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Vanishing Homology of Warped Product Submanifolds in Complex Space Forms and Applications

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Abstract: In this paper, we prove the nonexistence of stable integral currents in compact oriented warped product pointwise semi-slant submanifold M^n of a complex space form $\widetilde{M}(4\epsilon)$ under extrinsic conditions which involve the Laplacian, the squared norm gradient of the warped function, and pointwise slant functions. We show that i-the homology groups of M^n are vanished. As applications of homology groups, we derive new topological sphere theorems for warped product pointwise semi-slant submanifold M^n, in which M^n is homeomorphic to a sphere $\mathbb{S}^n$ if $n \geq 4$ and if $n = 3$, then M^3 is homotopic to a sphere $\mathbb{S}^3$ under the assumption of extrinsic conditions. Moreover, the same results are generalized for CR-warped product submanifolds.

Keywords: warped product submanifolds; complex space form; Homology groups; sphere theorem; stable currents; Dirichlet energy

MSC: 53C40; 53A20; 53C42; 53B25; 53Z05

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1. Introduction and Main Results

A traditional topic in Riemannian geometry is to find the geometrical and topological structures of submanifolds; there has been much progress in this field. For instance, the rigidity theorem was proved by Berger [1] for an even-dimensional complete simply connected manifold M with sectional curvature $\frac{1}{4} \leq K_M \leq 1$. Further, Gauhmen [2] considered even $n = 2m$-dimensional submanifolds minimally immersed in the unit sphere $\mathbb{S}^{n+1}$ with a co-dimension equal to one, and showed that if $||h(u,u)||^2 < 1$ for any unit vector u of M^n where h is the second fundamental form M^n, then M^n is totally geodesic in $\mathbb{S}^{n+1}$. If $max_{u\in M}\{||h(u,u)||^2\} = 1$, then M^n is $\mathbb{S}^m(\frac{1}{2}) \times \mathbb{S}^m(\frac{1}{2})$ minimally embedded in $\mathbb{S}^{2m+1}$, as described. A very famous result in this respect was formulated by Poincare [3], who stated that every simply connected closed 3-manifold is homeomorphic to a 3-sphere. Smale [4] generalized the Poincare conjecture and proved that for a closed C^∞-manifold M^n which has the homotopic types of an n-dimensional sphere greater than five, the manifold M^n is homeomorphic to $\mathbb{S}^n$. The differentiable sphere theorem was proven by Brendle and Schoen [5] under Ricci flow. In recent years, much attention has been paid to the classification of geometric function theory, topological sphere theorems, and differentiable sphere theorems (see [6–11]). In the sequelae, the homology groups of a manifold are important topological invariants that provide algebraic information about the manifold. Federer-Fleming [7] showed that any non-trivial integral homology class in $\mathbb{H}_p(M,\mathbb{G}))$ corresponds to a stable current. Motivated by the work of Federer and Fleming [7], Lawson and Simon [9], and Xin [11] proved the nonexistence of stable integral currents in a submanifold M^n and vanishing homology groups of M^n with non-negative sectional curvature according to the following theorem.

Theorem 1 ([9,11])**.** *Let M^n be a compact n-dimensional submanifold isometrically immersed in the space form $\widetilde{M}(c)$ of curvature $c \geq 0$ with the second fundamental form h. Let l_1, l_2 be any positive integers such that $l_1 + l_2 = n$ and*

$$\sum_{\alpha=1}^{l_1} \sum_{\beta=l_1+1}^{n} \Big\{ 2||h(e_\alpha, e_\beta)||^2 - g\big(h(e_\alpha, e_\alpha), h(e_\beta, e_\beta)\big) \Big\} < l_1 l_2 c, \tag{1}$$

for any $x \in M^n$ and an orthonormal frame $\{e_i\}_{1 \leq i \leq n}$ of the tangent space TM^n. Then, there do not exist stable l_1-currents in M^n and

$$\mathbb{H}_{l_1}(M^n, \mathbb{G}) = \mathbb{H}_{n-l_1=l_2}(M^n, \mathbb{G}) = 0,$$

where $\mathbb{H}_i(M^n, \mathbb{G})$ stands for i-the homology group of M^n and $\mathbb{G}$ is a finite abelian group with integer coefficients.

Due to these previous studies on large scales, a particular case we consider here is that of warped product pointwise semi-slant submanifolds of complex space form where 4ϵ is represented as a constant sectional curvature. In this regard, our motivation comes from the study of Sahin [12], where he discussed the warped product pointwise semi-slant submanifolds in a Kaehler manifold and showed that a warped product pointwise semi-slant submanifold of type $N_T^{l_1} \times_f N_\theta^{l_2}$ is nontrivial when angle θ is treated as a slant function. Furthermore, it was shown in [12] that the warped product pointwise semi-slant submanifold $N_T^{l_1} \times_f N_\theta^{l_2}$ of a Kaehler manifold is a natural generalization of CR-warped products [13]. Inspired by this notion, we define the extrinsic condition to prove nonexistence-stable integral l_1-currents and vanishing homology groups in a warped product pointwise semi-slant submanifold of complex space forms $\widetilde{M}^m(4\epsilon)$. We use Theorem 1 on this basis to arrive at our first result.

