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Hermite-Hadamard-Type Fractional Inclusions for Interval-Valued Preinvex Functions

Kin Keung Lai [1,*], Jaya Bisht [2], Nidhi Sharma [2] and Shashi Kant Mishra [2]

1 International Business School, Shaanxi Normal University, Xi'an 710119, China
2 Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi 221005, India; jaya.bisht10@bhu.ac.in (J.B.); nidhi.sharma10@bhu.ac.in (N.S.); shashikant.mishra@bhu.ac.in (S.K.M.)
* Correspondence: mskklai@outlook.com

Abstract: We introduce a new class of interval-valued preinvex functions termed as harmonically h-preinvex interval-valued functions. We establish new inclusion of Hermite–Hadamard for harmonically h-preinvex interval-valued function via interval-valued Riemann–Liouville fractional integrals. Further, we prove fractional Hermite–Hadamard-type inclusions for the product of two harmonically h-preinvex interval-valued functions. In this way, these findings include several well-known results and newly obtained results of the existing literature as special cases. Moreover, applications of the main results are demonstrated by presenting some examples.

Keywords: Hermite–Hadamard inequalities; harmonical convex functions; interval-valued functions; fractional integrals

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1. Introduction

It is well known that extensive literature on the class of integral inequalities is being introduced under various notions of convexity; see, for instance [1–6]. Inspired by the importance of convexity in multiple fields of pure and applied sciences, researchers generalized and extended the notion of convexity in various settings. A useful generalization of convex functions is introduced by Hanson [7] which is called invex functions. In 1986, Ben-Israel and Mond [8] proposed the notion of preinvex functions and showed that every differentiable preinvex function is invex, but the converse may not be true. Yang and Li [9] provided two conditions that determine the preinvexity of a function via an intermediate-point preinvexity check under conditions of upper and lower semicontinuity, respectively.

On the other hand, interval analysis was introduced to handle interval uncertainty in many mathematical or computer models of some deterministic real-world phenomena. Moore [10] was the first to propose the concept of interval analysis and extend the arithmetic of intervals to the computer. Moore et al. [11] discussed an arithmetic for intervals, integration of interval functions, and interval Newton methods. Bhurjee and Panda [12] provided a methodology to determine the efficient solution of general multi-objective interval fractional programming problem. Lupulescu [13] gave a theory of the fractional calculus for interval-valued functions using gH-difference for closed intervals. Further, Li et al. [14] introduced the concept of invexity using gH-derivative of interval-valued functions and derived Kuhn–Tucker optimality conditions for an interval-valued objective function. Interval analysis has applications in various fields such as experimental and computational physics, error analysis, computer graphics, robotics, numerical integration, and many other fields (see [15–19]).

2. Literature Survey

Işcan [20] proposed the concept of harmonically convex functions and presented some Hermite–Hadamard (H–H)-type inequalities for harmonically convex functions. Noor et al. [21] defined a new class of preinvex functions named h-preinvex functions and established H–H-type inequalities for these preinvex functions under certain conditions. Further, Noor et al. [22] introduced harmonic h-preinvex function and obtained Ostrowski type inequalities for harmonic h-preinvex functions using Riemann–Liouville (R–L) fractional integrals. In recent years, several integral inequalities for different type of preinvex functions are investigated by many authors; see, for instance [23–30].

Cano et al. [31] obtained some Ostrowski type inequalities for interval-valued functions using gH-derivative. Zhao et al. [32] investigated Riemann interval delta integrals for interval-valued functions on time scales and proved Jensen's, Hölder's, and Minkowski's inequalities using Riemann interval delta integrals. Budak et al. [33] defined right-sided R–L fractional integrals for interval-valued functions and obtained H–H-type inequalities for interval-valued R–L fractional integrals. Lou et al. [34] presented the notions of the Iq-integral and Iq-derivative and gave the Iq-H–H inequalities for interval-valued functions. Further, numerous concepts of quantum calculus for interval-valued functions have been investigated by [35–37].

Considering the importance of interval analysis, many researchers established relations between integral inequalities and different types of interval-valued functions. Zhao et al. [38] introduced the notion of harmonical h-convexity for interval-valued functions and proved some new H–H-type inequalities for the interval Riemann integral. Further, Zhao et al. [39,40] introduced the concept of interval-valued coordinated convexity and established H–H-type inequalities for newly defined interval-valued coordinated convex functions. Recently, Sharma et al. [41] introduced (h_1, h_2)-preinvex interval-valued function and derived fractional H–H-type inequalities for these class of interval-valued preinvex functions. Zhou et al. [42] derived H–H-type inequalities for interval-valued exponential type preinvex functions for R–L interval-valued fractional operator. For more inequalities for interval-valued functions, see references [43–49].

The work in this paper is mainly motivated by Zhao et al. [38] and Shi et al. [50]. We propose the concept of harmonically h-preinvex interval-valued function which includes harmonical h-convex interval-valued functions as a special case. We prove new fractional inclusions of H–H-type for harmonically h-preinvex interval-valued functions. We also present H–H-type inclusions for the product of two harmonically h-preinvex interval-valued functions for interval-valued R–L fractional integrals. Further, we discuss some special cases of our main results. The results obtained in this paper may be generalized for other kinds of interval-valued fractional integrals including harmonically h-preinvex interval-valued functions. As future directions, we can investigate the interval-valued preinvexity on coordinates and establish new inclusions of H–H-type for interval-valued coordinated preinvex functions.

