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The Greatest Common Decision Maker: A Novel Conflict and Consensus Analysis Compared with Other Voting Procedures

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Abstract: Consensus or conflict agreements, and how these change over time, have significant consequences for understanding the network behavior of human beings, especially when it is necessary to have agreements to move companies and countries forward peacefully. This paper proposes a new Greatest Common Decision Maker (GCDM) aggregation voting procedure applied to square preference matrices of n alternatives and n decision makers. An analysis of the mathematical combinatory ranking of consensus and conflicts generated by the GCDM is realized, and compared to the well-known Borda, Pluralism and Condorcet aggregation procedures to cover the entire class of dynamic accountable group decision-making phenomena. A classification for the family of magic squares is reviewed and it is determined that a conflict decision matrix corresponds to a Latin square. As an original contribution, a 2D color heatmap is generated as a visual tool to compare the consensus and conflict cases generated by the compared methods. Finally, a new consensus reaching model is proposed to compare these aggregation methods defining cost and effort change matrices to convert the cases of conflicts into consensus according to the change in individual preferences. The incorporation of social concepts into our research makes the results obtained stronger.

Keywords: multi-agent consensus; preference aggregation profile; Borda voting procedures; pluralism voting procedures; greatest common decision; Condorcet voting procedures

MSC: 91B12; 91B14; 91B10

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1. Introduction

The challenge of the 21st century, amid globalization, is to build better relations of integration, cooperation, and collaboration between all kinds of groups, organizations, countries, communities, and human beings. Understanding the internal behaviors of a network in relation to decision making is a global issue [1]. Consensus or conflict decisions and how they change over time become relevant and, therefore, it is necessary to negotiate agreements that satisfy all those involved [2,3]. How should these agreements and exchanges be led? Should they be conducted with blockades or in favor of one of the parties? Is conflict necessary for cooperation to exist or does it arise spontaneously [3–6]?

Decision-making analysis and theory provide answers to the question: "How should the decision problem be formulated to confront a decision maker?" The set of answers grows significantly in alternatives, outcomes, and function utilities. Decision making involves resolving between different options to act or not to act. [7,8].

Social choice theory emerged as a set of knowledge to be used in contexts where agents or decision makers need to take collective decisions over a combinatorial space of alternatives. This theory has introduced languages for preference representation and modeling decision making [3,9–12].

Social choice history can be traced back to Jeremy Bentham, Jean-Charles de Borda, and Marquis Condorcet in the 18th century. Later, in 1950, Fishburn and Kenneth Arrow proved that is impossible to aggregate preference relations into a collective (or social) rational preference relation or social choice function that satisfies certain natural axioms or democratic principles and values. This is called Arrow's impossibility theorem [13,14].

A classic impossibility result theorem also exists for judgment aggregation, similar to Arrow's impossibility theorem. It says no judgment aggregation function satisfies the complete combination of the rational social choice criteria [15]. Additionally, several other similar impossibility theorems exist, such as the ANR impossibility theorem, smoothed impossibility [16], Sen's and Muller–Satterthwaite theorems, and the Gibbard–Satterthwaite theorem [17].

Social agents, as decision makers, adopt distinct positions between cooperation and conflict that make it difficult to coordinate a single choice that satisfies their goals in a consensus, or deviate in a conflict or disagreement among them. In this cooperation process, the agents carry out a *consensus reaching process* (CRP) where they can modify their preferences to achieve a high level of agreement if they have a cooperation strategy before achieving a consensus. New paradigms and means of making group decisions have arisen in recent years [18–20].

The aggregation of individual preferences has been the subject of study for over two centuries. Initially used to theorize on elections of the eighteenth century, it has evolved into the social choice theory of today [21]. The most natural aggregation procedure in communities is the simple majority voting; for instance, simple games can provide a generalized interpretation of the notion of a "majority" [22].

The aggregation of individuals into collectivities is not straightforward: one cannot simply sum individual preferences into collective preferences [22]. "*Group decision making* (GDM) is viewed as a task to consolidate and aggregate preferences of a group of agents to find the best collective alternative solution. Consensus is not always the best solution, it becomes the most adequate cooperative answer or choice from a group to a decision problem, otherwise, it is a conflict" [18,20,23].

Conflict can be described as anything that causes a disagreement among people, a personal problem in a complicated situation, or an armed or violent confrontation or opinion contraposition. Conflict also occurs within and among organizations. [1,2,24].

Negotiations, conflict resolutions and consensus agreements prioritize the alternatives to be decided, with preference scales or rankings supported by the opinions, intentions, and thoughts of the decision makers. Conflict resolution and consensus require techniques to find the best nonviolent solution to a conflict between decision makers. [11,24]. The solution to a conflict is a process that seeks all viable alternatives; consensus seeks a dialogue until everyone agrees with a decision, which allows the course of one action to be followed, avoiding the existence of a winner or a loser [25].

The motivations we have are related to philosophy and with the emerging systems in nature and bio-societies. Do conflict and consensus arise naturally, or not? This is our social and psychological question, in the end, to be demonstrated as one of the hypotheses in this research.

Our first social question is deduced from the idea in [26] that "the science of politics combines the individuals in a natural condition of humanity, in a state of nature". This equality between individuals leads to conflict between them because they have a "natural right" to do what they want and what they believe [26–28]. Conflict or consensus can occur or not without the presence of leaders.

Second, our psychological question derives from the idea that, in science, systems, or art, the natural emergence of properties or behaviors occurs when an entity interacts with a wider whole [29]. The natural emergence of behaviors is what happens when a set of relatively simple entities organizes itself spontaneously and without explicit laws, until it gives rise to intelligent behavior [30].

According to Arrow's impossibility theorem, it is clear that an optimal aggregation strategy does not exist; furthermore, *group recommender systems* have confirmed that there is no ultimate winner [31,32].

Different standards and categories have been proposed to support the CRP (see [6,18,23,33–35]). This includes the most recent methods such as any linguistic, fuzzy or social network large-group decision making process that preserves individual consistency through personalized individual semantics (PISs) or uses incomplete hesitant fuzzy linguistic preference relations to control consensus reaching [36–41].

In addition, other vote procedures exist such as first past the post, single transferable vote, additional member system, alternative vote plus, two-round system, alternative vote, supplementary vote, Borda count [14], party list proportional representation [42], amendment, Copeland, Condorcet winner and loser criteria, Dogson, Schwartz, max–min, pluralism, approval, black runoff, Nanson, Hare, Coombs, mutual majority criteria, majority loser criteria, participation criteria, independence of clones, heritage criteria, polynomial time [43], and so on [44].

Main Idea

The main idea of this research is to propose *a simple methodology* that allows analyzing conflict and consensus in decision making and contributes to a better understanding of how they are related.

It is common to find situations in which a certain number of citizens elect among different alternatives and aggregate their preferences into a winner alternative, representative of the society´s choice. When the social choice consists of more than one alternative, usually we obtain tied winners, or a conflict [45]. Classical election methods such as Borda, pluralism, and Condorcet focus on finding only one winning candidate or alternative. Additionally, they have even been compared geometrically [46–49] to outline the optimal method, considering only the winning alternative, but not the ranking among the alternatives. Each method generates a certain quantity of total or partial conflicts. Therefore, the creation of novel methods which increase the consensus and reduce conflict results is an open field of research in the social choice literature.

We found structural conditions of the decisions among groups under equal conditions, where individuals only look to reach their private objectives, with an expectation of consensus and not pursuing a mutual goal group.

In our research, it is not only important to find the winning alternative, but to identify in what linear order the losing alternatives are so that *a ranking is built*, since analyzing the consensus among a group of decision makers leads us to understand the ranked choice of group alternatives. In other words, we aim to identify the group aggregation of the decision maker elections which satisfies their individual preferences, without causing a conflict among them, based on the theory of social choice.

Therefore, we decided to broaden our research perspective from analyzing a winning alternative to *ranking all alternatives. That is, we focus on the comparison between the rankings generated by the aggregation functions of the voting systems* to determine the consensus and conflicts among the decision makers.

There are other studies comparable with our idea [50]; it is important for the consensus reaching process not only to understand the ranking preferences but the disagreements that can lead to costly conflicts. In [51], the problem is approached with the support of a *cardinal ranking method* extended with indices and measures for conflict evaluations. This method collects negative and positive preferences before the alternative of *doing nothing*; this helps to make informed decisions in the face of decisions that may be prone to conflict.

Borda's method naturally gives us a ranking and, based on that ranking, we built our definitions of what *consensus* is (when *the alternatives are well ordered*), what *partial conflict* is (when *there are ties*), and what a *magic square* is (when *there is no ranking* because the group preference assigned to each alternative is the same).

In this article, we propose *a new method of the Greatest Common Decision Maker (GCDM)* as the method that best represents our expectations about the ranking of alternatives by achieving consensus with fewer conflicts when making a comparison with the other methods mentioned.

We propose the GCDM as a new aggregation preference function combining different visual arrangements to understand how to change their positions and preferences to reach a group consensus. This helps for better group decisions, avoiding disagreements such as interpersonal or intragroup conflicts among people gathered as a group, and increases productivity, always looking for negotiation and not competition, submission, or evasion.

The Greatest Common Decision Maker *generates a ranking that allows us to compare it, with the classic voting methods extended* to obtain linear orders or rankings for the group preference alternatives [3].

The purpose of this main idea has a reasonable logic:

We are not looking for a winning alternative, but rather to determine that all alternatives are winners at the same time, under a ranking. We do not want only some of the decision makers to feel satisfied by the group choice, obtaining a winning alternative, but rather that the group of decision makers accepts the consensus, as the one that best satisfies their preferences.

We are not comparing rankings either. *We are comparing the conflicts and consensuses which result from the rankings.* We are not comparing methods to define if they are better or more optimal than others to find the winning alternative or the ranking of alternatives, but to *analyze which methods foster conflict and which foster consensus*. Additionally, we hope to dynamically understand *how we could combine the different voting methods* so that they lead us to a conflict or consensus over a determined time; in other words, we aim to identify the combined application of voting methods that strategically leads us to a consensus or a conflict [3].

The combination of the different voting methods proposes a novel consensus reaching process framework based on effort (costs) or distance consensus changes to dynamically cluster decisions in maps that help to minimize the total consensus efforts or distance values in the consensus reaching process. This is similar to other studies such as [52,53], which are comparable with our idea.

In summary, the idea of *this innovative approach is to make it as simple as possible* to obtain a simple and quite easy-to-handle methodology to analyze consensus and conflict in the process of decision making under a new visual representation.

In Sections 2 and 3, we detail the problem core of our research and the main assumptions. In Section 4, we give and review definitions for the magic squares, the social terms, and the mathematical objects for the formalization of the voting procedures or aggregation functions. In Section 5, there are explanations about the frameworks, tools, and methods from the literature that bear the integration of our model and show the dynamic of our methodology. In Section 6, the main research results are given, and we explain our consensus reaching process, supported by a cost decision visualization. In Section 7, we present a discussion and explain our future work. In Section 8, we conclude the work. Additionally, and Appendix A are presented.

2. Problem Statement

Our research has the purpose of solving the following questions:

1. How, in **negotiation** among agents or groups, **can we help them** reach a consensus, **avoiding conflict** among them? **If each one does not know what the others think** when deciding or choosing an alternative, is it possible **to build an aggregation function** that leads most of the cases to consensus rather than to partial or total conflict?
2. **In self-controlled groups where there is no moderator** who reconciles a solution to conflicts, is it possible **to get out of a conflict by visualizing the advantages or disadvantages of changing decisions at a certain cost**?

There is no single study and software that can solve conflict in decision making, as far as we can tell after reviewing the state-of-the-art literature. To overwhelm the threshold of this problem, our research aims to develop **an innovative methodology to characterize consensus negotiation** [54] in an emerging environment of cooperation [55], or conflict **among multi-agent networks** at a personal or group level.

We think that under certain initial assumptions, the difficulty in modeling [12] such a solution can be overcome with the development of a new simplified preference aggregation and a new visual strategy of costs specifically designed to characterize conflict's preliminary and dynamic procedures.

We think this can add value to the state of the art because *the mathematical solution to a conflict is not always the best one for a real social decision-making problem*. Furthermore, we know that, according to Arrow´s theorem, no optimal maximum solution for a social choice function exists.

3. Assumptions

In this paper, we will focus on **interpersonal and intergroup decisions** (neither armed **nor violent conflicts**) under **descriptive models** with aggregation processes in an egalitarian context, i.e., decision aggregation methods in contexts of **equal conditions** and without advantages for decision makers.

The type of decisions we will be analyzing are those with certainty, finite criteria, and finite alternatives. We assume decision matrices of finite "n" number of alternatives and finite "n" decision makers organized in groups of agents, gathered in a **square matrix of n $\times$ n** or **magic squares** (as per classification systems or ranking systems [44], and not rectangular matrices of m $\times$ n or n $\times$ m). We expect to go through different preference profiles in the consensus reaching process, where consensus is measured by **conflict metrics**.

We assume preference relationships with agents in a group voting simultaneously with no privileged information. In other words, agents are not ordered, and all have the same rights and weights when their preferences are compared in the process of aggregation [56].

Strict and partial linear order relations are assumed, where the weights of preference profiles are linear orders that can be interpreted as follows: a low weight for object i and a high weight for object j implies that object i was periodized and/or voted later than j. Preference profiles are represented in a decision matrix.

The preference aggregation method is equal to **any voting system**. Additionally, we shall say we have a consensus when the aggregated collective preference imposes a linear order over the objects, and a conflict when there no such linear-selected preference exists. Moreover, we are not considering any consistency or consensus in any linguistic, fuzzy, or social network large-group decision making process during the CRP.

Agents are autonomous, have their utility functions, and look for their benefit only, and they are not looking to maximize the complete group social choice function [54–57]. Otherwise, agents are self-interested, and each one would like to obtain the decision that maximizes its utility [56].

The unquestionable properties that are usually required of voting systems to be considered as such are [43,58–60]: respect for anonymity, neutrality, majority criteria, independence from irrelevant alternatives (IAI), consistency or reinforcement [59,60], manipulability, no show paradox, monotony, the weak Pareto principle, and the reverse order paradox or reversal symmetry.

We follow the "rational" criteria of the theory of social choice: *universality* (any linear ordering), ***transitivity*** (the aggregation of group preferences is also a linear ordering), *unanimity* (if alternative "is preferred to" by all voters, then it is also preferred for the group aggregation), and ***absence of a "dictato"***, ***independence of irrelevant alternatives*** (the aggregated preference for a pair of alternatives "x" and "y" depends only on the preferences of the voters for the pair of alternatives "x" and "y").

4. Concepts and Definitions

4.1. Magic Square and Conflict Matrix Classification

Reviewing the literature, we notice that there is no clear agreement on how to classify and name the magic squares, so we proposed the following classifications:

The general magic square is a square matrix of order n × n, formed by "n^2" letters or different, non-negative integers. With the following properties, each letter or each number appears exactly once in each column or each row, or each main diagonal. In the case of numbers, the sum of "n" different numbers in each row, column, and the two main diagonals are the same. This common sum is known as a constant or **magic number** (see Figure 1) [61–66].

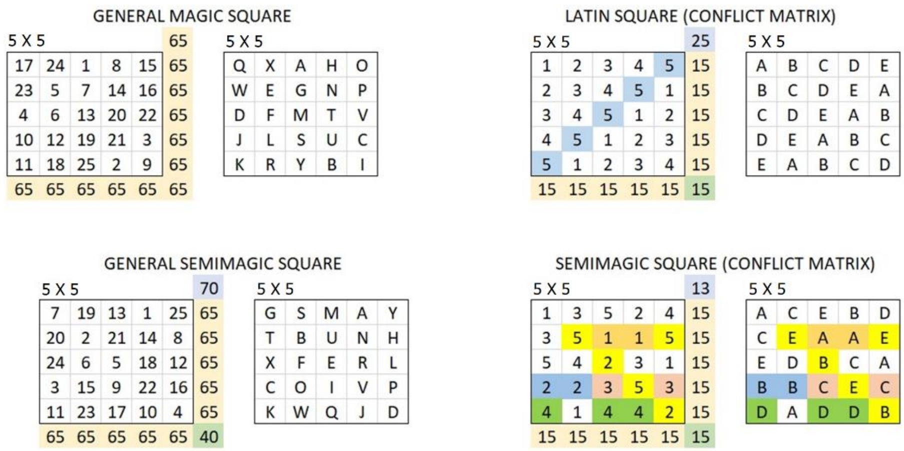


Figure 1. Different types of magic squares.

The general semi-magic square is a general magic square where the sum of the numbers of its main diagonals does not result in a magic number (see Figure 1) [61,62,65,66].

The Latin square (conflict matrix) is a general semi-magical square formed by "n" (and not "n^2") different letters or numbers per line or column, and arranged in some order such that they cannot be repeated either by column or by line. That is, no number appears twice in the same row or column (or in other words, each row and column is a permutation of the "n" different letters or numbers). Although, both the letters and the numbers could be repeated in the main diagonals (see Figure 1) [63,64,67–70].

The semi-magic square (conflict matrix), the object of our research (also known as a **normal magic square**) is a Latin square that allows letters or numbers to be repeated exclusively per line. It is a non-general semi-magic square (see Figure 1) [71,72].

To understand the relationships among the four classification squares, review Figure 1. Additionally, the sets and subsets of them are shown in Figure 2.

