mathematics

Review

Some Families of Generating Functions Associated with Orthogonal Polynomials and Other Higher Transcendental Functions

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Abstract: In this invited survey-cum-expository review article, we present a brief and comprehensive account of some general families of linear and bilinear generating functions which are associated with orthogonal polynomials and such other higher transcendental functions as (for example) hypergeometric functions and hypergeometric polynomials in one, two and more variables. Many of the results as well as the methods and techniques used for their derivations, which are presented here, are intended to provide incentive and motivation for further research on the subject investigated in this article.

Keywords: linear and bilinear generating functions; orthogonal polynomials; Jacobi, Laguerre and Hermite polynomials; Bessel polynomials; higher transcendental functions; Lagrange's expansion theorem; Bailey's bilinear generating function; Hille-Hardy formula; Mehler's formula; operational techniques; Laplace and inverse Laplace transforms; Riemann-Liouville fractional derivative; hypergeometric functions; hypergeometric polynomials

MSC: 33C20; 33C45; 26A33; 33B15; 33C60; 33C65; 44A10

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Citation: Srivastava, H.M. Some Families of Generating Functions Associated with Orthogonal Polynomials and Other Higher Transcendental Functions. *Mathematics* **2022**, *10*, 3730. https://doi.org/10.3390/math10203730

Academic Editor: Jan L. Cieśliński

Received: 15 September 2022
Accepted: 5 October 2022
Published: 11 October 2022

1. Introduction, Definitions and Preliminaries

As widely recognized, generating functions play an important role in the study of various potentially useful properties and characteristics of the sequences which they generate. In the form of Z-transforms, which essentially are the discrete counterparts of the Laplace transform, generating functions are also used in converting difference equations of discrete-time signals and systems into algebraic equations, thereby providing simplifications in discrete-time system analysis and also in a wide variety of problems involving sequential fractional-order difference operators, operations research and other areas of applied disciplines (including, for example, queuing theory and related stochastic processes) (see, for details, [1–6]). An effective use of generating functions involves the determination of the asymptotic behavior of the generated sequence $\{\mathfrak{f}_n\}_{n=0}^{\infty}$ by suitably adapting Darboux's method. Moreover, the existence of a generating function for a sequence $\{\mathfrak{f}_n\}_{n=0}^{\infty}$ of numbers or functions may be useful in finding $\sum\limits_{n=0}^{\infty} \mathfrak{f}_n$ by means of such summability methods as those due to Abel and Cesàro.

As observed and documented by Lando [7], modern combinatorics speaks the language of generating functions, the study of which does not require a bulky knowledge of numerous parts of mathematics, except for some preliminary acquaintance with calculus and algebra. Moreover, generating functions may prove to be extremely useful in furthering mathematical education because of their deep involvement in various mathematical activities, including computer science, and according to Wilf [8], generating functions provide a bridge between discrete mathematics, on the one hand, and continuous analysis (particularly, complex variable theory) on the other hand. One can study generating functions solely as tools for solving discrete problems. There is much in the study of generating functions that is powerful and magical in the way generating functions give unified methods for handling such problems. The full beauty of the subject of generating functions emerges only from tuning in on both channels: the discrete channel and the continuous channel. One can then see how they make the solution of difference equations into child's play.

This invited survey-cum-expository review article is motivated chiefly by the potential uses of generating functions in a wide variety of areas including the ones which we have indicated above. In this article, we aim at presenting a brief and comprehensive account of some general families of linear and bilinear generating functions which are associated with the classical orthogonal polynomials and such other higher transcendental functions as (for example) hypergeometric functions and hypergeometric polynomials in one and more variables. Many of the results presented here, and the methodology and techniques which are applied for their derivations, are intended to provide incentive and motivation for further research on the subject investigated in this article.

We choose to remark in passing that, for the purpose of this article, the term "*classical orthogonal polynomials*" is used in the traditional sense (see, for example, [9]). In the terminology of the basic or quantum (or $\mathfrak{q}$-) calculus, an orthogonal polynomial sequence is called classical if it is a special case or a limit case of the ${}_4\varphi_3$ polynomials known as the $\mathfrak{q}$-Racah polynomials or the Askey-Wilson polynomials (see, for details, ([10], p. 189)).

We begin by introducing the widely-accepted concepts of linear, bilinear (and multilinear) and bilateral (and multilateral) generating functions (see, for details, [11–14]). First of all, given a two-variable function $F(x,t)$ possessing at least *formal* power-series expansion in t in the following form:

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)\, t^n, \tag{1}$$

in which each member of the generated set $\{f_n(x)\}_{n=0}^{\infty}$ is independent of t, then $F(x,t)$ is called a *linear generating function* (or, simply, a generating function) of the set $\{f_n(x)\}_{n=0}^{\infty}$. This definition may be extended to include generating functions of the following type:

$$F^*(x,t) = \sum_{n=0}^{\infty} \alpha_n\, f_n(x)\, t^n, \tag{2}$$

where the coefficient sequence $\{\alpha_n\}_{n=0}^{\infty}$ may include the parameters of the set $\{f_n(x)\}_{n=0}^{\infty}$, but is independent of x and t. Similarly, we can define a *bilinear generating function* as follows:

$$G^*(x,t) = \sum_{n=0}^{\infty} \beta_n\, f_n(x)\, f_n(y)\, t^n \tag{3}$$

and a *bilateral generating function* in the following form:

$$H^*(x,t) = \sum_{n=0}^{\infty} \gamma_n\, f_n(x)\, g_n(y)\, t^n, \tag{4}$$

for two different function sequences $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(y)\}_{n=0}^{\infty}$, and in an analogous manner, the definitions (3) and (4) can be further extended to multilinear generating functions and multilateral (and mixed multilateral) generating functions involving products of several functions of the same or different or mixed function classes as the generated sets.

Next, in terms of the familiar (Euler's) Gamma function $\Gamma(z)$, we introduce the general Pochhammer symbol or the *shifted factorial* $(\lambda)_\nu$, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0),$$

which is defined (for $\lambda, \nu \in \mathbb{C}$), by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{5}$$

where it is understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists. Here, and in what follows, the following standard notations are freely used:

$$\mathbb{N} := \{1,2,3,\cdots\}, \quad \mathbb{N}_0 := \{0,1,2,3,\cdots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- := \{-1,-2,-3,\cdots\} = \mathbb{Z}_0^- \setminus \{0\} \qquad \left(\mathbb{Z}_0^- := \{0,-1,-2,-3,\cdots\}\right).$$

In addition,, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. Thus, as the most useful and the most fundamental special function of applicable and applied mathematical sciences, we introduce a generalized hypergeometric function, with p numerator parameters $\alpha_j \in \mathbb{C}$ $(j = 1,\cdots,p)$ and q denominator parameters $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1,\cdots,q)$, given by (see, for example, [15–19])

$$\begin{aligned} {}_pF_q\left[\begin{matrix} \alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q; \end{matrix}\ z\right] &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &=: {}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z) \end{aligned} \tag{6}$$

