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Article

Convexity, Starlikeness, and Prestarlikeness of Wright Functions

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Abstract: This article deals with the normalized Wright function and its geometric properties. In particular, we find sufficiency criteria for close-to-convexity with respect to starlike function $\frac{\varsigma}{1-\varsigma^2}$. We also find conditions such that the normalized Wright function is starlike. The convexity along the imaginary axis and starlikeness of a certain order is also a part of our discussion. Moreover, we study the bounded turning of the partial sums and prestarlikeness of this function. We use positivity techniques to obtain these results.

Keywords: starlike function; prestarlike functions; convexity along imaginary axis; close-to-convex function; Wright functions; positivity techniques

MSC: 30C45; 30C20; 33C10; 30C75

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1. Introduction

The Wright function is defined by

$$W_{a,b}(\varsigma) = \sum_{s=0}^{\infty} \frac{\varsigma^s}{s!\Gamma(b+as)}, \quad a > -1, \ b \in \mathbb{C}, \tag{1}$$

where Γ denotes the well-known Gamma function. The infinite series (1) converges absolutely in $\mathbb{C}$ for $a > -1$, whereas the series converges for $a = -1$, in open unit disc $\mathbb{U}$. It should be recalled that $W_{a,b}$ is an entire function. E. M. Wright, a British mathematician, was the first person to come up with the idea of this function while investigating the theory of partitions, see [1]. In [2], he studied integral representation for the derivation of the asymptotic behavior of this function and investigated the four-parameter Wright function in [3]. He only considered the case when $a > 0$. In [4], he studied the function $W_{a,b}$ and extended the range of the parameter a to $a > -1$. The distribution of zeros of the function $W_{a,b}$, and its order, type, and indicator function, indicates that it is an entire function of completely regular growth for every $a > -1$ [5]. It is worth mentioning that from the viewpoint of the theory of analytic functions, it is the Wright function and not the Mittag–Leffler function that is a natural fractional generalization of the exponential function.

Wright function plays an essential role in the theory of fractional partial differential equations [5–13]. The Wright function and its generalizations play an important role in an extension of the methods of the Lie groups to the partial fractional differential equations [14,15]. It has various applications in the Mikusiński operational calculus and in integral transforms of the Hankel type, see [16–19].

The generalizations of several functions such as the Array function and the Whittaker function, and entire auxiliary functions, can be found in the connection with Wright functions. In particular, the function $W_{1,v+1}(-\varsigma^2/4)$ can be connected with the Bessel function J_v as:

$$J_v(\varsigma) = \left(\frac{\varsigma}{2}\right)^2 W_{1,v+1}\left(\frac{-\varsigma^2}{4}\right) = \sum_{s=0}^{\infty} \frac{(-1)^s \varsigma^{2s+v}}{2^{2s+v} s! \Gamma(s+v+1)}.$$

For more details about the Wright function, see [20] (Chapter 7), [21] (Section 2.1), and [22] (Chapter 10).

Denote by $\mathbb{A}$, the well-known and most studied class of functions $\mathfrak{f}$, which are analytic and of the form

$$\mathfrak{f}(\varsigma) = \varsigma + \sum_{s=2}^{\infty} c_s \varsigma^s, \quad \varsigma \in \mathbb{U}. \tag{2}$$

Denote by $\mathcal{S}$ a class of univalent (one-to-one) functions in $\mathbb{A}$. A function is known as starlike if it maps $\mathbb{U}$ onto a domain that is starlike with respect to origin and convex if it maps $\mathbb{U}$ onto a convex domain. We denote by $\mathcal{S}^*$ and $\mathcal{C}$, respectively, the class of all starlike and convex univalent functions in $\mathbb{U}$. The generalizations of $\mathcal{S}^*$ and $\mathcal{C}$, which are denoted by $\mathcal{S}^*(\delta)$ (starlike) and $\mathcal{C}(\delta)$ (convex) of order $\delta \in [0,1)$ are, respectively, defined as

$$\mathcal{S}^*(\delta) = \left\{\mathfrak{f} : \Re\left(\frac{\varsigma \mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right) > \delta, \ \varsigma \in \mathbb{U}\right\},$$

and

$$\mathcal{C}(\delta) = \left\{\mathfrak{f} : \ \Re\left(1 + \frac{\varsigma \mathfrak{f}''(\varsigma)}{\mathfrak{f}'(\varsigma)}\right) > \delta, \ \varsigma \in \mathbb{U}\right\}.$$

The $\mathcal{K}(\delta)$ (close-to-convex) of order δ is defined as

$$\mathcal{K}(\delta) = \left\{\mathfrak{f} : \ \Re\left(\frac{\varsigma \mathfrak{f}'(\varsigma)}{\mathfrak{g}(\varsigma)}\right) > \delta, \ \varsigma \in \mathbb{U}, \ \mathfrak{g} \in \mathcal{S}^*(0)\right\}.$$

Let $\mathfrak{f} \in \mathbb{A}$. Then, the class $\mathcal{T}$ [23] of typically real function is defined as

$$\mathcal{T} = \{\Im(\varsigma)\Im\mathfrak{f}(\varsigma) > 0, \quad \varsigma \in \mathbb{U}\}.$$

A normalized univalent function $\mathfrak{f}$ will be considered in $\mathcal{CV}$, a class of convex functions in an imaginary axis direction if and only if $\mathfrak{f}(\mathbb{U})$ is a convex set in the same direction. In other words,

$$[w_1, w_2] \subset \mathfrak{f}(\mathbb{U}), \qquad w_1, \ w_2 \in \mathfrak{f}(\mathbb{U})$$

and $\Re w_1 = \Re w_2$. Robertson [23] has shown that a function $\mathfrak{f} \in \mathbb{A}$ having real coefficients is in the class $\mathcal{CV}$ if $\varsigma \mathfrak{f}'(\varsigma) \in \mathcal{T}$ and equivalently satisfies

$$\Re\left[\left(1 - \varsigma^2\right)\mathfrak{f}'(\varsigma)\right] > 0, \quad \varsigma \in \mathbb{U}.$$

If $\mathfrak{f} \in \mathcal{T}$ and satisfies $\Re\mathfrak{f}'(\varsigma) > 0$ for $\varsigma \in \mathbb{U}$, then $\mathfrak{f} \in \mathcal{S}^*$, see [24]. The extended definition with regard to order δ is due to Mondal and Swaminathan in [25].

For the functions $\mathfrak{f} \in \mathbb{A}$ given in (2) and $\mathfrak{g} \in \mathbb{A}$ having the following form:

$$\mathfrak{g}(\varsigma) = \varsigma + \sum_{s=2}^{\infty} c_s \varsigma^s,$$

the convolution or Hadamard product is denoted and defined as

$$(\mathfrak{f} * \mathfrak{g})(\varsigma) = \varsigma + \sum_{s=2}^{\infty} c_s c_s \varsigma^s \quad (\varsigma \in \mathbb{U}).$$

Using the concept of convolution, Ruscheweyh [26] introduced the class $\mathcal{R}_{\xi}$, which contains the prestarlike functions of order ξ as follows:

Let $\mathfrak{f} \in \mathbb{A}$. Then, $\mathfrak{f} \in \mathcal{R}_{\xi}$ if and only if

$$\begin{cases} \Re \frac{\mathfrak{f}(\varsigma)}{\varsigma} > 0, \quad \varsigma \in \mathbb{U} \quad \text{for} \quad \xi = 1, \\ \frac{\varsigma}{(1-\varsigma)^{2(1-\xi)}} * \mathfrak{f}(\varsigma) \in \mathcal{S}^*(\xi), \quad \varsigma \in \mathbb{U} \quad \text{for} \quad 0 \le \xi < 1. \end{cases}$$

In particular, when we put $\xi = 1/2$ then $\mathcal{C} = \mathcal{R}_0$ and $\mathcal{S}^*(1/2) = \mathcal{R}_{1/2}$. The class $\mathcal{R}_{\xi}$ was generalized to the class $\mathcal{R}[\alpha, \xi]$ by Sheil-Small et al. [27]. A function $\mathfrak{f} \in \mathcal{R}[\alpha, \xi]$ if $\mathfrak{f} * \mathcal{S}_{\alpha} \in \mathcal{S}^*(\xi)$, where $\mathcal{S}_{\alpha} = \frac{\varsigma}{(1-\varsigma)^{2-2\alpha}}$, $0 \le \alpha < 1$. It is easy to see that $\mathcal{R}[\xi, \xi] = \mathcal{R}_{\xi}$.