We found that **Arrow´s theorem** can be represented with the appliance of ***conflict matrices***. Conflict matrices or magic squares illustrate the fact of what this theorem states *"for a finite number of voters greater than one, and a number of alternatives greater than or equal to three is not possible to design a voting system that reflects the satisfactory aggregation of all individual preferences of a community into a group preference"* [73]. A conflict matrix does not allow a unique aggregated group preference *to be established* as a result of the disagreement among the individual preferences one by one in a group.

Therefore, to find optimal solutions to the process of seeking consensus negotiation, it is necessary to ***assemble voting systems*** [32].

4.2. Definitions for Voting Procedures

These are the definitions that we use and create inspired by the literature to build our methodology. The descriptions of variables and abbreviations are presented in the Appendix A.

Alternatives is a set of n objects or situations ($\mathbf{X} = \{x_1, x_2, \ldots, x_n\}$ = {alternatives x_i}, $n > 0$) ranked or ordered by agents or decision makers [2,8,20,23,56,57,59,60,74] (see Figure 3).

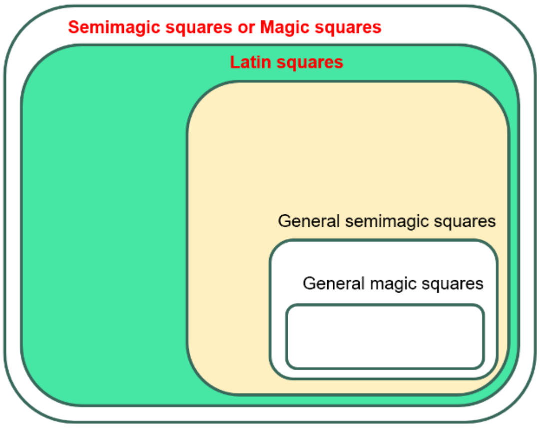
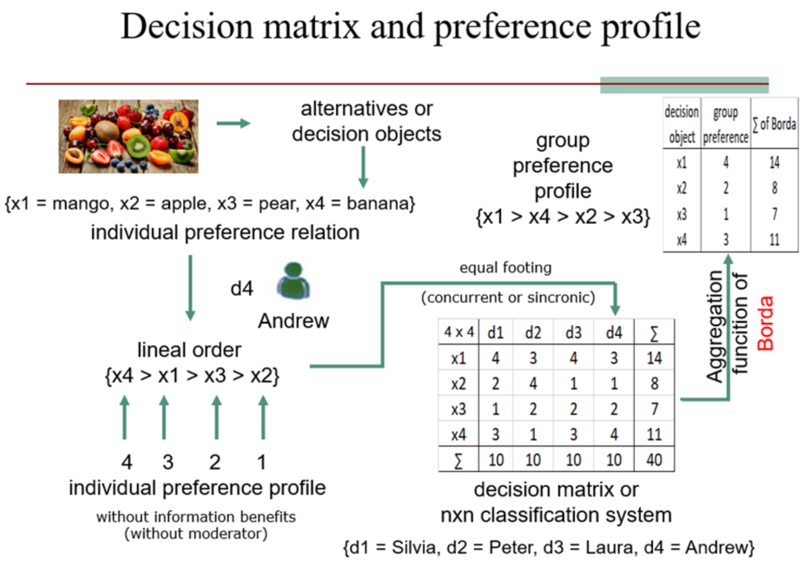


Figure 2. Sets and subsets of magic squares.



4 x 4	d1	d2	d3	d4	∑
x1	4	3	4	3	14
x2	2	4	1	1	8
x3	1	2	2	2	7
x4	3	1	3	4	11
∑	10	10	10	10	40

decision object	group preference	∑ of Borda
x1	4	14
x2	2	8
x3	1	7
x4	3	11

Figure 3. Decision matrix case 4×4 with Borda method.

Agents or **decision makers** are a panel of experts ($\mathbf{D} = \{d_1, d_2, \ldots, d_m\}$ = {agents or decision makers d_j}, $m > 0$) who express their preferences on the alternatives in **X** in the most natural way without any influence [2,9,20,23,54,56,57,73–75] (see Figure 3).

Preference weights are a set of values; $\mathbf{W} = \{w_1, w_2 \ldots, w_n,$ where $w_i \in \mathbb{N}\}$.$\mathbb{N}$ is the set of natural numbers. The weights $w_i \in \mathbf{W}$ are assigned to a series of alternatives x_j by a decision maker in **D**, where "i" could be equal or not equal to "j". Preferences have a linear order **R**, from the most preferred object to the least preferred object. The total number of different preferences is "n!" [2,23,56,57,59,60,74] (see Figure 3).

The most preferred alternative for an agent is known as the **agent's ideal**. The total number of distinct ideal preferences is "n". Additionally, the total number of distinct ideal preference patterns is "n^m", where "m" is the number of agents

$\mathbf{R_j} = \{w_{1j}, w_{2j}, \ldots, w_{nj}\}$ represents one of the possible **individual preferences** (or permutation without repetition over the alternatives of X) with $w_{ij} \in \mathbf{W}$, which is associated with the decision maker d_j in D (see Figure 3) [2,8,56,57].

$\check{\mathbf{R}}_\mathbf{k} = \{w_{1k}, w_{2k}, \ldots, w_{nk}\}$ represents one of the possible **group preferences** with $w_{ij} \in \mathbf{W}$, which is associated with the collective preference of the "m" agents in set D, with $1 \le k \le n!^m$ (see Appendix B). This corresponds to the output of the preference aggregation rule process or function [60] described before (see Figure 3) [2,56].

The preference profile or **pattern** $M = [P_1, P_2, \ldots, P_m]$ is an array that represents the preference data collected from "m" agents, where $0 < j \leq m$, and the jth agent uses the linear order or preference order $\mathbf{P_j}$ to represent his or her preferences [2,23,57,76]. The total number of different profiles of a group of agents is equal to "$n!^m$".

Decision matrix [77] is a selection matrix or **preference profile M** that helps to evaluate and prioritize a list of alternatives in a group. The decision makers determine a preference profile over each alternative to be elected under their criteria, and this is added as a column to the decision matrix (see Figure 3).

M is a decision matrix of dimension nxm, and it is made up of "n" rows (the alternatives) and "m" columns (the decision makers), such that each column P_j of the matrix **M** has dimension nx1 in the following way: $P_j = {R_j}^T$, where $R_j = (x_{1j}, x_{2j}, \ldots, x_{nj})$, that is, $M = [P_1, P_2, \ldots, P_m] = [{R_1}^T, {R_2}^T, \ldots, {R_m}^T] = [x_{ij}]_{(n \times m)}$

$\mathbf{M_k}$ is one of the possible decision matrices (or group choice profile) which represents the individual preference arrangement of a set of agents D in a decision event where $k = 1, 2, \ldots, n!^m$ (see Figure 3).

Unanimity occurs when the "m" agents or decision makers have exactly the same preferences. Additionally, consensus on a proposal does not mean that everyone is unanimously in agreement.

Consensus is a mutual agreement between the members of a group or network where the decision or consensus satisfies almost all the goals and objectives of the individuals, but not necessarily their choices. Consensus implies consent among all the participants, and tries to solve and reduce the objections of the majority or the minority at the same time to reach the most convenient decision for the whole group [2,18,60].

In the consensus process, decision makers exchange judgments, opinions, and relevant information to expose arguments and deliberate the reasons for a certain alternative [15,20]. **Consensus grade** is the level of the preference values or imprecision coefficient regarding mutual agreement, i.e., cooperative or conflicting behavior. Consensus is not the election or decision that looks for the maximum element that maximizes all the individual agents' utility functions, but the greatest one under a unified negotiation protocol [54]. It is proven that there several nonequivalent consensuses may exist [78].

A **preference consensus** occurs if the agents have identical or similar preference rankings [2,60], and an **ideal consensus** occurs if the agents have identical or similar ideals. A **conflict** occurs if there is no consensus.

Conflict is the opposition or total disagreement among people or groups due to the contradictory coexistence among the participants, i.e., whenever participants do not suit one another. The input for modeling a conflict [12] is the preference that each decision maker has with respect to a variety of feasible objects, alternatives, states, or outcomes that could be elected [2,23].

Partial consensus or **partial conflict** or **dissent**: The opposite of consensus is dissent. **Dissent** is not synonymous with confrontation or conflict, but the tolerance of the opinion of the majority regarding the common good. **Partial consensus** or **partial conflict** is not an agreement among all the participants [60].

It is necessary to determine when a group of decision makers is under a consensus or conflict, after applying an aggregation function to their preferences over the alternatives.

Types of decision matrices are those that, transformed by an **aggregation function**, represent the **preference profile of a group** through a **group preference** $\check{\mathbf{R}}_\mathbf{k}$. The **consensus matrix** *establishes* a strict linear order relationship for group preference. The **conflict matrix** *does not establish* a strict linear order relationship for group preference. A **consensus or partial conflict matrix** is the one that collects the cases of matrices *that do not comply with consensus or conflict matrices.*

O(v) represents a **conflict metric** or **function** that calculates the **degree of conflict–consensus** on a weight vector "v" of dimension (n × 1), resulting from the application of an aggregation function to a decision matrix, as follows:

$$\mathbf{O} : \mathbb{N}^{\mathrm{n}} \rightarrow \{1,2,3\}, \ \mathbb{N} \text{ is the set of natural numbers}$$

$$\mathbf{O}(\mathrm{v}) = \mathbf{O}\begin{pmatrix} \mathrm{v1} \\ \mathrm{v2} \\ \cdots \\ \mathrm{vn} \end{pmatrix} = \begin{cases} 1 \text{ if vi} \neq \text{vj} \ \forall \ \text{i} \neq \text{j (consensus)} \\ 3 \text{ if vi} = \text{vj} \forall \ \text{i} \neq \text{j (conflict)} \\ 2 \text{ otherwise (partial conflict or consensus)} \end{cases}$$

Preference aggregation (or consensus reaching process CRP) is a decision rule aggregation approach (process, method, or algorithm) that combines or aggregates a set of preference profiles of individual agents in one group with only one group preference $\check{\mathbf{R}}_{\mathbf{k}}$ or social choice [44]. Preference aggregation is known in the literature as **the aggregation function** [8,9,20,44,59,73,74] **or aggregation operator** [20] that maps the set of n x m dimensional decision matrices M to the set of individual or group preferences [35,79,80], electing one as the group preference $\check{\mathbf{R}}_{\mathbf{k}}$ or social choice. Therefore, a **social choice function** is a preference aggregation that maps an elected collective preference profile into a single preference order [6,44]. The aggregation criteria or algorithm may be a utility function or a voting process or system, or a particular election theory or methodology agreed upon by multiple agents or the group, with certain properties, principles or rules, certainty or uncertainty, with an adversary or without an adversary (see Figure 3).

The process of modeling the preference profiles and the decision matrix shown in Figure 3 starts from the alternatives or decision objects. In this example, fruits that the agent Andrew will order linearly according to his taste and preference are: banana > mango > pear > apple. This will be integrated into a decision matrix in equal conditions to those of the other participating agents: Silvia, Peter, and Laura. Additionally, through an aggregation function related to a voting system, in particular, the group preference is also ordered linearly (mango > banana > apple > pear) according to the result of the aggregation function. In this example, the aggregation function is the sum used in the Borda method.

We decided to standardize the voting procedure concepts with the help of mathematical objects to systematize the traditional definitions in the literature such as the Borda, pluralism, and Condorcet aggregation functions [44]. This helped us to understand the computation algorithms behind each voting system we studied. This section could also be seen as part of our results, because we proposed a new aggregation function that we called **the Greatest Common Decision Maker (GCD)**.

4.3. Borda Voting Procedure

Borda is a positional voting procedure where each voter provides a ranking in the group decision problem [14,77,81,82]. An individual preference profile is expressed in terms of an ordinal ranking over the available alternatives to choose from [23]. The purpose is to combine them into a group preference or consensus ordinal ranking [21,83].

B($\mathbf{M_k}$) represents the **Borda aggregation function** applied to a matrix of decision $\mathbf{M_k}$ of dimension (n × n) and $\mathrm{I}_{(\mathrm{n} \times 1)}$ = (1, 1, . . . , 1)$^{\mathrm{T}}$ in the following way:

$$\mathbf{B} : \mathbb{N}^{\mathrm{n} \times \mathrm{n}} \rightarrow \mathbb{N}^{\mathrm{n}}, \text{ where } \mathbb{N} \text{ is the set of natural numbers}$$

$$\mathbf{B}(\mathbf{M_k}) = \mathrm{M_k}\ \mathrm{I}_{(\mathrm{n} \times 1)} = \begin{pmatrix} \sum_{j=1}^{n} x1j \\ \sum_{j=1}^{n} x2j \\ \cdots \\ \sum_{j=1}^{n} xnj \end{pmatrix} = \begin{pmatrix} \mathrm{B1} \\ \mathrm{B2} \\ \cdots \\ \mathrm{Bn} \end{pmatrix} = \check{\mathrm{R}}_{\mathrm{k}} \text{ and } \mathrm{O}\begin{pmatrix} \mathrm{B1} \\ \mathrm{B2} \\ \cdots \\ \mathrm{Bn} \end{pmatrix}$$

The Borda voting method supposes that there are "n" candidates. The voters assign "1" to the candidate ranked last, and "n" to the candidate ranked first according to their preferences on a ballot. The Borda procedure consists of the sum of the ranks assigned by each decision maker to each candidate, and the one with the most points is the winner, or Borda social choice [42,83,84]. Therefore, the relative magnitudes of each sum induce a new ordinal ordering over the alternatives [21,23,81,85]. The compromise is to obtain the best agreement for the assignment of a collective ordinal ranking.

In other words, the rows and weights of the decision matrix are added to obtain the points each candidate receives as the last column. This column of sums represents the group ranking preference linear order assigned to the candidates, or alternatives by Borda. The degree of conflict–consensus is calculated. The Borda aggregation matrix crosses the alternatives against the decision makers.

4.4. Stronger Alternative (Candidate) Profile Anatomy Matrix

In the pluralism and majority voting system, the winner of an election is the candidate that received more votes than the others [86]. We propose here a **stronger alternative (candidate) profile anatomy matrix** to identify the anatomy votes each candidate receives, identifying not only the winner but a **function to obtain the "r" preference weight profile submatrices,** where "**r** represents the $\mathbf{w_r}$ value in the set **W** of *preference weights*".

$\mathbf{SUB^r(M_k)}$ represents the **function to obtain the** $\mathbf{^rSB^k}$ **weight profile submatrix** of dimension (n × n) for the set of *preference weight* values $\mathbf{W} = \{w_1, w_2 \ldots, \mathbf{w_r}, \ldots, w_n\}$ of the decision matrix $\mathbf{M_k}$, with elements "$\mathbf{^rsub^k{}_{ij}}$", in the following way:

$\mathbf{^rSB^k}$ is the *weight submatrix profile* with "i, j = 1, 2, . . . , n", and elements "$\mathbf{^rsub^k{}_{ij}}$".

$$\mathbf{SUB^r} : \mathbb{N}^{n\times n} \to \mathbb{N}^{n\times n}, \text{ where } \mathbb{N} \text{ is the set of natural numbers}$$

$$\mathbf{^rsub^k{}_{ij}} = \begin{cases} 0 \text{ if } m_{ij} \neq r \\ r \text{ if } m_{ij} = r \end{cases}$$

$$\mathbf{SUB^r}(\mathbf{M_k}) = {}^r\mathrm{SB^k} = \begin{pmatrix} ^r\mathrm{sub^k{}_{11}} & ^r\mathrm{sub^k{}_{12}} & \ldots & ^r\mathrm{sub^k{}_{1n}} \\ ^r\mathrm{sub^k{}_{21}} & ^r\mathrm{sub^k{}_{22}} & \ldots & ^r\mathrm{sub^k{}_{2n}} \\ \ldots & \ldots & \ldots & \ldots \\ ^r\mathrm{sub^k{}_{n1}} & ^r\mathrm{sub^k{}_{n2}} & \ldots & ^r\mathrm{sub^k{}_{nn}} \end{pmatrix}$$

$\mathbf{SAPA(M_k)}$ represents the **function to obtain the** $\mathbf{SP^k}$ **stronger alternative (candidate) profile matrix** of dimension (n × n) by weighting the alternatives of a decision matrix $\mathbf{M_k}$ and $I_{(n \times 1)} = (1, 1, \ldots, 1)^T$.

$\mathbf{SP^k}$ is the *stronger alternative (candidate) profile matrix* with "i, j = 1, 2, . . . , n", and elements " $\mathbf{sp^k{}_{ij}}$ ".