The presentation sequence of the proposed work is the following. In Section 3, we consider some basic definitions and notions of interval analysis. Additionally, we discuss the related results required for this paper. In Section 4, we define harmonically h-preinvexity of interval-valued functions and prove fractional H–H-type inclusions for harmonically h-preinvex interval-valued functions. Some special cases of these results are also discussed in Section 4. In Section 5, we discuss the results obtained by us in this paper. Finally, in Section 6, conclusions and future directions of this study are given.

3. Preliminaries

Let X_I be the collection of all closed intervals of $\mathbb{R}$ and $\Delta \in X_I$. Then, interval Δ is defined by:

$$\Delta = [\underline{\Delta}, \overline{\Delta}] = \{u \in \mathbb{R} | \ \underline{\Delta} \leq u \leq \overline{\Delta}\}, \ \underline{\Delta}, \ \overline{\Delta} \in \mathbb{R}.$$

We say Δ is positive if $\underline{\Delta} > 0$ or negative if $\overline{\Delta} < 0$. We denote the set of all positive closed intervals by X_I^+ and the set of all negative closed intervals by X_I^-. The following binary operations for intervals $\Delta_1 = [\underline{\Delta_1}, \overline{\Delta_1}]$ and $\Delta_2 = [\underline{\Delta_2}, \overline{\Delta_2}]$ are given by [17].

$$\Delta_1 + \Delta_2 = [\underline{\Delta_1}, \overline{\Delta_1}] + [\underline{\Delta_2}, \overline{\Delta_2}] = [\underline{\Delta_1} + \underline{\Delta_2}, \overline{\Delta_1} + \overline{\Delta_2}],$$

$$\Delta_1 - \Delta_2 = [\underline{\Delta_1}, \overline{\Delta_1}] - [\underline{\Delta_2}, \overline{\Delta_2}] = [\underline{\Delta_1} - \overline{\Delta_2}, \overline{\Delta_1} - \underline{\Delta_2}],$$

$$\Delta_1.\Delta_2 = [min\{\underline{\Delta_1}\,\underline{\Delta_2}, \underline{\Delta_1}\,\overline{\Delta_2}, \overline{\Delta_1}\,\underline{\Delta_2}, \overline{\Delta_1}\,\overline{\Delta_2}\}, max\{\underline{\Delta_1}\,\underline{\Delta_2}, \underline{\Delta_1}\,\overline{\Delta_2}, \overline{\Delta_1}\,\underline{\Delta_2}, \overline{\Delta_1}\,\overline{\Delta_2}\}],$$

$$1/\Delta = \{1/u : 0 \neq u \in \Delta\} = [1/\overline{\Delta}, 1/\underline{\Delta}],$$

$$\Delta_1/\Delta_2 = \Delta_1.(1/\Delta_2) = \{u.(1/v) : u \in \Delta_1, 0 \neq v \in \Delta_2\},$$

$$\rho\Delta = \rho[\underline{\Delta}, \overline{\Delta}] = \begin{cases} [\rho\underline{\Delta}, \rho\overline{\Delta}], & \text{if } \rho > 0, \\ \{0\}, & \text{if } \rho = 0, \\ [\rho\overline{\Delta}, \rho\underline{\Delta}], & \text{if } \rho < 0, \end{cases}$$

where $\rho \in \mathbb{R}$.

Definition 1 ([17]). *A function ψ is called an interval-valued function on $[\omega_1, \omega_2]$ if it assigns a nonempty interval to each $u \in [\omega_1, \omega_2]$ and*

$$\psi(u) = [\underline{\psi}(u), \overline{\psi}(u)],$$

where $\underline{\psi}$ and $\overline{\psi}$ are real-valued functions.

Theorem 1 ([11]). *Let $\psi : [\omega_1, \omega_2] \to X_I$ be an interval-valued function such that $\psi(u) = [\underline{\psi}(u), \overline{\psi}(u)]$. Then, ψ is interval Riemann integrable ($IR-$integrable) on $[\omega_1, \omega_2]$ if and only if $\underline{\psi}(u)$ and $\overline{\psi}(u)$ are Riemann integrable ($R-$integrable) on $[\omega_1, \omega_2]$ and*

$$(IR)\int_{\omega_1}^{\omega_2} \psi(u)du = \left[(R)\int_{\omega_1}^{\omega_2} \underline{\psi}(u)du, (R)\int_{\omega_1}^{\omega_2} \overline{\psi}(u)du\right].$$

The collection of all R-integrable and IR-integrable functions on $[\omega_1, \omega_2]$ denoted by $R_{([\omega_1,\omega_2])}$ and $IR_{([\omega_1,\omega_2])}$, respectively.

Definition 2 ([51]). *Let $\psi \in L_1[\omega_1, \omega_2]$. The R–L fractional integrals $J^{\alpha}_{\omega_1^+}\psi$ and $J^{\alpha}_{\omega_2^-}\psi$ of order $\alpha > 0$ with $\omega_1 \geq 0$ are defined by*

$$J^{\alpha}_{\omega_1^+}\psi(u) = \frac{1}{\Gamma(\alpha)}\int_{\omega_1}^{u} (u - \epsilon)^{(\alpha-1)}\psi(\epsilon)d\epsilon, \ \ u > \omega_1$$

and

$$J^{\alpha}_{\omega_2^-}\psi(u) = \frac{1}{\Gamma(\alpha)}\int_{u}^{\omega_2} (\epsilon - u)^{(\alpha-1)}\psi(\epsilon)d\epsilon, \ \ u < \omega_2,$$

respectively. Here, $\Gamma(.)$ is the Gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\epsilon}\epsilon^{\alpha-1}d\epsilon.$$