$\left(p \leqq q+1;\ p < q+1 \text{ and } |z| < \infty;\ p = q+1 \text{ and } z \in \mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}\right).$

Since

$$(-n)_k = \begin{cases} \dfrac{(-1)^n\, n!}{(n-k)!} & (0 \leqq k \leqq n) \\ 0 & (k \geqq n+1), \end{cases} \tag{7}$$

whenever one of the numerator parameters is a negative integer or zero, the generalized hypergeometric series in (6) would terminate, and we are led to a generalized hypergeometric polynomial of the following type:

$$\begin{aligned} &{}_{p+1}F_q\left[\begin{matrix} -n,\alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q; \end{matrix}\ z\right] \\ &\quad := \sum_{k=0}^{n} \frac{(-n)_k\,(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{z^k}{k!} \\ &\quad = \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} (-z)^n \\ &\qquad \cdot {}_{q+1}F_p\left[\begin{matrix} -n,1-\beta_1-n,\cdots,1-\beta_q-n; \\ 1-\alpha_1-n,\cdots,1-\alpha_p-n; \end{matrix}\ \frac{(-1)^{p+q}}{z}\right] \qquad (n \in \mathbb{N}_0). \end{aligned} \tag{8}$$

Most (if not all) of the familiar families of the classical orthogonal polynomials and several other polynomial systems are essentially one form or the other of the generalized hypergeometric polynomials given by (8).

2. Generating Functions of Orthogonal Polynomials

Many of the classical orthogonal polynomials happen to be special or limit cases of the celebrated Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined by

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &:= \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+k+\alpha+\beta}{k} \left(\frac{x-1}{2}\right)^k \\ &= \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k} \\ &= \binom{\alpha+n}{n} \, {}_2F_1\left(-n, \alpha+\beta+n+1; \alpha+1; \frac{1-x}{2}\right). \end{aligned} \tag{9}$$

The Gegenbauer (or ultraspherical) polynomials C_n^{ν}, the Legendre (or spherical) polynomials $P_n(x)$ and the Chebyshev polynomials $T_n(x)$ and $U_n(x)$, correspond to the special cases of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ when $\alpha=\beta=\nu-\frac{1}{2}$, $\alpha=\beta=0$ and $\alpha=\beta=\mp\frac{1}{2}$, respectively.

The other members of the relatively more familiar family of orthogonal polynomials include (for example) the Laguerre polynomials $L_n^{(\alpha)}(x)$ for which we have (see, for details, [9]; see also [20,21])

$$\begin{aligned} L_n^{(\alpha)}(x) &:= \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \\ &= \binom{\alpha+n}{n} \, {}_1F_1(-n; \alpha+1; x) \\ &= \lim_{|\beta|\to\infty} \left\{ P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) \right\} \end{aligned} \tag{10}$$

and the Hermite polynomials $H_n(x)$ for which

$$\begin{aligned} H_n(x) &:= \sum_{k=0}^{[n/2]} \frac{(-1)^k \, n!}{k! \, (n-2k)!} \, x^{n-2k} \\ &= (2x)^n \, {}_2F_0\left(-\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n; \underline{\quad}; -\frac{1}{x^2}\right), \end{aligned} \tag{11}$$

$[\mu]$ being the largest integer in $\mu \in \mathbb{R}$, so that

$$H_{2n}(x) = \lim_{|\epsilon|\to\infty} \left\{ (-1)^n \, n! \, 2^{2n} \, P_n^{(\frac{1}{2},-\epsilon)}\left(1+\frac{2x^2}{\epsilon}\right) \right\}$$

and

$$H_{2n+1}(x) = \lim_{|\epsilon|\to\infty} \left\{ (-1)^n \, n! \, 2^{2n+1} \, x \, P_n^{(-\frac{1}{2},-\epsilon)}\left(1+\frac{2x^2}{\epsilon}\right) \right\}.$$

Yet another member of the family of the classical orthogonal polynomials, which is a limit case of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined by (9), happens to be the *two-parameter* Bessel polynomials $y_n(x,a,b)$ given by

$$\begin{aligned} y_n(x,a,b) &:= \sum_{k=0}^{n} \binom{n}{k} \binom{a+n+k-2}{k} \, k! \left(\frac{x}{b}\right)^k \\ &= {}_2F_0\Big(-n, a+n-1; \text{———}; -\frac{x}{b}\Big) \\ &= \lim_{|\epsilon|\to\infty} \left\{ \frac{n!}{(\epsilon)_n} \, P_n^{(\epsilon-1, a-\epsilon-1)}\left(1+\frac{2\epsilon x}{b}\right) \right\}. \end{aligned} \tag{12}$$

Together with the *simple* Bessel polynomials $y_n(x)$ given by

$$y_n := \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \, k! \left(\frac{x}{2}\right)^k = y_n(x,2,2), \tag{13}$$

the importance of the Bessel polynomials can be appreciated by the fact that they arise rather naturally in several seemingly diverse contexts including (for example) in connection with the solution of the wave equation in spherical polar coordinates (see [22]), in network synthesis and design (see [23]), in a representation of the energy spectral functions for a family of isotropic turbulence fields (see [24,25]), in developing a matrix technique applicable in solving some multi-order pantograph differential equations of fractional order (see [26]) and so on (see also the monograph on the subject of the Bessel polynomials by Grosswald [27] for further details about these polynomials).

One of the fundamental generating functions in the theory of the classical orthogonal polynomials happens to be Jacobi's generating function of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ (see ([28], pp. 193–194) and ([29], p. 172, Equation 10.8 (29))):

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) \, t^n = 2^{\alpha+\beta} \, R^{-1} \, (1-t+R)^{-\alpha} \, (1+t+R)^{-\beta}, \tag{14}$$

where for convenience,

$$R = R(x,t) := \left(1-2xt+t^2\right)^{\frac{1}{2}} \qquad (R=1 \text{ when } t=0).$$

In the literature, one can find many interesting proofs of Jacobi's generating function (14). Some of these proofs were given by (for example) Pólya and Szegö ([30], pp. 147 and 346, Problem 219), Szegö ([9], Section 4.4), Rainville ([12], Section 140), Carlitz [31], Askey [32], and Foata and Leroux [33] (see also [34,35] and ([14], p. 82)). More recently, an independent proof of Jacobi's generating function (14) was given by Srivastava [36], which was based upon some elementary results from the theory of the Gauss hypergeometric function ${}_2F_1$ such as the transformation formula ([12], p. 60, Theorem 20):

$${}_2F_1(a,b;c;z) = (1-z)^{-a} \, {}_2F_1\left(a, c-b; c; -\frac{z}{1-z}\right) \tag{15}$$

$$\left(\max\left\{|z|, \left|\frac{z}{1-z}\right|\right\} < 1\right),$$

the hypergeometric reduction formula ([12], p. 70, Problem 10):

$${}_2F_1\left(a, a+\frac{1}{2}; 2a; z\right) = \frac{1}{\sqrt{1-z}} \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2a} \qquad (|z|<1), \tag{16}$$

and the binomial expansion given by ([12], p. 58, Equation (1))

$$ {}_1F_0(a; \underline{\quad}; z) := \sum_{n=0}^{\infty} \binom{a+n-1}{n} z^n = (1-z)^{-a} \qquad (|z| < 1). \tag{17} $$

