}$*-group* (i.e., a group in which normality is a transitive relation in every subgroup). However, the converse is false, as an example due to Kuzennyi and Subbotin [11] shows. We point out incidentally that in the universe of groups with no infinite simple sections the property $\overline{T}$ for a group G is equivalent to saying that every subgroup of G is *weakly normal* (see [12]). A tool which is useful to control pronormal subgroups of a group G is the *pronorm* of G, which is defined as the set $P(G)$ of all elements g of G such that X and X^g are conjugate in $\langle X, X^g \rangle$ for any subgroup X of G. The consideration of the alternating group A_5 shows that the pronorm of a group need not be in general a subgroup. On the other hand, the pronorm of a $\overline{T}$-group G with no infinite simple sections is a subgroup of G which coincides with the set $L(G)$ consisting of all elements $g \in G$ such that, if H is a subgroup of G, then g normalizes a subgroup of finite index of H (see [13], Theorem 2.2). The last result of this section shows in particular that a $\overline{T}$-group G with no infinite simple sections has all subgroups pronormal whenever G belongs to the class $\mathcal{FW}$.

Corollary 3. *Let G be a group. Then, $FW(G)$ is contained in $L(G)$. In particular, if G is a $\overline{T}$-group with no infinite simple sections, $FW(G)$ is a subgroup of $P(G)$.*

Proof. By Theorem 1, for every element g of $FW(G)$ we may find a normal subgroup $W(g)$ of finite index of G on which g acts as a power automorphism. If we let H be a subgroup of G, then the subgroup $H \cap W(g)$ of $W(g)$ is normalized by g, has finite index in H and this proves our claim. $\square$

3. Groups with Non-Trivial Cyclicizer

It is straightforward to see that a group with non-trivial cyclicizer is either torsion-free or periodic. Therefore, it is natural to inspect the cases in which the groups are either torsion-free or primary groups. As some arguments can be unified, in the following elements of infinite order will be said *elements of order 0* and torsion-free groups will be called 0-*groups*.

Lemma 2. *Let G be a p-group where p is a prime or* 0. *If the cyclicizer $Cyc(G)$ of G is not trivial, then it coincides with the centre $Z(G)$ of G.*

Proof. Assume for a contradiction that $Cyc(G)$ is a proper subgroup of $Z(G)$. Then, we may find an element x of G and an element $y \in Z(G)$ such that $\langle x, y\rangle = \langle x\rangle \times \langle y\rangle$. Let now c be a non-trivial element of $Cyc(G)$. As the subgroups $\langle x, c\rangle$ and $\langle y, c\rangle$ are cyclic, there is a power of c which belongs to $\langle x\rangle \cap \langle y\rangle = \{1\}$. It follows that $Cyc(G)$ is periodic, so that also G is periodic and hence the subgroups $\langle x, c\rangle$ and $\langle y, c\rangle$ have a unique subgroup of order p for a prime p dividing the order of, say, $\langle x, c\rangle$. In particular, the intersection $\langle x\rangle \cap \langle y\rangle$ is not trivial. This contradiction completes the proof. □

The consideration of the direct product of a group of order 3 and a dihedral group of order 8 shows that there exists a (finite) group G whose order is divided by only two primes and such that $\{1\} \neq Cyc(G) < Z(G)$.

Let $A = \langle a\rangle$ be a cyclic group of order 4, let B be a group of type 2^∞ and let b be an element of order 4 of B. Consider the semidirect product $H = A \ltimes B$ where a acts as the inversion on B. Take $K = \langle a^2 b^2\rangle$ and put $G = H/K$. Clearly, every finite non-abelian subgroup of G is a generalized quaternion group. Therefore, in analogy with the locally dihedral 2-group D_{2^∞}, we call G a *locally generalized quaternion group* and we denote it with Q_{2^∞}.

Here we give a first extension of Theorem 8 in [5].

Lemma 3. *Let G be a locally finite p-group for some prime p. Then, the cyclicizer of G is not trivial if and only if*

(1) G is locally cyclic or

(2) G is isomorphic with a subgroup of Q_{2^∞}.

In particular, if G is finite and non-abelian, then G is a generalized quaternion group.

Proof. Assume that the cyclicizer C of G contains a non-trivial element c of order p. If G is abelian, then Lemma 2 yields that G coincides with its cyclicizer and then G is locally cyclic. Assume thus that there exists a finite non-abelian subgroup H of G and let x be an element of $\langle H, c\rangle$ of order p. As $\langle x, c\rangle$ is cyclic, one has that x is a power of c, namely $\langle H, c\rangle$ contains a unique subgroup of order p. By a well-known characterization (see, for instance, [9], 5.3.6) we have that $\langle H, c\rangle$ is a generalized quaternion group. As this property holds for every finite subgroup of G containing $\langle H, c\rangle$ and the set of finite subgroups of G containing $\langle H, c\rangle$ is a direct system of G, we can clearly assume that G is infinite. Therefore, it is possible to find in G a subgroup Q which is isomorphic with Q_{2^∞}. Let g be any element of G, let P be the Prüfer 2-subgroup of Q and let y be an element of order $n > 4$ of P. As $\langle g, y\rangle = \langle g, y, c\rangle$ is either a cyclic or a generalized quaternion group, we have in any case that $\langle y\rangle$ is normalized by g and hence the whole P is normalized by g. Moreover, $\langle g\rangle$ has non-trivial intersection with P, as both must contain c. Then, g has to be contained in Q, otherwise $\langle g, Q\rangle$ would contain a direct product of two cyclic subgroups of order 2. From this it immediately follows that G is isomorphic with Q_{2^∞}.