It is noted that the function $\mathcal{W}_{a,b}$ is not in class $\mathbb{A}$; therefore, we assume the following function:

$$\mathcal{W}_{a,b}(\varsigma) = \varsigma W_{a,b}(\varsigma)\Gamma(b) = \varsigma + \sum_{s=1}^{\infty} \frac{\Gamma(b)}{s!\Gamma(b+as)} \varsigma^{s+1}, \qquad a > -1,\ b > 0.$$

We also recall here the Schwarz reflection principle and the minimum principle of harmonic functions.

The Schwarz Reflection Principle: It states that if an analytic function is defined on the upper half-plane and has well-defined (non-singular) real values on the real axis, then it can be extended to the conjugate function on the lower half-plane. In notation, if $\mathfrak{f}$ is a function that satisfies the above requirements, then its extension to the rest of the complex plane is given by

$$\mathfrak{f}(\overline{\varsigma}) = \overline{\mathfrak{f}(\varsigma)}. \tag{3}$$

The extension Formula (3) is an analytic continuation to the whole complex plane [28].

Minimum Principle of Harmonic Functions: a harmonic function u cannot have either a minimum or a maximum at an interior point unless it is constant, see [29].

In the last few years, some researchers have shown considerable interest in the geometric properties of certain special functions. For further detail, see [30–37]. Parajapat [31] was the first who studied the starlikeness and convexity of the function $\mathcal{W}_{a,b}$. The main tools of his investigation were the functional inequalities of this function. Later, Raza et al. [38] studied the starlikeness and convexity of order α for the function $\mathcal{W}_{a,b}$. They also investigated Hardy spaces and the close-to-convexity of the function. The radii of starlikeness and the convexity of some normalized forms of the Wright functions were discussed by Baricz et al. [39]. Maharana et al. [40] discussed the close-to-convexity with respect to certain starlike functions and strongly starlike functions of the function $\mathcal{W}_{a,b}$.

In recent years, by using the positivity technique, the geometric properties of hypergeometric functions were studied by Sangal and Swaminathan [41].

In this work, we focus on certain geometric properties of $\mathcal{W}_{a,b}$ by using the results of [41]. We complete the study of $\mathcal{W}_{a,b}$ by discussing starlikeness, close-to-convexity, convexity in the direction of the imaginary axis, and prestarlikeness. The main tools of our study are the positivity techniques.

2. Preliminaries

We use the following lemmas to obtain our main results.

Lemma 1 ([25])**.** *Let $\mathfrak{f} \in \mathbb{A}$ be such that $\mathfrak{f}'$ and $\mathfrak{f}'(\varsigma) - \delta\frac{\mathfrak{f}(\varsigma)}{\varsigma}$ both belong to $\mathcal{T}$. Additionally, suppose that $\Re\, \mathfrak{f}'(\varsigma) > 0$ and $\Re\left(\mathfrak{f}'(\varsigma) - \delta\frac{\mathfrak{f}(\varsigma)}{\varsigma}\right) > 0$. Then, $\mathfrak{f} \in \mathcal{S}^*(\delta)$, $0 \le \delta < 1$.*

Lemma 2 ([42])**.** *Let $\kappa \ge 0$, $\lambda \in \mathbb{R}$ such that $0 < \lambda + \kappa < 1$ and $m \in \mathbb{N}$. If $c_0 = c_1 = 1$ and $c_{2s} = c_{2s+1} = \frac{(\kappa+\lambda)_s}{s!} \cdot \frac{m!(\kappa+1)_{m-s}}{(\kappa+1)_m (m-s)!}$ for $1 \le s \le m$. Then,*

(i) $\sum\limits_{s=0}^{m} \cos(s\theta)c_s > 0 \Leftrightarrow \lambda + \kappa \leq \lambda^*\left(\frac{1}{2}\right) = 0.691556\ldots,$

(ii) $\sum\limits_{s=1}^{2m+1} \sin(s\theta)c_s > 0 \Leftrightarrow \lambda + \kappa \leq \lambda^*\left(\frac{1}{2}\right),$

(iii) $\sum\limits_{s=1}^{2m} \sin(s\theta)c_s > 0$ for $\lambda \leq \frac{1-\kappa}{2},$

where $\lambda^*(\tau)$, $\tau \in (0,1]$ is the solution of

$$\int\limits_0^{(\tau+1)\pi} \frac{\sin(x-\tau\pi)}{x^{1-\lambda}}dx = 0$$

which is unique in $[0,1]$. It is observed that $\lambda^*(\tau)$ was first obtained by Koumandos and Ruscheweyh [42]. In this work, we use $\lambda^*\left(\frac{1}{2}\right) = \lambda_0^*$.

Lemma 3 ([41])**.** *Let $\kappa \geq 0$, $\lambda \in \mathbb{R}$ such that $0 < \lambda + \kappa < 1$ and $m \in \mathbb{N}$. If $\{c_s\}_{s\geq 1}$ is a sequence of decreasing numbers that are non-negative such that $c_0 > 0$ and*

$$s(m-s+1+\kappa)c_{2s} \leq (s+\lambda+\kappa-1)(m-s+1)c_{2s-1}, \quad \textit{for } 1 \leq s \leq m, \tag{4}$$

then for all $0 < \theta < \pi$

$$\sum_{s=0}^{m} c_s \cos s\theta > 0 \Leftrightarrow \lambda + \kappa \leq \lambda_0^*.$$

Lemma 4 ([41])**.** *Let $0 \leq \kappa \leq 2\lambda_0^* - 1$, $\lambda \in \mathbb{R}$ such that $0 < \lambda + \kappa < 1$ and $m \in \mathbb{N}$. If $\{c_s\}_{s\geq 1}$ is a sequence of decreasing numbers that are non-negative such that $c_0 > 0$ and*

$$s(1-s+m+\kappa)c_{2s} \leq (\lambda+\kappa+s-1)(1-s+m)c_{2s-1}, \quad \textit{for } 1 \leq s \leq m,$$

then for all $0 < \theta < \pi$

$$\sum_{s=0}^{m} c_s \sin s\theta > 0 \Leftrightarrow \lambda + \kappa \leq \frac{1+\kappa}{2}.$$

Lemma 5 ([41])**.** *Let $0 \leq \kappa \leq 2\lambda_0^* - 1$ and $-\kappa < \lambda \leq \frac{1-\kappa}{2}$, $c_1 = 1$, $c_s \geq 0$ satisfy*

$$\begin{aligned} &[s(1+\kappa)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]c_s \\ \geq\ &[(s+1)(1+\kappa)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]c_{s+1}, \end{aligned} \tag{5}$$

$$\begin{aligned} &(\lambda+s+\kappa-1)(1-s+m)[2s(\kappa+1)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]c_{2s} \\ \geq\ &s(1-s+m+\kappa)[(1+\kappa)(2s+1)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]c_{2s+1}, \end{aligned} \tag{6}$$

for $1 \leq s \leq m$. Then, $\mathfrak{f}_m(\varsigma) = \sum\limits_{s=1}^{m} c_s\varsigma^s \in \mathcal{S}^*(\delta)$, where $\delta = \frac{1-2\lambda-\kappa}{(1+\kappa)(1-\lambda-\kappa)}$. Moreover, in the limiting case, $\mathfrak{f}(\varsigma) = \lim_{m\to\infty} \mathfrak{f}_m(\varsigma) = \sum\limits_{s=1}^{\infty} c_s\varsigma^s \in \mathcal{S}^*(\delta)$ if $\{c_s\}$ satisfies (5) and in addition

$$\begin{aligned} &(s+\lambda+\kappa-1)[(1-\lambda-\kappa)2s(1+\kappa) + 2\lambda + \kappa - 1]c_{2s} \\ \geq\ &s[(1+\kappa)(2s+1)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]c_{2s+1}, \textit{ for } s \geq 1. \end{aligned} \tag{7}$$