We next recall following analogues of the generating function (14), each of which was derived by Srivastava [37]:

$$ \sum_{n=0}^{\infty} P_n^{\left(\alpha-n, \beta-(b+1)n\right)}(x)\, t^n = \frac{(1+w)^{-\alpha-\beta}}{1-bw} \left(1 + \frac{2w}{1-x}\right)^{\alpha} \tag{18} $$

$$ \left(w = w(x,t) = \frac{1}{2}(1-x)t(1+w)^{b+1} \quad \text{with} \quad w(x,0) = 0 \right) $$

and

$$ \sum_{n=0}^{\infty} P_n^{\left(\alpha+bn, \beta-(b+1)n\right)}(x)\, t^n = \frac{(1+\zeta)^{\alpha+1}}{1-b\zeta} \left[1 + \frac{1}{2}(1-x)\zeta\right]^{-\alpha-\beta-1} \tag{19} $$

$$ \left(\zeta = \zeta(t) = t(1+\zeta)^{b+1} \quad \text{with} \quad \zeta(0) = 0\right). $$

Each of the generating functions (18) and (19) include, as their particular cases, a large number of generating functions for the Jacobi polynomials which were considered by (for example) Brown [38], Calvez and Génin [39], Carlitz [40], Feldheim [41] and other authors (see also [42]). Furthermore, since

$$ \begin{aligned} P_n^{(\alpha,\beta)}(x) &= (-1)^n\, P_n^{(\beta,\alpha)}(-x) \\ &= \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-2n-1, \beta)}\left(\frac{x+3}{x-1}\right), \end{aligned} \tag{20} $$

it is fairly straightforward to verify that the following rather specialized generating functions are equivalent to one another:

$$ \sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta)}(x)\, t^n = (1+t)^{\alpha} \left[1 - \frac{1}{2}(x-1)t\right]^{-\alpha-\beta-1}, \tag{21} $$

$$ \sum_{n=0}^{\infty} P_n^{(\alpha,\beta-n)}(x)\, t^n = (1-t)^{\beta} \left[1 - \frac{1}{2}(x-1)t\right]^{-\alpha-\beta-1} \tag{22} $$

and

$$ \sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)\, t^n = \left[1 + \frac{1}{2}(x+1)t\right]^{\alpha} \left[1 + \frac{1}{2}(x-1)t\right]^{\beta}. \tag{23} $$

A probabilistic proof of this last generating (23) was given by Milch [43].

We now choose to introduce a potentially useful method of finding generating functions, which is based upon Lagrange's expansion theorem. Indeed, if the function $\varphi(z)$ is holomorphic at $z = z_0$ and $\varphi(z_0) \neq 0$, and if

$$ z = z_0 + w\varphi(z), \tag{24} $$

then an analytic function $f(z)$, which is holomorphic at $z = z_0$, can have a power-series expansion in w by the celebrated Lagrange expansion theorem ([44], p. 133):

$$ f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left\{ f'(z)[\varphi(z)]^n \right\} \Big|_{z=z_0}. \tag{25} $$

Upon having differentiated both sides of (25), if we use the relation (24) and replace $f'(z)\varphi(z)$ in the resulting equation by $f(z)$, the Lagrange expansion theorem (25) can be rewritten in the following elegant and easy-to-use form ([30], p. 146, Problem 207):

$$\frac{f(z)}{1-w\varphi'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n}\{f(z)[\varphi(z)]^n\}\bigg|_{z=z_0}. \tag{26}$$

The Lagrange expansion theorem (26) has been applied widely and extensively in the study of generating functions and in other areas. In particular, as a widely-cited unification and generalization of all of the above generating functions of the Jacobi polynomials, it was used by Srivastava and Singhal [45] (see also [46]), who showed that

$$\sum_{n=0}^{\infty} P_n^{(\alpha-\lambda n,\beta-\mu n)}(x)\ t^n = \frac{(1+u)^{\alpha}\ (1+v)^{\beta}}{1+(1-\lambda)u+(1-\mu)v}, \tag{27}$$

where the parameters α, β, λ and μ are unrestricted, in general, and u and v are functions of x and t defined *implicitly* by

$$\begin{cases} u = u(x,t) = -\frac{1}{2}(x+1)t(1+u)^{\lambda}\ (1+v)^{\mu-1} \\ v = v(x,t) = -\frac{1}{2}(x-1)t(1+u)^{\lambda-1}\ (1+v)^{\mu}. \end{cases} \tag{28}$$

Equivalently, upon setting

$$u = -\frac{\xi}{1+\xi} \qquad \text{and} \qquad v = -\frac{\eta}{1+\eta},$$

the Srivastava-Singhal generating function (27) assumes the following relatively more elegant form:

$$\sum_{n=0}^{\infty} P_n^{(\alpha-\lambda n,\beta-\mu n)}(x)\ t^n = \frac{(1+\xi)^{\alpha+1}\ (1+\eta)^{\beta+1}}{1+\lambda\xi+\mu\eta-(1-\lambda-\mu)\xi\eta}, \tag{29}$$

where

$$\begin{cases} \xi = \xi(x,t) = \frac{1}{2}(x+1)t(1+\xi)^{1-\lambda}\ (1+\eta)^{1-\mu} \\ \eta = \eta(x,t) = \frac{1}{2}(x-1)t(1+\xi)^{1-\lambda}\ (1+\eta)^{1-\mu}, \end{cases} \tag{30}$$

which, for $\lambda = -a$ and $\mu = -b$, immediately yields

$$\sum_{n=0}^{\infty} P_n^{(\alpha+an,\beta+bn)}(x)\ t^n = \frac{(1+\xi)^{\alpha+1}\ (1+\eta)^{\beta+1}}{1-a\xi-b\eta-(1+a+b)\xi\eta}, \tag{31}$$

where

$$\begin{cases} \xi = \xi(x,t) = \frac{1}{2}(x+1)t(1+\xi)^{a+1}\ (1+\eta)^{b+1} \\ \eta = \eta(x,t) = \frac{1}{2}(x-1)t(1+\xi)^{a+1}\ (1+\eta)^{b+1}, \end{cases} \tag{32}$$

By applying the approach used by Foata and Leroux [33] in their combinatorial proof of Jacobi's generating function (14), Strehl [47] gave an interesting combinatorial proof of the Srivastava-Singhal generating function (29). On the other hand, Chen and Ismail [48] made use of Darboux's method in conjunction with the Srivastava-Singhal generating function (31) in order to derive the asymptotics of the Jacobi polynomials $P_n^{(\alpha+an,\beta+bn)}(x)$ as $n \to \infty$ when the parameters α, β, a and b, as well as the argument x, are fixed. Gawronski and Shawyer [49] used (31) to calculate the asymptotic distribution of the zeros of the Jacobi polynomials $P_n^{(\alpha+an,\beta+bn)}(x)$ as $n \to \infty$. Many other developments, which have emerged essentially from the Srivastava-Singhal generating functions (27), (29) and (31), include those presented in (for example) [42,50,51] and also in [52–54].