Definition 3 ([13,33]). *Let $\psi : [\omega_1, \omega_2] \to X_I$ be an interval-valued function and $\psi \in IR_{([\omega_1,\omega_2])}$. The interval-valued R–L fractional integrals of function ψ are defined by*

$$J^{\alpha}_{\omega_1^+}\psi(u) = \frac{1}{\Gamma(\alpha)}(IR)\int_{\omega_1}^{u} (u - \epsilon)^{(\alpha-1)}\psi(\epsilon)d\epsilon, \ \ u > \omega_1, \alpha > 0$$

and

$$J^{\alpha}_{\omega_2^-}\psi(u) = \frac{1}{\Gamma(\alpha)}(IR)\int_{u}^{\omega_2} (\epsilon - u)^{(\alpha-1)}\psi(\epsilon)d\epsilon, \ \ u < \omega_2, \alpha > 0,$$

where $\Gamma(\alpha)$ is the Gamma function.

Corollary 1 ([33]). *If $\psi : [\omega_1, \omega_2] \to X_I$ is an interval-valued function such that $\psi(u) = [\underline{\psi}(u), \overline{\psi}(u)]$ with $\underline{\psi}(u), \overline{\psi}(u) \in R_{([\omega_1,\omega_2])}$, then we have*

$$J^{\alpha}_{\omega_1^+}\psi(u) = [J^{\alpha}_{\omega_1^+}\underline{\psi}(u),\ J^{\alpha}_{\omega_1^+}\overline{\psi}(u)]$$

and

$$J^{\alpha}_{\omega_2^-}\psi(u) = [J^{\alpha}_{\omega_2^-}\underline{\psi}(u),\ J^{\alpha}_{\omega_2^-}\overline{\psi}(u)].$$

Definition 4 ([52]). *A set $I = [\omega_1, \omega_2] \subseteq \mathbb{R}\backslash\{0\}$ is called a harmonic convex set if*

$$\frac{uv}{tu + (1-t)v} \in I, \quad \forall u, v \in I, \quad t \in [0,1].$$

Definition 5 ([20]). *A function $\psi : I = [\omega_1, \omega_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is called harmonic convex, if*

$$\psi\left(\frac{uv}{tu + (1-t)v}\right) \leq (1-t)\psi(u) + t\psi(v), \ \forall u, v \in I, \ t \in [0,1].$$

Now we consider some concepts for harmonic preinvex functions. Let $\psi : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ and $\eta(.,.) : I \times I \to \mathbb{R}$ be continuous functions.

Definition 6 ([53]). *A set $I = [\omega_1, \omega_1 + \eta(\omega_2, \omega_1)] \subseteq \mathbb{R}\backslash\{0\}$ is called a harmonic invex with respect to $\eta(.,.)$, if*

$$\frac{u(u + \eta(v,u))}{u + (1-t)\eta(v,u)} \in I, \quad \forall u, v \in I, \quad t \in [0,1].$$

It is well known that every harmonic convex set is harmonic invex with respect to $\eta(v,u) = v - u$ but not conversely.

Definition 7 ([53]). *A function $\psi : I = [\omega_1, \omega_1 + \eta(\omega_2, \omega_1)] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonic preinvex with respect to the bifunction $\eta(.,.)$, if*

$$\psi\left(\frac{u(u + \eta(v,u))}{u + (1-t)\eta(v,u)}\right) \leq (1-t)\psi(u) + t\psi(v), \ \forall u, v \in I, \ t \in [0,1].$$

Condition C [54]. Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta(.,.)$. Then, function η holds the condition C if for any $t \in [0,1]$ and any $u, v \in I$,

$$\eta(v, v + t\eta(u,v)) = -t\eta(u,v),$$

$$\eta(u, v + t\eta(u,v)) = (1-t)\eta(u,v).$$

Note that $\forall\ t_1, t_2 \in [0,1],\ u, v \in I$ and from condition C, we have

$$\eta(v + t_2\eta(u,v), v + t_1\eta(u,v)) = (t_2 - t_1)\eta(u,v).$$

Theorem 2 ([55]). *Let $\psi : I = [\omega_1, \omega_1 + \eta(\omega_2, \omega_1)] \subseteq \mathbb{R} \to (0, \infty)$ be a preinvex function on I and $\omega_1, \omega_2 \in I$ with $\omega_1 < \omega_1 + \eta(\omega_2, \omega_1)$. Then*

$$\psi\left(\frac{2\omega_1 + \eta(\omega_2, \omega_1)}{2}\right) \leq \frac{1}{\eta(\omega_2, \omega_1)} \int_{\omega_1}^{\omega_1 + \eta(\omega_2, \omega_1)} \psi(u)du \leq \frac{\psi(\omega_1) + \psi(\omega_2)}{2},$$

which is called the H–H-Noor inequality.

Definition 8 ([41]). *If $I \subseteq \mathbb{R}$ is an invex set with respect to $\eta(.,.)$, $\psi(u) = [\underline{\psi}(u), \overline{\psi}(u)]$ is an interval-valued function on I. Then ψ is preinvex interval-valued function on I with respect to $\eta(.,.)$ if*

$$\psi(v + t\eta(u,v)) \supseteq t\psi(u) + (1-t)\psi(v), \ \ \forall\, t \in [0,1] \text{ and } \forall\, u,v \in I.$$

4. Main Results

In this section, first, we define harmonically h-preinvex interval-valued function and discuss some special cases of harmonically h-preinvex interval-valued function.