Lemma 6 ([41])**.** *Let $\lambda \in \mathbb{R}$ and $\kappa \geq 0$ such that $0 < \lambda + \kappa < 1$ and let $c_1 = 1$ and $c_s \geq 0$ satisfy*

$$0 \leq mc_m \leq \cdots \leq (s+1)c_{s+1} \leq sc_s \leq \cdots \leq 3c_3 \leq 2c_2 \leq \frac{\lambda+\kappa}{\lambda_0^*}, \quad \lambda + \kappa \in (0, \lambda_0^*], \tag{8}$$

and

$$2(\mathrm{s}-1+\lambda+\kappa)(m-\mathrm{s}+1)\mathrm{c}_{2\mathrm{s}} \geq (m+\kappa-\mathrm{s}+1)(1+2\mathrm{s})\mathrm{c}_{2\mathrm{s}+1}, \quad 1 \leq \mathrm{s} \leq \left[\frac{m}{2}\right]. \tag{9}$$

Then, $\mathfrak{f}_m(\varsigma) = \varsigma + \sum_{\mathrm{s}=2}^{m} \mathrm{c}_\mathrm{s}\varsigma^\mathrm{s}$ *satisfies* $\Re(\mathfrak{f}'_m(\varsigma)) > 1 - \frac{\lambda+\kappa}{\lambda_0^*}$.

Lemma 7 ([25])**.** *Consider the sequence* $\{\mathrm{c}_\mathrm{s}\}_{\mathrm{s}=1}^{\infty}$ *of positive real number such that* $\mathrm{c}_1 = 1$. *Let* $\mathrm{c}_1 \geq 8\mathrm{c}_2$ *and* $(\mathrm{s}-1)\mathrm{c}_\mathrm{s} - (1+\mathrm{s})\mathrm{c}_{\mathrm{s}+1} \geq 0$, $\forall \mathrm{s} \geq 2$. *Then,* $\mathfrak{f}(\varsigma) = \varsigma + \sum_{\mathrm{s}=2}^{\infty} \mathrm{c}_\mathrm{s}\varsigma^\mathrm{s} \in \mathcal{K}$ *with respect to starlike function* $\frac{\varsigma}{1-\varsigma^2}$.

Lemma 8 ([25])**.** *Consider the sequence* $\{\mathrm{c}_\mathrm{s}\}_{\mathrm{s}=1}^{\infty}$ *of a positive real number such that* $\mathrm{c}_1 = 1$. *Let* $0 \leq \lambda < 1$ *and*

(i) $(1-\delta)\mathrm{c}_1 \geq (2-\delta)\mathrm{c}_2 \geq 2^{(\delta+1)}(3-\delta)\mathrm{c}_3$,
(ii) $(\mathrm{s}-1-\delta)(\mathrm{s}-\delta)\mathrm{c}_\mathrm{s} \geq s(s-\delta+1)\mathrm{c}_{\mathrm{s}+1}$, $\forall \mathrm{s} \geq 3$.

Then, $\mathfrak{f}(\varsigma) = \varsigma + \sum_{\mathrm{s}=2}^{\infty} c_\mathrm{s}\varsigma^\mathrm{s} \in \mathcal{S}^*(\delta)$.

Lemma 9 ([43])**.** *If the function* $\mathfrak{f}(\varsigma) = \sum_{\mathrm{s}=1}^{\infty} \mathrm{c}_\mathrm{s}\varsigma^{\mathrm{s}-1}$, *where* $\mathrm{c}_1 = 1$ *and* $\mathrm{c}_\mathrm{s} \geq 0$, $\forall \mathrm{s} \geq 2$ *is analytic in* $\mathbb{U}$, *and if* $\{\mathrm{c}_\mathrm{s}\}_{\mathrm{s}=1}^{\infty}$ *is a convex decreasing sequence, i.e.,* $\mathrm{c}_{\mathrm{s}+2} - 2\mathrm{c}_{\mathrm{s}+1} + \mathrm{c}_\mathrm{s} \geq 0$ *and* $\mathrm{c}_\mathrm{s} - \mathrm{c}_{\mathrm{s}+1} \geq 0$, $\forall \mathrm{s} \geq 1$, *then*

$$\Re\mathfrak{f}(\varsigma) > \frac{1}{2}, \ \forall \varsigma \in \mathbb{U}.$$

Lemma 10 ([43])**.** *If* $\mathrm{c}_\mathrm{s} \geq 0$, $\{\mathrm{s}\mathrm{c}_\mathrm{s}\}$ *and* $\{\mathrm{s}\mathrm{c}_\mathrm{s} - (\mathrm{s}+1)\mathrm{c}_{\mathrm{s}+1}\}$ *are non-increasing, then* $\mathfrak{f}$ *defined by* $\mathfrak{f}(\varsigma) = \varsigma + \mathrm{c}_2\varsigma^2 + \mathrm{c}_3\varsigma^3 + \ldots$, $(\varsigma \in \mathbb{U})$ *is in* $\mathcal{S}^*$.

3. Main Results

Theorem 1. *Let* $a \geq 1$, $b \geq 1$, *and*

$$\Gamma(a+b) \geq 8\Gamma(b), \ 2\Gamma(2a+b) \geq 3\Gamma(a+b)$$

are satisfied. Then, $\mathcal{W}_{a,b} \in \mathcal{K}$ *with respect to starlike function* $\frac{\varsigma}{1-\varsigma^2}$.

Proof. Consider

$$\mathcal{W}_{a,b}(\varsigma) = \varsigma + \sum_{\mathrm{s}=2}^{\infty} \mathrm{c}_\mathrm{s}\varsigma^\mathrm{s},$$

where $\mathrm{c}_\mathrm{s} = \frac{\Gamma(b)}{(\mathrm{s}-1)!\Gamma(a(\mathrm{s}-1)+b)}$, $\forall \mathrm{s} \geq 2$. We have to show that c_s satisfies the hypothesis of Lemma 7. It is clear that, for $a \geq 1$ and $b \geq 1$, the inequality $\Gamma(a+b) \geq 8\,\Gamma(b)$ is satisfied. Additionally,

$$\mathrm{c}_1 = 1 \text{ and } \mathrm{c}_1 \gtrapprox 8\mathrm{c}_2.$$

Again for $\mathrm{s} \geq 2$, consider

$$(\mathrm{s}-1)\mathrm{c}_\mathrm{s} - (\mathrm{s}+1)\mathrm{c}_{\mathrm{s}+1} = A(\mathrm{s})M(\mathrm{s}),$$

where

$$\begin{aligned} A(\mathrm{s}) &= \frac{\mathrm{c}_\mathrm{s}}{\mathrm{s}\Gamma(a\mathrm{s}+b)}, \\ M(\mathrm{s}) &= \mathrm{s}(\mathrm{s}-1)\Gamma(a\mathrm{s}+b) - (\mathrm{s}+1)\Gamma(a(\mathrm{s}-1)+b). \end{aligned}$$

One can easily observe that $A(\mathrm{s})$ is non-negative for $a \geq 1, b \geq 1$ and $M(\mathrm{s})$ is non-negative for $a \geq 1, b \geq 1$ if $2\Gamma(2a+b) \geq 3\Gamma(a+b)$. It is clear that $\{\mathrm{c_s}\}_{\mathrm{s}=1}^{\infty}$ satisfies Lemma 7. This completes the result. □

Theorem 2. *Let $a \geq 1, b \geq 1$, and*

$$\Gamma(a+b) > \Gamma(b),\ \{2\Gamma(2a+b)+\Gamma(b)\}\Gamma(a+b) > 4\Gamma(b)\Gamma(2a+b),$$

are satisfied. Then,

$$\Re\left\{\frac{\mathcal{W}_{a,b}(\varsigma)}{\varsigma}\right\} > \frac{1}{2},\ for\ \varsigma \in \mathbb{U}.$$