In view of the limit relationship in (10), upon replacing x in any form of the Srivastava-Singhal generating functions (27), (29) and (31) by $1-\frac{2x}{\beta}$, if we first proceed to the limit when $|\beta| \to \infty$ and then set $\lambda = -\beta$, we arrive at the Carlitz's generating function for the Laguerre polynomials $L_n^{(\alpha)}(x)$ (see [55]):

$$\sum_{n=0}^{\infty} L_n^{(\alpha+\beta n)}(x)\, t^n = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\, \mathrm{e}^{-x\zeta}, \tag{33}$$

where ζ is given *implicitly* by

$$\zeta = \zeta(t) = t(1+\zeta)^{\beta+1} \qquad \text{and} \qquad \zeta(0) = 0. \tag{34}$$

The generating function (33) would also follow as a similar limit case of Srivastava's generaling function (19) with, of course, some obvious parameter adjustments.

From among the bilinear generating functions of the classical orthogonal polynomials, we first recall the following result which is popularly known as Bailey's bilinear generating function of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ (see ([56], p. 9, Equation (2.1)) and ([15], p. 102, Example 19); see also [57,58]):

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{n!\,(\alpha+\beta+1)_n}{(\alpha+1)_n\,(\beta+1)_n}\, P_n^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(y)\, t^n &= (1+t)^{-\alpha-\beta-1} \\ &\cdot F_4\left[\frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2); \frac{(1-x)(1-y)t}{(1+t)^2}, \frac{(1+x)(1+y)t}{(1+t)^2}\right],\end{aligned} \tag{35}$$

where F_4 denotes the two-variable Appell function of the fourth kind defined by (see, for details, [59,60])

$$F_4[a,b;c,c';x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}}{(c)_m\,(c')_n}\, \frac{x^m}{m!}\, \frac{y^n}{n!} \qquad \left(\sqrt{|x|}+\sqrt{|y|}<1\right). \tag{36}$$

Bailey's proof of (35) was based upon Watson's transformation which expresses an Appell function F_4 as the product of two hypergeometric ${}_2F_1$ series. By following the lines of Askey's proof of Jacobi's generating function (14) in [32], Stanton [61] derived the bilinear generating function (35) by using the orthogonality property of the Jacobi polynomials in conjunction with a quadratic transformation for a well-poised hypergeometric ${}_3F_2$ series. Subsequently, Srivastava [62] presented an elementary proof of (35), which is based upon the hypergeometric definition in (9), the following equivalent form of (15):

$$ {}_2F_1(a,b;c;z) = (1-z)^{-b}\, {}_2F_1\left(c-a, b; c; -\frac{z}{1-z}\right) \tag{37}$$

$$\left(\max\left\{|z|, \left|\frac{z}{1-z}\right|\right\} < 1\right),$$

and the celebrated Chu-Vandermonde sum:

$$\sum_{k=0}^{n} \frac{(-n)_k\,(b)_k}{k!\,(c)_k} = \frac{(c-b)_n}{(c)_n} \qquad \left(c \notin \mathbb{Z}_0^-\right).$$

In fact, Srivastava [62] also proved a basic (or quantum or q-) analogue of Bailey's bilinear generating function (35), for which he later provided an elementary proof in [63]. For other developments related to Bailey's bilinear generating function (35), the reader may be referred to [14,64].

For the Laguerre polynomials $L_n^{(\alpha)}(x)$, the following widely-investigated bilinear generating function is popularly known as the Hille-Hardy formula (see, for details, Hille [65], Hardy ([66], p. 192) and Watson (see ([67], p. 190) and [68])):

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(x)\, L_n^{(\alpha)}(y)\, t^n$$
$$= (1-t)^{-1}\,(xyt)^{-\frac{1}{2}\alpha}\, \exp\left(-\frac{(x+y)t}{1-t}\right) I_\alpha\left(\frac{2\sqrt{xyt}}{1-t}\right), \tag{38}$$

where $I_\nu(z)$ denotes the modified Bessel function defined by (see, for example, [69])

$$I_\nu(z) := \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}$$
$$= \frac{\left(\frac{1}{2}z\right)^{\nu}}{\Gamma(\nu+1)}\, {}_0F_1\left(\text{——};\nu+1;\frac{z^2}{4}\right).$$

A number of potentially useful extensions and generalizations of various families of bilinear, bilateral and multilinear generating functions, which are applicable to Jacobi, Laguerre and other special functions and polynomials, can be found in (for example) [70–92] (see also [14]). In particular, Weisner [92] made use of group-theoretic (or Lie algebraic) method in his investigation of such generating functions as we have mentioned above (see also [14,93–96]).

We conclude this section by remarking that, in the case of the Hermite polynomials $H_n(x)$, the following bilinear generating function is known as Mehler's formula:

$$\sum_{n=0}^{\infty} H_n(x)\, H_n(y)\, \frac{t^n}{n!} = \left(1-4t^2\right)^{-\frac{1}{2}} \exp\left(\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right), \tag{39}$$

which was proved combinatorially by Foata [97] (see also [98,99]). For several extensions and generalizations of the Mehler formula (39), the reader is referred to [100–105]).

3. Operational Techniques and Hypergeometric Polynomials

In this section, we propose to consider some operational techniques, which are based upon the Laplace and the inverse Laplace transforms as well as upon the Riemann-Liouville fractional derivative, with a view to deriving some general families of generating functions for hypergeometric polynomials (see, for example, ([106], Chapters 4 and 5) and [107]; see also ([108], Chapter 13), [109–111]).