Theorem 2. *Let $M^{l_1+l_2} = N_T^{l_1} \times_f N_\theta^{l_2}$ be a compact warped product pointwise semi-slant submanifold of a complex space form $\widetilde{M}^m(4\epsilon)$. If the following condition is satisfied*

$$\big(\csc^2\theta + \cot^2\theta + l_2\big)||\nabla f||^2 + f\Delta f + \frac{f^2}{l_2}\|h_\mu\|^2 < 3l_1\epsilon f^2, \tag{2}$$

then there do not exist stable integral l_1-currents in $M^{l_1+l_@}$ and

$$\mathbb{H}_{l_1}(M^{l_1+l_2}, \mathbb{G}) = \mathbb{H}_{l_2}(M^{l_1+l_2}, \mathbb{G}) = 0,$$

where $\mathbb{H}_i(M^{l_1+l_2}, \mathbb{G})$ stands for i-the homology group of $M^{l_1+l_2}$ with integer coefficients, ∇f and Δf are the gradient and the Laplacian of the warped function f, respectively, and h_μ represents the components of the second fundamental form h in an invariant subspace μ.

Our next result is in accordance with Lemma 3.1 in [12], which states that the inner product of the second fundamental form of $N_T^{l_1}$ and F-components of $N_\theta^{l_1}$ is equal to zero. To be precise, we have the following result.

Theorem 3. *Let $M^{l_1+l_2} = N_T^{l_1} \times_f N_\theta^{l_2}$ be a compact warped product pointwise semi-slant submanifold of a complex space form $\widetilde{M}^m(4\epsilon)$. If the inequality*

$$\|\nabla f\|^2 < \left\{ \frac{\big(4l_1 l_2\epsilon - \|h_\mu\|^2\big)f^2}{2l_2\big(\csc^2\theta + \cot^2\theta\big)} \right\}, \tag{3}$$

holds, then there do not exist stable integral l_1-currents in $M^{l_1+l_2}$ and

$$\mathbb{H}_{l_1}(M^{l_1+l_2}, \mathbb{G}) = \mathbb{H}_{l_2}(M^{l_1+l_2}, \mathbb{G}) = 0.$$

The notation is the same as in Theorem 2.

To apply Theorems 2 and 3 in [14], let the slant function θ become globally constant, setting $\theta = \frac{\pi}{2}$ in Theorems 2 and 3. Then, the pointwise slant submanifold $N_\theta^{l_1}$ is turned into a totally real submanifold $N_\perp^{l_2}$. Thus, a warped product pointwise semi-slant submanifold $M^{l_1+l_2} = N_T^{l_1} \times_f N_\theta^{l_2}$ becomes CR-warped products in a Kaehler manifold of type $M^n = N_T^{l_1} \times_f N_\perp^{l_2}$. Therefore, following to the motivation of Chen [13], we deduce the following result from Theorem 2 for the nonexistence of stable integral l_1-currents and vanishing homology in a CR-warped product submanifold of complex space forms $\widetilde{M}^m(4\epsilon)$.

Corollary 1. *Let $M^{l_1+l_2} = N_T^{l_1} \times_f N_\perp^{l_2}$ be a compact CR-warped product submanifold of complex space form $\widetilde{M}^{2m}(4\epsilon)$. If the following condition is satisfied*

$$(1 + l_2)||\nabla f||^2 + f\Delta f + \frac{f^2}{l_2}\|h_\mu\|^2 < 3l_1\epsilon f^2, \tag{4}$$

then there do not exist stable integral l_1-currents in $M^{l_1+l_2}$ and $\mathbb{H}_{l_1}(M^{l_1+l_2}, \mathbb{G}) = \mathbb{H}_{l_2}(M^{l_1+l_2}, \mathbb{G}) = 0$.

As an immediate consequence of Theorem 3, we have

Corollary 2. *Let $M^{l_1+l_2} = N_T^{l_1} \times_f N_\perp^{l_2}$ be a compact CR-warped product submanifold of complex space form $\widetilde{M}^{2m}(4\epsilon)$ satisfying the following inequality*

$$\|\nabla f\|^2 < \left\{ \frac{(4l_1 l_2\epsilon - \|h_\mu\|^2) f^2}{2l_2} \right\}.$$

Then, there do not exist stable integral l_1-currents in $M^{l_1+l_2}$ and we have the trivial homology groups, i.e.,

$$\mathbb{H}_{l_1}(M^{l_1+l_2}, \mathbb{G}) = \mathbb{H}_{l_1}(M^{l_1+l_2}, \mathbb{G}) = 0.$$

Our next motivation comes from Calin [15] who studied geometric mechanics on Riemannian manifolds and defined a positive differentiable function φ ($\varphi \in \mathcal{F}(M^n)$) on a compact Riemannian manifold M^n. The *Dirichlet energy* of a function φ is defined in [15] (see p. 41) as follows:

$$\mathbb{E}(\varphi) = \frac{1}{2}\int_{M^n} ||\nabla\varphi||^2 dV \qquad 0 < E(\varphi) < \infty. \tag{5}$$

In view of the kinetic energy formula (5) for a compact oriented manifold without boundary along with Theorem 2, we arrive at the following result.