$\mathbf{SAPA} : \mathbb{N}^{n\times n} \to \mathbb{N}^{n\times n}$, where $\mathbb{N}$ is the set of natural numbers, and given "r", then

$$\mathbf{colsp^k{}_r} = \begin{pmatrix} \mathrm{sp^k{}_{1r}} \\ \mathrm{sp^k{}_{2r}} \\ \ldots \\ \mathrm{sp^k{}_{nr}} \end{pmatrix} = {}^r\mathrm{SB^k}\, I_{(n\times 1)} = \begin{pmatrix} ^r\mathrm{sub^k{}_{11}} & ^r\mathrm{sub^k{}_{12}} & \ldots & ^r\mathrm{sub^k{}_{1n}} \\ ^r\mathrm{sub^k{}_{21}} & ^r\mathrm{sub^k{}_{22}} & \ldots & ^r\mathrm{sub^k{}_{2n}} \\ \ldots & \ldots & \ldots & \ldots \\ ^r\mathrm{sub^k{}_{n1}} & ^r\mathrm{sub^k{}_{n2}} & \ldots & ^r\mathrm{sub^k{}_{nn}} \end{pmatrix} I_{(n\times 1)} = \begin{pmatrix} \sum_{j=1}^{n} {}^r\mathrm{sub^k{}_{1j}} \\ \sum_{j=1}^{n} {}^r\mathrm{sub^k{}_{2j}} \\ \ldots \\ \sum_{j=1}^{n} {}^r\mathrm{sub^k{}_{nj}} \end{pmatrix}$$

$$\mathbf{SAPA}(\mathbf{M_k}) = \mathbf{SP^k} = [\mathrm{colsp^k{}_1}, \mathrm{colsp^k{}_2}, \ldots, \mathrm{colsp^k{}_n}] = \begin{pmatrix} \mathrm{sp^k{}_{11}} & \mathrm{sp^k{}_{12}} & \ldots & \mathrm{sp^k{}_{1n}} \\ \mathrm{sp^k{}_{21}} & \mathrm{sp^k{}_{22}} & \ldots & \mathrm{sp^k{}_{2n}} \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{sp^k{}_{n1}} & \mathrm{sp^k{}_{n2}} & \ldots & \mathrm{sp^k{}_{nn}} \end{pmatrix}$$

The *$^rSB^k$ weight submatrix profile* identifies the allocated or positioned *weight values* assigned by the decision makers to the alternatives in the *M_k decision matrix*. Therefore, each column in the *SP stronger alternative (candidate) profile matrix* represents the assigned weight rank to the preferences of decision makers in the *M_k decision matrix*. The SP profile matrix will support our explanations for pluralism and the pluralism ranking voting procedures.

4.5. Pluralism Voting Procedure

A voting method in which the candidate or alternative with the most votes wins is called a **simple majority rule or method**. Plurality is a well-known voting system where each voter is asked only for their first preference (or none if they abstain) [43,58,60,73], and a candidate wins when they receive the greatest number of votes with the most first-place votes [87]. The winning candidate only needs to obtain more votes than any other opponent; a majority is not required, but more votes than the opposition combined [73,84,88].

Plu($^{\mathbf{r}}$SB$^{\mathbf{k}}$) represents the **pluralism aggregation** function applied to a $^{\mathbf{r}}\mathbf{SB}^{\mathbf{k}}$ matrix of dimension (n × n), where "r" corresponds to the maximum weight preference of $\mathbf{M_k}$ in the following way:

$$\mathbf{Plu}\ :\ \mathbb{N}^{n\times n} \rightarrow\ \mathbb{N}^{n},\ \text{where } \mathbb{N} \text{ is the set of natural numbers}$$

$$\mathbf{Plu}\left({}^{\mathbf{r}}\mathbf{SB}^{\mathbf{k}}\right)\ = {}^{r}SB^{k}\ I_{(n\,\times\,1)} = \begin{pmatrix} sp^{k}{}_{1r} \\ sp^{k}{}_{2r} \\ \dots \\ sp^{k}{}_{nr} \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ \dots \\ P_n \end{pmatrix} = \check{R}_k \text{ and } O \begin{pmatrix} P_1 \\ P_2 \\ \dots \\ P_n \end{pmatrix}$$

The pluralism group preference social choice is calculated based on the pluralism aggregation function and the preference weight profile submatrix that represents the most first-place votes of each candidate. Therefore, the obtained vector (column) of values shows the won number of votes each candidate has as a first-place option, i.e., the candidates are ranked according to the number of first-place votes [73,84,88].

Both the majority and the plurality winners are able to be determined. The degree of conflict–consensus is calculated, and the pluralism aggregation social choice vector crosses the alternatives against the weights. Although the plurality voting procedure is a method that looks for a winner, it can be seen as an incomplete candidate ranking of the most first-place votes [84].

4.6. Pluralism Ranking Voting Procedure

Unlike *the simple pluralism voting procedure*, we propose a new *pluralism ranking method* to determine not only the winner of the ballot but the linear order or ranking of the candidates based on the weight group preferences and vote place assigned to candidates; note that, to the best of our knowledge, we could not find a natural extended procedure for plurality in the literature, except Hare and Coombs majority procedures, which can be adapted to pluralism [84].

Plura(SP$^{\mathbf{k}}$) represents the **pluralism ranking aggregation** function applied to an $\mathbf{SP}^{\mathbf{k}}$ of dimension (n × n) and $I_{(n\,\times\,1)} = (1, 1, \dots, 1)^T$ in the following way:

$$\mathbf{Plura}\ :\ \mathbb{N}^{n\times n} \rightarrow\ \mathbb{N}^{n},\ \text{where } \mathbb{N} \text{ is the set of natural numbers}$$

$$\mathbf{Plura}\left(\mathbf{SP}^{\mathbf{k}}\right)\ = \max\left(SP^{k}\right) = \begin{pmatrix} \max\left\{sp^{k}{}_{11},\ sp^{k}{}_{12},\ \dots,\ sp^{k}{}_{1n}\right\} \\ \max\left\{sp^{k}{}_{21},\ sp^{k}{}_{22}, \dots,\ sp^{k}{}_{2n}\right\} \\ \dots \qquad \dots \quad \dots \quad \dots \\ \max\left\{sp^{k}{}_{n1},\ sp^{k}{}_{n2}, \dots,\ sp^{k}{}_{nn}\right\} \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ \dots \\ P_n \end{pmatrix} = \check{R}_k \text{ and } O \begin{pmatrix} P_1 \\ P_2 \\ \dots \\ P_n \end{pmatrix}$$

The pluralism ranking is calculated based on the *stronger alternative (candidate) profile matrix* that represents the anatomy of the group preference votes of each candidate, according to their *weights* in the *decision matrix* and each *weight submatrix profile.* Therefore, the obtained vector (column) of values combines the number of votes for each weight and the related candidates obtained in the ballot, and represents the preference order assigned to the candidates. The degree of conflict–consensus is calculated. The purpose is not to obtain the majority winner, but the complete ranking votes of the candidates [84].

4.7. Condorcet's Ranking Voting Procedure

Condorcet (1743–1794) postulated his own model based on philosophical arguments; he was Borda's main antagonist. According to his model, the winning candidate is the one who defeats the rest of the candidates by a simple majority in pairwise comparisons, which implies that there will not always be a winner [58,73,82,83,89].

Condorcet is a voting method where the candidate who is preferred over all others always wins. Here, as in the other methods, the voters or decision makers rank candidates in order of preference. The Condorcet ranking of candidates is obtained by counting the number of wins for each candidate when all candidates are run head-to-head in *simple majority* elections [73,83,84,90]. Similarly to Borda´s voting procedure, this method also naturally gives us a ranking based on the comparisons completed to obtain each candidate´s number of votes.

$\mathbf{M^k}_{\mathbf{co}}$ represents the **Condorcet matrix** of dimension $(n \times n) \subseteq X^2$, obtained from a decision matrix $\mathbf{M_k}$ of dimension $(n \times n)$ in the following way:

$$\mathbf{coSub^k}_{\mathbf{rp}} = \{x_{rj} \in M_k \mid x_{rj} > x_{pj};\ \forall j = 1, \ldots, n; d_j \in D\};\ \forall\, r, p = 1, \ldots, n$$

then

$$\mathbf{M^k}_{\mathbf{co}} = \left[\#\mathrm{coSub^k}_{rp}\right] = [\mathrm{mco^k}_{rp}],\ \forall\, r, p = 1, \ldots, n$$

Co($\mathbf{M^k}_{\mathbf{co}}$) represents the **Condorcet aggregation function** applied to a matrix of Condorcet $\mathbf{M^k}_{\mathbf{co}}$ of dimension $(n \times n)$ and $I_{(n \times 1)} = (1, 1, \ldots, 1)^T$ in the following way:

$$\mathbf{Co} : \mathbb{N}^{n \times n} \to \mathbb{N}^n,\ \text{where } \mathbb{N} \text{ is the set of natural numbers}$$

$$\mathbf{Co}(\mathbf{M^k}_{\mathbf{co}}) = M^k{}_{co}\, I_{(n \times 1)} = \begin{pmatrix} \sum_{j=1}^{n} \mathrm{mco1j} \\ \sum_{j=1}^{n} \mathrm{mco2j} \\ \cdots \\ \sum_{j=1}^{n} \mathrm{mconj} \end{pmatrix} = \begin{pmatrix} \mathrm{co1} \\ \mathrm{co2} \\ \cdots \\ \mathrm{con} \end{pmatrix} = \check{R}_k \text{ and } O \begin{pmatrix} \mathrm{co1} \\ \mathrm{co2} \\ \cdots \\ \mathrm{con} \end{pmatrix}$$

The Condorcet aggregation matrix crosses alternatives against alternatives. A value in a row assigned to one candidate shows the counting number of times that the row candidate has a greater rank compared with the candidate represented in each column. The last column is calculated with the sums of the values by rows and helps to assign a group preference ranking linear order by Condorcet to the candidates or alternatives. The degree of conflict–consensus is calculated.

4.8. Greatest Common Decision Maker Ranking Voting Procedure

We propose this new voting procedure or aggregation function as a composition of Condorcet and Borda voting procedures in an innovative and strategical mixture that, as far as we know, is not yet reported in the literature. We call it the Greatest Common Decision Maker because it represents the main purpose of how to obtain the group preference. We wanted to find out the most coincidences among the decision makers. This means that the greatest common agent preference could be considered to **represent the group preference profile**. We called it the Greatest Common Decision Maker because it resembles the **greatest common divisor** procedure in mathematics.

In philosophy, the Greatest Common Decision Maker could be related to "the general will of society, that ... generalizes ... not so much the number of votes as the common interest that unites them" [91] (pp. 111–112). "Populism (or pluralism), by contrast, is a majority that would seek to unfairly appropriate the private goods because it does not seek the common good" [91] (p. 115), or the common decision of society.

$\mathbf{M^k}_{\mathbf{mcd}}$ represents the **Mix matrix** of dimension (m × n), where m = n(n − 1)/2 and is obtained from a decision matrix $\mathbf{M_k}$ of dimension (n × n) in the following way:

$$\sup(x_{ij},\ x_{rj}) = \begin{cases} 1 \text{ if } x_{ij}\ >\ x_{rj},\ i \neq r \\ 0 \text{ if } x_{ij}\ <\ x_{rj}, i \neq r \end{cases}$$

$$\mathbf{M^k}_{\mathbf{mcd}} = \begin{pmatrix} \sup(x_{1j},\ x_{2j}) \\ \sup(x_{1j},\ x_{3j}) \\ \ldots \\ \sup(x_{1j},\ x_{nj}) \\ \sup(x_{2j},\ x_{3j}) \\ \sup(x_{2j},\ x_{4j}) \\ \ldots \\ \sup(x_{2j},\ x_{nj}) \\ \ldots \\ \sup\left(x_{ij},\ x_{i(+1j)}\right) \\ \sup\left(x_{ij},\ x_{i(+2j)}\right) \\ \ldots \\ \sup(x_{ij},\ x_{nj}) \\ \ldots \\ \sup\left(x_{n\ (-2j)},\ x_{n\ (-1j)}\right) \\ \sup\left(x_{n\ (-2j)},\ x_{nj}\right) \\ \sup\left(x_{n\ (-j)},\ x_{nj}\right) \end{pmatrix} = \begin{pmatrix} {mix^k}_{1j} \\ {mix^k}_{2j} \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ {mix^k}_{ij} \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ {mix^k}_{m\ (-1j)} \\ {mix^k}_{mj} \end{pmatrix} = [{mix^k}_{ij}]; \forall j = 1,\ \ldots,\ n;\ d_j \in D;\ \forall\ i = 1,\ \ldots,\ m$$

$\mathbf{M^k}_{\mathbf{mcdco}}$ represents the **Condorcet mix matrix** of dimension (n × n) $\subseteq D^2$, obtained from the transpose of a Mix matrix $\mathbf{M^k}_{\mathbf{mcd}}$ of dimension (m × n), m = n(n − 1)/2, in the following way:

$$\mathbf{mcdcoSub^k}_{\mathbf{rp}} = \left\{x_{rj}\ \in\ {\mathbf{M^k}_{\mathbf{mcd}}}^{\mathbf{T}}\ \middle|\ x_{rj}\ >\ x_{pj}\ ;\ \forall j = 1, \ldots,\ m; d_r, d_p \in D\right\},\ \forall\ r, p\ = 1, \ldots, n$$

$$\mathbf{M^k}_{\mathbf{mcdco}}\ = \left[\#{mcdcoSub^k}_{rp}\right] = [{mcdco^k}_{rp}],\ \forall\ r, p = 1,\ \ldots,\ n$$

The Condorcet mix matrix crosses alternatives against decision makers. It counts, using the columns or the decision maker, how many times each alternative has a greater rank compared with the other alternatives in the decision matrix. The last column is calculated with the sums of the values by rows, and we propose that it helps to build the Condorcet matrix.

GCD($\mathbf{M^k}_{\mathbf{mcdco}}$) represents the **aggregation function of the Greatest Common Decision Maker** applied to a Condorcet mix matrix $\mathbf{M^k}_{\mathbf{mcdco}}$ of dimension (n × n) and $I_{(n\ \times\ 1)}$ = (1, 1, . . . , 1)^T in the following way:

$$\mathbf{GCD}\ :\ \mathbb{N}^{n\times n} \rightarrow\ \mathbb{N}^n,\ \text{where } \mathbb{N} \text{ is the set of natural numbers}$$

$$\mathbf{GCD}(\mathbf{M^k}_{\mathbf{mcdco}}) = {M^k}_{mcdco}\ I_{(n\ \times\ 1)} = \begin{pmatrix} \sum_{j=1}^{n} \text{mcdco1j} \\ \sum_{j=1}^{n} \text{mcdco2j} \\ \ldots \\ \sum_{j=1}^{n} \text{mcdconj} \end{pmatrix} = \begin{pmatrix} \text{GCD1} \\ \text{GCD2} \\ \ldots \\ \text{GCDn} \end{pmatrix} = \check{R}_k \text{ and } O'$$

Once the Condorcet mix matrix is obtained, based on it, the Greatest Common Decision Maker matrix is built. Each decision maker column is compared with other decision makers and the coincidences in the row values are counted one-to-one; each decision maker row value in the GCD matrix shows the coincidences with the column decision maker. Therefore, we found the Greatest Decision Maker by adding the coincidence values of the columns, and the column that accumulates the greatest value compared with the others is the chosen one. The decision maker assigned to this column in the decision matrix represents the group preference by GCD. The degree of conflict–consensus is calculated.

Note that we are using O′ instead of O as a conflict metric because the O metric needs to be adjusted to identify the underlying or subjacent parallel conflicts and ties among decision maker pairs. The O conflict metric is useful only for Borda, pluralism, and Condorcet. *Here, our sense of consensus and conflict needs to turn from weighted alternatives to weighted decision makers, as comparisons will be performed among decision makers to determine the common choice*. However, calculating the degree of conflict–consensus implies understanding in detail the behavior of the underlying rectangle matrices that could represent ties or not, and this is out of the scope of this article review, so will be left for future work. This is one of the disadvantages of the GCM aggregation function.

We will be explaining the O′ conflict metric for 3 × 3 and 4 × 4 decision matrix dimensions cases to show its procedures, but the remaining cases for n × n decision matrix dimensions will be left for future work.

To build an appropriate O′ conflict metric, it is necessary to define the $\mathbf{M^{k}{}_{cod}}$ matrix that represents the **Condorcet decision maker's matrix** of dimension $(n \times n) \subseteq D^2$, obtained from a decision matrix $\mathbf{M_k}$ of dimension (n × n) in the following way:

$$\mathbf{codSub^{k}{}_{rp}} = \{x_{ip} \in M_k \mid x_{ir} = x_{ip};\ \forall i = 1, \ldots, n; x_i \in X\};\ \forall\, r,\ p = 1, \ldots, n$$

then

$$\mathbf{M^{k}{}_{cod}} = \left[\#codSub^{k}{}_{rp}\right] = [mcod^{k}{}_{rp}],\ \forall\, r, p = 1, \ldots, n$$

CoD($\mathbf{M^{k}{}_{cod}}$) represents the **Condorcet decision maker's aggregation function** applied to a matrix of Condorcet decision makers $\mathbf{M^{k}{}_{cod}}$ of dimension (n × n) and $I_{(n \times 1)} = (1, 1, \ldots, 1)^T$ in the following way:

$$\mathbf{CoD} : \mathbb{N}^{n\times n} \to \mathbb{N}^{n}, \text{ where } \mathbb{N} \text{ is the set of natural numbers}$$

$$\mathbf{CoD}(\mathbf{M^{k}{}_{cod}}) = M^{k}{}_{cod}\, I_{(n \times 1)} = \begin{pmatrix} \sum_{j=1}^{n} mcod1j \\ \sum_{j=1}^{n} mcod2j \\ \ldots \\ \sum_{j=1}^{n} mcodnj \end{pmatrix} = \begin{pmatrix} CoD1 \\ CoD2 \\ \ldots \\ CoDn \end{pmatrix} = \check{R}_k$$

The Condorcet decision maker's aggregation matrix crosses decision makers against decision makers. A value in a row assigned to one decision maker shows the counting number of times that the row decision maker has equal rank compared with the decision maker represented in each column. The last column is calculated with the sums of the values by rows and helps to assign the group preference ranking order given by Condorcet to the decision makers.