Given a function $f(t)$, which is at least piecewise continuous on every finite closed interval $[0,T]$ $(T>0)$ and for which the following asymptotic property holds true:

$$f(t) = O\left(e^{\sigma t}\right) \qquad (t \to \infty) \tag{40}$$

for some σ, the classical Laplace transform is defined, as usual, by

$$\mathcal{L}\{f(t):s\} := \int_0^\infty e^{-st}\, f(t)\, dt =: \mathfrak{F}(s) \qquad \left(\Re(s) > \sigma\right). \tag{41}$$

On the other hand, if the function $f(t)$ is continuous for each $t \geqq 0$ and satisfies the asymptotic property (40), the inverse Laplace transform is given by

$$\mathcal{L}^{-1}\{\mathfrak{F}(s):t\} = \frac{1}{2\pi\mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{st}\, \mathfrak{F}(s)\, \mathrm{d}s = f(t) \qquad \left(\gamma > \sigma;\ \mathrm{i} := \sqrt{-1}\right). \tag{42}$$

Computations of the Laplace transform (41) and the inverse Laplace transform (42) can be accomplished by appropriately applying the following Eulerian integral:

$$\mathcal{L}\{t^{\lambda-1}:s\} = \int_0^\infty t^{\lambda-1}\, \mathrm{e}^{-st}\, \mathrm{d}t = \frac{\Gamma(\lambda)}{s^\lambda} \qquad \left(\min\{\Re(s), \Re(\lambda)\} > 0\right) \tag{43}$$

and by Hankel's contour integral (see, for details, ([44], p. 245, Example 1) and ([112], p. 17, Equation (1))):

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{s}\, s^{-z}\, \mathrm{d}s \qquad \left(\gamma > 0;\ \Re(z) > 0;\ \ \mathrm{i} := \sqrt{-1}\right). \tag{44}$$

It is regrettable to see that, in a large number of amateurish-type publications, many trivial and inconsequential parametric and argument variations of the classical Laplace transform (41) and its s-multiplied (or, equivalently, the Laplace-Carson) version are being claimed to be a "new" integral transform in the present-day literature. Such trends and tendencies, and obviously false claims, ought to be discouraged by all means (see, for details, ([113], pp. 1508–1510) and ([114], pp. 36–38)).

When applied to the generalized hypergeometric function (6), the operators $\mathcal{L}$ and $\mathcal{L}^{-1}$ yield ([106], p. 219, Entry 4.23 (17))

$$\mathcal{L}\left\{ t^{\lambda-1}\ {}_pF_q\left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{matrix}\ zt\right] : s \right\} = \frac{\Gamma(\lambda)}{s^\lambda}\ {}_{p+1}F_q\left[\begin{matrix} \lambda, \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{matrix}\ \frac{z}{s}\right] \tag{45}$$

$$\left(\Re(\lambda) > 0;\ \Re(z) > 0\ \text{ if }\ p < q;\ \Re(s) > \Re(z)\ \text{ if }\ p = q\right)$$

and ([106], p. 297, Entry 5.21 (1))

$$\mathcal{L}^{-1}\left\{ s^{-\mu}\ {}_pF_q\left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{matrix}\ \frac{z}{s}\right] : t \right\} = \frac{t^{\mu-1}}{\Gamma(\mu)}\ {}_pF_{q+1}\left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q, \mu; \end{matrix}\ zt\right] \tag{46}$$

$$\left(\Re(\mu) > 0;\ p \leqq q+1\right).$$

We note that the operator $\mathcal{L}$ is capable of augmenting one numerator parameter of the hypergeometric function ${}_pF_q$ and also that the operator $\mathcal{L}^{-1}$ augments one denominator parameter of the hypergeometric function ${}_pF_q$. For a reasonably detailed presentation of various other processes of augmentation of parameters of the hypergeometric function ${}_pF_q$ such as those by (for example) the Euler and Beta type transformations, we choose to refer the reader to Rainville ([12], Section 56) and to Srivastava and Manocha ([14], Section 4.3).

In the case of generating functions for hypergeometric polynomials, the above-mentioned process of augmentation of parameters can be fruitfully applied in conjunction with the principle of mathematical induction to produce a generating function for a higher-order hypergeometric polynomial by starting with a known generating function for a lower-order hypergeometric polynomial. For example, upon writing Carlitz's generating function (33) in the following equivalent hypergeometric form:

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} {}_1F_1 \left[\begin{array}{c} -n; \\ \\ \alpha+\beta n+1; \end{array} x \right] t^n$$
$$= \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta} \, {}_0F_0 \left[\begin{array}{c} \text{———}; \\ \\ \text{———}; \end{array} -x\zeta \right], \tag{47}$$

where ζ is a function of t given by (34).

The following generating function for hypergeometric polynomials was derived by using the above operational technique based upon parametric augmentation and mathematical induction by Srivastava $\big($[37], p. 591, Equation (9)$\big)$ (see also [115] for much more general families of generating functions for hypergeometric polynomials):

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} {}_{p+1}F_{q+1} \left[\begin{array}{c} -n, \alpha_1, \cdots, \alpha_p; \\ \\ \alpha+\beta n+1, \beta_1, \cdots, \beta_q; \end{array} x \right] t^n$$
$$= \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta} \, {}_pF_q \left[\begin{array}{c} \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} -x\zeta \right], \tag{48}$$

which led eventually to the following general family of generating functions:

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} t^n \sum_{k=0}^{n} \frac{(-n)_k \, \Omega_k}{(\alpha+\beta n+1)_k} \frac{x^k}{k!}$$
$$= \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta} \sum_{k=0}^{\infty} \Omega_k \frac{(-x\zeta)^k}{k!}. \tag{49}$$

This last result (49) holds true for a suitably bounded sequence $\{\Omega_n\}_{n=0}^{\infty}$ of real or complex numbers.

In its special case when $\beta = 0$ and $\beta = -1$, the hypergeometric generating function (48) would correspond, respectively, to the following results of Chaundy [116]:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, {}_{p+1}F_q \left[\begin{array}{c} -n, \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} x \right] t^n$$
$$= (1-t)^{-\lambda} \, {}_{p+1}F_q \left[\begin{array}{c} \lambda, \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} -\frac{xt}{1-t} \right] \tag{50}$$
$$(\lambda \in \mathbb{C}; \; |t| < 1)$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, {}_{p+1}F_{q+1} \left[\begin{array}{c} -n, \alpha_1, \cdots, \alpha_p; \\ \\ 1-\lambda-n, \beta_1, \cdots, \beta_q; \end{array} x \right] t^n$$
$$= (1-t)^{-\lambda} \, {}_pF_q \left[\begin{array}{c} \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} xt \right] \tag{51}$$

$$(\lambda \in \mathbb{C};\ |t| < 1).$$

In fact, Chaundy [116] also gave the following companion of his results (50) and (51):

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_{p+2}F_q \left[\begin{array}{c} -n, \lambda+n, \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} x \right] t^n \\ = (1-t)^{-\alpha}\ {}_{p+2}F_q \left[\begin{array}{r} \frac{1}{2}\lambda, \frac{1}{2}\lambda+\frac{1}{2}, \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} -\frac{4xt}{(1-t)^2} \right] \end{aligned} \tag{52}$$

$$(\lambda \in \mathbb{C};\ |t| < 1),$$

which provides a generalization of a known generating function for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ given by

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\ P_n^{(\alpha,\beta)}(x)\ t^n \\ = (1-t)^{-\alpha-\beta-1}\ {}_2F_1 \left[\begin{array}{r} \frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2); \\ \\ \alpha+1; \end{array} \frac{2(x-1)t}{(1-t)^2} \right]. \end{aligned}$$