Definition 9. *Let $h : [0,1] \subseteq J \to \mathbb{R}$ be a non-negative function such that $h \not\equiv 0$, and $I \subseteq \mathbb{R}\backslash\{0\}$ be a harmonic invex set with respect to $\eta(.,.)$. Let $\psi : I \subseteq \mathbb{R}\backslash\{0\} \to X_I^+$ be an interval-valued function on set I, then ψ is called harmonically h-preinvex interval-valued function with respect to $\eta(.,.)$ if*

$$\psi\left(\frac{u(u+\eta(v,u))}{u+(1-t)\eta(v,u)}\right) \supseteq h(1-t)\psi(u) + h(t)\psi(v), \ \ \forall\, t \in [0,1] \text{ and } \forall\, u,v \in I.$$

Now, we consider some special cases of harmonically h-preinvex interval- valued functions.

For $h(t) = 1$, function ψ is called a harmonically $P-$preinvex interval-valued function.
For $h(t) = t$, function ψ is called a harmonically preinvex interval-valued function.
If $h(t) = t^s$, $s \in (0,1)$, then we find the definition of Breckner type of $s-$harmonically preinvex interval-valued functions.
If $h(t) = t^{-s}$, $s \in (0,1)$, then we find the definition of Godunova–Levin type of $s-$harmonically preinvex interval-valued functions.

Example 1. *Let $I = [1,2] \subset \mathbb{R}\backslash\{0\}$, $\psi(u) = \left[1 - \frac{1}{2u^2}, 1 + \frac{1}{2u}\right]$, $\eta(v,u) = v - 2u$, $h(t) = t$ then ψ is harmonically h-preinvex interval-valued function on I.*

Now, we establish fractional inclusion of H–H for harmonically h-preinvex interval-valued functions.

Theorem 3. *Let $h : [0,1] \to \mathbb{R}$ be a non-negative function such that $h(\frac{1}{2}) \neq 0$. Let $\psi : I = [\omega_1, \omega_1 + \eta(\omega_2,\omega_1)] \subseteq \mathbb{R}\backslash\{0\} \to X_I^+$ be a harmonically h-preinvex interval-valued function such that $\psi = [\underline{\psi}, \overline{\psi}]$ and $\omega_1, \omega_2 \in I$ with $\omega_1 < \omega_1 + \eta(\omega_2,\omega_1)$. If $\psi \in L[\omega_1, \omega_1 + \eta(\omega_2,\omega_1)]$, $\alpha > 0$ and η holds condition C, then*

$$\begin{aligned}
&\frac{1}{\alpha h(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right) \\
&\supseteq \Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}(\psi o\Omega)\left(\frac{1}{\omega_1}\right) + J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right] \\
&\supseteq [\psi(\omega_1) + \psi(\omega_1+\eta(\omega_2,\omega_1))]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]dt,
\end{aligned}$$

where $\Omega(u) = \frac{1}{u}$ and $\psi o\Omega$ is defined by $\psi o\Omega(u) = \psi(\Omega(u))$, $\ \forall\, u \in \left[\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}, \frac{1}{\omega_1}\right]$.

Proof. As ψ is harmonically h-preinvex interval-valued function on $[\omega_1, \omega_1 + \eta(\omega_2,\omega_1)]$, we have

$$\frac{1}{h(\frac{1}{2})}\psi\left(\frac{2u(u+\eta(v,u))}{2u+\eta(v,u)}\right) \supseteq \psi(u) + \psi(v), \ \forall\, u,v \in [\omega_1, \omega_1+\eta(\omega_2,\omega_1)]. \quad (1)$$

Let $u = \frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}$ and $v = \frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}$. Then, using Condition C in (1), we find

$$\frac{1}{h(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\supseteq\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)+\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right). \tag{2}$$

Multiplying (2) by $t^{\alpha-1}$, $\alpha>0$ and integrating over $[0,1]$ with respect to t, we have

$$\begin{aligned}\frac{1}{h(\frac{1}{2})}(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)dt &\supseteq (IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)dt\\ &+(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)dt.\end{aligned} \tag{3}$$

Applying Theorem 1 in above relation, we find

$$\begin{aligned}&(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)dt\\ &=\left[(R)\int_0^1 t^{\alpha-1}\underline{\psi}\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)dt,(R)\int_0^1 t^{\alpha-1}\overline{\psi}\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)dt\right]\\ &=\left[\frac{1}{\alpha}\underline{\psi}\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right),\frac{1}{\alpha}\overline{\psi}\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\right]\\ &=\frac{1}{\alpha}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right),\end{aligned} \tag{4}$$

$$\begin{aligned}&(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)dt\\ &=\left[(R)\int_0^1 t^{\alpha-1}\underline{\psi}\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)dt,(R)\int_0^1 t^{\alpha-1}\overline{\psi}\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)dt\right]\\ &=\Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}\underline{\psi}o\Omega\left(\frac{1}{\omega_1}\right),J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}\overline{\psi}o\Omega\left(\frac{1}{\omega_1}\right)\right]\\ &=\Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}(\psi o\Omega)\left(\frac{1}{\omega_1}\right).\end{aligned} \tag{5}$$

Similarly,

$$(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)dt=\Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right). \tag{6}$$

Using (4)–(6) in (3), we have

$$\begin{aligned}\frac{1}{\alpha h(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\supseteq\Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\Bigg[&J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)\\ &+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\Bigg].\end{aligned} \tag{7}$$