$\mathbf{O'_{3\times 3}(GCD(M^{3}{}_{mcdco}),\ CoD(M^{3}{}_{cod}))}$ represents a **conflict metric** or **function** that calculates the **GCD degree of conflict–consensus** on a series of weighted vectors "v" of dimension (3 × 1), resulting from the application of the GCD method to a decision matrix of dimension 3 × 3, as follows:

$$\mathbf{O'_{3\times3}} : \mathbb{N}^{3\times3} \to \{1,2,3\}, \text{ where } \mathbb{N} \text{ is the set of natural numbers}$$

$$O'_{\mathbf{3\times3}} = \begin{cases} \text{if } CoDi = CoDj = 0, \forall\, i \neq j \text{ then } 3 \text{ (conflict)} \\ \text{else if } CoDi = CoDj = n(n-1), \forall\, i \neq j \text{ then } 1 \text{ (consensus)} \\ \text{else if } GCDi \neq GCDj\ \forall\, i \neq j \text{ then } 1 \text{ (consensus)} \\ \text{else if } mode(max(GCDi) \leq 2\ \forall\, i \text{ then } 1 \text{ (consensus)} \\ \text{otherwise then } 2 \text{ (partial conflict)} \end{cases}$$

$\mathbf{O'_{4\times 4}(GCD(M^{4}{}_{mcdco}),\ CoD(M^{4}{}_{cod}))}$ represents a **conflict metric** or **function** that calculates the **GCD degree of conflict–consensus** on a series of weighted vectors "v" of dimension (4 × 1), resulting from the application of the GCD method to a decision matrix of dimension 4 × 4, as follows:

$$\mathbf{O}'_{\mathbf{4}\times\mathbf{4}} : \mathbb{N}^{4\times 4} \rightarrow \{1,2,3\}, \text{ where } \mathbb{N} \text{ is the set of natural numbers}$$

$$\mathrm{O}'_{\mathbf{4}\times\mathbf{4}} = \begin{cases} \text{if CoDi} = \text{CoDj} = 0, \forall\, i \neq j \text{ then 3 (conflict)} \\ \text{else if CoDi} = \text{CoDj} = n(n-1), \forall\, i \neq j \text{ then 1 (consensus)} \\ \text{else if CoDi} = \text{CoDj} = 4 \text{ or } 6, \forall\, i \neq j \text{ then 2 (partial conflict)} \\ \text{else if GCDi} = \text{GCDj} = 8\ \forall\, i \neq j \text{ then 2 (partial conflict)} \\ \text{else if GCDi} = \text{GCDj} = 6\ \forall\, i \neq j \text{ then 2 (partial conflict)} \\ \text{else if GCDi} = \text{GCDj}\ \forall\, i \neq j \text{ then 2 (partial conflict)} \\ \text{else if GCDi} \neq \text{GCDj}\ \forall\, i \neq j \text{ then 1 (consensus)} \\ \text{else if mode(max(GCDi)} \leq 3\ \forall\, i \text{ then 1 (consensus)} \\ \text{otherwise then 2 (partial conflict or consensus)} \end{cases}$$

The Borda, plurality, Hare, and Coombs methods are described as "democratic" voting methods, while dictatorship social welfare is just a ranking of the candidates submitted by the dictator [84]. Because Hare and Coombs majority voting procedures break ties randomly [84], we decided to only compare Borda, pluralism, Condorcet, and the Greatest Common Decision Maker rankings.

4.9. Examples of Voting Rule Dynamics

These are examples of the voting procedures that show the dynamics of the voting rules and help in understanding the concepts.

To illustrate some of the concepts, we use a **3 × 3 order matrix**, which means analyzing individual preference accommodations in decision matrices (with a total of 216). We identify as **alternatives** the set **X** of x1, x2, and x3. As agents or **decision makers,** the set **D** comprises d1, d2, and d3. Additionally, the values of the preferences are the set **W,** comprising w1, w2, and w3, and their respective **weights,** 1, 2, and 3 (see Figure 4).

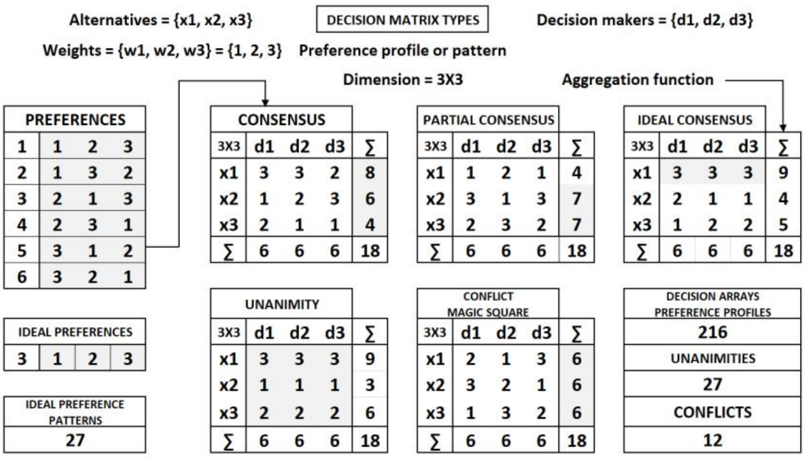


Figure 4. Example of Borda voting procedure.

In this 3 × 3 structural dimension, there are six different types of individual preferences $\mathbf{R_j}$ that can be constructed, which can in turn be organized in **decision matrices M** called **consensus** when the Borda voting system sums per row are different, **partial consensus,** or **partial conflict** when there are some rows with equal Borda sums, and **conflict** when the Borda sum of the three rows of the matrix are equal, which leads us to not being able to have a representative group preference $\check{\mathbf{R}}_{\mathbf{k}}$.

Notice, we do not allow repeated values in the column preferences or **individual preferences** of the decision makers since one of our assumptions is that preferences establish strict linear orderings among the decision alternatives, as mentioned above.

Conflict matrices are magic squares or Latin squares used in the design of statistical experiments or Sudokus, in which the numbers accommodated in them must not be repeated by columns or rows; they must add the same amount identified as the **magic number**, which in this dimension corresponds to 6 in a total of 12 conflict profiles.

There is another type of matrix, the **unanimity matrix,** that shows the preferences of the agents are the same in a total of 27 matrices; in addition, this matrix is also a consensus.

We identify **aggregation functions** or voting systems such as **Borda**, **pluralism**, majority, **Greatest Common Decision Maker**, Copeland, single-member majority vote, **Condorcet**, etc.

The **ballot dynamic** entails starting by defining a **voting matrix** and crossing **the weights of the preferences against the agents** to indicate what position each of the agents gives to the alternatives with respect to the values of their preference (see Figure 5).

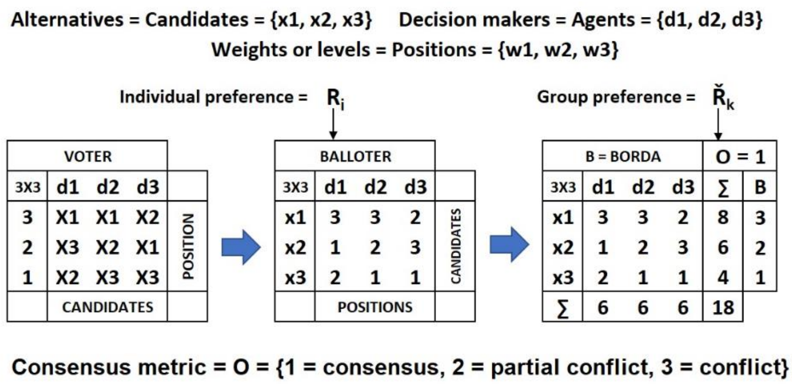


Figure 5. Ballot dynamic.

From there, we pass to an **equivalent decision matrix** crossing **the alternatives against the agents** to indicate the weight of the preference that each of the agents assigned to the alternatives of choice.

Next, we apply the aggregation function of the voting system to the **decision matrix** $\mathbf{M_k}$, in this case, **Borda**, ordering the results of the sums by weight {8, 6, 4} obtaining $\mathbf{\check{R}_k}$ into {3, 2, 1}, which represents the preference order assigned to the candidates, and finally we assign the consensus metric **O** as **1**, because the group preference $\mathbf{\check{R}_k}$ corresponds to a **group consensus**.

In Figure 6, we can observe the construction of matrices of **order 5** under several aggregation functions; in practice, this means we must analyze a total of 24,883,200,000 matrices. We have already explained the **Borda** aggregation function.

Pluralism represents the aggregation of preferences when the number of agents is not monitored and the winning alternative achieves 51% or more. In Figure 6, we observe a simple **pluralism** aggregation applied to the $\mathbf{^{r}SB^{k}}$ *weight submatrix* where "**r = 5**" corresponds to the maximum weight preference of the **decision matrix** $\mathbf{M_k}$ and $\mathbf{\check{R}_k}$ = $\begin{pmatrix} sp^{k}{}_{15} \\ sp^{k}{}_{25} \\ sp^{k}{}_{35} \\ sp^{k}{}_{45} \\ sp^{k}{}_{55} \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 5 \\ 15 \\ 0 \end{pmatrix}$ is the column (vector) obtained by multiplication of matrices $^{5}SB^{k}\, I_{(5\times 1)}$ $= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, where **Plu** represents the preference order assigned to the candidates. Furthermore, in Figure 7, we need to obtain the $\mathbf{^{r}SB^{k}}$ *weight submatrices* for all "**r = 1, . . . , 5**" weight preference values of the **decision matrix** $\mathbf{M_k}$ to build the $\mathbf{SP^{k}}$ *stronger* *alternative profile matrix* by columns from the vectors $\begin{pmatrix} {sp^k}_{1r} \\ {sp^k}_{2r} \\ \ldots \\ {sp^k}_{nr} \end{pmatrix}$ after the simple **pluralism** aggregation is applied to each "**r**" in the following way:

X = {Alternatives} D = {Agents or decision makers} W = {Weights or positions}

VOTING MATRIX

W\D	D1	D2	D3	D4	D5
5	C	D	B	D	D
4	D	E	C	C	C
3	B	C	E	E	B
2	E	A	D	A	E
1	A	B	A	B	A

DECISIÓN MATRIX

X\D	D1	D2	D3	D4	D5
A	1	2	1	2	1
B	3	1	5	1	3
C	5	3	4	4	4
D	4	5	2	5	5
E	2	4	3	3	2
∑	15	15	15	15	15

BORDA B(M$_k$) O = 1

X\D	D1	D2	D3	D4	D5	$\check{R}_k$	B
A	1	2	1	2	1	7	1
B	3	1	5	1	3	13	2
C	5	3	4	4	4	20	4
D	4	5	2	5	5	21	5
E	2	4	3	3	2	14	3
∑	15	15	15	15	15		

PLURALISM RANKING Plura(SP^k) O = 2

X\W	5	4	3	2	1	$\check{R}_k$	Plura
A	3	4	0	0	0	4	2
B	2	0	6	0	5	6	3
C	0	0	3	12	5	12	4
D	0	2	0	4	15	15	5
E	0	4	6	4	0	6	3
∑	5	10	15	20	25		

r = 5 PLURALISM Plu(^{r}SB^k) O = 2

X\W	5	4	3	2	1	$\check{R}_k$	Plu
A	0	0	0	0	0	0	3
B	0	0	5	0	0	5	4
C	5	0	0	0	0	5	4
D	0	5	0	5	5	15	5
E	0	0	0	0	0	0	3
∑	5	5	5	5	5		

CONDORCET Co(M^k$_{co}$) O = 1

X\X	A	B	C	D	E	$\check{R}_k$	Co
A	0	2	0	0	0	2	1
B	3	0	1	1	3	8	2
C	5	4	0	2	4	15	4
D	5	4	3	0	4	16	5
E	5	2	1	1	0	9	3
∑	18	12	5	4	11		

MIX M^k$_{mcd}$

X\D	D1	D2	D3	D4	D5	Mi	~Mi
A>B	0	1	0	1	0	2	3
A>C	0	0	0	0	0	0	5
A>D	0	0	0	0	0	0	5
A>E	0	0	0	0	0	0	5
B>C	0	0	1	0	0	1	4
B>D	0	0	1	0	0	1	4
B>E	1	0	1	0	1	3	2
C>D	1	0	1	0	0	2	3
C>E	1	0	1	1	1	4	1
D>E	1	1	0	1	1	4	1
∑	4	2	5	3	3		

GCD(M^k$_{mcdco}$) O' = 1

D\D	D1	D2	D3	D4	D5	$\check{R}_k$	coMi	X	GCD
D1	0	6	7	7	9	29	4	A	1
D2	6	0	3	9	7	25	2	B	3
D3	7	3	0	4	6	20	1	C	4
D4	7	9	4	0	8	28	3	D	5
D5	9	7	6	8	0	30	5	E	2
∑	29	25	20	28	30				

Dimension = 5x5
Decision arrays total = 24,883,200,000

COMPARISON

X	B	Plura	Plu	Co	GCD	
A	1	2	3	1	1	3
B	2	3	4	2	3	2
C	4	4	4	4	4	5
D	5	5	5	5	5	5
E	3	3	3	3	2	4
O	1	2	2	1	1	

Figure 6. Comparison of the voting procedures, $^rSB^k$ weight submatrix where "r = 5".

r = 1 PLURALISM Plu(^{r}SB^k) O = 2

X\W	5	4	3	2	1	$\check{R}_k$	Plu
A	1	0	1	0	1	3	5
B	0	1	0	1	0	2	4
C	0	0	0	0	0	0	3
D	0	0	0	0	0	0	3
E	0	0	0	0	0	0	3
∑	1	1	1	1	1		

r = 2 PLURALISM Plu(^{r}SB^k) O = 2

X\W	5	4	3	2	1	$\check{R}_k$	Plu
A	0	2	0	2	0	4	5
B	0	0	0	0	0	0	3
C	0	0	0	0	0	0	3
D	0	0	2	0	0	2	4
E	2	0	0	0	2	4	5
∑	2	2	2	2	2		

r = 3 PLURALISM Plu(^{r}SB^k) O = 2

X\W	5	4	3	2	1	$\check{R}_k$	Plu
A	0	0	0	0	0	0	3
B	3	0	0	0	3	6	5
C	0	3	0	0	0	3	4
D	0	0	0	0	0	0	3
E	0	0	3	3	0	6	5
∑	3	3	3	3	3		

r = 4 PLURALISM Plu(^{r}SB^k) O = 2

X\W	5	4	3	2	1	$\check{R}_k$	Plu
A	0	0	0	0	0	0	3
B	0	0	0	0	0	0	3
C	0	0	4	4	4	12	5
D	4	0	0	0	0	4	4
E	0	4	0	0	0	4	4
∑	4	4	4	4	4		

r = 5 PLURALISM Plu(^{r}SB^k) O = 2

X\W	5	4	3	2	1	$\check{R}_k$	Plu
A	0	0	0	0	0	0	3
B	0	0	5	0	0	5	4
C	5	0	0	0	0	5	4
D	0	5	0	5	5	15	5
E	0	0	0	0	0	0	3
∑	5	5	5	5	5		

Figure 7. Comparison of the voting procedures, $^rSB^k$ weight submatrix where "r = 1, . . . , 5".

To obtain the **pluralism ranking** aggregation as shown in Figure 6, $\check{\mathbf{R}}_{\mathbf{k}}$ is the column (vector) of the maximum values of each row (or alternative) in the $\mathbf{SP^k}$ matrix, where **Plura** represents the preference order assigned to the candidates.

Condorcet aggregation represents, by row, the count of the winning alternatives when each one is compared against all other preferences of the agents. To be specific, if we take the alternative A row and compare if it is greater than B in the voting matrix for all the agent columns, we can observe that it is in the preferences of agents D2 and D4, where the position of A is greater than B. Or, if we take the row of alternative D and compare it against alternative A, we observe that five agents prefer alternative D to alternative A.

The **mix matrix** is an intermediate step to achieve the GCD where the breakdown by agents of the Condorcet matrix is found by columns. Let us observe how it identifies the two cases in which A > B; these are D2 and D4, who have the same opinion. The **Mi** column summarizes what is represented in the Condorcet aggregation example by rows.

We created the **"Greatest Common Decision Maker"** preference aggregation, **GCD**, and it consists of finding the agent whose preference behavior pattern intersects the most with the other decision makers based on the **mix matrix** descriptions.

The **GCD** matrix is similar to the Condorcet matrix, but instead of applying it to the alternatives, the comparison is made among the decision makers. If we compare the columns of agents D1 and D2 in the mix matrix, we observe that the first five zeros and the final one are the same in both columns, that is, in a coincidence count of 6. Then, we can discover agent D5 as the greatest decision maker, i.e., the one that accumulates 30 counts more than all the other decision makers. This means that D5 is the GCD because it is the one that has the most coincidences with all the otheOKr decision makers; hence, the individual preference of D5 becomes the representative of the group preference ranking.

To obtain the degree of conflict–consensus for the GCD function, the Condorcet decision maker's matrix is $\mathbf{M^5}_{\mathbf{cod}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 2 & 2 \\ 0 & 3 & 2 & 0 & 2 \\ 3 & 1 & 2 & 2 & 0 \end{pmatrix}$ and the Condorcet decision maker's aggregation function is $\mathbf{CoD(M^5}_{\mathbf{cod}}) = \begin{pmatrix} 4 \\ 4 \\ 5 \\ 7 \\ 8 \end{pmatrix}$, but because $\mathbf{GCD(M^5}_{\mathbf{mcdco}}) = \begin{pmatrix} 29 \\ 25 \\ 20 \\ 28 \\ 30 \end{pmatrix}$ has $\mathrm{GCD_i} \neq \mathrm{GCD_j}$ $\forall \mathrm{I} \neq \mathrm{j}$ with i, j = 1, 2, 3, 4, 5, the degree of the conflict–consensus metric is $\mathbf{O'}_{\mathbf{5} \times \mathbf{5}} = 1$, i.e., we have a consensus.