Moreover, for the Bessel polynomials $y_n(x,\alpha,\beta)$ given by (12), the following *divergent* generating function follows readily from Chaundy's result (52):

$$\begin{aligned} (1-t)^{1-\alpha}\ {}_2F_0 \left[\begin{array}{c} \frac{1}{2}(\alpha-1), \frac{1}{2}\alpha; \\ \\ \underline{\qquad\qquad}; \end{array} \frac{4xt}{\beta(1-t)^2} \right] \\ \approx \sum_{n=0}^{\infty} \frac{(\alpha-1)_n}{n!}\ y_n(x,\alpha,\beta)\ t^n. \end{aligned} \tag{53}$$

Indeed, for the Bessel polynomials $y_n(x,\alpha,\beta)$ given by (12), we recall Burchnall's generating function in the following *corrected* form (see ([117], p. 67) and ([14], p. 84)):

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x,\alpha,\beta)\ \frac{t^n}{n!} = \frac{1}{\sqrt{1-\dfrac{4xt}{\beta}}} \left(\frac{2}{1+\sqrt{1-\dfrac{4xt}{\beta}}} \right)^{\alpha-2} \\ \cdot \exp\left(\frac{2t}{1+\sqrt{1-\dfrac{4xt}{\beta}}} \right). \end{aligned} \tag{54}$$

A limit case of Chaundy's result (50) when t is replaced by $\frac{t}{\lambda}$ and $|\lambda| \to \infty$ yields the following generating function:

$$\sum_{n=0}^{\infty} {}_{p+1}F_q\left[\begin{array}{c} -n, \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} x\right] \frac{t^n}{n!}$$
$$= \mathrm{e}^t \; {}_pF_q\left[\begin{array}{c} \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} -xt\right], \tag{55}$$

which is usually attributed to Rainville (see, for example, ([11], p. 267)). As a matter of fact, Rainville [118] rediscovered Chaundy's result (52) and also derived the following interesting generalization of Burchnall's generating function (54) for the Bessel polynomials $y_n(x, \alpha, \beta)$:

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \; {}_{q+2}F_p\left[\begin{array}{c} -n, \lambda+n, 1-\beta_1-n, \cdots, 1-\beta_q-n; \\ \\ 1-\alpha_1-n, \cdots, 1-\alpha_p-n; \end{array} (-1)^{p+q+1}\, x\right] \frac{t^n}{n!}$$
$$= \frac{1}{\sqrt{1-4xt}} \left(\frac{2}{1+\sqrt{1-4xt}}\right)^{\lambda-1} {}_pF_q\left[\begin{array}{c} \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} \frac{2t}{1+\sqrt{1-4xt}}\right] \tag{56}$$
$$\left(\sqrt{1-4xt} \to 1 \;\; \text{when} \;\; t \to 0\right),$$

The following general form of Chaundy's result (50) was given by Brafman [119]:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \; {}_{m+p}F_q\left[\begin{array}{c} \Delta(m; -n), \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} x\right] t^n$$
$$= (1-t)^{-\lambda} \; {}_{m+p}F_q\left[\begin{array}{c} \Delta(m; \lambda), \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} x\left(-\frac{t}{1-t}\right)^m\right] \tag{57}$$
$$(\lambda \in \mathbb{C};\; m \in \mathbb{N};\; |t| < 1),$$

where for convenience, $\Delta(m; \lambda)$ denotes the set of m parameters:

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \cdots, \frac{\lambda+m-1}{m} \qquad (\lambda \in \mathbb{C};\; m \in \mathbb{N}).$$

Indeed, for the Gould-Hopper polynomials $g_n^m(x, h)$ defined by (see, for details, [120])

$$g_n^m(x, h) := \sum_{k=0}^{[n/m]} \frac{n!}{k!\,(n-mk)!}\, h^k\, x^{n-mk}$$
$$= x^n \; {}_mF_0\left[\begin{array}{c} \Delta(m; -n); \\ \\ \overline{}; \end{array} \left(-\frac{m}{x}\right)^m h\right], \tag{58}$$

the following *divergent* generating function follows from (57):

$$(1-xt)^{-\lambda}\ {}_mF_q\left[\begin{array}{c}\Delta(m;\lambda);\\ \\ \underline{\qquad\quad};\end{array}\ \left(\frac{mt}{1-xt}\right)^m h\right]$$
$$\approx \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ g_n^m(x,h)\ t^n. \tag{59}$$

Since the Hermite polynomials $H_n(x)$ is related to the Gould-Hopper polynomials as follows:

$$H_n(x)=g_n^2(2x,-1),$$

we can apply the result (59) to derive the following *divergent* generating function for the Hermite polynomials:

$$(1-2xt)^{-\lambda}\ {}_2F_0\left[\begin{array}{c}\Delta(2;\lambda);\\ \\ \underline{\qquad\quad};\end{array}\ -\frac{4t^2}{(1-2xt)^2}\right]$$
$$\approx \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ H_n(x)\ t^n. \tag{60}$$

Next, in connection with the general family of generating functions, given by (49), from [121–123] (see also [124]) we recall the Fox-Wright function ${}_p\Psi_q$ $(p,q\in\mathbb{N}_0)$ or ${}_p\Psi_q^*$ $(p,q\in\mathbb{N}_0)$, with p numerator parameters $\alpha_1,\cdots,\alpha_p$ and q denominator parameters $\beta_1,\cdots,\beta_q$ such that

$$\alpha_j\in\mathbb{C}\quad(j=1,\cdots,p)\qquad\text{and}\qquad\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-\quad(j=1,\cdots,q).$$

By definition, we have (see, for details, ([16], p. 183) and ([19], p. 21); see also ([109], p. 56), ([125], p. 65) and ([126], p. 19))

$${}_p\Psi_q^*\left[\begin{array}{c}(\alpha_1,A_1),\cdots,(\alpha_p,A_p);\\ \\ (\beta_1,B_1),\cdots,(\beta_q,B_q);\end{array}\ z\right]$$
$$:=\sum_{n=0}^{\infty}\frac{(\alpha_1)_{A_1n}\cdots(\alpha_p)_{A_pn}}{(\beta_1)_{B_1n}\cdots(\beta_q)_{B_qn}}\frac{z^n}{n!}$$
$$=:\frac{\Gamma(\beta_1)\cdots\Gamma(\beta_q)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}\ {}_p\Psi_q\left[\begin{array}{c}(\alpha_1,A_1),\cdots,(\alpha_p,A_p);\\ \\ (\beta_1,B_1),\cdots,(\beta_q,B_q);\end{array}\ z\right] \tag{61}$$

$$\left(\Re(A_j)>0\ (j=1,\cdots,p);\ \Re(B_j)>0\ (j=1,\cdots,q);\ \Re\left(\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\right)\geqq -1\right),$$

where the equality in the convergence condition holds true only for suitably bounded values of $|z|$ given by

$$|z|<\nabla:=\left(\prod_{j=1}^{p}A_j^{-A_j}\right)\cdot\left(\prod_{j=1}^{q}B_j^{B_j}\right).$$