As ψ is an harmonically h-preinvex interval-valued function on $[\omega_1,\omega_1+\eta(\omega_2,\omega_1)]$, we have

$$\begin{aligned}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)&=\psi\left(\frac{(\omega_1+\eta(\omega_2,\omega_1))(\omega_1+\eta(\omega_2,\omega_1)+\eta(\omega_1,\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+\eta(\omega_2,\omega_1)+t\eta(\omega_1,\omega_1+\eta(\omega_2,\omega_1))}\right)\\ &\supseteq h(t)\psi(\omega_1+\eta(\omega_2,\omega_1))+h(1-t)\psi(\omega_1)\end{aligned} \tag{8}$$

and

$$\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)=\psi\left(\frac{(\omega_1+\eta(\omega_2,\omega_1))(\omega_1+\eta(\omega_2,\omega_1)+\eta(\omega_1,\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+\eta(\omega_2,\omega_1)+(1-t)\eta(\omega_1,\omega_1+\eta(\omega_2,\omega_1))}\right)$$
$$\supseteq h(1-t)\psi(\omega_1+\eta(\omega_2,\omega_1))+h(t)\psi(\omega_1). \tag{9}$$

Adding (8) and (9), we have

$$\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)+\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\supseteq[h(t)+h(1-t)][\psi(\omega_1)+\psi(\omega_1+\eta(\omega_2,\omega_1))]. \tag{10}$$

Multiplying (10) by $t^{\alpha-1}$ and integrating over $[0,1]$ with respect to t, we have

$$(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)dt+(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)dt$$
$$\supseteq(IR)\int_0^1 t^{\alpha-1}[h(t)+h(1-t)][\psi(\omega_1)+\psi(\omega_1+\eta(\omega_2,\omega_1))]dt.$$

This implies

$$\Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]$$
$$\supseteq[\psi(\omega_1)+\psi(\omega_1+\eta(\omega_2,\omega_1))]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]dt. \tag{11}$$

From (7) and (11), we find

$$\frac{1}{\alpha h(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)$$
$$\supseteq\Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]$$
$$\supseteq[\psi(\omega_1)+\psi(\omega_1+\eta(\omega_2,\omega_1))]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]dt.$$

□

Example 2. *Let* $I=[\omega_1,\omega_1+\eta(\omega_2,\omega_1)]=[1,2]$, $\eta(\omega_2,\omega_1)=\omega_2-2\omega_1$. *Let* $\alpha=1$ *and* $h(t)=t\;\forall\,t\in[0,1]$,
$\psi:I\to X_I^+$ *be defined by*

$$\psi(u)=\left[-\frac{1}{u}+2,\frac{1}{u}+2\right],\ \forall\,u\in I.$$

We find

$$\frac{1}{\alpha h(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)=2\psi\left(\frac{4}{3}\right)=\left[\frac{5}{2},\frac{11}{2}\right], \tag{12}$$

$$\Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]$$
$$=\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\int_{\omega_1}^{\omega_1+\eta(\omega_2,\omega_1)}\frac{\psi(u)}{u^2}du=2\int_1^2\frac{1}{u^2}\left[-\frac{1}{u}+2,\frac{1}{u}+2\right]du=\left[\frac{5}{2},\frac{11}{2}\right] \tag{13}$$

and

$$[\psi(\omega_1)+\psi(\omega_1+\eta(\omega_2,\omega_1))]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]dt=[\psi+\psi]=\left[\frac{5}{2},\frac{11}{2}\right]. \tag{14}$$

From (12)–(14), we see Theorem 3 is verified.

Remark 1. *If we put $\eta(\omega_2,\omega_1)=\omega_2-\omega_1$ in the above theorem, we obtain Theorem 5 of [50].*

Remark 2. *If we put $\eta(\omega_2,\omega_1)=\omega_2-\omega_1$ and $\alpha=1$ in the above theorem, we obtain Theorem 1 of [38].*

Remark 3. *If we put $\eta(\omega_2,\omega_1)=\omega_2-\omega_1$ and $h(t)=t$ in the above theorem, we obtain Theorem 3.6 of [56].*

Now we present some particular cases of Theorem 3.

Corollary 2. *If $\alpha=1$, then Theorem 3 gives the following result:*

$$\begin{aligned}\frac{1}{h(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)&\supseteq\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\int_{\omega_1}^{\omega_1+\eta(\omega_2,\omega_1)}\frac{\psi(u)}{u^2}du\\&\supseteq[\psi(\omega_1)+\psi(\omega_1+\eta(\omega_2,\omega_1))]\int_0^1[h(t)+h(1-t)]dt.\end{aligned}$$

Corollary 3. *If $h(t)=t$, then Theorem 3 gives the following result:*

$$\begin{aligned}&\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\\&\supseteq\frac{\Gamma(\alpha+1)}{2}\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}\psi\left(\frac{1}{\omega_1}\right)+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}\psi\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]\\&\supseteq\frac{\psi(\omega_1)+\psi(\omega_1+\eta(\omega_2,\omega_1))}{2}.\end{aligned}$$

Next, we prove fractional inclusions of H–H-type for the product of two harmonically h-preinvex interval-valued functions.

Theorem 4. *Let $h_1,h_2:[0,1]\to\mathbb{R}$ be non-negative functions and $h_1,h_2\not\equiv 0$. Let $\psi,\varphi: I=[\omega_1,\omega_1+\eta(\omega_2,\omega_1)]\subseteq\mathbb{R}\backslash\{0\}\to X_I^+$ be two harmonically h_1- and h_2-preinvex interval-valued functions, respectively, such that $\psi=[\underline{\psi},\overline{\psi}]$, $\varphi=[\underline{\varphi},\overline{\varphi}]$ and $\omega_1,\omega_2\in I$ with $\omega_1<\omega_1+\eta(\omega_2,\omega_1)$. If $\psi\varphi\in L[\omega_1,\omega_1+\eta(\omega_2,\omega_1)]$, $\alpha>0$ and η holds condition C, then*

$$\begin{aligned}&\Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1}\right)\right.\\&\left.+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]\\&\supseteq F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1[t^{\alpha-1}+(1-t)^{\alpha-1}]h_1(t)h_2(t)dt\\&+G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1[t^{\alpha-1}+(1-t)^{\alpha-1}]h_1(1-t)h_2(t)]dt,\end{aligned} \tag{15}$$

where $F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))=\psi(\omega_1)\varphi(\omega_1)+\psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1+\eta(\omega_2,\omega_1))$,
$N(\omega_1,\omega_1+\eta(\omega_2,\omega_1))=\psi(\omega_1)\varphi(\omega_1+\eta(\omega_2,\omega_1))+\psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1)$
and $\Omega(u)=\frac{1}{u}$.