Theorem 4. *Let $M^{l_1+l_2} = N_T^{l_1} \times_f N_\perp^{l_2}$ be a compact warped product pointwise semi-slant submanifold of a complex space form $\widetilde{M}^{2m}(4\epsilon)$ without boundary. If the following condition is satisfied*

$$\mathbb{E}(f) < \left\{ \frac{\int_{M^n} \left(3l_1 l_2\epsilon - \|h_\mu\|^2\right) f^2 dV}{2l_2(2\csc^2\theta + l_2)} \right\}, \tag{6}$$

where $\mathbb{E}(f)$ *is the Dirichlet energy of the warping function* f *with respect to the volume element* dV*, then there do not exist stable integral* l_1*-currents in* $M^{l_1+l_2}$ *and* $\mathbb{H}_{l_1}(M^{l_1+l_2}, \mathbb{G}) = \mathbb{H}_{l_2}(M^{l_1+l_2}, \mathbb{G}) = 0$.

An important concept relates to the geometrical and topological properties on Riemannian manifolds when considering the pinched condition on its metric. It is interesting to investigate the curvature and topology of submanifolds in a Riemannian manifold and the usual sphere theorems in Riemannian geometry. For instance, using the nonexistence of stable currents on compact submanifolds, Lawson and Simon [9] obtained their striking sphere theorem, which proved that for an n-dimensional compact-oriented submanifold M^n in a unit sphere $\mathbb{S}^{n+k}$ with the second fundamental form bounded above by a constant which depends on the dimension n, then M^n is homeomorphic to a sphere $\mathbb{S}^n$ when $n \neq 3$ and M^3 are homotopic to a sphere $\mathbb{S}^3$.

Making use of Lawson and Simon [9], Leung [16] proved that for a compact connected oriented submanifold M^n in the unit sphere $\mathbb{S}^{n+k}$ such that $\|h(X,X)\|^2 < \frac{1}{3}$, when $n \neq 3$ and M^3 are homotopic to a sphere $\mathbb{S}^3$, then M^n is homeomorphic to a sphere $\mathbb{S}^n$. Recently, it has been shown in [17] that if the sectional curvature satisfies some pinching condition $K_M \geq \frac{l_1.sign(l_1-1)}{2(l_1+1)}$ for n-dimensional compact oriented minimal submanifold M in the unit sphere $\mathbb{S}^{n+l_1}$ with co-dimension l_1, then M is either a totally geodesic sphere, one of the Clifford minimal hyper-surfaces $\mathbb{S}^k(\frac{k}{n}) \times \mathbb{S}^{n-k}(\frac{n-k}{n})$ in $\mathbb{S}^{n+1}$ for $k = 1, \ldots, n-1$, or a Veronese surface in $\mathbb{S}^4$. More recently, several results have been derived on topological and differentiable structures of submanifolds when imposing certain conditions on the second fundamental form, Ricci curvatures, and sectional curvatures in a series of articles [4,10,11,18–23] by different geometers. For the warped product structure, we refer to [20,24–30].

The second target of note is to establish topological sphere theorems from the viewpoint of warped product submanifold geometry with positive constant sectional curvature and pinching conditions in terms of the squared norm of the warping function and Laplacian of the warped function as extrinsic invariants. In this sense, we work with conditions on the extrinsic curvature (second fundamental form, warping function), which have the advantage of being invariant under rigid motions. Motivated by Lawson and Simon [9], (p. 441, Theorem 4), we consider a warped product pointwise semi-slant submanifold in a complex space form $\widetilde{M}^{2m}(4\epsilon)$ such that the constant holomorphic sectional curvature is 4ϵ, and state our main theorem of this paper.

Theorem 5. *Let* $M^{l_1+l_2} = N_T^{l_1} \times_f N_\theta^{l_2}$ *be a compact warped product pointwise semi-slant submanifold in a complex space form* $\widetilde{M}^{2m}(4\epsilon)$ *satisfying the condition* (2)*. Then,* $M^{l_1+l_2}$ *is homeomorphic to sphere* $\mathbb{S}^{l_1+l_2}$ *when* $l_1 + l_2 \geq 4$*, while* M^3 *is homotopic to a sphere* $\mathbb{S}^3$.

Remark 1. *As a consequence of Theorem 5, we obtain the following sphere theorem for a compact CR-warped product submanifold in a complex space form* $\widetilde{M}^{2m}(4\epsilon)$*, thanks to Chen [13].*

Corollary 3. *Let* $M^{l_1+l_2} = N_T^{l_1} \times_f N_\perp^{l_2}$ *be a compact CR-warped product submanifold in a complex space form* $\widetilde{M}^{2m}(4\epsilon)$ *satisfying the pinching condition* (4)*. Then,* $M^{l_1+l_2}$ *is homeomorphic to a sphere* $\mathbb{S}^{l_1+l_2}$ *when* $l_1 + l_2 \geq 4$*, and* M^3 *is homotopic to a sphere* $\mathbb{S}^3$.