Proof. To obtain our result, we first prove that the sequence

$$\{\mathrm{c_s}\}_{\mathrm{s}=1}^{\infty} = \left\{\frac{\Gamma(b)}{(\mathrm{s}-1)!\Gamma(a(\mathrm{s}-1)+b)}\right\}_{\mathrm{s}=1}^{\infty},$$

is decreasing. Since

$$\mathrm{s}!\Gamma(a\mathrm{s}+b) > (\mathrm{s}-1)!\Gamma(a(\mathrm{s}-1)+b)\ \ (\forall\ \mathrm{s} \geq 1,\ a \geq 1 \text{ and } b \geq 1).$$

Therefore,

$$\frac{\Gamma(b)}{(\mathrm{s}-1)!\Gamma(a(\mathrm{s}-1)+b)} > \frac{\Gamma(b)}{\mathrm{s}!\Gamma(a\mathrm{s}+b)}\ \ (\forall\ \mathrm{s} \geq 1,\ a \geq 1 \text{ and } b \geq 1).$$

Now, we prove that the sequence $\{\mathrm{c_s}\}_{\mathrm{s}=1}^{\infty}$ is convex and decreasing. For this, we prove that $\mathrm{c_s} + \mathrm{c_{s+2}} - 2\mathrm{c_{s+1}} \geq 0$. Take

$$\mathrm{c_{s+2}} - 2\mathrm{c_{s+1}} + \mathrm{c_s} =$$
$$\Gamma(b)\left[\frac{\begin{array}{c}(\mathrm{s}+1)!\{\mathrm{s}!\Gamma(a\mathrm{s}+b) - 2(\mathrm{s}-1)!\Gamma(a(\mathrm{s}-1)+b)\}\Gamma(a(\mathrm{s}+1)+b) \\ +\mathrm{s}!\Gamma(a\mathrm{s}+b)\{(\mathrm{s}-1)!\Gamma(a(\mathrm{s}-1)+b)\}\end{array}}{\mathrm{s}!\Gamma(a\mathrm{s}+b)\{(\mathrm{s}-1)!\Gamma(a(\mathrm{s}-1)+b)\}\{(\mathrm{s}+1)!\Gamma(a(\mathrm{s}+1)+b)\}}\right]. \quad (10)$$

The expression (10) is non-negative for all $\mathrm{s} \geq 1$, $a \geq 1$ and $b \geq 1$, if $2\Gamma(2a+b) + \Gamma(b) > \frac{4\Gamma(b)\Gamma(2a+b)}{\Gamma(a+b)}$, which shows that $\{\mathrm{c_s}\}_{\mathrm{s}=1}^{\infty}$ is a convex decreasing sequence. Now, from the Lemma 9 $\{\mathrm{c_s}\}_{\mathrm{s}=1}^{\infty}$ satisfy

$$\Re\left(\sum_{\mathrm{s}=1}^{\infty} \mathrm{c_s}\varsigma^{\mathrm{s}-1}\right) > \frac{1}{2},\ \text{for all } \varsigma \in \mathbb{U},$$

therefore,

$$\Re\left(\frac{\mathcal{W}_{a,b}(\varsigma)}{\varsigma}\right) > \frac{1}{2},\ \text{for all } \varsigma \in \mathbb{U}.$$

Hence, the result follows. □

Theorem 3. *Let $a \geq 1, b \geq 1$, and*

$$\Gamma(a+b) > 2\Gamma(b)\ ,\ \{2\Gamma(2a+b)+3\Gamma(b)\}\Gamma(a+b) > 8\Gamma(b)\Gamma(2a+b),$$

are satisfied. Then the normalized Wright function $\mathcal{W}_{a,b} \in \mathcal{S}^*$.

Proof. To prove that $\mathcal{W}_{a,b} \in \mathcal{S}^*$, we show that $\{sc_s\}$ and $\{sc_s - (s+1)c_{s+1}\}$ both are non-increasing since $c_s \geq 0$ for $\mathcal{W}_{a,b}(\varsigma)$ under the given conditions. So, consider

$$\begin{aligned} sc_s - (s+1)c_{s+1} &= \frac{\Gamma(b)}{(s-1)!}\left\{\frac{s^2\Gamma(as+b) - (s+1)\Gamma(a(s-1)+b)}{s\Gamma(as+b)\Gamma(a(s-1)+b)}\right\} \\ &> 0\ (\forall\, s \geq 1,\ a \geq 1 \text{ and } b \geq 1). \end{aligned}$$

Now,

$$\begin{aligned} &(s+2)c_{s+2} - 2(s+1)c_{s+1} + sc_s \\ &= \frac{\Gamma(b)}{(s-1)!}\left[\frac{\begin{array}{c}\Gamma(a(s+1)+b)(s+1)\{s^3\Gamma(as+b) - 2s(s+1)\Gamma(a(s-1)+b)\} \\ +s(s+2)\Gamma(as+b)\Gamma(a(s-1)+b)\end{array}}{s^2(s+1)\Gamma(a(s+1)+b)\Gamma(as+b)\Gamma(a(s-1)+b)}\right]. \end{aligned} \tag{11}$$

The expression (11) is non-negative for all $s \geq 1$, $a \geq 1$ and $b \geq 1$, if $2\Gamma(2a+b) + 3\Gamma(b) > \frac{8\Gamma(b)\Gamma(2a+b)}{\Gamma(a+b)}$. So, from Lemma 10 $\mathcal{W}_{a,b}(\varsigma)$ is starlike in $\mathbb{U}$. □

Remark 1. *This result improves the result of Prajapat [31] (Theorem 2.7 p. 4.).*

Theorem 4. *Let $0 \leq \kappa \leq 2\lambda_0^* - 1$, $-\kappa < \lambda \leq \frac{1-\kappa}{2}$, $2\lambda + \kappa > 1$, $a \geq 1$ and $b \geq 2$. If $M_1 = (1+\kappa)(1-\lambda-\kappa) > 0$ and $M_2 = -1 + 2\lambda + \kappa > 0$, then $\mathcal{W}_{a,b}$ is starlike of order $\frac{1-2\lambda-\kappa}{(1+\kappa)(1-\lambda-\kappa)}$.*

Proof. It is observed that $(\mathcal{W}_{a,b})_m(\varsigma) = \sum\limits_{s=1}^{m} c_s \varsigma^s$ provides $c_1 = 1$ and $c_s = \frac{\Gamma(b)}{(s-1)!\Gamma(a(s-1)+b)}$ for $s \geq 2$. The relation between c_s and c_{s+1} is

$$c_{s+1} = \frac{\Gamma(a(s-1)+b)}{s\Gamma(as+b)}c_s, \quad \text{for } s \geq 1.$$

To proceed the proof of this result, it would be enough to prove the assertion that $\{c_s\}_{s=1}^{\infty}$ satisfies the conditions (5) and (7) of Lemma 5. Making use of the above relation followed by simple computation leads us to

$$\begin{aligned} &[s(1-\lambda-\kappa)(1+\kappa) + 2\lambda + \kappa - 1]c_s - [(s+1)(1+\kappa)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]c_{s+1} \\ &= \frac{\Gamma(a(s-1)+b)c_s}{s\Gamma(as+b)}h(s), \end{aligned}$$

where h is defined as

$$\begin{aligned} h(s) &= \frac{s\Gamma(as+b)}{\Gamma(a(s-1)+b)}[s(1-\lambda-\kappa)(1+\kappa) + 2\lambda + \kappa - 1] \\ &\quad -[(s+1)(1-\lambda-\kappa)(1+\kappa) + 2\lambda + \kappa - 1] \\ &= \frac{s\Gamma(as+b)}{\Gamma(a(s-1)+b)}(sM_1 + M_2) - [(s+1)M_1 + M_2] \\ &= \left[\frac{s^2\Gamma(as+b)}{\Gamma(a(s-1)+b)} - (s+1)\right]M_1 + \left[\frac{s\Gamma(as+b)}{\Gamma(a(s-1)+b)} - 1\right]M_2. \end{aligned} \tag{12}$$