Proof. As ψ and φ are two harmonically h_1- and h_2-preinvex interval-valued functions on $[\omega_1, \omega_1 + \eta(\omega_2, \omega_1)]$, respectively. Therefore,

$$\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right) = \psi\left(\frac{(\omega_1+\eta(\omega_2,\omega_1))(\omega_1+\eta(\omega_2,\omega_1)+\eta(\omega_1,\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+\eta(\omega_2,\omega_1)+t\eta(\omega_1,\omega_1+\eta(\omega_2,\omega_1))}\right) \supseteq h_1(t)\psi(\omega_1+\eta(\omega_2,\omega_1)) + h_1(1-t)\psi(\omega_1) \tag{16}$$

and

$$\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right) = \varphi\left(\frac{(\omega_1+\eta(\omega_2,\omega_1))(\omega_1+\eta(\omega_2,\omega_1)+\eta(\omega_1,\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+\eta(\omega_2,\omega_1)+t\eta(\omega_1,\omega_1+\eta(\omega_2,\omega_1))}\right) \supseteq h_2(t)\varphi(\omega_1+\eta(\omega_2,\omega_1)) + h_2(1-t)\varphi(\omega_1). \tag{17}$$

As $\psi(u), \varphi(u) \in X_I^+$, $\forall\, u \in [\omega_1, \omega_1 + \eta(\omega_2, \omega_1)]$, then from (16) and (17), we obtain

$$\begin{aligned}&\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\\ &\supseteq h_1(t)h_2(t)\psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1+\eta(\omega_2,\omega_1)) + h_1(1-t)h_2(1-t)\psi(\omega_1)\varphi(\omega_1)\\ &\quad + h_1(t)h_2(1-t)\psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1) + h_1(1-t)h_2(t)\psi(\omega_1)\varphi(\omega_1+\eta(\omega_2,\omega_1)).\end{aligned} \tag{18}$$

Similarly,

$$\begin{aligned}&\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\\ &\supseteq h_1(1-t)h_2(1-t)\psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1+\eta(\omega_2,\omega_1)) + h_1(t)h_2(t)\psi(\omega_1)\varphi(\omega_1)\\ &\quad + h_1(1-t)h_2(t)\psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1) + h_1(t)h_2(1-t)\psi(\omega_1)\varphi(\omega_1+\eta(\omega_2,\omega_1)).\end{aligned} \tag{19}$$

Adding (18) and (19), we have

$$\begin{aligned}&\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\\ &\quad + \psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\\ &\supseteq [h_1(t)h_2(t) + h_1(1-t)h_2(1-t)][\psi(\omega_1)\varphi(\omega_1) + \psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1+\eta(\omega_2,\omega_1))]\\ &\quad + [h_1(t)h_2(1-t) + h_1(1-t)h_2(t)][\psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1) + \psi(\omega_1)\varphi(\omega_1+(\omega_2,r))]\\ &= F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))[h_1(t)h_2(t) + h_1(1-t)h_2(1-t)]\\ &\quad + G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))[h_1(1-t)h_2(t) + h_1(t)h_2(1-t)].\end{aligned} \tag{20}$$

Multiplying (20) by $t^{\alpha-1}$ and integrating over $[0,1]$ with respect to t, we have

$$\begin{aligned}&(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)dt\\ &\quad + (IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)dt\\ &\supseteq (IR)\int_0^1 t^{\alpha-1}F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))[h_1(t)h_2(t) + h_1(1-t)h_2(1-t)]dt\\ &\quad + (IR)\int_0^1 t^{\alpha-1}G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))[h_1(1-t)h_2(t) + h_1(t)h_2(1-t)]dt.\end{aligned} \tag{21}$$

As

$$(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)dt$$

$$= \Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha} J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1}\right), \tag{22}$$

$$\begin{aligned}(IR)&\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)dt \\ &= \Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha} J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right).\end{aligned} \tag{23}$$

Using (22), (23) in (21), we have

$$\begin{aligned}&\Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1}\right)\right.\\ &\left.+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]\\ &\supseteq F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1[t^{\alpha-1}+(1-t)^{\alpha-1}]h_1(t)h_2(t)dt\\ &+G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1[t^{\alpha-1}+(1-t)^{\alpha-1}]h_1(1-t)h_2(t)]dt.\end{aligned}$$

□

Remark 4. *If we put $\eta(\omega_2,\omega_1)=\omega_2-\omega_1$ in the above theorem, we obtain Theorem 6 of [50].*

Remark 5. *If we put $\eta(\omega_2,\omega_1)=\omega_2-\omega_1$ and $\alpha=1$ in the above theorem, we obtain Theorem 3 of [38].*