Using Theorem 4 and 5, we can now obtain an important result.

Corollary 4. *Let* $M^{p+q} = N_T^p \times_f N_\theta^q$ *be a compact warped product pointwise semi-slant submanifold of complex space form* $\widetilde{M}^{2m}(4\epsilon)$*. If* (6) *is satisfied, then* $M^{l_1+l_2}$ *is homeomorphic to sphere* $\mathbb{S}^{l_1+l_2}$ *when* $l_1 + l_2 \geq 4$ *and* M^3 *is homotopic to a sphere* $\mathbb{S}^3$.

Remark 2. *The principle behind Cheng's eigenvalue comparison theorem (see [31]) forms the basis of the following finding. With the help of the first non-zero eigenvalue of the Laplacian operator, Cheng has demonstrated that if M is complete and isometric to the sphere of the standard unit then the following theorem can be inferred using the maximum principle for the first non-zero eigenvalue* λ_1*, provided that* $Ric(M) geq 1$ *and* $d(M) = \pi$.

Theorem 6. *Let* $M^{l_1+l_2} = N_T^{l_1} \times_f N_\theta^{l_2}$ *be a compact warped product pointwise semi-slant submanifold of a complex space form* $\widetilde{M}^{2m}(4\epsilon)$ *with f being a non-constant eigenfunction of the first non-zero eigenvalue* λ_1 *such that the following inequality is satisfied:*

$$\lambda_1 < \frac{3l_1 l_2 \epsilon - \|h_\mu\|}{l_1\left(2\csc^2\theta + l_2\right)}. \tag{7}$$

Then, $M^{l_1+l_2}$ *is homeomorphic to sphere* $\mathbb{S}^{l_1+l_2}$ *when* $l_1 + l_2 \geq 4$ *and* M^3 *is homotopic to a sphere* $\mathbb{S}^3$ *when* $l_1 + l_2 = 3$.

Motivated by Bochner's formula [32], we arrive at the following result.

Theorem 7. *Let* $M^{l_1+l_2} = N_T^{l_1} \times_f N_\theta^{l_2}$ *be a compact warped product pointwise semi-slant submanifold of a complex space form* $\widetilde{M}^{2m}(4\epsilon)$ *such that following inequality holds:*

$$\|\nabla^2 f\|^2 + Ric(\nabla f, \nabla f) > \left\{\frac{\left(\|h_\mu\|^2 - 3l_1 l_2 \epsilon\right) f\Delta f}{\left(2\csc^2\theta + l_2\right)}\right\}, \tag{8}$$

where $\|\nabla^2 f\|^2$ *denotes the Hessian form of the warping function f and Ric denotes the Ricci curvature along the base manifold* $N_T^{l_1}$*. Then,* $M^{l_1+l_2}$ *is homeomorphic to sphere* $\mathbb{S}^{l_1+l_2}$ *when* $l_1 + l_2 \geq 4$ *and* M^3 *is homotopic to a sphere* $\mathbb{S}^3$ *when* $l_1 + l_2 = 3$.

2. Preliminaries

Let $M^{2m}(4\epsilon)$ be a complex space form with the complex dimension $\dim_{\mathbb{R}} M = 2m$. Then, the curvature tensor R of $M^{2m}(4\epsilon)$ with constant holomorphic sectional curvature 4ϵ is expressed as

$$\begin{aligned} R(X_2, Y_2)Z_2 = c\Big(&g(X_2, Z_2)Y_2 - g(Y_2, Z_2)X_2 + g(X_2, JZ_2)JY_2 \\ &- g(Y_2, JZ_2)X_2 + 2g(X_2, JY_2)JZ_2\Big). \end{aligned} \tag{9}$$

The Gauss and Weingarten formulas for transforming submanifold M^n into an almost Hermitian manifold $\widetilde{M}^{2m}$ are provided by

$$\begin{aligned} \widetilde{\nabla}_{X_2} Y_2 &= \nabla_{X_2} Y_2 + h(X_2, Y_2), \\ \widetilde{\nabla}_{X_2} N &= -A_N X_2 + \nabla^{\perp}_{X_2} N, \end{aligned}$$

for each $X_2, Y_2 \in \mathfrak{X}(TM)$ and $N \in \mathfrak{X}(T^{\perp}M)$ such that the second fundamental form and the shape operator are denoted by h and A_N. They are connected as $g(h(U,V), N) = g(A_N U, V)$. Now, for any $X_2 \in \mathfrak{X}(M)$ and $N \in \mathfrak{X}(T^{\perp}M)$, we have

$$(i)\ JX_2 = TX_2 + FX_2, \quad (ii)\ \ JN = tN + fN, \tag{10}$$

where $TX_2(tN)$ and $FX_2(fN)$ are the tangential and normal components of $JX_2(JN)$, respectively.