If we compare the five aggregation functions in the table, there is one tie between C and D, with five coincidences; E has four coincidences and A has three coincidences. Meanwhile, B has two double coincidences between "B-Co" and "Plura-GCD", becoming the last alternative most preferred without consensus.

5. Frameworks, Tools, and Methods

We divided our methodology into two phases: one static and the other dynamic. The **static** approach aggregates the preferences of the individuals or agents involved in an innovative way: invariant to the structure of the agent and the network topology [13,54,92]. We found how the magic number and the Borda aggregation function are related to conflict matrices. Additionally, how different arrangements of graphs formed fractals allowed us to calculate the Borda frequency function and tables to understand larger decision matrices. The voting systems' properties were explained in the previous section.

Additionally, the **dynamic** allows analyzing and creating strategic agendas that facilitate a satisfactory consensus among the parties when they are in conflict, with variation in the structure of the agent and the topology of the network. We identified how visual conflict maps (or heatmaps [93]) are useful to understand the group decision making problem, the consensus reaching process, and the conflict–cooperation dynamic process that satisfies individual goals, under mutual agreement but not necessarily unanimity. The heatmap is a mosaic of colors that represents the hierarchical structure of data by rows and columns of a matrix, and has been used by statisticians since the end of the 19th century [94]. The methodology is based on voting processes combined with spaces for talking among decision makers to understand each other and move preferences for a consensus in each cycle.

6. Results

The size of our research problem in the static phase, namely, in the case of 4 × 4 order matrices, makes a total number of 331,776 matrices to review, within which there are

576 Latin squares and 1944 semi-magic squares, from a total of 2520 conflict matrices, 166,752 conflict or partial consensus matrices, and 162,504 consensus matrices (see Figure 8).

Borda voting procedure							
Matrix order (Ideal preferences)	Alternatives permutations (Unanimities)	Preference profiles (Decision arrays)	Conflict arrays (Semimagic squares)			Partial conflict or partial consensus arrays	Consensus arrays (no conflict at all)
			Latin squares	Semimagic squares	Total		
2 X 2	2	4	2	0	2	0	2
3 X 3	6	216	12	0	12	36	168
4 X 4	24	331,776	576	1944	2520	166,752	162,504
5 X 5	120	24,883,200,000	161,280	6,854,400	7,015,680	14,163,825,600	10,712,358,720

Figure 8. Summary of Borda voting procedure matrices size with n = 2, 3, 4, 5.

This allows us to conjecture, that this problem is one of computational complexity of the NP type [78] if we observe the magnitude in which the number of matrices grows as the matrix order changes. These numbers were simulated under the Borda voting procedure with the help of an Excel spreadsheet.

The Borda aggregation procedure sums total per row values, and can vary from "n" to "n^2", i.e., there are "$n(n-1)+1$" different values. In the case of 3×3 order matrices, there are 8 distinct categories of matrices plus their row combinations, making a total of 216 different Borda matrices whose sums are in the set {3, 4, 5, 6, 7, 8, 9} (see Figure 9c). Additionally, a unique category of matrices "with conflict" exists whose sums are equal to 6, the **magic number** (see Figure 9b). Alongside Figure 9b, we can observe the pattern of the frequency graph of the totals of rows sums. In this case, if we take the marked category, Figure 9a, it represents a decision matrix with 2 row sum totals equal to 5 and 1 row sum total equal to 8, in other words, a partial conflict decision matrix.

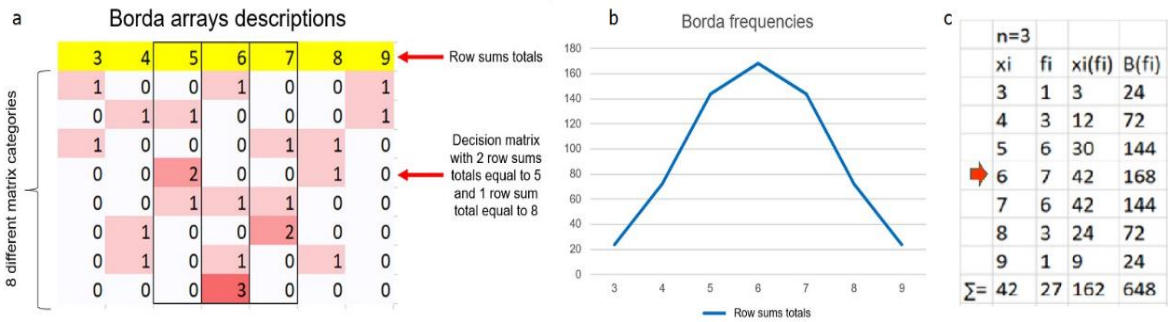


Figure 9. Frequency graph of the row sum totals in Borda aggregation method. (**a**) Borda arrays descriptions, (**b**) Borda frequencies and (**c**) Table of Borda frequencies.

The **magic number** is the base to determine what will be the equal value for the row sum totals in the conflict matrices. Then, we can define a **property of n × n conflict matrices for the Borda aggregation method** (our first general result) as follows: "*a decision matrix of order $n \times n$ is a conflict matrix or magic square, if and only if, its row sum totals are equal to*":

$$\frac{n(n+1)}{2}$$

In Figure 10, we find a table for the magic numbers of the Borda decision matrices from dimensions 2 to 8 and their related row sum total sets.

We calculated Borda frequency tables for the row sum totals of the matrix dimensions, aiming to develop the Borda distribution function for order $n \times n$ (see Appendix C).

We carried out other types of accommodations for the decision matrices' group preferences. In Figure 11, we can observe accommodation of the total 27 different possible mathematical combinations of rows of **alternatives** in the case of the 3×3 matrix order; therefore, the cells represent the total 729 (27^2) array cases. The empty cells represent the matrices whose $\mathbf{R_j}$ **individual preferences** have at least for one decision maker and a repeated **preference weight** value per column. The filled cells represent the matrices whose $\mathbf{R_j}$ **individual preferences** do not have repeated **preference weight** values per column.

Dimension order	n	n^2	Set of rows sums totals	Total different values	Magic number
2 X 2	2	4	{2, 3, 4}	3	3
3 X 3	3	9	{3, 4, ..., 9}	7	6
4 X 4	4	16	{4, 5, ..., 16}	13	10
5 X 5	5	25	{5, 6, ..., 25}	21	15
6 X 6	6	36	{6, 7, ..., 36}	31	21
7 X 7	7	49	{7, 8, ..., 49}	43	28
8 X 8	8	64	{8, 9, ..., 64}	57	36

Figure 10. Magic numbers of Borda matrices for n = 2, . . . , 8.

				1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
				1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3
				1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
				1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
1	2	3	4																											
R	C1	C2	C3																											
1	1	1	1														333	332		323	322					233	232		223	222
2	1	1	2													333		331	323		321				233		231	223		221
3	1	1	3													332	331		322	321					232	231		222	221	
4	1	2	1											333	332					313	312		233	232					213	212
5	1	2	2										333		331				313		311	233		231				213		211
6	1	2	3										332	331					312	311		232	231					212	211	
7	1	3	1											323	322		313	312					223	222		213	212			
8	1	3	2										323		321	313		311				223		221	213		211			
9	1	3	3										322	321		312	311					222	221		212	211				
10	2	1	1					333	332		323	322														133	132		123	122
11	2	1	2				333		331	323		321													133		131	123		121
12	2	1	3				332	331		322	321														132	131		122	121	
13	2	2	1		333	332					313	312											133	132					113	112
14	2	2	2	333		331				313		311										133		131				113		111
15	2	2	3	332	331					312	311											132	131					112	111	
16	2	3	1		323	322		313	312														123	122		113	112			
17	2	3	2	323		321	313		311													123		121	113		111			
18	2	3	3	322	321		312	311														122	121		112	111				
19	3	1	1					233	232		223	222					133	132		123	122									
20	3	1	2				233		231	223		221				133		131	123		121									
21	3	1	3				232	231		222	221					132	131		122	121										
22	3	2	1		233	232					213	212		133	132					113	112									
23	3	2	2	233		231				213		211	133		131				113		111									
24	3	2	3	232	231					212	211		132	131					112	111										
25	3	3	1		223	222		213	212					123	122		113	112												
26	3	3	2	223		221	213		211				123		121	113		111												
27	3	3	3	222	221		212	211					122	121		112	111													

Figure 11. Accommodation of the combinations of rows of 3 × 3 matrices.

In Figure 11, for example, the decision matrix $\begin{bmatrix} 222 \\ 313 \\ 131 \end{bmatrix}$ is formed with the combination of row 14 values "222", row 21 values "313", and row 7 values "131", corresponding to the last row of the decision matrix, but also to the cell position, color and conflict–consensus value in the color maps (or heatmaps) of Figures 12–14.

Figure 12 is divided into two quadrants. Each quadrant is built accordingly, as indicated in Figure 11. In the left quadrant, we can observe the 216 preference profile matrix cases of the 3 × 3 matrix order of **Borda**, where the decision maker preferences are not repeated per column (remember they are linear orders). This also applies to the **simple pluralism ranking** in the right quadrant.

The colored cell points represent the **O(v) conflict metric value**. The **blue color** with a value equal to 1 represents a matrix where the group preference is a **consensus**. The **yellow color** with a value equal to 2 represents a matrix where the group preference is a **partial consensus** or partial conflict. Additionally, the **red color** with a value equal to 3 represents a matrix where the group preference is a **conflict**.

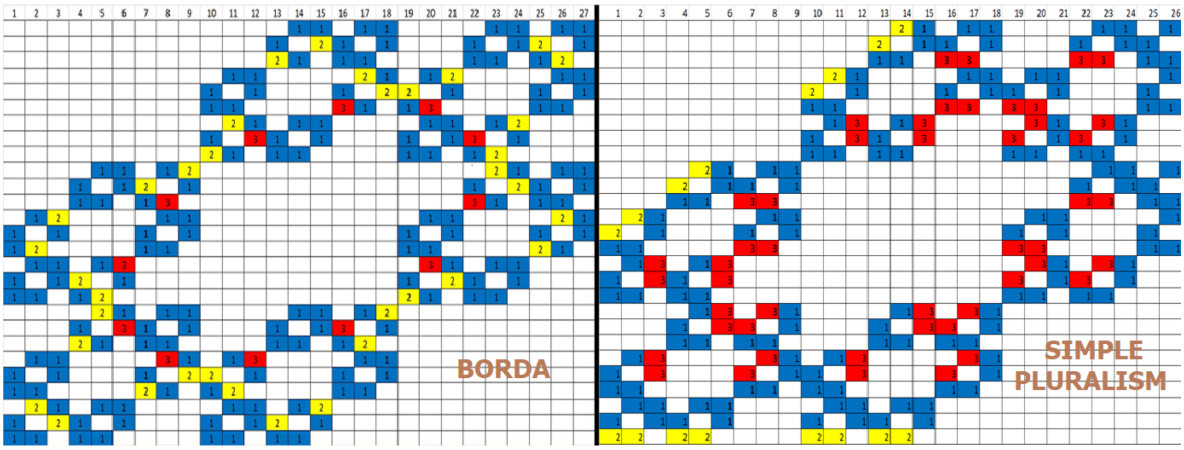


Figure 12. Comparison between the color maps (or heatmaps) for the Borda and simple pluralism methods for n = 3.

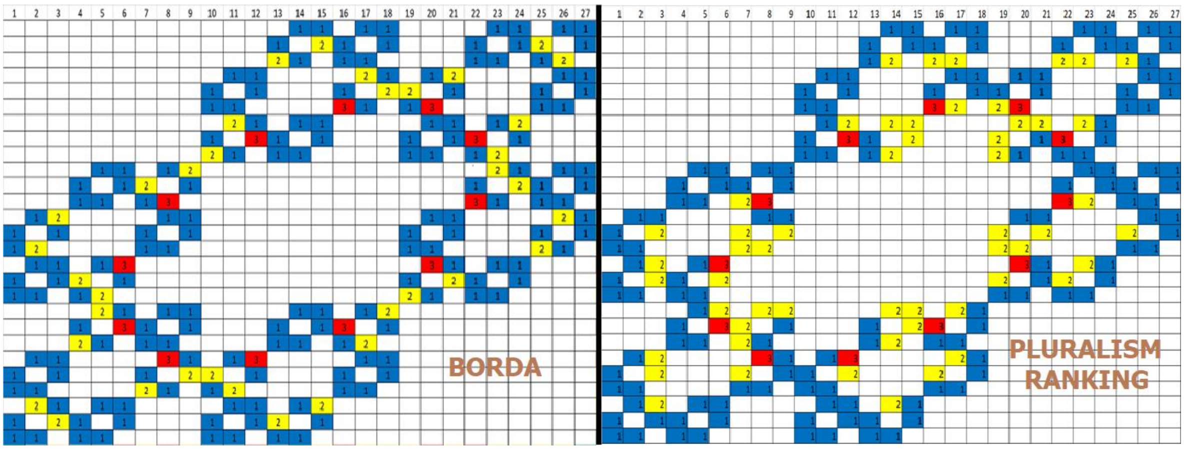


Figure 13. Comparison between the color maps (or heatmaps) for the Borda and pluralism ranking methods for n = 3.

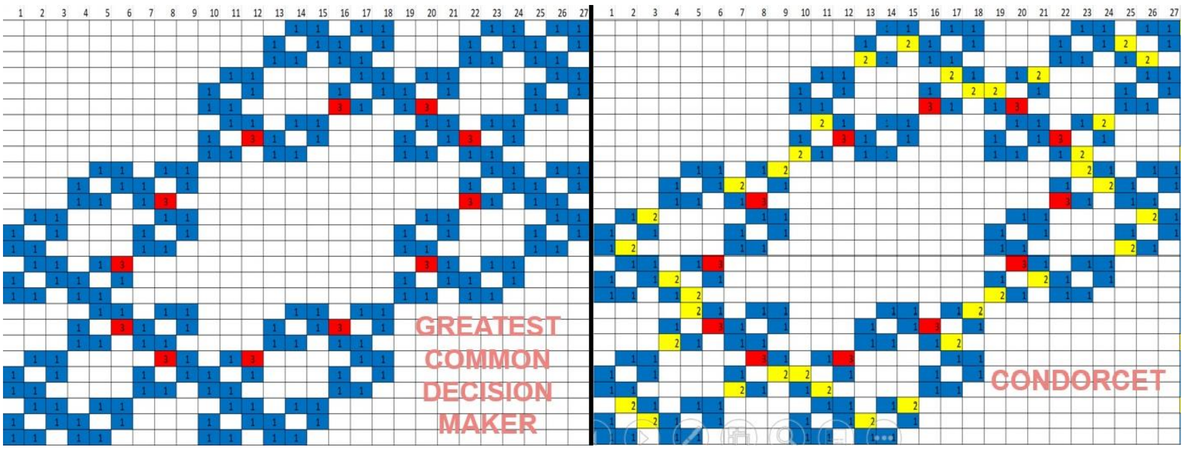


Figure 14. Comparison between the color maps (or heatmaps) for the Greatest Common Decision Maker and Condorcet methods for n = 3.

In the quadrants for Borda and simple pluralism in Figure 12, it was observed that the geometric arrangements form helices or snowflakes where the conflicts in red form an ellipse and represent a reduced frequency, in addition to the yellow and blue ones. Additionally, this is also noted in the fractal patterns.

It is also appreciated that even though the cases of partial conflicts are reduced in the simple pluralism method, pluralism produces more conflicts and fewer consensuses than Borda. Additionally, some asymmetries are observed between both procedures inside the quadrants.

Now, in Figure 13, we compare the Borda and pluralism rankings and we observe that the conflicts are the same; meanwhile, the pluralism ranking produces more partial conflicts and fewer consensuses than Borda.

We conducted the same comparison for the Greatest Common Decision Maker and Condorcet, which is illustrated in Figure 14. Notice that the GCD does not generate a partial consensus, and only total conflicts are respected; this confirms our conjecture that conflicts are structural, and not voting method-dependent for the matrices in our research. That is, conflicts depend only on the matrix order.

After these comparisons of voting methods, we conjecture that conflicts are structurally independent of the methods, and partial conflicts such as consensuses are not, and depend on the method used.

The Borda case for the 4 $\times$ 4 matrix order is shown in Figure 15. Again, a fractal behavior of the 331,776 matrices is observed. We augmented the behavior with a magnifying glass to better understand it (see Figure 16).