Corollary 4. *If $\alpha=1$, then Theorem 4 gives the following result:*

$$\begin{aligned}&\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\int_{\omega_1}^{\omega_1+\eta(\omega_2,\omega_1)}\frac{\psi(u)\varphi(u)}{u^2}du\\ &\supseteq F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1 h_1(t)h_2(t)dt+G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1 h_1(1-t)h_2(t)]dt.\end{aligned}$$

Corollary 5. *If $h_1(t)=h_2(t)=t$, then Theorem 4 gives the following result:*

$$\begin{aligned}&\Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1}\right)\right.\\ &\left.+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]\\ &\supseteq F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1 t^2[t^{\alpha-1}+(1-t)^{\alpha-1}]dt\\ &+G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1 t(1-t)[t^{\alpha-1}+(1-t)^{\alpha-1}]dt\\ &=\frac{\alpha^2+\alpha+2}{\alpha(\alpha+1)(\alpha+2)}F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))+\frac{2}{(\alpha+1)(\alpha+2)}G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\\ &=\frac{(\alpha^2+\alpha+2)F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))+2\alpha G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))}{\alpha(\alpha+1)(\alpha+2)}.\end{aligned}$$

Theorem 5. *Let $h_1,h_2:[0,1]\to\mathbb{R}$ be non-negative functions and $h_1(\frac{1}{2})h_2(\frac{1}{2})\neq 0$. Let $\psi,\varphi: I=[\omega_1,\omega_1+\eta(\omega_2,\omega_1)]\subseteq\mathbb{R}\backslash\{0\}\to X_I^+$ be two harmonically h_1- and h_2-preinvex* *interval-valued functions, respectively, such that* $\psi = [\underline{\psi}, \overline{\psi}]$, $\varphi = [\underline{\varphi}, \overline{\varphi}]$ *and* $\omega_1, \omega_2 \in I$ *with* $\omega_1 < \omega_1 + \eta(\omega_2, \omega_1)$. *If* $\psi\varphi \in L[\omega_1, \omega_1 + \eta(\omega_2, \omega_1)]$, $\alpha > 0$ *and* η *holds condition C, then*

$$\begin{aligned}
&\frac{1}{\alpha h_1(\frac{1}{2})h_2(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\\
&\supseteq \Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^+}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1}\right)\right.\\
&\left.+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^-}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]\\
&+F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1(t^{\alpha-1}+(1-t)^{\alpha-1})h_1(t)h_2(1-t)dt\\
&+G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1(t^{\alpha-1}+(1-t)^{\alpha-1})h_1(t)h_2(t)dt,
\end{aligned}$$

where $F(\omega_1, \omega_1 + \eta(\omega_2, \omega_1))$ *and* $G(\omega_1, \omega_1 + \eta(\omega_2, \omega_1))$ *are defined as previous.*

Proof. As ψ is harmonically h_1-preinvex interval-valued function on $[\omega_1, \omega_1 + \eta(\omega_2, \omega_1)]$, we have

$$\frac{1}{h_1(\frac{1}{2})}\psi\left(\frac{2u(u+\eta(v,u))}{2u+\eta(v,u)}\right) \supseteq \psi(u)+\psi(v),\ \forall\, u,v \in [\omega_1,\omega_1+\eta(\omega_2,\omega_1)]. \tag{24}$$

Let $u = \frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}$ and $v = \frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}$. Then, using Condition C in (24), we find

$$\frac{1}{h_1(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right) \supseteq \psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)+\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right). \tag{25}$$

Similarly,

$$\frac{1}{h_2(\frac{1}{2})}\varphi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right) \supseteq \varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)+\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right). \tag{26}$$

From (25) and (26), we find

$$\begin{aligned}
&\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\\
&\supseteq \left[\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)+\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\right]\\
&\quad\times\left[\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)+\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\right]\\
&=\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\\
&\quad+\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\\
&\quad+\left[\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\right.\\
&\quad\left.+\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\right].
\end{aligned} \tag{27}$$

As $\psi(u)$ and $\varphi(u) \in X_I^+$, $\forall u \in [\omega_1, \omega_1 + \eta(\omega_2, \omega_1)]$ are two harmonically h_1- and h_2-preinvex interval-valued functions, respectively. Therefore,

$$
\begin{aligned}
&\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\\
&\supseteq h_1(t)h_2(t)\psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1)+h_1(1-t)h_2(1-t)\psi(\omega_1)\varphi(\omega_1+\eta(\omega_2,\omega_1))\\
&\quad+h_1(t)h_2(1-t)\psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1+\eta(\omega_2,\omega_1))+h_1(1-t)h_2(t)\psi(\omega_1)\varphi(\omega_1).
\end{aligned}\tag{28}
$$

Similarly,

$$
\begin{aligned}
&\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\\
&\supseteq h_1(t)h_2(t)\psi(\omega_1)\varphi(\omega_1+\eta(\omega_2,\omega_1))+h_1(1-t)h_2(1-t)\psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1)\\
&\quad+h_1(t)h_2(1-t)\psi(\omega_1)\varphi(\omega_1)+h_1(1-t)h_2(t)\psi(\omega_1+\eta(\omega_2,\omega_1))\varphi(\omega_1+\eta(\omega_2,\omega_1)).
\end{aligned}\tag{29}
$$

Adding (28) and (29), we obtain

$$
\begin{aligned}
&\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\\
&\quad+\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\\
&\supseteq G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]\\
&\quad+F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))[h_1(1-t)h_2(t)+h_1(t)h_2(1-t)].
\end{aligned}\tag{30}
$$

From (27) and (30), we have

$$
\begin{aligned}
&\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\\
&\supseteq\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\\
&\quad+\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\\
&\quad+G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]dt\\
&\quad+F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))[h_1(1-t)h_2(t)+h_1(t)h_2(1-t)]dt.
\end{aligned}\tag{31}
$$