The Gauss equation for a submanifold M^n is defined as

$$\tilde{R}(X_2, Y_2, Z_2, W_2) = R(X_2, Y_2, Z_2, W_2) + g\big(h(X_2, Z_2), h(Y_2, W_2)\big) - g\big(h(X_2, W_2), h(Y_2, Z_2)\big), \tag{11}$$

for any $X_2, Y_2, Z_2, W_2 \in \mathfrak{X}(TM)$, where $\widetilde{R}$ and R are the curvature tensors on $\widetilde{M}^{2m}$ and M^n, respectively.

The norm of second fundamental form h for an orthonormal frame $\{e_1, e_2, \cdots e_n\}$ of the tangent space TM on M^n is defined by

$$h^r_{ij} = g(h(e_i, e_j), e_r), \quad ||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{12}$$

Let $\{e_1, \ldots, e_n\}$ be an local orthonormal frame of vector field M^n. Then, we have

$$\nabla\varphi = \sum_{i=1}^{n} e_i(\varphi)e_i.$$

and

$$\|\nabla\varphi\|^2 = \sum_{i=1}^{n} \big((\varphi)e_i\big)^2, \tag{13}$$

where $\nabla\varphi$ and $||\nabla\varphi||^2$ are the gradient of function φ and its squared norm.

The following classifications can be provided as:

(i) If $J(T_xM) \subseteq T_xM$ for every $x \in M^n$, then M^n is a holomorphic submanifold.
(ii) If $J(T_xM) \subseteq T^{\perp}M$ for each $x \in M^n$, then M^n is a totally real submanifold.

There are four types of submanifolds of a Kaehler manifold, namely, the CR-submanifold, slant submanifold, semi-slant submanifold, pointwise slant submanifold, and pointwise semi-slant submanifold. The definitions and classifications of such submanifolds are discussed in [12,13]. Moreover, for examples of a pointwise semi-slant submanifold in a Kaehler manifold and related problems, we refer to [12]. It follows from Definition 3.1 in [12] that if we denote as l_1 and l_2 the dimensions of a complex distribution $\mathcal{D}^T$ and pointwise slant distribution $\mathcal{D}^\theta$ of a pointwise semi-slant submanifold in a Kaehler manifold $\widetilde{M}^{2m}$, then the following remarks hold:

Remark 3. *M^n is invariant if $l_1 = 0$ and pointwise slant if $l_2 = 0$.*

Remark 4. *If we consider the slant function $\theta : M^n \to R$ as globally constant on M^n and $\theta = \frac{\pi}{2}$, then M^n is a CR-submanifold.*

Remark 5. *An invariant subspace μ under J of normal bundle $T^{\perp}M$, is defined as $T^{\perp}M = F\mathcal{D}^\theta \oplus \mu$.*

3. Warped Product Submanifolds

A product manifold of the type $M^n = N_1^{l_1} \times_f N_2^{l_2}$ is a warped product manifold if the metric is defined as $g = g_1 + f^2 g_2$, where $N_1^{l_1}$ and $N_2^{l_2}$ are two Riemannian manifolds and their Riemannian metrics are g_1 and g_2, respectively. It was discovered by Bishop and O'Neill [33] that the warping function f is a smooth function defined on base $N_1^{l_1}$. The following properties are a direct consequence of the warped product manifold $M^n = N_1^{l_1} \times_f N_2^{l_2}$:

(i) $\nabla_Z X = \nabla_X Z = \frac{(Xf)}{f} Z$,
(ii) $\nabla_Z W = \nabla'_Z W - \frac{g(Z,W)}{f} \nabla f$,

for any $X, Y \in \mathfrak{X}(TN_1)$ and $Z, W \in \mathfrak{X}(TN_2)$, where ∇ and ∇' denote the Levi-Civita connection on M^n and N_2, respectively.

The gradient ∇f of f is written as

$$g(\nabla \ln f, X_2) = X_2(\ln f). \tag{14}$$

The following relation is an interesting property of warped products:

$$\mathcal{R}(X_2, Z_2)Y_2 = \frac{\mathcal{H}^f(X_2, Z_2)}{f}Y_2, \tag{15}$$

where $\mathcal{H}^f$ is a Hessian tensor of f; the remarks below follow as a consequence.

Remark 6. *A warped product manifold $M^n = N_1^{l_1} \times_f N_2^{l_2}$ is said to be trivial or simply a Riemannian product manifold if the warping function f is a constant function along $N_1^{l_1}$.*

Remark 7. *If $M^n = N_1^{l_1} \times_f N_2^{l_2}$ is a warped product manifold, then $N_1^{l_1}$ is totally geodesic and $N_2^{l_2}$ is a totally umbilical submanifold of M^n, respectively.*

4. Non-Trivial Warped Product Pointwise Semi-Slant Submanifolds $N_T^{l_1} \times_f N_\theta^{l_2}$

It is well known that warped product submanifolds of types

$$(i)\ N_\theta^{l_2} \times_f N_T^{l_1}, \quad and \quad (ii)\ N_T^{l_1} \times_f N_\theta^{l_2},$$

are called warped product pointwise semi-slant submanifolds, which were discovered in [12]. They contain holomorphic and pointwise slant submanifolds of a Kähler manifold. The first case, with $M^n = N_\theta^{l_2} \times_f N_T^{l_1}$ in a Kähler manifold, is trivial. The second is non-trivial. Before proceeding to the second case, let us recall the following result [12].