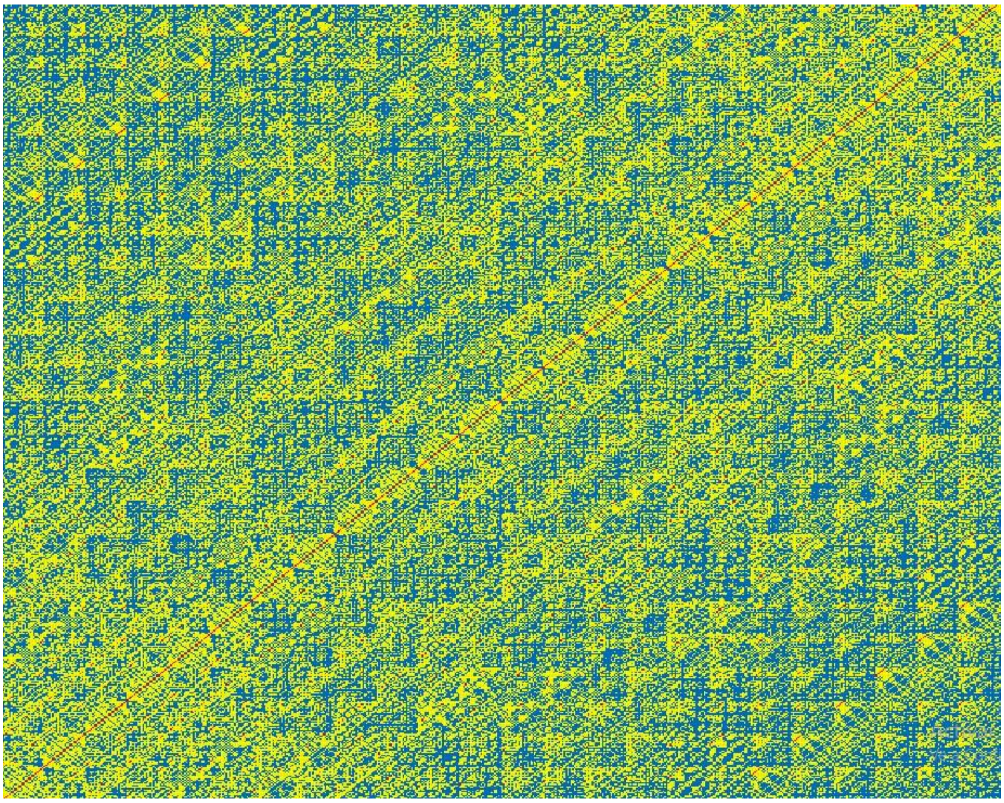


Figure 15. Color map (or heatmap) for the Borda method with n = 4.

The accommodation in Figure 15 is different to Figure 11; we combined 256 (24^2) columns and 256 (24^2) rows to represent the 331,776 matrices or $\check{\mathbf{R}}_\mathbf{k}$ **group preferences** (details are in Figure 17). Here, the total 24 different permutations of columns represent the **decision maker preferences** in the case of the 4 $\times$ 4 matrix order; therefore, if we combine them in a matrix of 24^4 cells, they represent the total 331,776 matrices cases whose $\mathbf{R_j}$ **individual preferences** do not have repeated **preference weight** values per column.

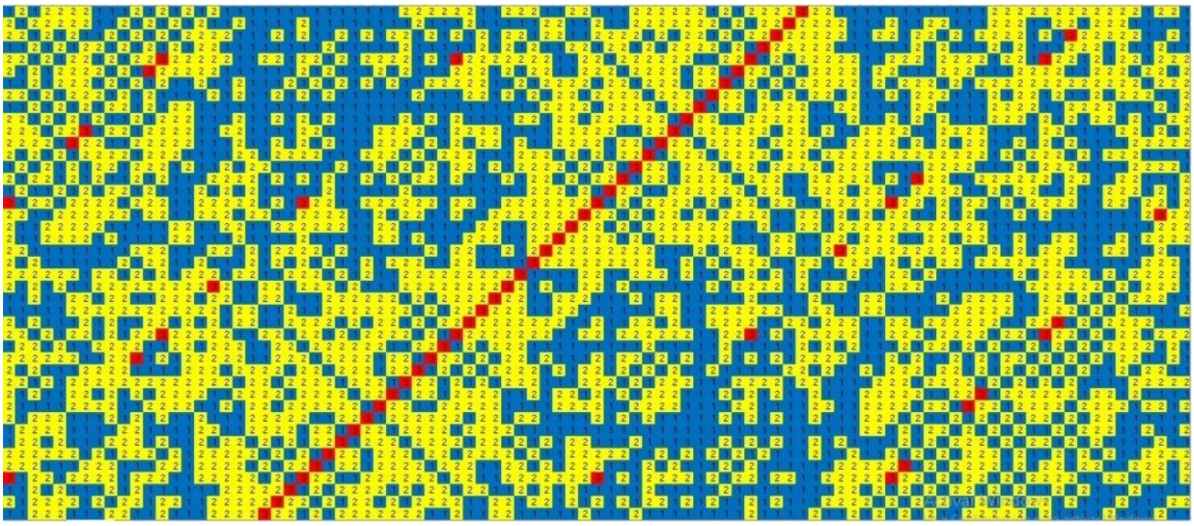


Figure 16. Detail of color map (or heatmap) fractal for the Borda method with n = 4.

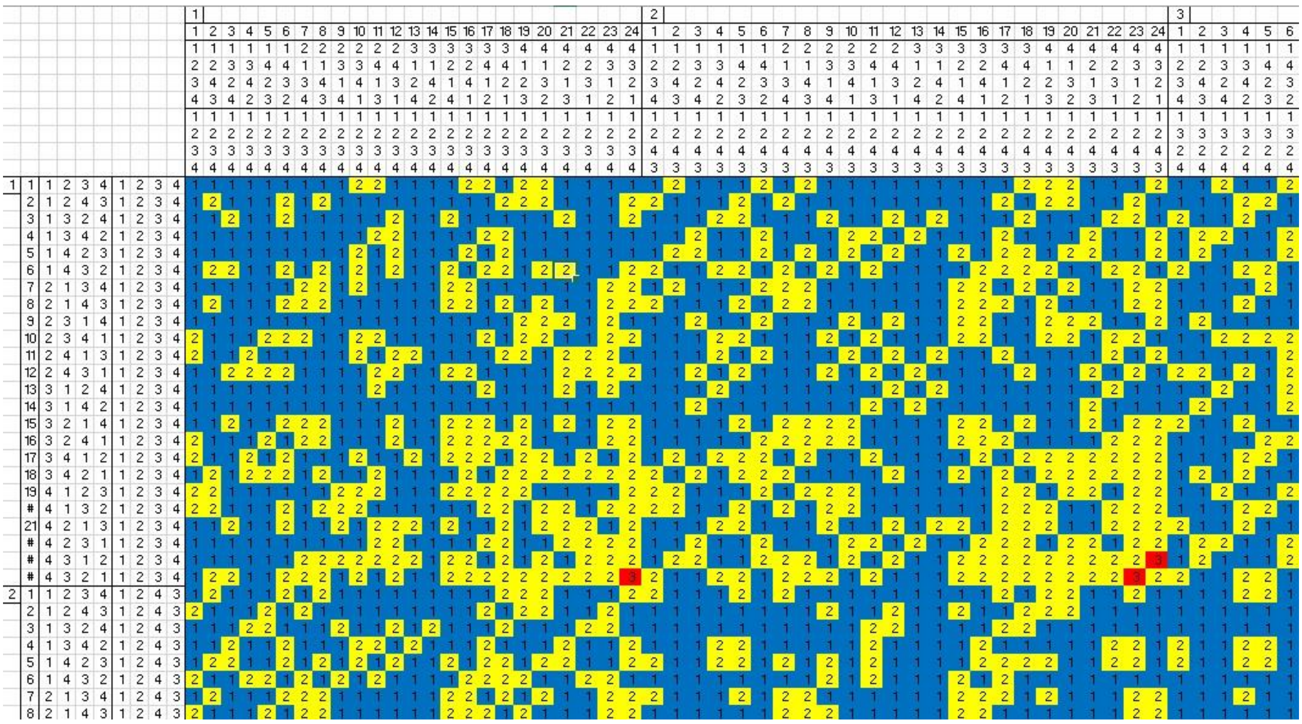


Figure 17. Accommodation of the combinations of columns of 4 × 4 matrices.

If we adopt the accommodation of Figure 17 for the case of the 3 × 3 matrices, we can combine the 6 possible individual **decision maker preferences** as columns and the 36 (6^2) rows to represent the 216 = 36(6) matrix cases or $\check{\mathbf{R}}_{\mathbf{k}}$ **group preferences**. Additionally, all methods are compared in Figure 18.

In Figure 19, we can observe the comparison of all voting methods for the 4 × 4 matrices. The fractal behaviors are, again, remarkable.

In Figure 20, we can observe the augmented behavior details of the Greatest Common Decision Maker with a magnifying glass.

In Figure 21, we can observe a table with all the number comparisons for the 3 × 3 and 4 × 4 matrices and all the voting methods we are reviewing. Associated with the table in Figure 22, the figure shows increases and decreases among the distributions of the voting methods. Notice, that pluralism methods increase conflicts and the GCDM method decreases partial consensus and conflicts. Borda and Condorcet behave the same way, at least for 3×3 and 4×4 matrices cases. In contrast, simple pluralism eliminates all consensuses (case 4×4) and GCDM eliminates all partial conflicts (case 3×3).

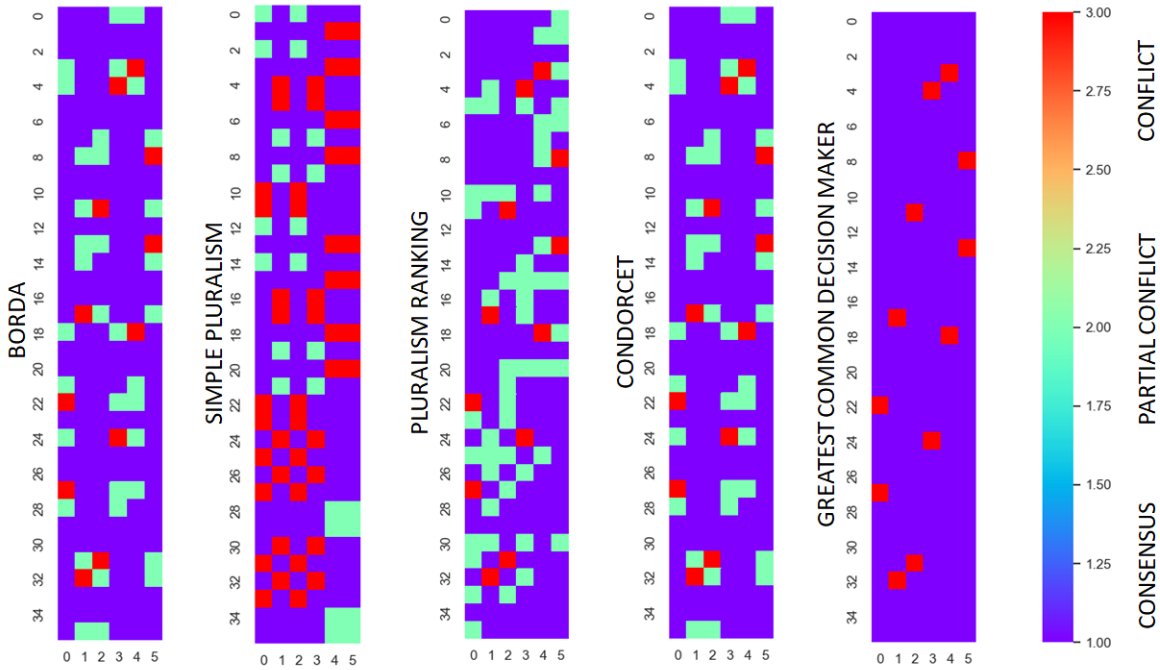


Figure 18. Comparison of all voting methods for 3×3 matrices.

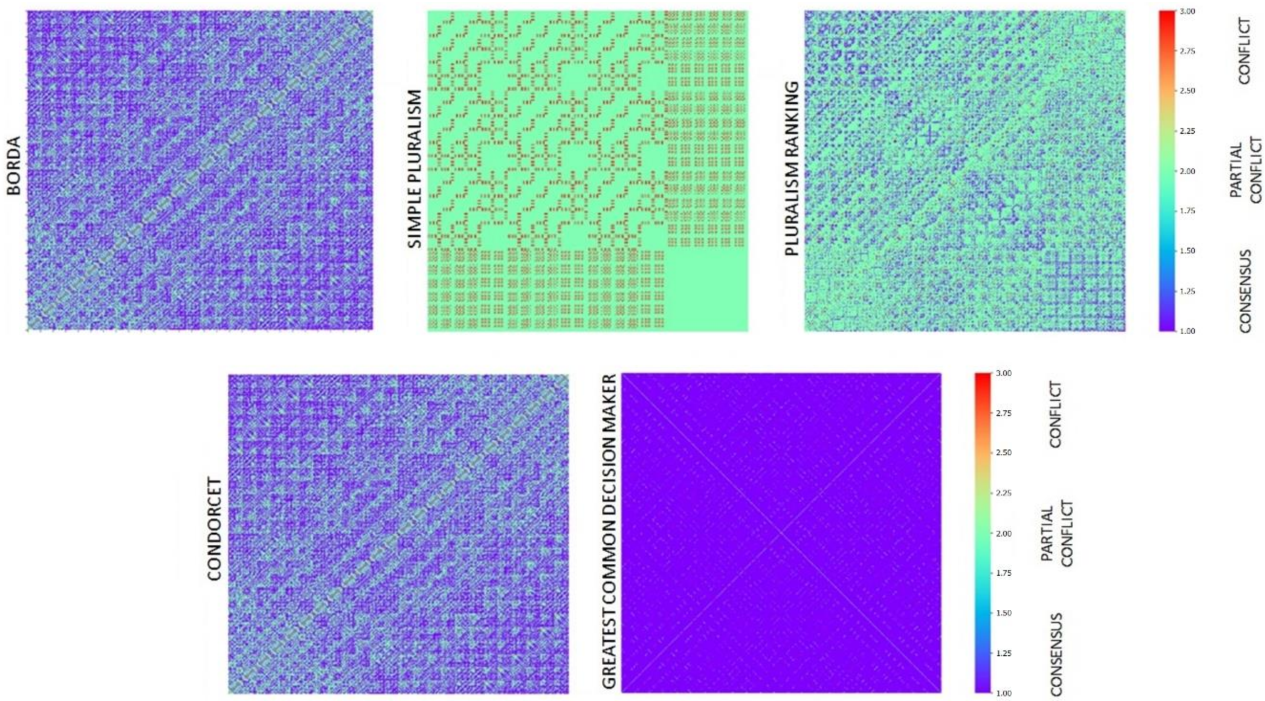


Figure 19. Comparison of all voting methods for 4×4 matrices.

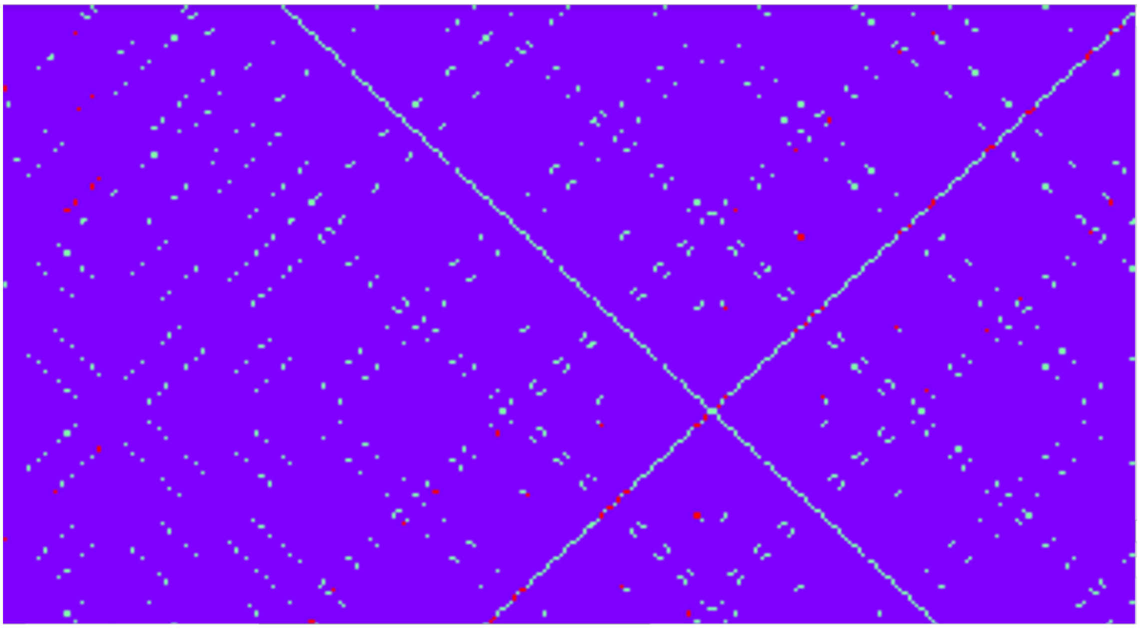


Figure 20. Greatest Common Decision Maker details for 4 × 4 matrices.

		Conflict	Partial conflict	Consensus	Totals
3 X 3	Borda	12	36	168	216
	Simple pluralism	48	24	144	216
	Pluralism ranking	12	54	150	216
	Condorcet	12	36	168	216
	GCDM	12	0	204	216
4 X 4	Borda	2520	166,752	162,504	331,776
	Simple pluralism	31,104	300,672	0	331,776
	Pluralism ranking	3456	265,992	62,328	331,776
	Condorcet	2520	166,752	162,504	331,776
	GCDM	576	6120	325,080	331,776

Figure 21. Comparison among voting procedures for 3 × 3 and 4 × 4 matrices.

In Figures 23 and 24, we can observe how the GCDM ranking method transforms the conflicts, partial conflicts, and consensus generated by the Borda method. It is noteworthy that the GCDM helps to mark off important *preference profile patterns* such as unanimities, Latin squares, and ties between pairs.