As observed by (for example) Srivastava [127], the widely-applied Mittag-Leffler function $E_\alpha(z)$ and its two-parameter version $E_{\alpha,\beta}(z)$ (see [128–130]), as well as most (if not all) of their multi-parameter generalizations and extensions, are contained, as obvious special cases, of the general Fox-Wright function ${}_p\Psi_q$ or ${}_p\Psi_q^*$. Moreover, for Wright's generalized Bessel function $J_\nu^\mu(z)$ (see [131]), we readily have

$$J_\nu^\mu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(\mu n+\nu+1)} = {}_0\Psi_1\left[\begin{array}{c} \overline{}\,; \\ (\nu+1,\mu); \end{array} -z\right] =: \phi(\mu,\nu+1;-z),$$

so that, obviously, Wright's generalized Bessel function $J_\nu^\mu(z)$ corresponds to the familiar Bessel function $J_\nu(z)$ given by (see [58]):

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}.$$

Thus, if in the general family of generating functions given by (49), we set

$$\Omega_n = \frac{(\alpha_1)_{A_1 n}\cdots(\alpha_p)_{A_p n}}{(\beta_1)_{B_1 n}\cdots(\beta_q)_{B_q n}} \qquad (n\in\mathbb{N}_0)$$

and apply the definition (61), we can deduce the following extension of Chaundy's result (50):

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{\alpha+\beta n+1}{n}\, {}_{p+1}\Psi_{q+1}^*&\left[\begin{array}{c}(-n,1),(\alpha_1,A_1),\cdots,(\alpha_p,A_p);\\ (\alpha+\beta n+1,1),(\beta_1,B_1),\cdots,(\beta_q,B_q);\end{array} x\right] t^n\\ &= \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\, {}_p\Psi_q^*\left[\begin{array}{c}(\alpha_1,A_1),\cdots,(\alpha_p,A_p);\\ (\beta_1,B_1),\cdots,(\beta_q,B_q);\end{array} -x\zeta\right],\end{aligned} \tag{62}$$

where $\zeta=\zeta(t)$ is defined, as before, by (34).

Finally, we turn to yet another operational technique in the study of generating functions of hypergeometric functions and hypergeometric polynomials, which is facilitated by the Riemann-Liouville fractional derivative operator $\mathfrak{D}_z^\mu$ defined by ([108], Chapter 13) (see also the recent developments reported in [109,111,113])

$$\begin{aligned}&\mathfrak{D}_z^\mu\{f(z)\}\\ &\quad=\begin{cases}\dfrac{1}{\Gamma(-\mu)}\displaystyle\int_0^z (z-t)^{-\mu-1} f(t)\,\mathrm{d}t & (\Re(\mu)<0)\\[3mm] \dfrac{\mathrm{d}^m}{\mathrm{d}z^m}\{\mathfrak{D}_z^{\mu-m}\{f(z)\}\} & \big(m-1\leqq\Re(\mu)<m\ \ (m\in\mathbb{N})\big).\end{cases}\end{aligned} \tag{63}$$

Since

$$\mathfrak{D}_z^\mu\left\{z^\lambda\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\, z^{\lambda-\mu} \qquad (\Re(\lambda)>-1),$$

it is easily seen from the definition (6) that

$$\mathfrak{D}_z^{\lambda-\mu}\left\{z^{\lambda-1}\,{}_pF_q\left[\begin{array}{c}\alpha_1,\cdots,\alpha_p;\\ \\ \beta_1,\cdots,\beta_q;\end{array}\,z\right]\right\}$$
$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\,z^{\mu-1}\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\lambda,\alpha_1,\cdots,\alpha_p;\\ \\ \mu,\beta_1,\cdots,\beta_q;\end{array}\,z\right] \tag{64}$$
$$\big(\Re(\lambda)>0;\ |z|<\infty\ \text{ when }\ p\leqq q;\ |z|<1\ \text{ when }\ p=q+1\big).$$

Judging by the above formula (64), the Riemann-Liouville fractional derivative operator $\mathfrak{D}_z^{\mu}$ is advantageous over the operators $\mathcal{L}$ and $\mathcal{L}^{-1}$ in the sense that its one single operation of $\mathfrak{D}_z^{\lambda-\mu}$ is capable of augmenting the two parameters λ and μ *simultaneously*, that is, the parameter λ in the numerator and the parameter μ in the denominator of the hypergeometric function involved.

For a simple illustrative example involving the use of the Riemann-Liouville fractional derivative operator $\mathfrak{D}_z^{\mu}$ in deriving generating functions for hypergeometric functions and hypergeometric polynomials, we consider the following identity:

$$[1-(1-z)t]^{-\lambda}=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,[(1-z)t]^n$$
$$=(1-t)^{-\lambda}\left(1+\frac{zt}{1-t}\right)^{-\lambda}\qquad\big(|t|<|1-z|^{-1}\big). \tag{65}$$

Upon multiplying both sides of the second equation in (65) by $z^{\alpha-1}\,(1-z)^{-\rho}$, we obtain

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,z^{\alpha-1}\,(1-z)^{n-\rho}\,t^n=z^{\alpha-1}\,(1-z)^{-\rho}\,(1-t)^{-\lambda}\left(1+\frac{zt}{1-t}\right)^{-\lambda} \tag{66}$$
$$\big(|t|<|1-z|^{-1}\big).$$

Now, in terms of Lauricella's fourth hypergeometric function $F_D^{(n)}$ of n complex variables $z_1,\cdots,z_n$, defined by (see [60,132])

$$F_D^{(n)}[\alpha,\beta_1,\cdots,\beta_n;\gamma;z_1,\cdots,z_n]$$
$$=\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}\,(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}}\,\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!} \tag{67}$$
$$\big(\max\{|z_1|,\cdots,|z_n|\}<1\big),$$

it is known for the Riemann-Liouville fractional derivative operator $\mathfrak{D}_z^{\lambda-\mu}$ that ([14], p. 303, Problem 1)

$$\mathfrak{D}_z^{\lambda-\mu}\left\{z^{\lambda-1}\prod_{j=1}^{n}(1-a_j)^{-\alpha_j}\right\}$$
$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\,F_D^{(n)}[\lambda,\alpha_1,\cdots,\alpha_n;\mu;a_1z,\cdots,a_nz] \tag{68}$$
$$\big(\Re(\lambda)>0;\ \max\{|a_1z|,\cdots,|a_nz|\}<1\big),$$

Since the Lauricella function $F_D^{(2)}$ corresponds to the relatively more familiar Appell function F_1 in two variables, upon applying the Riemann-Liouville fractional derivative operator $\mathfrak{D}_z^{\lambda-\mu}$ on both sides of (66) and the two-variable case of (68), we obtain

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, {}_2F_1\left[\begin{array}{c} \rho-n, \alpha; \\ \\ \beta; \end{array} z\right] t^n = (1-t)^{-\lambda} \, F_1\left(\alpha, \rho, \lambda; \beta; z, -\frac{zt}{1-t}\right) \tag{69}$$

$$\left(|t| < \frac{1}{1+|z|}\right),$$

where we have also applied the principle of analytic continuation for determining the radius of convergence of the series in (69).