Lemma 1. *Let $M^n = N_T^{l_1} \times_f N_\theta^{l_2}$ be a warped product pointwise semi-slant submanifold of a Kähler manifold $\widetilde{M}^m$. Then,*

$$g(h(X_2, Z_2), FTZ_2) = -(X_2 \ln f) \cos^2\theta ||Z_2||^2, \tag{16}$$

$$g(h(Z_2, JX_2), FZ_2) = (X_2 \ln f) ||Z_2||^2, \tag{17}$$

for any $X_2, Y_2 \in \mathfrak{X}(TN_T)$ and $Z_2 \in \mathfrak{X}(TN_\theta)$.

5. Proof of Main Results

5.1. Proof of Theorem 2

The crucial point of this paper is to derive an upper bound for

$$\sum_{i=1}^{l_1} \sum_{j=l_1+1}^{n} \Big\{2||h(e_i, e_j)||^2 - g\big(h(e_i, e_i), h(e_j, e_j)\big)\Big\}$$

in terms of Δf and $||\nabla f||^2$.

Let $M = N_T^{l_1} \times_f N_\theta^{l_2}$ be an $n = l_1 + l_2$-dimensional warped product pointwise semi-slant submanifold with $\dim N_T^{l_1} = l_1 = 2\alpha$ and $\dim N_\theta^{l_2} = l_2 = 2\beta$, where $N_\theta^{l_1}$ and $N_T^{l_1}$ are integral manifolds of $\mathcal{D}^\theta$ and $\mathcal{D}$, respectively. Thus, we consider $\{e_1, e_2, \cdots e_\alpha, e_{\alpha+1} = Je_1, \cdots e_{2\alpha} = Je_\alpha\}$ and $\{e_{2\alpha+1} = e_1^*, \cdots e_{2\alpha+\beta} = e_\beta^*, e_{2\alpha+\beta+1} = e_{\beta+1}^* = \sec\theta Pe_1^*, \cdots e_{l_1+l_2} = e_{l_2}^* = \sec\theta Pe_\beta^*\}$ to be orthonormal frames of TN_T and TN_θ, respectively. Thus the orthonormal frames of the normal sub-bundles $F\mathcal{D}^\theta$ and μ are $\{e_{n+1} = \bar{e}_1 = \csc\theta Fe_1^*, \cdots e_{n+\beta} =$

$\bar{e}_\beta = \csc\theta F e_1^*, e_{n+\beta+1} = \bar{e}_{\beta+1} = \csc\theta\sec\theta F P e_1^*, \cdots e_{n+2\beta} = \bar{e}_{2\beta} = \csc\theta\sec\theta F P e_\beta^*\}$ and $\{e_{n+2\beta+1}, \cdots e_{2m}\}$, respectively. Then, from the Gauss Equation (11), we have

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2} g\big(R(e_i,e_j)e_i,e_j\big) = \sum_{i=1}^{l_1}\sum_{j=1}^{l_2} g\big(\widetilde{R}(e_i,e_j)e_i,e_j\big) + ||h(e_i,e_j)||^2 - \sum_{i=1}^{l_1}\sum_{j=1}^{l_2} g\big(h(e_i,e_j),h(e_i,e_j)\big).$$

By adding the squared norm of the second fundamental terms in both side of the above equation, we obtain

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2} g\big(R(e_i,e_j)e_i,e_j\big) + ||h(e_i,e_j)||^2 = \sum_{i=1}^{l_1}\sum_{j=1}^{l_2} g\big(\widetilde{R}(e_i,e_j)e_i,e_j\big) - \sum_{i=1}^{l_1}\sum_{j=1}^{l_2} g\big(h(e_j,e_j),h(e_i,e_i)\big) + 2||h(e_i,e_j)||^2. \tag{18}$$

Using the orthonormal frames $\{e_i\}_{1\le i\le l_1}$ and $\{e_j\}_{1\le j\le l_2}$ of $N_T^{l_1}$ and $N_\theta^{l_2}$, respectively, in (15), we derive

$$R(e_i,e_j)e_i = \frac{e_j}{f}\mathcal{H}^f(e_i,e_i).$$

Summing up with an orthonormal frame $\{e_j\}_{1\le j\le l_2}$ (here it should be pointed out that we have adopted the opposite sign from the usual sign convention for the Laplacian), then

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2} g\big(R(e_i,e_j)e_i,e_j\big) = -\frac{l_2}{f}\sum_{i=1}^{l_1} g\big(\nabla_{e_i}\nabla f, e_i\big). \tag{19}$$

Thus, from Equations (18) and (19), we can derive

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\Big\{2||h(e_i,e_j)||^2 - g\big(h(e_j,e_j),h(e_i,e_i)\big)\Big\} + \sum_{i=1}^{l_1}\sum_{j=1}^{l_2} g\big(\widetilde{R}(e_i,e_j)e_i,e_j\big) = -\frac{l_2}{f}\sum_{i=1}^{l_1} g\big(\nabla_{e_i}\nabla f, e_i\big) + \sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\big(h_{ij}^r\big)^2. \tag{20}$$