Method vs Method *		Borda	Simple pluralism	Pluralism ranking	Condorcet	GCDM
Borda	1		▼	▼	▬	▲
	2		▼	▲	▬	▼
	3		▲	▬	▬	▬
Simple pluralism	1	▲		▲	▲	▲
	2	▲		▲	▲	▼
	3	▼		▼	▼	▼
Pluralism ranking	1	▲	▼		▲	▲
	2	▼	▼		▼	▼
	3	▬	▲		▬	▬
Condorcet	1	▬	▼	▼		▲
	2	▬	▼	▲		▼
	3	▬	▲	▬		▬
GCDM	1	▼	▼	▼	▼	
	2	▲	▲	▲	▲	
	3	▬	▲	▬	▬	

1 - Consensus 2 - Partial conflict 3 - Conflict

▲ Increase ▬ Equal ▼ Decrease

* Column method compaired against raw method

Figure 22. Increments and decrements among the distributions of the voting methods.

4 x 4			Borda			
GCDM		Totals	Conflict	Partial conflict	Consensus	Totals
Conflict	Latin squares	576	576	0	0	576
Partial conflict	Ties between pairs	216	216	0	0	216
	Partial conflict	5904	1728	3744	432	5904
Consensus	Unanimities	24	0	0	24	24
	Consensus	325,056	0	163,008	162,048	325,056
	Totals	331,776	2520	166,752	162,504	331,776

Figure 23. GCDM transformation of Borda conflicts and consensus for 4 × 4 matrices.

3 x 3			Borda			
GCDM		Totals	Conflict	Partial conflict *	Consensus	Totals
Conflict	Latin squares	12	12	0	0	12
Partial conflict	Ties between pairs	0	0	0	0	0
	Partial conflict	0	0	0	0	0
Consensus	Unanimities	6	0	0	6	6
	Consensus	198	0	36	162	198
	Totals	216	12	36	168	216

* The 36 Borda partial conflicts are ties between pairs

Figure 24. GCDM transformation of Borda conflicts and consensus for 3 × 3 matrices.

Consensus Reaching Process in the Dynamic Phase, a Cost Decision Visualization

To start the dynamic phase, inspired by [2,20,74,75,95], we decided to follow the consensus reaching process of Figure 25, where temporary group preferences are generated from the agent's decision matrices; each decision maker reviews their individual preference in a dialogue event to evaluate if it is acceptable or not to reach their individual goals under this context, and this becomes the moment in which each agent can modify their preferences, to build another decision matrix to evaluate subsequent group preferences in a cycle, until the definitive group preference is found.

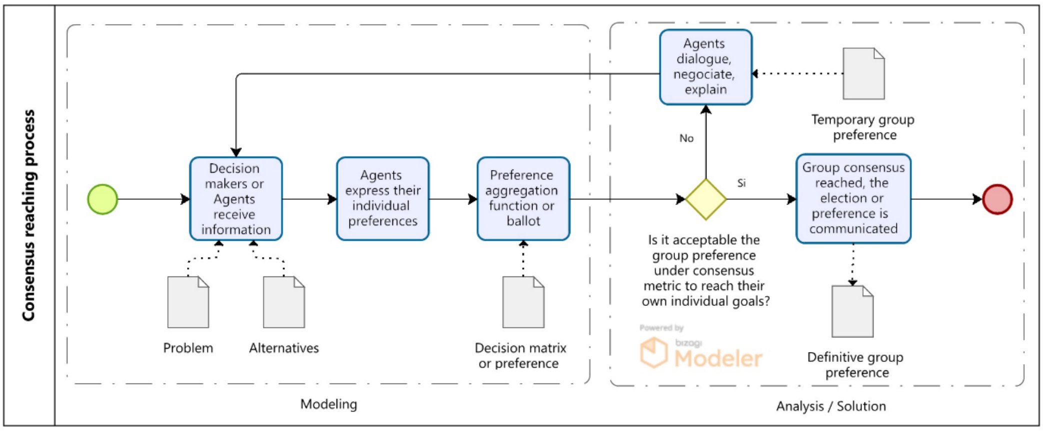


Figure 25. Consensus reaching process.

After reviewing the literature, we aimed to create a visual representation to allow us to comparatively evaluate the criteria of consensus–conflict against costs decisions using visualization maps; we found different diagrams such as those of Yee [96], Zlotkin–Rosenschein [54], Saari and Xu–Hipel–Kilgour–Fang [2].

Our proposal of a visual representation is depicted in Figure 26. Therefore, to support the evaluation of the preference aggregation methods, we carried out a new visual arrangement of the decision matrices that we will call color map (or heatmap), as shown in Figure 26c, with the case of the 3 × 3 matrix order as an example. This shows the 216 consensus metric values: the 168 values in blue represent consensus, the 36 values in yellow represent partial consensus, and the 12 values in red represent conflicts. Note that the conflict represented in the upper-right part of the box, on the fourth line and second column from right to left, is what we will call a pivot point (a4, a1, a5) because it plays the role of a reference against to which to compare its consensus metric value with the rest of the slack point consensus metric values; we will use it later in the following explanations. In the example, the pivot value corresponds to the O(v) consensus metric value, and equals 3 in red for the decision matrix or preference profile $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} a4 & a1 & a5 \end{bmatrix}$ = (a4, a1, a5). The whole map is constructed following the same procedure.

First, to evaluate the effort involved in changing the decision or preference of a decision agent, it is necessary to define the effort change matrix of the preferences [56], as shown in Figure 26a. Remember that in the case of the 3 × 3 decision matrix order, we have six different types of preferences, here represented by "a1" to "a6".

Suppose that a certain company has a striking problem, and we identify that the choice alternatives vary from stopping a strike to improving salaries, or continuing production, and the decision makers involved are the director, the labor union, and the owner. Depending on the context and particular situation of the company, we can build the change effort matrix, located in the left-middle part of Figure 26b. Additionally, to understand how this matrix works, read the next explanation. In Figure 26d, for the director to reach a favorable consensus for all employees, the labor union and the owner will have to value the effort, which could mean changing their initial preference "a1" to preference "a5". In the preference position "a1", the director or agent "d1", gives greater importance with a value of 3 to continue with production, a value of 2 to review the salary of the employees, and with a value of 1, with less importance, to attending the demands of the strike. In preference position "a5", a value of 3 is given to strike requests, a value of 2 to continuing production, and a value of 1 to reviewing salary. The change in position or individual preferences could mean an effort of 17; while in the opposite direction, changing from position "a5" to "a1", could mean no effort, because it is easy for the agent to stop attending strike requests and return to production. Notice that if we want to reduce the research scope, instead of calibrating a preference change effort matrix, we could adopt Kendall Tau distance [97] and prepare a Kendall Tau distance change matrix of preferences, or other kinds of metrics [98], to evaluate the weight change from one position to the others among the six preferences "a1" to "a6". In Figure 27, we calculate the Kendall Tau-normalized distance change matrix of the preferences, as a reference.

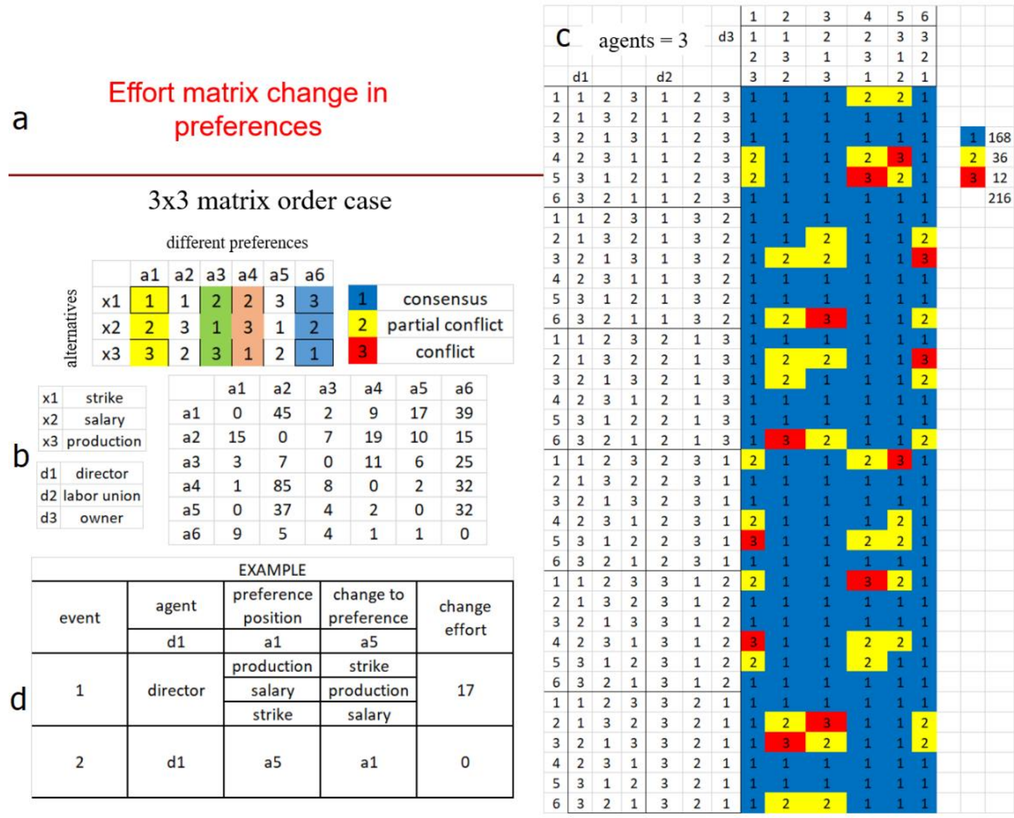


Figure 26. Cost decision conflict heatmap to achieve consensus. (**a**) Effort matrix change in preferences, (**b**) Example of a change effort matrix, (**c**) Example of a color map (or heatmap) associated to a decision matrix and (**d**) Example built over change effort matrix of (**b**).

Once the Kendall Tau-normalized distance change matrix for the preferences has been built, we are able to develop the following visual conflict maps, as shown in Figure 28. Additionally, for explanations, will use the third map of the right part of Figure 28c. The idea of a visual conflict map consists of the arrangement of a map explained in Figure 26c, but instead of coloring and numbering consensus metric values according to the O(v) consensus metric, first, the pivot point value is assigned to zero, because there is no effort or distance to change from *pivot point* (a4, a1, a5) to *pivot point* (a4, a1, a5), and second, *there exists an effort or distance to change* from *pivot point* (a4, a1, a5) to *slack point* (a1, a1, a1) with a value of "1.32". This is calculated by adding the effort or distance change needed to move from "a4" to "a1", "a1" to "a1", and "a5" to "a1", respectively; in this example, consulting the Kendall Tau-normalized distance change matrix of the preferences, the distance change is equal to "0.66 + 0 + 0.66 = 1.32". Moreover, the point values of the visual conflict map are colored with the original map point colors corresponding to the consensus metric. Therefore, the pivot point (a4, a1, a5) value is 0 and red in color. Additionally, the slack point (a1, a1, a1) value is 1.32 and blue in color.

	a1	a2	a3	a4	a5	a6
a1	0	0.33	0.33	0.66	0.66	1
a2	0.33	0	0.66	1	0.33	0.66
a3	0.33	0.66	0	0.33	1	0.66
a4	0.66	1	0.33	0	0.66	0.33
a5	0.66	0.33	1	0.66	0	0.33
a6	1	0.66	0.66	0.33	0.33	0

Figure 27. Kendall Tau-normalized distance change matrix in preferences.

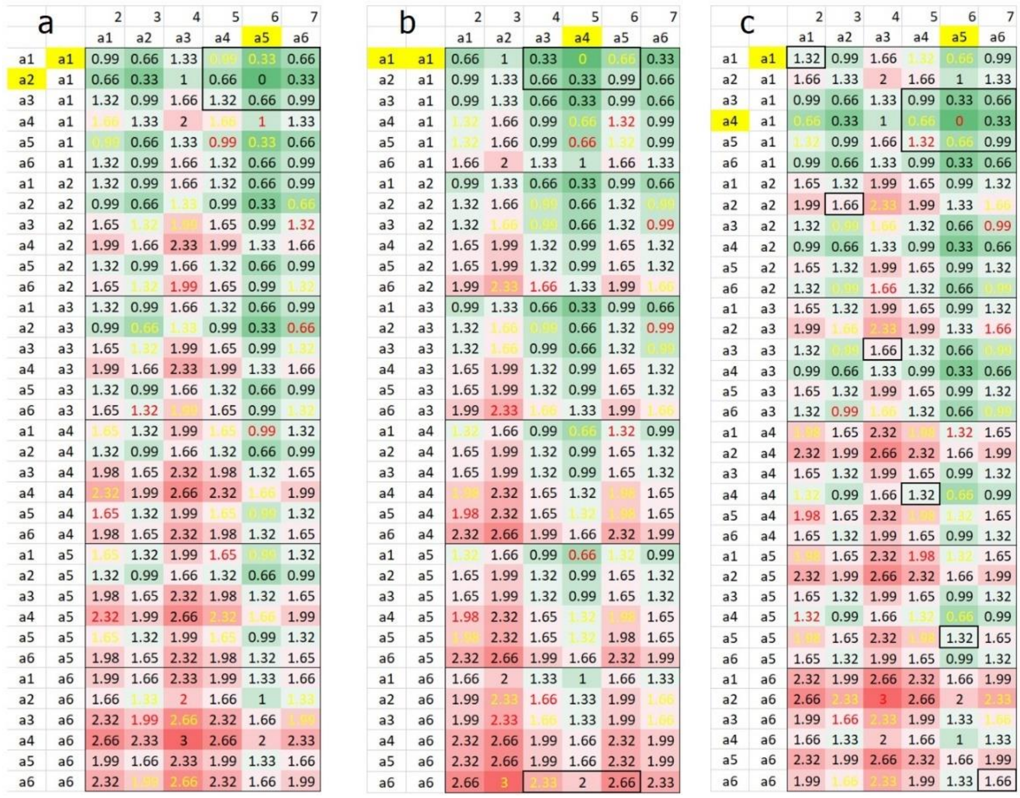


Figure 28. Conflict heatmap with the Kendall Tau method. (**a**) Conflict map for the pivot point (a2, a1, a5), (**b**) Conflict map for the slack point (a6, a6, a4) and (**c**) Conflict map for the pivot point (a4, a1, a5).

In Figure 28a, we also find the visual conflict maps for the pivot point (a2, a1, a5), (Figure 28a, second row, and fifth column) and the pivot point (a1, a1, a4) (Figure 28b, first row and fourth column). Note that, on the right map, the points of unanimities are also marked by squares. The 3 × 3 marked squares show the nearest neighborhoods of the pivot points in the visual conflict maps; slack point (a6, a6, a4) (Figure 28b, last row and fourth column) belongs to the nearest neighborhood of pivot point (a1, a1, a4) (Figure 28b, first row and fourth column) as a reference only.

From previous explanations, we can conclude that if we want to change from conflict pivot point (a4, a1, a5) (Figure 28c) to consensus, unanimity, partial conflict or partial consensus slack points, there are a plurality of slack points with different distances and different consensus metric values. This, in turn, makes us also calculate the costs that will mean staying in a conflict or consensus map point, to be able to evaluate visually and mathematically speaking the consensus point we want to reach to achieve our individual goals in a group decision-making problem or multi-agent election.

Next, to complete our research results, we include an explanation for the costs due to loss or utility [56,99,100] that occur from staying at each pivot or slack map point position (see Figure 29). Once the loss or profit costs have been defined for each of the positions "a1" to "a6", as per the cost table shown in Figure 29c, we can calculate the cost of staying at consensus point (a1, a1, a1) with a loss value of "1 + 1 + 1 = 3" or a profit value of "10 + 10 + 10 = 30". The point of conflict (a4, a1, a5), Figure 28c, mentioned above would have a loss value of "0 + 1 + 5 = 6" or a profit value of "20 + 10 + 70 = 37", and so on for all the map points; see Figure 29b for the cost map. In Figure 29e, the example also shows the loss and profit cost calculation value for the consensus slack point (a2, a1, a6).

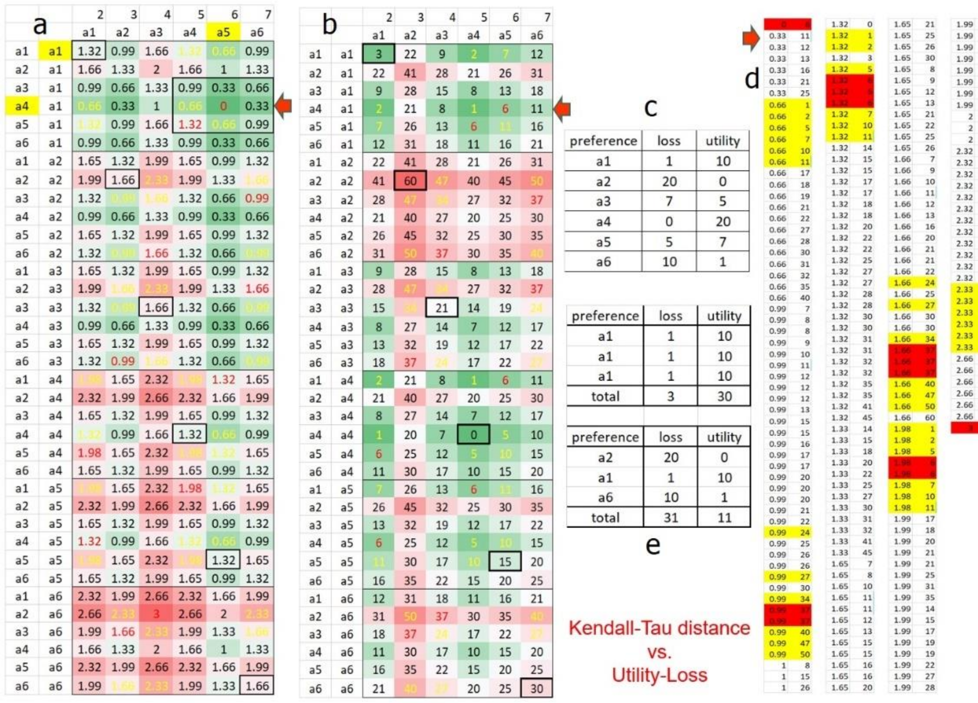


Figure 29. Conflict map vs. utility loss list. (**a**) Distance change from slack point (a4, a1, a6) to its right, (**b**) Loss change from slack point (a4, a1, a6) to its right, (**c**) Cost of staying at consensus point (a1, a1, a1), (**d**) The point couple that implies the least distance and the least loss and (**e**) The loss and profit cost calculation value for the consensus slack point (a2, a1, a6).