It is observed that under the certain conditions, expression (12) is positive for $s \geq 1$ but with (7) to verify further. Now,

$$\begin{aligned} &(s+\lambda+\kappa-1)[2s(1+\kappa)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]c_{2s} \\ \geq\ &s[(2s+1)(1+\kappa)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]c_{2s+1}. \end{aligned}$$

Clearly,

$$\begin{aligned}(\mathrm{s}+\lambda+\kappa-1)[2\mathrm{s}(1-\lambda-\kappa)(1+\kappa)+2\lambda+\kappa-1]\mathrm{c}_{2\mathrm{s}}\\ -\mathrm{s}[(2\mathrm{s}+1)(1+\kappa)(1-\lambda-\kappa)-1+2\lambda+\kappa]\mathrm{c}_{2\mathrm{s}+1}=\frac{\Gamma(a(2\mathrm{s}-1)+b)\mathrm{c}_{2\mathrm{s}}}{2\Gamma(a(2\mathrm{s})+b)}\mathfrak{g}(\mathrm{s}),\end{aligned}$$

where $\mathfrak{g}$ is defined as

$$\begin{aligned}\mathfrak{g}(\mathrm{s})=&\,2\frac{\Gamma(a(2\mathrm{s})+b)}{\Gamma(a(2\mathrm{s}-1)+b)}(\mathrm{s}+\lambda+\kappa-1)[2\mathrm{s}(1-\lambda-\kappa)(1+\kappa)+2\lambda+\kappa-1)]\\ &-[(2\mathrm{s}+1)(1-\lambda-\kappa)(1+\kappa)-1+2\lambda+\kappa]\\ =&\,\frac{\Gamma(a(2\mathrm{s})+b)}{\Gamma(a(2\mathrm{s}-1)+b)}2(\mathrm{s}+\lambda+\kappa-1)[2\mathrm{s}M_1+M_2]-[(2\mathrm{s}+1)M_1+M_2]\\ =&\left[\frac{\Gamma(a(2\mathrm{s})+b)}{\Gamma(a(2\mathrm{s}-1)+b)}4\mathrm{s}(\mathrm{s}+\lambda+\kappa-1)-(2\mathrm{s}+1)\right]M_1\\ &+\left[\frac{\Gamma(a(2\mathrm{s})+b)}{\Gamma(a(2\mathrm{s}-1)+b)}2(\mathrm{s}+\lambda+\kappa-1)-1\right]M_2.\end{aligned}\tag{13}$$

It is observed that the expression (13) is positive under the given conditions for $\mathrm{s}\geq 1$, which proves the hypothesis. □

Theorem 5. *Let* $0\leq\kappa\leq 2\lambda_0^*-1$, $2\lambda+\kappa>1$, $a\geq 1$, $b\geq 2$ *and* $c_1=1$, $\mathrm{c_s}\geq 0$ *satisfy*

$$\begin{aligned}&\mathrm{sc_s}-(\mathrm{s}+1)\mathrm{c}_{\mathrm{s}+1}\geq 0,\quad \mathrm{s}=1,2,3,...,s-1,\\ &(m-\mathrm{s}+1)(\mathrm{s}+\lambda+\kappa-1)(2\mathrm{s}-1)\mathrm{c}_{2\mathrm{s}-1}\geq 2\mathrm{s}^2(m-\mathrm{s}+1+\kappa)\mathrm{c}_{2\mathrm{s}},\ \ \mathrm{s}=4,5,...,\left[\frac{m+3}{2}\right],\end{aligned}\tag{14}$$

for $s\geq 4$, $-\kappa<\lambda\leq\frac{1-\kappa}{2}$. *Then,* $(\mathcal{W}_{a,b})_m$ *is convex along the imaginary axis.*

Proof. To proove the result, we need to show that $\varsigma(\mathcal{W}_{a,b})_m'(\varsigma)$ is typically real and $(\mathcal{W}_{a,b})_m(\varsigma)$ has real coefficients. Set

$$\varsigma(\mathcal{W}_{a,b})_m'(\varsigma)=\varsigma+\sum_{\mathrm{s}=2}^{m}\frac{\mathrm{s}\Gamma(b)}{(\mathrm{s}-1)!\Gamma(a(\mathrm{s}-1)+b)}\varsigma^{\mathrm{s}},$$

where $\mathrm{c_s}=\frac{\Gamma(b)}{(\mathrm{s}-1)!\Gamma(a(\mathrm{s}-1)+b)}$. To obtain the result, it is required that the coefficients of Wright functions $\mathrm{c_s}$ must fullfil the conditions mentioned in Lemma 4. Consider,

$$sc_{\mathrm{s}}-(\mathrm{s}+1)\mathrm{c}_{\mathrm{s}+1}=\frac{\Gamma(b)}{(\mathrm{s}-1)!}\left[\frac{\mathrm{s}^2\Gamma(a\mathrm{s}+b)-(\mathrm{s}+1)\Gamma(a(\mathrm{s}-1)+b)}{\mathrm{s}\Gamma(a(\mathrm{s}-1)+b)\Gamma(a\mathrm{s}+b)}\right]>0,$$

for $\mathrm{s}=1,2,3,\ldots,m-1$. Take

$$(m-\mathrm{s}+1)(2\mathrm{s}-1)(\mathrm{s}+\lambda+\kappa-1)\mathrm{c}_{2\mathrm{s}-1}-2\mathrm{s}^2(m-\mathrm{s}+\kappa+1)\mathrm{c}_{2\mathrm{s}}=\frac{\mathrm{c}_{2\mathrm{s}}\Gamma(a(2\mathrm{s}-1)+b)}{\Gamma(a(2\mathrm{s}-2)+b)}q(\mathrm{s}),$$

here q is defined as

$$q(\mathrm{s})=(m-\mathrm{s}+1)(2\mathrm{s}-1)^2(\mathrm{s}+\lambda+\kappa-1)-2\mathrm{s}^2(m-\mathrm{s}+\kappa+1)\frac{\Gamma(a(2\mathrm{s}-2)+b)}{\Gamma(a(2\mathrm{s}-1)+b)}.\tag{15}$$

Since Γ is an increasing function in $\left[\frac{3}{2},\infty\right)$, therefore (15) becomes positive when $m\geq\mathrm{s}$, $a\geq 1$ and $b\geq 2$. Thus, the sequence $\{\mathrm{c_s}\}_{\mathrm{s}=1}^{\infty}$ satisfies the conditions of Lemma 4.

Therefore, by following the minimum principle for harmonic functions with the conditions $\lambda + \kappa \in \left(0, \frac{1+\kappa}{2}\right]$

$$\Im\big(\varsigma\mathfrak{f}'_m(\varsigma)\big) = \sum_{\mathrm{s}=1}^{m} \mathrm{c_s} r^{\mathrm{s}} \sin \mathrm{s}\theta > 0, \quad \text{where } \theta \in [0,\pi] \text{ and } r \in [0,1]$$

and

$$\Im\big(\varsigma\mathfrak{f}'_m(\varsigma)\big) = 0 \text{ for } \varsigma \in (0,1).$$

The Schwarz reflection principle provides that $\Im(\varsigma\mathfrak{f}'_m(\varsigma)) < 0$ for $\theta \in (\pi, 2\pi)$. So, $\varsigma\mathfrak{f}'_m(\varsigma)$ is typically real, which leads to the required result. □