First, we figure out the term Δf for M^n, which is the Laplacian of f.

$$\Delta f = -\sum_{i=1}^{n} g\big(\nabla_{e_i} grad f, e_i\big) = -\sum_{\alpha=1}^{l_1} g\big(\nabla_{e_\alpha} grad f, e_\alpha\big) - \sum_{\beta=1}^{l_2} g\big(\nabla_{e_\beta} grad f, e_\beta\big).$$

The above equation can be expressed as components of N_θ^q from adapted orthonormal framel in this way, we obtain

$$\Delta f = -\sum_{\alpha=1}^{l_1} g(\nabla_{e_\alpha} gradf, e_\alpha) - \sum_{j=1}^{\beta} g(\nabla_{e_j} gradf, e_j)$$
$$- \sec^2\theta \sum_{j=1}^{\beta} g(\nabla_{Te_j} gradf, Te_j).$$

Benefiting from ∇ being a Levi-Civita connection on M^n, we derive

$$\Delta f = -\sum_{\alpha=1}^{l_1} g(\nabla_{e_\alpha} gradf, e_\alpha) - \sum_{j=1}^{\beta} \Big(e_j g(gradf, e_j) - g(\nabla_{e_j} e_j, gradf) \Big).$$
$$- \sec^2\theta \sum_{j=1}^{\beta} \Big(Te_j g(gradf, Te_j) - g(\nabla_{Te_j} Te_j, gradf) \Big).$$

From the property of the gradient of function (14), we obtain

$$\Delta f = -\sum_{\alpha=1}^{l_1} g(\nabla_{e_\alpha} gradf, e_\alpha) - \sum_{j=1}^{\beta} \Big(e_j(e_j f) - (\nabla_{e_j} e_j f) \Big)$$
$$- \sec^2\theta \sum_{j=1}^{\beta} \Big(Te_j(Te_j(f)) - (\nabla_{Te_j} Te_j f) \Big).$$

After computation, we have

$$\Delta f = -\sum_{\alpha=1}^{l_1} g(\nabla_{e_\alpha} gradf, e_\alpha) - \sum_{j=1}^{\beta} \Big(e_j\big(g(gradf, e_j)\big) - g(\nabla_{e_j} e_j, gradf) \Big)$$
$$- \sec^2\theta \sum_{j=1}^{\beta} \Big(Te_j\big(g(gradf, Te_j)\big) - g(\nabla_{Te_j} Te_j, gradf) \Big).$$

Starting from the hypothesis of a warped product pointwise semi-slant submanifold, $N_T^{l_1}$ is totally geodesic in M^n. This implies that $gradf \in \mathfrak{X}(TN_T)$, and from (i)–(ii) in Section 3, we obtain

$$\Delta f = -\frac{1}{f} \sum_{j=1}^{\beta} \Big(g(e_j, e_j) \|\nabla f\|^2 + \sec^2\theta g(Te_j, Te_j) \|\nabla f\|^2 \Big)$$
$$- \sum_{i=1}^{l_1} g(\nabla_{e_i} gradf, e_i).$$

By multiplying the above equation by $\frac{1}{f}$, from (3.7) of Corollary 3.1 in [12] we obtain

$$\frac{\Delta f}{f} = -\frac{1}{f} \sum_{i=1}^{l_1} g(\nabla_{e_i} gradf, e_i) - l_2 \|\nabla(\ln f)\|^2.$$

It is not difficult to check that

$$-\frac{1}{f} \sum_{i=1}^{l_1} g(\nabla_{e_i} gradf, e_i) = \frac{\Delta f}{f} + l_2 ||\nabla \ln f||^2.$$

This combines with (20) to yield

$$l_2^2||\nabla(\ln f)||^2+\frac{l_2\Delta f}{f}+\sum_{\alpha=1}^{l_1}\sum_{\beta=1}^{l_2}(h^r_{\alpha\beta})^2$$
$$=\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\Big\{2||h(e_i,e_j)||^2-g\big(h(e_j,e_j),h(e_i,e_i)\big)\Big\}$$
$$+\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}g\big(\widetilde{R}(e_i,e_j)e_i,e_j\big). \tag{21}$$

On taking $X=e_i$ and $Z=e_j$ for $1\leq i\leq l_1$ and $1\leq j\leq l_2$, respectively, we have

$$\sum_{r=n+1}^{2m}\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}(h^r_{ij})^2=\sum_{r=n+1}^{n+2\beta}\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}g\big(h(e_i,e_j^*),e_r\big)^2$$
$$+\sum_{r=n+2\beta+1}^{2m}\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}g\big(h(e_i,e_j^*),e_r\big)^2.$$