The dynamic process to reach the definitive consensus point is more convenient for the decision makers' group, and would consist of obtaining all the paired map point couples, one of the distance maps (in the left part of the figure) and the other of the cost maps (in the middle part of the figure), and finding the point couple that implies the least distance and the least loss [92,93], as can be seen from the whole list of the 216 ordered pair points from smallest to largest (Figure 29d). In the example list, conflict pair values are marked in red, the partial consensus pair values are in yellow, and the consensus pair values are in blue. The first pair in the list is the pair value for the pivot point. Hence, the most convenient change from the conflict pivot point to a consensus slack point with the minimum distance and loss (inspired by [61]) is the slack point (a4, a1, a6) to its right, where its distance value is "0.33" (see Figure 29a) and its loss value is "11" (see Figure 29b). We must clarify that even though we found slack point (a4, a1, a6) as the best slack point to move the decision makers' positions, it might not be the best for the agents' individual goals. A dynamic voting strategy agenda will need to be built for each agent until they reach a consensus map point to satisfy their goals under a mutual agreement, but not necessarily unanimity.

7. Discussion

A comparison among the five aggregation preference processes showed that pluralism is a complex method for defining a group preference, considering that as a classical rule it only accepts the first alternative voted to be ranked, and the others are not the main targets to be ranked. However, even though it was difficult to modify, we built and completed the rule to obtain a linear order over the alternatives. This voting method generates a lot of conflict compared with the others.

The Borda aggregation preference, on the other hand, can be organized and characterized systematically. The property found is an important result that will help to understand how conflict appears in the aggregation preference processes and understand behavior during the simulations for the consensus negotiation [54] reaching process among the multi-agent networks. This voting system does not generate as many partial conflicts as pluralism, and its behavior is similar to that of the Condorcet voting [101]; both can be managed easily.

The Greatest Common Decision Maker aggregation preference generates the minimum structural number of conflicts and no partial conflicts of partial consensus. This aggregation preference is shown to be too versatile to achieve consensus in the static phase because it finds the decision maker whose individual preference has the greatest coincidence with the others. The decision maker's individual preference becomes the representative group preference.

The way the decision maker assigns its preferences and how they are aggregated is the base for understanding conflict in the group decision making process.

Our methodology intends to use different points of view, abstractions, theories, and practice granularities to dynamically simulate the world or particular regions and individual agents' environments to find a consensus balance among all the actors in the decision-making elections, problems, and scenarios.

The incorporation of social concepts into our research makes the results obtained stronger.

In the end, one of the advantages of this methodology is that it is simpler than the others and obtains the same or better results. The Greatest Common Decision Maker method is an aggregation process containing the simplest and smallest set of assumptions that have sufficient generality to cover the entire class of accountable decision-making phenomena. We are aware that a consensus cannot be reached without a conflict to overwhelm support by cooperation.

We faced several challenges during the development of the present research:

- The need to organize a large number of decision matrices or preference profiles. In real-life environments, it is not usual to face voting selections among large amounts of objects, but few of them.
- The amount of time needed to organize all possible combinations of objects to be elected from the alternative set. Additionally, performing them naturally to evaluate the strategies for each agent and map conflict was also difficult.
- In the static phase, the aggregation preference procedures played a significant role, while in the dynamic phase the conflict maps did.
- It is necessary to know all the types of conflict matrices to understand what role they play in the process of reaching consensus negotiation [54].
- Our research allowed us to identify that the decision problems under our assumptions have fractal behavior patterns and computational complexity of the NP type [78].

We found that the following **principles** apply to our research:

- Higher frequency is better than higher weight.
- Majority is not consensus. Majority is democracy.
- Ordering means decision-making agent election order.
- Weight does not mean a majority.
- Assigning a weight is relative to each person and it is an individual opinion.

- A decision matrix, preference matrix, or conflict matrix is a semi-magic square, ordinary magic square, or a Latin square.

There are many ways to decide. In certain circumstances, having one person decide for a whole group speeds up the organizing process, if that is the most important variable. Is this procedure fair, does it achieve good decisions? Therefore, it is important to involve everyone who is affected by the group decision process so that it will reflect the will of the entire group and not just that of the leaders because the concerns that arise are resolved by the entire group.

Reaching a consensus on a proposal does not mean that everyone agrees.

Future Work

We will measure the computational complexity of algorithms and their comparison. We need to determine what results allow us to explain the ordering and behavior of an agent strategy agenda that generates an agreement or consensus. Voting synchronically or asynchronically will also be investigated.

We will explore not only square matrix profiles, but also rectangular profiles in which there are more agents than alternatives. We will design hypercubic matrices to analyze packages of alternatives against individual decision makers or packages of decision makers against individual alternatives. Moreover, we will identify the hidden paired conflicts [102,103] that appear in the underlying matrices to determine the conflict degree metric in the proposed Greatest Common Decision Maker voting method.

We will improve the consensus reaching process for the decision-making aggregation problem of network groups under trust networks [104–110], apply rough sets [111] and concurrency [75], and identify group preference biases, opinion preference changes, and chaos. Additionally, we will consider any consistency and consensus in any linguistic, fuzzy, or social network large-group decision making process during the CRP.

We will use the statistics and the Borda probability distribution function found to carry out the simulations of case studies [112–115]. Additionally, we will explore the relationship that exists between raw frequencies and conflict–consensus matrix frequencies, as well as what kind of fractals could be formed.

We will apply the results and methodology to real case studies such as aquifer basins and medical autoimmune diseases.

8. Conclusions

The proposed Greatest Common Decision Maker aggregation preference ranking procedure makes it easy to automate the consensus reaching process more efficiently. Furthermore, conflict maps, effort or distance change matrices, and cost maps developed over different visual arrangements have sufficient generality to cover the entire class of accountable group decision-making phenomena or multi-agent elections.

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Appendix A. Abbreviations

CRP	Consensus reaching process
GDM	Group decision making
$(n, n-1, \ldots, 1)$	Ordinal ranking
$\mathbf{X}$	Set of alternatives
x_i	Alternative "i"
i	Represents the number of the alternative
$\mathbf{D}$	Panel of experts or decision makers
d_i	Decision maker or agent "i"
m	Represents the number of the decision maker
$\mathbf{W}$	Set of preference weights
W_i	Weight assigned to alternative "i"
$x > y$	"x" is preferred to "y"
$\mathbf{R}$	A linear order
$\mathbf{R}_j$ or $\mathbf{P}_j$	An individual preference or linear order associated with decision maker d_j
$\check{\mathbf{R}}_\mathbf{k}$	A preference or linear order which is associated as a group preference of the "m" decision makers in the set **D**
M	An array that represent the preference data collected from "m" agents
$\lvert L(X) \rvert$	Cardinality of set L(X)
#A	Cardinality of a set A
${R_j}^T$	Transposed matrix of $\mathbf{R}_j$
$\mathbf{M}_\mathbf{k}$	Decision matrix (or group choice profile) which represents the individual preference arrangement of a set of agents D
$^{\mathbf{r}}\mathbf{SB}^{\mathbf{k}}$	**Weight submatrix profile**
$\mathbf{SP}^{\mathbf{k}}$	**Stronger alternative (candidate) profile matrix**
$\mathbf{M^k}_{\mathbf{co}}$	**Condorcet matrix**
$\mathbf{M^k}_{\mathbf{cod}}$	**Condorcet decision maker's matrix**
$\mathbf{M^k}_{\mathbf{mcd}}$	**Mix matrix**
$\mathbf{M^k}_{\mathbf{mcdco}}$	**Condorcet mix matrix**
O(v)	**Conflict metric** or **function** that calculates the **degree of conflict–consensus**
O′(v)	**Conflict metric** or **function** that calculates the **degree of conflict-consensus** for the Greatest Common Decision Maker
v	Vector of weights or $\check{\mathbf{R}}_\mathbf{k}$
B	**Borda** aggregation function
Plu	**Pluralism** aggregation function
Plura	**Pluralism ranking** aggregation function
$\mathbf{SUB^r}$	Weight submatrix profile function
SAPA	Stronger alternative profile function
GCD	**Greatest Common Decision Maker** aggregation function
Co	**Condorcet** aggregation function
CoD	**Condorcet decision maker's** aggregation function

Appendix B. Demonstration that $n!^m$ Is the Total Cardinality Number of the Set of Decision Matrices

In other words, $n!^m$ is the total number of different matrices of dimension "$n \times m$", where "m" is the number of agents and "n" is the number of alternatives.

1. Definition $\check{\mathbf{R}}_\mathbf{k} = \{w_{1k}, w_{2k}, \ldots, w_{nk}\}$ represents one of the aggregation **group preferences of different matrices of dimension "$\mathbf{n \times m}$"**, where "m" is the number of decision makers and "n" is the number of alternatives. In other words, one group preference $\check{\mathbf{R}}_\mathbf{k}$ is associated one-to-one with a decision matrix.
2. From the definition, the set **X** has "n" alternatives; therefore, n! different orders or permutations of this set exist based on mathematical combinatory. Each permutation is associated one-to-one with a $\mathbf{P}_\mathbf{j}$ individual preference.
3. By definition, if we build a decision matrix **M,** it has to be a combination of "m" decision makers in this way $M = [P_1, P_2, \ldots, P_m]$.

4. Then, the total permutations allowing repetitions necessary to build the complete set of decision matrices is the multiplication of the "m" decision maker total preferences where each decision maker has, in turn, n! alternatives to select.
5. Therefore, the total cardinality number of the set of decision matrices is $\mathbf{n!^m}$ based on mathematical combinatory QED.

Appendix C. Borda Frequency Tables and Borda Distribution Function

We calculated Borda frequency tables for the row sum totals of matrix dimensions 2 to 8 under different procedures; one of them (*described in Appendix D*) is a formula made with the *worlds of math & physics* site and blog [116]. See tables for dimensions 4 and 5 in Figure A1.

	n = 4			
	xi	fi	xi(fi)	B(fi)
	4	1	4	5184
	5	4	20	20,736
	6	10	60	51,840
	7	20	140	103,680
	8	31	248	160,704
	9	40	360	207,360
	10	44	440	228,096
	11	40	440	207,360
	12	31	372	160,704
	13	20	260	103,680
	14	10	140	51,840
	15	4	60	20,736
	16	1	16	5184
∑=	130	256	2560	1,327,104

	n = 5			
	xi	fi	xi(fi)	B(fi)
	5	1	5	39,813,120
	6	5	30	199,065,600
	7	15	105	597,196,800
	8	35	280	1,393,459,200
	9	70	630	2,786,918,400
	10	121	1210	4,817,387,520
	11	185	2035	7,365,427,200
	12	255	3060	10,152,345,600
	13	320	4160	12,740,198,400
	14	365	5110	14,531,788,800
	15	381	5715	15,168,798,720
	16	365	5840	14,531,788,800
	17	320	5440	12,740,198,400
	18	255	4590	10,152,345,600
	19	185	3515	7,365,427,200
	20	121	2420	4,817,387,520
	21	70	1470	2,786,918,400
	22	35	770	1,393,459,200
	23	15	345	597,196,800
	24	5	120	199,065,600
	25	1	25	39,813,120
∑=	315	3125	46,875	124,416,000,000

Figure A1. Row sum totals in Borda distribution for n = 4, 5.

To understand the behavior of the row sum totals, we made different arrangements for them, aiming to develop the Borda distribution function for order n × n. Additionally, we found that the sum total values fulfilled the Hankel matrices (see Figure A2d) for diagonals extending from left to right, as can be seen in Figure A2b, for the matrix dimension 4 × 4 case where the magic number is 10 (see Figure A2a).

Additionally, we found that the behavior pattern is similar to one of the fractals. This can be observed in the way the colors expand in the main square, boxes, and sub-boxes (see Figure A2c).

Based on our results, we normalized the Borda frequencies to the interval [0, 1] to make their graphic representation easy. In Figure A3. we see the Borda distributions for orders from 3 × 3 to 8 × 8 centered on the magic number 36 for 8 × 8 matrices. Blue corresponds to the distribution of the order 3 × 3, green to order 8 × 8, and so on. Notice that as the order of the matrix increases, the distributions tend to spread out on the x-axis towards "minus infinity" and "plus infinity", but also flatten towards a constant. This allows us to conjecture that, for large matrix orders, the number of consensuses and conflicts tends to proportionally balance. Additionally, we performed normality hypothesis tests on the

Borda distributions, and we found that it fits them as a statistic norm distribution with an error of $p << 0.005$.

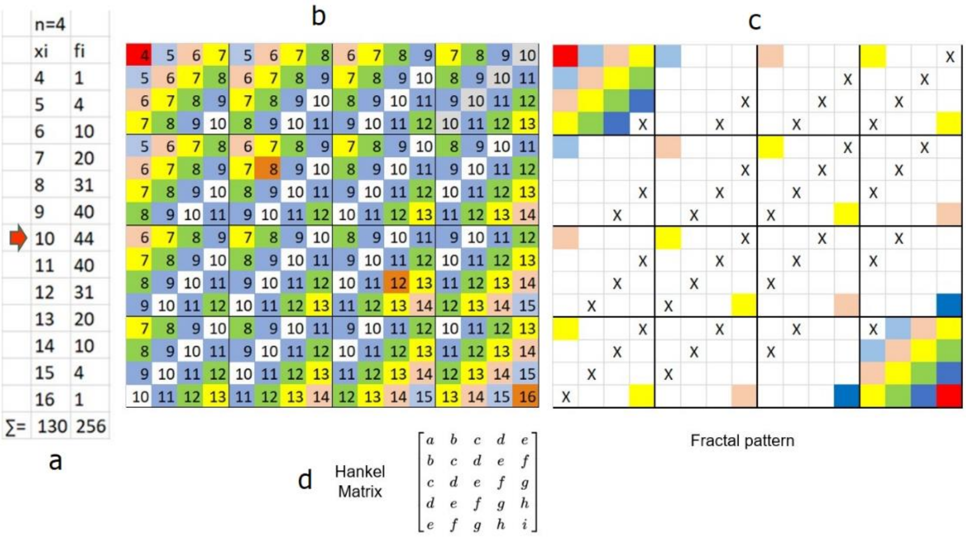

Figure A2. Fractal behavior pattern of row sum totals in Borda distribution for n = 4. (**a**) Reduced table of frequencies for matrix dimension 4 × 4, (**b**) Fractal example for matrix dimension 4 × 4 case where magic number is 10, (**c**) Fractal pattern for matrix dimension 4 × 4 case and (**d**) Example of a Hankel matrix.

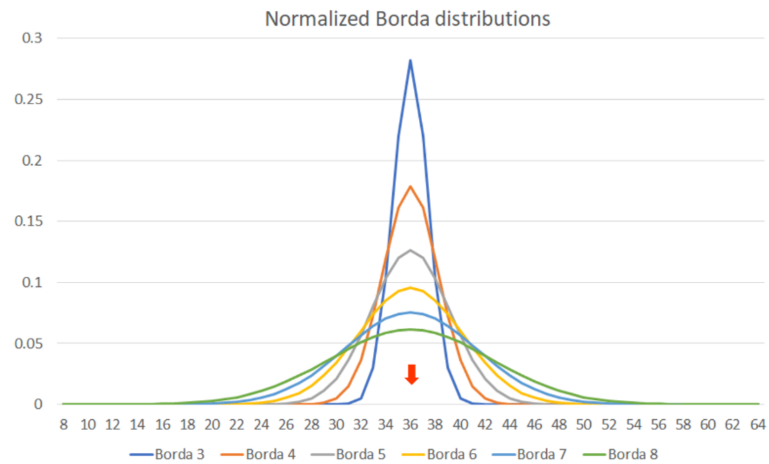

Figure A3. Normalized Borda distribution frequencies for n = 3, . . . , 8.

Appendix D. Borda Distribution Function Algorithm

Worlds of math & physics site and blog Borda distribution function definition [116]:

$P(p, n, s) = \frac{1}{s^n} \sum_{k=0}^{k_{max}} (-1)^k \binom{n}{k} \binom{p-(sk)-1}{p-(sk)-n}$ where $k_{max} = \left[\frac{p-n}{s}\right]$ and [x] is the floor function that gives the largest integer less than or equal to x (i.e., [6.83] = 6).

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