Multiplying (31) by $t^{\alpha-1}$, then integrating over $[0,1]$ with respect to t, we find

$$
\begin{aligned}
&\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)dt\\
&\supseteq(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+(1-t)\eta(\omega_2,\omega_1)}\right)dt\\
&\quad+(IR)\int_0^1 t^{\alpha-1}\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\omega_1+t\eta(\omega_2,\omega_1)}\right)dt\\
&\quad+G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))(IR)\int_0^1 t^{\alpha-1}[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]dt\\
&\quad+F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))(IR)\int_0^1 t^{\alpha-1}[h_1(1-t)h_2(t)+h_1(t)h_2(1-t)]dt.
\end{aligned}
$$

This implies

$$
\frac{1}{\alpha h_1(\frac{1}{2})h_2(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)
$$

$$
\begin{aligned}
&\supseteq \Gamma(\alpha)\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^{+}}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)\varphi o\Omega\left(\frac{1}{\omega_1}\right)\right.\\
&\left.+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^{-}}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]\\
&+F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1(t^{\alpha-1}+(1-t)^{\alpha-1})h_1(t)h_2(1-t)]dt\\
&+G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1(t^{\alpha-1}+(1-t)^{\alpha-1})h_1(t)h_2(t)dt.
\end{aligned}
$$

□

Remark 6. *If we put $\eta(\omega_2,\omega_1)=\omega_2-\omega_1$ in the above theorem, we obtain Theorem 7 of [50].*

Remark 7. *If we put $\eta(\omega_2,\omega_1)=\omega_2-\omega_1$ and $\alpha=1$ in the above theorem, we obtain Theorem 4 of [38].*

Corollary 6. *If $\alpha=1$, then Theorem 5 gives the following result:*

$$
\begin{aligned}
&\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\\
&\supseteq \frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\int_{\omega_1}^{\omega_1+\eta(\omega_2,\omega_1))}\frac{\psi(u)\varphi(u)}{u^2}du+F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1 h_1(t)h_2(1-t)dt\\
&+G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1 h_1(t)h_2(t)dt.
\end{aligned}
$$

Corollary 7. *If $h_1(t)=h_2(t)=t$, then Theorem 5 gives the following result:*

$$
\begin{aligned}
&\frac{4}{\alpha}\psi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\varphi\left(\frac{2\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{2\omega_1+\eta(\omega_2,\omega_1)}\right)\\
&\supseteq \Gamma(\alpha)\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^{+}}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)\varphi o\Omega\left(\frac{1}{\omega_1}\right)\right.\\
&\left.+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^{-}}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]\\
&+F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1 t(1-t)(t^{\alpha-1}+(1-t)^{\alpha-1})dt\\
&+G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\int_0^1 t^2(t^{\alpha-1}+(1-t)^{\alpha-1})dt\\
&=\Gamma(\alpha)\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^{+}}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)\varphi o\Omega\left(\frac{1}{\omega_1}\right)\right.\\
&\left.+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^{-}}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]\\
&+\frac{2}{(\alpha+1)(\alpha+2)}F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))+\frac{\alpha^2+\alpha+2}{\alpha(\alpha+1)(\alpha+2)}G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))\\
&=\Gamma(\alpha)\psi\left(\frac{\omega_1(\omega_1+\eta(\omega_2,\omega_1))}{\eta(\omega_2,\omega_1)}\right)^{\alpha}\left[J^{\alpha}_{\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)^{+}}(\psi o\Omega)\left(\frac{1}{\omega_1}\right)\varphi o\Omega\left(\frac{1}{\omega_1}\right)\right.\\
&\left.+J^{\alpha}_{\left(\frac{1}{\omega_1}\right)^{-}}(\psi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)(\varphi o\Omega)\left(\frac{1}{\omega_1+\eta(\omega_2,\omega_1)}\right)\right]
\end{aligned}
$$

$$+\frac{2\alpha F(\omega_1,\omega_1+\eta(\omega_2,\omega_1))+(\alpha^2+\alpha+2)G(\omega_1,\omega_1+\eta(\omega_2,\omega_1))}{\alpha(\alpha+1)(\alpha+2)}.$$

5. Results and Discussions

After illustrating the concept of interval-valued functions, this paper proposes a new definition of harmonically h-preinvex interval-valued functions. Further, with the help of the proposed harmonically h-preinvexity for interval-valued functions, we have proven H–H-type inclusions for interval-valued R–L fractional integrals. From the definition of harmonically h-preinvex interval-valued function, we can see that every harmonical h-convex interval-valued function is harmonically h-preinvex interval-valued function with respect to $\eta(v,u) = v-u$. The results obtained in this paper are generalization of the results of Zhao et al. [38] and Shi et al. [50]. Moreover, some particular cases of our main outcomes are considered.

6. Conclusions and Future Directions

In this paper, we have introduced harmonically h-preinvex interval-valued functions which include harmonical h-convex interval-valued functions and harmonical convex interval-valued functions as special cases. We have obtained H–H-type fractional inclusions for harmonically h-preinvex interval-valued functions. After that, we have proven fractional H–H-type inclusions for the product of two harmonically h-preinvex interval-valued functions. The results obtained in this paper may be extended for other kinds of interval-valued fractional integrals including harmonically h-preinvex interval-valued functions. In the future, we can investigate the interval-valued preinvexity on coordinates and establish new inclusions of H–H-type for interval-valued coordinated preinvex functions. It is expected that current work will motivate researchers working in fractional calculus, interval analysis, and other related areas.

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