Let us prove the converse. If G is locally cyclic the result is clear. On the other hand, take G to be a subgroup of Q_{2^∞} which is not locally cyclic. Then, G is not abelian, so that it is either the whole Q_{2^∞} or a generalized quaternion group. In both cases $Z(G)$ is the only subgroup of G of order 2 and therefore it coincides with the cyclicizer of G, which is then non-trivial. □

This result gives a generalization to the locally finite case of the already quoted result about finite p-groups [9], 5.3.6.

Corollary 4. *Let p be a prime. A locally finite p-group G contains exactly one subgroup of order p if and only if it satisfies one of the following conditions:*

(1) G is locally cyclic;
(2) G is isomorphic with a generalized quaternion group;
(3) G is isomorphic with Q_{2^∞}.

In [7], it is proved that if G is a torsion-free group such that cyclicizer $Cyc(G)$ is not trivial, then $Cyc(G) = Z(G)$ and if $Z(G)$ is divisible, then G is locally cyclic. One may ask whether a torsion-free or a p-group with non-trivial cyclicizer is locally cyclic. In general, these questions can be answered in the negative because of two results by Olšanskiĭ (see in [14], Theorem 31.4 and Theorem 31.5). On the other hand, our next result shows that for a wide class of generalized soluble groups the statement is true.

A group G is said to be *weakly radical* if it contains an ascending (normal) series all of whose factors are either locally soluble or locally finite.

Theorem 2. *Let G be a locally weakly radical group such that $|\pi(G)| \leq 1$. Then, G has non-trivial cyclicizer if and only if*

(1) G is locally cyclic or
(2) G is isomorphic with a subgroup of Q_{2^∞}.

Proof. Let C be the cyclicizer of G. If $C \neq \{1\}$, it follows from Lemma 2 that $C = Z(G)$. Moreover, as already pointed out, G is either torsion-free or periodic. By Lemma 3, we may also suppose that G is torsion-free. Let c be a non-trivial element of C. If x is an element of G, then the subgroup $E = \langle x, c\rangle$ of G is cyclic and hence there exists a positive integer n such that x^n belongs to $\langle c\rangle$. Thus the factor group G/C is periodic and so even locally finite since G is locally weakly radical. Now an easy application of a famous theorem by Schur (see, for instance, Corollary to Theorem 4.12 in [10]) shows that the commutator subgroup of G is locally finite and hence G is abelian. In particular, G is locally cyclic.

The converse is an immediate consequence of Lemma 3. □

Corollary 5. *Let G be a locally weakly radical group such that $|\pi(G)| \leq 1$. If G has non-trivial cyclicizer, then it is an FW-group.*

A straightforward application of Theorem 2 and of [9], 12.1.1 is the following.

Corollary 6. *Let G be a locally nilpotent group. Then G has non-trivial cyclicizer if and only if either it is locally cyclic or G is periodic and there is a prime number p such that the p-component G_p of G either is locally cyclic or is isomorphic with a subgroup of Q_{2^∞}.*

A well-known result of Baer (see, for instance, in [10], Theorem 4.16) states that a group is central-by-finite if and only if it has a finite covering consisting of abelian subgroups. Furthermore, we have already quoted the theorem by Schur that ensures that a central-by-finite group is finite-by-abelian. In the following we rephrase these results replacing the centre $Z(G)$ of G by the cyclicizer $Cyc(G)$. Recall that a collection Σ of subgroups of a group G is said to be a *covering* of G if each element of G belongs to at least one subset in Σ.

Theorem 3. *Let G be a group and let C be the cyclicizer of G. Then, the following hold:*

(1) If C has finite index in G, then G is finite-by-(locally cyclic);
(2) The factor group G/C is finite if and only if G has a finite covering consisting of locally cyclic subgroups.

Proof. (1) As $C \leq Z(G)$, then G is central-by-finite and hence the commutator subgroup G' of G is finite. Clearly, we may assume that G is infinite, so that C too is infinite and, by replacing G with G/G', we may suppose that G is abelian. Moreover, as C is non-trivial, then G is either torsion-free or periodic. In the former case, G is locally cyclic by Proposition 2. Assume hence that G is periodic. In this case, as we aim to show that G is locally cyclic, we may also suppose that G is a p-group for a prime p. However, C is locally cyclic and hence of type p^∞. It follows that G can be decomposed as $G = C \times H$ where H is a subgroup of G. If c and h are elements of order p of C and H, respectively, then the subgroup $\langle c, h \rangle$ is not cyclic. This contradiction shows that H is trivial and hence $G = C$ is locally cyclic.

(2) First assume that the factor group G/C is finite. Choose a (left) transversal to C in G, say $\{x_1, \ldots, x_n\}$. Then, for any element g of G, we can write $g = x_i c$ where c is an element of C. Therefore, g belongs to $\langle x_i, C \rangle$, which is locally cyclic, and G is covered by the subgroups $\langle x_i, C \rangle$ with $i = 1, \ldots, n$.

Conversely, assume that G is covered by finitely many locally cyclic subgroups. Then by a result of Neumann (see in [10], Lemma 4.17) G is covered by finitely many locally cyclic subgroups of finite index. Let L be their intersection. Clearly, L is contained in C and $|G : L|$ is finite. It follows that G/C is finite. □

We remark that the cyclicizer of the direct product of $\mathbb{Z}_2 \times \mathbb{Q}$ is trivial, so that the converse of point (1) of Theorem 3 is not true.

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