Theorem 6. *Let* $0 \leq \kappa \leq 2\lambda_0^* - 1$, $-\kappa < \lambda \leq \frac{1-\kappa}{2}$ *with* $2\lambda + \kappa > 1$, $a \geq 1$, $b \geq \frac{3}{2}$ *and* $\mathrm{c_1} = 1$, $\mathrm{c_s} \geq 0$ *satisfy*

$$\begin{aligned} &[s(1+\kappa)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]c_s \geq \\ &[(s+1)(1+\kappa)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]c_{s+1}, \end{aligned} \tag{16}$$

$$\begin{aligned} &(m-\mathrm{s}+1)(\mathrm{s}+\lambda+\kappa-1)[2\mathrm{s}(1+\kappa)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]\mathrm{c_{2s}} \geq \\ &\mathrm{s}(m-\mathrm{s}+1+\kappa)[(2\mathrm{s}+1)(1+\kappa)(1-\lambda-\kappa) - 1 + 2\lambda + \kappa]\mathrm{c_{2s+1}}, \end{aligned} \tag{17}$$

for $1 \leq \mathrm{s} \leq m$. *Then,* $(\mathcal{W}_{a,b})_m \in \mathcal{S}^*(\delta)$, *where* $\delta = \frac{1-2\lambda-\kappa}{(1+\kappa)(1-\lambda-\kappa)}$. *Furthermore, the limiting function* $\mathfrak{f}(\varsigma) = \lim_{m\to\infty}(\mathcal{W}_{a,b})_m(\varsigma) = \sum_{\mathrm{s}=1}^{\infty} \mathrm{c_s}\varsigma^{\mathrm{s}} \in \mathcal{S}^*(\delta)$ *if* $\{\mathrm{c_s}\}_{\mathrm{s}=1}^{\infty}$ *satisfies* (16) *with*

$$\begin{aligned} &(\mathrm{s}+\lambda+\kappa-1)[2\mathrm{s}(1-\lambda-\kappa)(1+\kappa) + 2\lambda + \kappa - 1]\mathrm{c_{2s}} \\ \geq\ &\mathrm{s}[(2\mathrm{s}+1)(1-\lambda-\kappa)(1+\kappa) + 2\lambda + \kappa - 1]\mathrm{c_{2s+1}}, \textit{ for } \mathrm{s} \geq 1. \end{aligned} \tag{18}$$

Proof. Let $(\mathcal{W}_{a,b})_m(\varsigma) = \varsigma + \sum_{\mathrm{s}=2}^{m} \mathrm{c_s}\varsigma^{\mathrm{s}}$ and $\delta = \frac{1-2\lambda-\kappa}{(1+\kappa)(1-\lambda-\kappa)}$. Then,

$$\mathfrak{g}_m(\varsigma) = (\mathcal{W}_{a,b})'_m(\varsigma) + \frac{(\mathcal{W}_{a,b})_m(\varsigma)}{\varsigma} = \mathrm{c_0} + \sum_{\mathrm{s}=1}^{m-1} \mathrm{c_s}\varsigma^{\mathrm{s}}.$$

Here, $\mathrm{c_0} = (1-\delta)$ and $\mathrm{c_s} = (\mathrm{s}+1-\delta)\mathrm{c_{s+1}}$, for $1 \leq \mathrm{s} \leq m$, and $\mathrm{c_{s+1}} = \frac{\Gamma(\kappa)}{(\mathrm{s}-1)!\Gamma(\alpha(\mathrm{s}-1)+\kappa)}$. It see that $c_0 > 0$. Now, to prove that $\{\mathrm{c_s}\}_{\mathrm{s}=1}^{\infty}$ is decreasing we will show that

$$(\mathrm{s}-\delta)\mathrm{c_s} \geq (\mathrm{s}+1-\delta)\mathrm{c_{s+1}}$$

or equivalently

$$\left(\mathrm{s} - \frac{1-2\lambda-\kappa}{(1+\kappa)(1-\lambda-\kappa)}\right)\mathrm{c_s} \geq \left(\mathrm{s}+1 + \frac{1-2\lambda-\kappa}{(1+\kappa)(1-\lambda-\kappa)}\right)\mathrm{c_{s+1}}. \tag{19}$$

We obtain (16) by rearranging (19). Now, to show that $\{\mathrm{c_s}\}_{\mathrm{s}=1}^{\infty}$ satisfies (4), we have

$$(\mathrm{s}+\lambda+\kappa-1)(m-\mathrm{s}+1)(2\mathrm{s}-\delta)\mathrm{c_{2s}} \geq \mathrm{s}(2\mathrm{s}+1-\delta)(m-\mathrm{s}+1+\kappa)\mathrm{c_{2s+1}},$$

$$\begin{aligned} &(\mathrm{s}+\lambda+\kappa-1)(m-\mathrm{s}+1)\left(2\mathrm{s} - \frac{1-\kappa-2\lambda}{(1+\kappa)(1-\lambda-\kappa)}\right)\mathrm{c_{2s}} \\ \geq\ &\mathrm{s}(m-\mathrm{s}+1+\kappa)\left(2\mathrm{s}+1 - \frac{1-\kappa-2\lambda}{(1+\kappa)(1-\lambda-\kappa)}\right)\mathrm{c_{2s+1}}. \end{aligned}$$

Under the given conditions and for $a \geq 1$, $b \geq \frac{3}{2}$ above relation is satisfied. This shows that $\{c_s\}_{s=1}^{\infty}$ satisfies the conditions of Lemmas 3 and 4. Now, by using the minimum principle of harmonic functions, we have

$$\Re(\mathfrak{g}_m(\varsigma)) = \sum_{s=0}^{m} c_s r^s \cos(s\theta) > 0, \quad \text{for } \theta \in (0,\pi) \text{ and } r \in [0,1).$$

Furthermore, $\Im(\mathfrak{g}_m(\varsigma)) = \sum_{s=1}^{m} c_s r^s \sin(s\theta) \equiv 0$, if $-1 < \varsigma = x + iy < 1$ and $\Im(\mathfrak{g}_m(\varsigma)) > 0$ in $D \cap \{\varsigma : \Im(\varsigma) > 0\}$. By the reflection principle $\Im(\mathfrak{g}_m(\varsigma)) < 0$ in the above-mentioned domain. This implies $\mathfrak{g}_m(\varsigma)$ is typically real. For other part of the theorem, it is enough to show that $(\mathcal{W}_{a,b})'_m$ is typically real with real part. Now,

$$(\mathcal{W}_{a,b})'_m(\varsigma) = 1 + \sum_{s=1}^{m-1} (s+1) c_{s+1} \varsigma^s.$$

Clearly, by the same arguments $\Re(\mathcal{W}_{a,b})'_m(\varsigma) > 0$ and typically real. Therefore by Lemma 1, it is clear that $(\mathcal{W}_{a,b})_m(\varsigma)$ is starlike of order $\frac{1-2\lambda-\kappa}{(1+\kappa)(1-\lambda-\kappa)}$. For the case $m \to \infty$ (17) becomes (18), so we have $\mathcal{W}_{a,b}(\varsigma) = \lim_{m\to\infty}(\mathcal{W}_{a,b})_m(\varsigma)$ is starlike of order $\frac{1-2\lambda-\kappa}{(1+\kappa)(1-\lambda-\kappa)}$ if $\{c_s\}$ satisfy (16) and (18). □

Theorem 7. *Let $\lambda \in \mathbb{R}$ and $\kappa \geq 0$ such that $2\lambda + \kappa > 1$, and let $c_1 = 1$ and $c_s \geq 0$ satisfy*

$$0 \leq mc_m \leq \ldots \leq (s+1)c_{s+1} \leq sc_s \leq \ldots 3c_3 \leq 2c_2 \leq \frac{\lambda+\kappa}{\lambda_0^*}, \quad \lambda + \kappa \in (0, \lambda_0^*], \tag{20}$$

and

$$2(s+\lambda+\kappa-1)(m-s+1)c_{2s} \geq (2s+1)(m-s+1+\kappa)c_{2s+1}, \quad 1 \leq s \leq \left[\frac{m}{2}\right]. \tag{21}$$

If $a \geq 1$ and $b \geq 2$, then $(\mathcal{W}_{a,b})_m$ satisfies $\Re(\mathfrak{f}'_m(\varsigma)) > 1 - \frac{\lambda+\kappa}{\lambda_0^*}$.