It is fairly straightforwad to appropriately apply the operators $\mathcal{L}$ and $\mathcal{L}^{-1}$, together with the method based upon augmentation of parameters and mathematical induction, in order to derive the following general form of (69):

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, {}_{p+1}F_q\left[\begin{array}{c} \rho-n, \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} z\right] t^n \\ &\qquad = (1-t)^{-\lambda} \, F_{q:0;0}^{p:1;1}\left[\begin{array}{c} \alpha_1, \cdots, \alpha_p : \rho; \lambda; \\ \\ \beta_1, \cdots, \beta_q : -; -; \end{array} z, -\frac{zt}{1-t}\right], \end{aligned} \tag{70}$$

where $F_{q:m;\mathfrak{s}}^{p:\ell;\mathfrak{r}}$ denotes a general Kampé de Fériet type two-variable hypergeometric function defined by (see, for details, [60,133])

$$\begin{aligned} &F_{q:\mathfrak{m};\mathfrak{s}}^{p:\ell;\mathfrak{r}}\left[\begin{array}{c} \alpha_1, \cdots, \alpha_p : \rho_1, \cdots, \rho_\ell; \lambda_1, \cdots, \lambda_\mathfrak{r}; \\ \\ \beta_1, \cdots, \beta_q : \sigma_1, \cdots, \sigma_\mathfrak{m}; \mu_1, \cdots, \mu_\mathfrak{s}; \end{array} x, y\right] \\ &\qquad = \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_{m+n} \cdots (\alpha_p)_{m+n} \, (\rho_1)_m \cdots (\rho_\ell)_m \, (\lambda_1)_n \cdots (\lambda_\mathfrak{r})_n}{(\beta_1)_{m+n} \cdots (\beta_q)_{m+n} \, (\sigma_1)_m \cdots (\sigma_\mathfrak{m})_m \, (\mu_1)_n \cdots (\mu_\mathfrak{s})_n} \, \frac{x^m}{m!} \, \frac{y^n}{n!}, \end{aligned} \tag{71}$$

so that, clearly, we have

$$F_1(\alpha, \beta, \beta'; \gamma; x, y) = F_{1:0;0}^{1:1;\mathfrak{r}}\left[\begin{array}{c} \alpha : \beta; \beta'; \\ \\ \gamma : —; —; \end{array} x, y\right].$$

For $\gamma = \beta + \beta'$, the Appell function F_1 is reduced to a hypergeometric ${}_2F_1$ function (see ([60], p. 24, Equation (28))). So, by also using the Pfaff-Kummer transformation (37), we can rewrite the generating function (70) as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, {}_2F_1\left[\begin{array}{c} \lambda+n, \alpha; \\ \\ \beta; \end{array} z\right] t^n \\ &\qquad = (1-t)^{-\lambda} \, {}_2F_1\left[\begin{array}{c} \lambda, \alpha; \\ \\ \beta; \end{array} \frac{z}{1-t}\right] \qquad (|t| < 1). \end{aligned} \tag{72}$$

By the method of augmentation of parameters and the principle of mathematical induction based appropriately upon the operators $\mathcal{L}$ and $\mathcal{L}^{-1}$, this last result (72) can indeed be generalized to the following form:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \,{}_{p+1}F_q \left[\begin{array}{c} \lambda+n, \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} z \right] t^n$$
$$= (1-t)^{-\lambda} \,{}_{p+1}F_q \left[\begin{array}{c} \lambda, \alpha_1, \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} \frac{z}{1-t} \right] \qquad (|t| < 1). \tag{73}$$

It should be remarked in passing that two interesting consequences of the hypergeometric identity (73) were applied by Srivastava [134] in Probability Theory involving bivariate normal distribution and in the Theory of Eigenfunction Expansions involving the classical Hermite polynomials $H_n(x)$.

4. Concluding Remarks and Observations

Many different methods and techniques are used in the theory of generating functions associated with various families of special functions and special polynomials in addition, of course, to the series iteration technique which is based usually upon the following series identities for a suitably bounded double sequence $\{A(k,n)\}_{k,n\in\mathbb{N}_0}$ of essentially arbitrary real or complex numbers (see ([14], p. 101, Lemma 3)):

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} A(k,n-mk) \qquad (m \in \mathbb{N}) \tag{74}$$

and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n+mk) \qquad (m \in \mathbb{N}), \tag{75}$$

provided that each member of the double-series identities (74) and (75) exists, $[\kappa]$ being the greatest integer in $\kappa \in \mathbb{R}$. In our present survey-cum-expository review article, we have aimed at presenting a brief survey of some general families of linear and bilinear generating functions which are associated with the classical orthogonal Jacobi, Laguerre, Hermite and Bessel polynomials, and such other higher transcendental functions as (for example) hypergeometric functions and hypergeometric polynomials in one, two and more variables. The results as well as the methods and techniques used for their derivations, which we have considered here, are intended to provide incentive and motivation for further research on the subject investigated in this article.

The study presented in this article could possibly be extended to the classical orthogonal polynomials of a discrete variable in the sense of Nikiforov, Suslov and Uvarov (see, for details, [135]) or to the classical orthogonal polynomials in the sense of Atakishisyev et al. ([10], p. 189). Some other potential applications of generating functions include (for example) determination of the number of alignments between DNA sequences (see also [8]).

For the theory and applications of the various families of special functions and special polynomials, the interested reader should familiarize also with several other book and monographs such as [135–148]. Each of these books and monographs, together with some other recent developments (see [149–153]), as well as the hypergeometric functions of one and more variables studied in (for example) [154–156], will provide potential situations for the derivation and application of various families of generating functions.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: It gives me great pleasure in expressing my appreciation and sincere thanks to the colleagues in the Mathematics Editorial Office for kindly inviting me to contribute this survey-cum-expository review article to *Mathematics*. This article is being dedicated to the memory of each of the mathematical scientists whose invaluable works and contributions to its subject-matter have been used and cited herein. It was indeed my proud privilege to have personally met many times and in many countries, and also to have discussed mathematical research, especially on various families of higher transcendental functions and related topics, with Joseph Kampé de Fériet (1893–1982), Ralph Philip Boas, Jr. (1912–1992), Bille Chandler Carlson (1924–2013), Leonard Carlitz (1907–1999), Arthur Erdéyi (1908–1977), Charles Fox (1897–1977), Emil Grosswald (1912–1989), Einar Hille (1894–1980), Anatolii Aleksandrovich Kilbas (1948–2010), Harry LeVerne Krall (1907–1994), Yudell Leo Luke (1918–1983), Frank William John Olver (1924–2013), George Pólya (1887–1985), Lucy Joan Slater (1922–2008), Ian Naismith Sneddon (1919–2000), David Vernon Widder (1898–1990) and Edward Maitland Wright (1906–2005). Presumably, their researches and contributions to the field, which we have developed in this article, as well as to various other related fields, will continue to inspire and encourage future researchers in each of these fields.

Conflicts of Interest: The author declares that there is no conflict of interest.

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