Proof. Let $\sigma - 1 = -\frac{\lambda+\kappa}{\lambda_0^*}$ and $(\mathcal{W}_{a,b})_m(\varsigma) = \varsigma + \sum_{s=2}^{m} c_s \varsigma^s$, where $c_s = \frac{\Gamma(b)}{(s-1)!\Gamma(a(s-1)+b)}$. Then,

$$\frac{(\mathcal{W}_{a,b})'_m(\varsigma) - \sigma}{1-\sigma} = \sum_{s=0}^{m-1} c_s \varsigma^s,$$

where $c_s = \frac{(s+1)c_{s+1}}{1-\sigma}$ and $c_0 = 1$ for $1 \leq s \leq m-1$. It is observed that for $a \geq 1$ and $b \geq 2$, the coefficients c_s are positive. Therefore, $c_s > 0$ for $s \geq 1$. To prove the assertion, we need to prove that the coefficients' sequence $\{c_s\}$ is decreasing and satisfies (9). For this, consider

$$(s+1)c_{s+1} - (s+2)c_{s+2} = \frac{\Gamma(b)}{(s-1)!}\left[\frac{(s+1)^2\Gamma(a(s+1)+b) - (s+2)\Gamma(as+b)}{(s+1)\Gamma(a(s+1)+b)\Gamma(as+b)}\right] > 0,$$

for $s = 1, 2, 3, \ldots, m-2$. This shows that the coefficients' sequence of thw Wright function is decreasing and $c_1 < c_0 \Rightarrow 2c_2 < 1 - \sigma$. Now, to prove (9), consider

$$
\begin{aligned}
& 2\mathrm{s}(\lambda+\kappa+\mathrm{s}-1)(m-\mathrm{s}+1)\mathrm{c}_{2\mathrm{s}}-\mathrm{s}(m-\mathrm{s}+\kappa+1)(2\mathrm{s}+1)\mathrm{c}_{2\mathrm{s}+1} \\
= & \frac{2\mathrm{s}(\lambda+\kappa+\mathrm{s}-1)(m-\mathrm{s}+1)\Gamma(b)}{(2\mathrm{s}-1)!\Gamma(a(2\mathrm{s}-1)+b)}-\frac{\mathrm{s}(m-\mathrm{s}+1+\kappa)(2\mathrm{s}+1)\Gamma(b)}{(2\mathrm{s})!\Gamma(a(2\mathrm{s})+b)} \\
= & \frac{\mathrm{s}\Gamma(b)}{(2\mathrm{s}-1)!}\left[\frac{\begin{array}{c}4\mathrm{s}^2(\lambda+\kappa+\mathrm{s}-1)(m-\mathrm{s}+1)(2\mathrm{s})!\Gamma(a(2\mathrm{s})+b)\\ -(m-\mathrm{s}+\kappa+1)(2\mathrm{s}+1)\Gamma(a(2\mathrm{s}-1)+b)\end{array}}{2\mathrm{s}\Gamma(a(2\mathrm{s})+b)\Gamma(a(2\mathrm{s}-1)+b)}\right] \\
> & 0,\ \text{for } 1\le \mathrm{s}\le \left[\frac{m}{2}\right].
\end{aligned}
$$

It is evident that

$$2\mathrm{s}(\lambda+\kappa+\mathrm{s}-1)(m-\mathrm{s}+1)\mathrm{c}_{2\mathrm{s}}-\mathrm{s}(m-\mathrm{s}+\kappa+1)(2\mathrm{s}+1)\mathrm{c}_{2\mathrm{s}+1}>0$$

for $m\ge \mathrm{s}$, $a\ge 1$ and $b\ge 2$. Additionally, this yields the (21) due to the increasing behavior of Γ in $\left[\frac{3}{2},\infty\right)$. By similar arguments, the usage of the principle of minimum for harmonic functions, we obtain the result. □

4. Prestarlikeness of Wright Functions

Theorem 8. *Let $a\ge 1$ and $b\ge 1$. Then, $\mathcal{W}_{a,b}\in\mathcal{R}[\alpha,\xi]$ if for $0\le\xi<1$*

$$\frac{\Gamma(a+b)}{\Gamma(b)}\ge\begin{cases}\mathrm{T}_1(\alpha,\xi), & 0\le\alpha\le\alpha_0(\xi)\\ \max\{\mathrm{T}_1(\alpha,\xi),\ \mathrm{T}_2(\alpha,\xi)M_1,\ \mathrm{T}_3(\alpha,\xi)\}M_2, & \alpha_0(\xi)\le\alpha\le 1\end{cases},$$

where

$$
\begin{aligned}
\mathrm{T}_1(\alpha,\xi) &= \frac{2(2-\xi)(1-\alpha)}{1-\xi},\quad \mathrm{T}_2(\alpha,\xi)=\frac{2^{\xi+1}(3-\xi)(3-2\alpha)}{4(2-\xi)} \\
\mathrm{T}_3(\alpha,\xi) &= \frac{(4-\xi)(4-2\alpha)}{3(2-\xi)(3-\xi)},\quad \alpha_0(\xi)=1-\frac{2(1-\xi)(4-\xi)}{6(2-\xi)^2(3-\xi)-2(1-\xi)(4-\xi)} \\
M_1 &= \frac{\{\Gamma(a+b)\}^2}{\Gamma(2a+b)\Gamma(b)},\quad M_2=\frac{\Gamma(a(s-1)+b)}{\Gamma(as+b)}\frac{\Gamma(a+b)}{\Gamma(b)}
\end{aligned}
$$

Proof. Consider the function $h(\varsigma)=\varsigma+\sum\limits_{\mathrm{s}=2}^{\infty}\mathrm{c}_\mathrm{s}\varsigma^\mathrm{s}$, c_s which is provided as

$$\mathrm{c}_1=1,\qquad \mathrm{c}_{\mathrm{s}+1}=\frac{(\mathrm{s}+1-2\alpha)}{\mathrm{s}^2}\frac{\Gamma(a(\mathrm{s}-1)+b)}{\Gamma(as+b)}\mathrm{c}_\mathrm{s},\quad \forall \mathrm{s}\ge 1.$$

For $0\le\xi<1$, $\alpha<1$, $a\ge 1$ and $b\ge 1$

$$\frac{\Gamma(a+b)}{\Gamma(b)}\ge\max\{\{\mathrm{s}_1(\alpha,\xi),\ \mathrm{s}_2(\alpha,\xi)M_1,\ \mathrm{s}_3(\alpha,\xi)\}M_2\}.$$

Clearly, $\Gamma(a+b)\ge \mathrm{s}_1(\alpha,\xi)\Gamma(b)$, which is equivalent to $(1-\xi)\Gamma(a+b)\ge 2(2-\xi)(1-\alpha)\Gamma(b)$. Hence, $(1-\xi)\mathrm{c}_1-(2-\xi)\mathrm{c}_2=\frac{1}{\Gamma(a+b)}[(1-\xi)\Gamma(a+b)-2(2-\xi)(1-\alpha)\Gamma(b)]$. Again,

$$(2-\xi)\mathrm{c}_2-2^{\xi+1}(3-\xi)\mathrm{c}_3=\frac{\mathrm{c}_2}{4\Gamma(2a+b)}\Big[4(2-\xi)\Gamma(2a+b)-2^{\xi+1}(3-\xi)(3-2\alpha)\Gamma(a+b)\Big].$$

Take

$$\begin{aligned}
4(2-\xi)\Gamma(2a+b)-2^{\xi+1}(3-\xi)(3-2\alpha)\Gamma(a+b) &\geq 0\\
4(2-\xi)\Gamma(2a+b) &\geq 2^{\xi+1}(3-\xi)(3-2\alpha)\Gamma(a+b)\\
\frac{\Gamma(a+b)}{\Gamma(b)} &\geq \mathrm{s}_2(\alpha,\xi)M_1.
\end{aligned}$$

Let us consider

$$A(\alpha,\xi) = \Gamma(a\mathrm{s}+b) \tag{22}$$
$$B(\alpha,\xi) = (8-2\xi)\Gamma(a\mathrm{s}+b)-\Gamma(a(\mathrm{s}-1)+b) \tag{23}$$
$$C(\alpha,\xi) = \{(5-\xi)(3-\xi)+3(2-\xi)\}\Gamma(a\mathrm{s}+b)-(8-2\alpha-\xi)\Gamma(a(\mathrm{s}-1)+b) \tag{24}$$
$$D(\alpha,\xi) = 3(2-\xi)(3-\xi)\Gamma(a\mathrm{s}+b)-(4-\xi)(4-2\alpha)\Gamma(a(\mathrm{s}-1)+b). \tag{25}$$

Since Γ is an increasing function in $\left[\frac{3}{2},\infty\right)$, therefore (22)–(25) become positive under the conditions $a\geq 1$ and $b\geq 1$. Now, if $\frac{\Gamma(a+b)}{\Gamma(b)}\geq \mathrm{s}_3(\alpha,\xi).M_2$, then clearly $D(\alpha,\xi)\geq 0$, and it can easily be observed that

$$A(\alpha,\xi)=\Gamma(a\mathrm{s}+b)>0.$$

Additionally, we can observe that

$$\begin{aligned}
B(\alpha,\xi) &= (8-2\xi)\Gamma(a\mathrm{s}+b)-\Gamma(a(\mathrm{s}-1)+b)\\
&> 6\Gamma(a\mathrm{s}+b)-\Gamma(a(\mathrm{s}-1)+b)>0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
C(\alpha,\xi) &= \{(5-\xi)(3-\xi)+3(2-\xi)\}\Gamma(a\mathrm{s}+b)-(8-2\alpha-\xi)\Gamma(a(\mathrm{s}-1)+b)\\
&> 11\Gamma(a\mathrm{s}+b)-5\Gamma(a(\mathrm{s}-1)+b)>0.
\end{aligned}$$

Now, for $\mathrm{s}\geq 3$, consider

$$(\mathrm{s}-1-\xi)(\mathrm{s}-\xi)\mathrm{c}_\mathrm{s}-\mathrm{s}(\mathrm{s}+1-\xi)\mathrm{c}_{\mathrm{s}+1}=A(\mathrm{s})M(\mathrm{s}),$$

where

$$A(\mathrm{s})=\frac{\mathrm{c}_\mathrm{s}}{\mathrm{s}^2\Gamma(a\mathrm{s}+b)}$$

and

$$\begin{aligned}
M(\mathrm{s}) &= \mathrm{s}^2(\mathrm{s}-1-\xi)(\mathrm{s}-\xi)\Gamma(a\mathrm{s}+b)-\mathrm{s}(\mathrm{s}+1-\xi)(\mathrm{s}+1-2\alpha)\Gamma(a(\mathrm{s}-1)+b)\\
&= (\mathrm{s}-3)^3A(\alpha,\xi)+(\mathrm{s}-3)^2\mathrm{c}(\alpha,\xi)+(\mathrm{s}-3)C(\alpha,\xi)+D(\alpha,\xi).
\end{aligned} \tag{26}$$

Here, $A(\alpha,\xi)$, $B(\alpha,\xi)$, $C(\alpha,\xi)$, and $D(\alpha,\xi)$ are positive expressions as given in (22)–(25), respectively. Since in the polynomial of $M(\mathrm{s})$ the coefficients of $(\mathrm{s}-3)$ and constant $D(\alpha,\xi)$ are positive, we have $M(\mathrm{s})$ to be an increasing function for $\mathrm{s}\geq 3$. Since $M(3)$ is positive, thus we have

$$(\mathrm{s}-1-\xi)(\mathrm{s}-\xi)\mathrm{c}_\mathrm{s}\geq \mathrm{s}(\mathrm{s}+1-\xi)\mathrm{c}_{\mathrm{s}+1}.$$

It is clear that c_s satisfies the hypothesis of Lemma 8. This shows that $\mathfrak{g} \in S^*(\xi)$. After tittle simplification, we can conclude that $\mathfrak{g}(\varsigma) = \mathcal{W}_{a,b}(\varsigma) * \frac{\varsigma}{(1-\varsigma)^{2-2\alpha}}$. Therefore, by the definition of $\mathcal{R}[\alpha,\xi]$, we have $\mathcal{W}_{a,b}(\varsigma) \in \mathcal{R}[\alpha,\xi]$. Now,

$$\begin{aligned} T_3(\alpha,\xi) - T_1(\alpha,\xi) &= \frac{(4-\xi)(4-2\alpha)}{3(2-\xi)(3-\xi)} - \frac{2(2-\xi)(1-\alpha)}{1-\xi} \\ &= \frac{(1-\xi)(4-\xi)(4-2\alpha) - 6(2-\xi)^2(3-\xi)(1-\alpha)}{3(1-\xi)(2-\xi)(3-\xi)}. \end{aligned}$$

We can easily observe that for $0 \le \alpha \le \alpha_0(\xi)$, the numerator is negative for all ξ and hence $T_3(\alpha,\xi) \le T_1(\alpha,\xi)$. Similarly, if $0 \le \alpha \le \alpha_1(\xi)$, $T_3(\alpha,\xi) \le T_2(\alpha,\xi)$ for all ξ. Here,

$$\begin{aligned} \alpha_0(\xi) &= 1 - \frac{2(1-\xi)(4-\xi)}{6(2-\xi)^2(3-\xi) - 2(1-\xi)(4-\xi)}, \\ \alpha_1(\xi) &= 1 - \frac{3.2^{\xi}(2-\xi)(3-\xi)^2 - 4(2-\xi)(4-\xi)}{4(2-\xi)(4-\xi) - 6.2^{\xi}(2-\xi)(3-\xi)^2}. \end{aligned}$$

From the above calculation, it is clear that $\min_{i=1,2,3}\{T_i(\alpha,\xi)\} = T_3(\alpha,\xi)$. Additionally, we can conclude that, for $0 \le \alpha \le \min\{\alpha_0(\xi),\ \alpha_1(\xi)\}$. To complete the proof, we only need to check that $\min\{\alpha_0(\xi),\ \alpha_1(\xi)\}$. For this, consider

$$\begin{aligned} \alpha_0 - \alpha_1 &= \frac{2(1-\xi)(4-\xi)}{6(2-\xi)^2(3-\xi) - 2(1-\xi)(4-\xi)} - \frac{3.2^{\xi}(2-\xi)(3-\xi)^2 - 4(2-\xi)(4-\xi)}{4(2-\xi)(4-\xi) - 6.2^{\xi}(2-\xi)(3-\xi)^2} \\ &= \frac{M(\xi)}{\{6(2-\xi)^2(3-\xi) - 2(1-\xi)(4-\xi)\}\{4(2-\xi)(4-\xi) - 6.2^{\xi}(2-\xi)(3-\xi)^2\}} \end{aligned}$$

where

$$\begin{aligned} M(\xi) &= \{2(1-\xi)(4-\xi)\}\left\{4(2-\xi)(4-\xi) - 6.2^{\xi}(2-\xi)(3-\xi)^2\right\} \\ &\quad -\left\{3.2^{\xi}(2-\xi)(3-\xi)^2 - 4(2-\xi)(4-\xi)\right\}\left\{6(2-\xi)^2(3-\xi) - 2(1-\xi)(4-\xi)\right\} \\ &< 0. \end{aligned}$$

Therefore, $\alpha_0(\xi) = \min\{\alpha_0(\xi),\ \alpha_1(\xi)\}$, which completes the proof. □

5. Conclusions

In this article, we have studied the geometric properties of the normalized Wright function. We have mainly focused on the close-to-convexity, starlikeness, convexity in the direction of the imaginary axis, and prestarlikeness. We have obtained the conditions such that $\Re\left\{\mathcal{W}'_{a,b}(\varsigma)\right\} > \frac{1}{2}$ and $\Re\left(\frac{\mathcal{W}_{a,b}(\varsigma)}{\varsigma}\right) > \frac{1}{2}$ in $\mathbb{U}$. The main tools of our investigations are positivity techniques.

The techniques used in this work can be utilized to study the geometric properties of certain other special functions.

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