In the above equation, the first term on the right hand side is the $F\mathcal{D}^\theta$-component and the second term is the μ-component for the orthonormal frame for vector fields of $N_T^{l_1}$ and $N_\theta^{l_2}$. Summing over the vector fields of $N_T^{l_1}$ and $N_\theta^{l_2}$ and using (16) and (17) from Lemma 1 in the last equation, we are able to find that

$$\sum_{r=n+1}^{2m}\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}(h^r_{ij})^2=2\Big(\csc^2\theta+\cot^2\theta\Big)\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\big(e_i\ln f)\big)^2g(e_j^*,e_j^*)^2$$
$$+2\Big(\csc^2\theta+\cot^2\theta\Big)\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\big(Je_i\ln f)\big)^2g(e_j^*,e_j^*)^2$$
$$+\sum_{r=n+2\beta+1}^{2m}\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}g\big(h(e_i,e_j^*),e_r\big)^2.$$

From the adapted orthonormal frame for N_T, the last equation can then be expressed as follows:

$$\sum_{r=n+1}^{2m}\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}(h^r_{ij})^2=2\Big(\csc^2\theta+\cot^2\theta\Big)\sum_{i=1}^{l_1}\big(e_i(\ln f)\big)^2\sum_{j=1}^{l_2}g(e_j^*,e_j^*)^2$$
$$+\sum_{r=n+2\beta+1}^{2m}\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}g\big(h(e_i,e_j^*),e_r\big)^2.$$

Together with the definition of the squared norm of the gradient function f from (13), the above implies that

$$\sum_{r=n+1}^{2m}\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}(h^r_{ij})^2=l_2\Big(\csc^2\theta+\cot^2\theta\Big)||\nabla\ln f||^2+\|h_\mu\|^2. \tag{22}$$

Following (21) and (22), we arrive at

$$\begin{aligned}\frac{l_2\Delta f}{f}&+l_2^2||\nabla(\ln f)||^2+l_2(1+2\cot^2\theta)||\nabla(\ln f)||^2+\|h_\mu\|^2\\&=\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\left\{2||h(e_i,e_j)||^2-g\big(h(e_j,e_j),h(e_i,e_i)\big)\right\}\\&+\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}g\big(\widetilde{R}(e_i,e_j)e_i,e_j\big).\end{aligned}$$

Because we have the following relation for symmetry of the curvature tensor R,

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}g\big(\widetilde{R}(e_i,e_j)e_i,e_j\big)=\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\widetilde{R}(e_i,e_j,e_i,e_j). \tag{23}$$

Next, we use the curvature tensor from Formula (9) for the complex space form $\widetilde{M}^m(4\epsilon)$, which can be simply written as

$$\begin{aligned}\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\widetilde{R}(e_i,e_j,e_i,e_j)=\epsilon\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\Big\{&g(e_i,e_j)g(e_i,e_j)-g(e_i,e_i)g(e_j,e_j)\\&-g(Je_i,e_i)g(Je_j,e_j)\\&+3g(Je_i,e_j)g(Je_j,e_i)\Big\}.\end{aligned} \tag{24}$$

As we know that $e_i\in\mathfrak{X}(TN_T)$ and $e_j\in\mathfrak{X}(TN_\theta)$, then $g(e_i,e_j)=0$, and $g(Je_i,e_i)=0(resp, g(Je_j,e_i)=0)$ by the fact that for $Je_i\perp e_i(Je_j\perp e_j)$, respectively. Similarly, from (10)i, we can derive that $g(Je_i,e_j)=g(Te_i+Fe_i,e_j)=0$ for $Te_i\in\mathfrak{X}(TN_T)$ and $e_j\in\mathfrak{X}(TN_\theta)$; thus, (24) implies that

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\widetilde{R}(e_i,e_j,e_i,e_j)=-\epsilon\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}g(e_i,e_i)g(e_j,e_j).$$

After computation using the above equation, we can derive

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\widetilde{R}(e_i,e_j,e_i,e_j)=-l_1l_2\epsilon, \tag{25}$$

Therefore, following (23) and (25), we finally obtain

$$\begin{aligned}\frac{l_2\Delta f}{f}&+l_2^2||\nabla(\ln f)||^2+l_2\big(\csc^2\theta+\cot^2\theta\big)||\nabla(\ln f)||^2+\|h_\mu\|^2+l_1l_2\epsilon\\&=\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\left\{2||h(e_i,e_j)||^2-g\big(h(e_j,e_j),h(e_i,e_i)\big)\right\}.\end{aligned} \tag{26}$$

If the pinching condition (2) is satisfied, then from (26) we have

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\left\{2||h(e_i,e_j)||^2-g\big(h(e_j,e_j),h(e_i,e_i)\big)\right\}<4l_1l_2\epsilon$$

By applying Theorem 1 with $c = 4\epsilon > 0$, we obtain the following:

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\left\{2||h(e_i,e_j)||^2 - g\big(h(e_j,e_j),h(e_i,e_i)\big)\right\} < l_1 l_2 c.$$

This completes the proof of Theorem 2, as the assertion follows from Theorem 1.

5.2. Proof of Theorem 4

If we consider M^n as the compact-oriented Riemannian manifold without boundary $\partial M^n = \emptyset$, then we are able to prove the strong result in terms of tthe *Dirichlet energy* and pointwise slant immersion as follows. Taking the integration along the volume element dV in (2), we obtain

$$\int_{M^n}\big(\csc^2\theta + \cot^2\theta + l_2\big)||\nabla f||^2 dV + \int_{M^n} f\Delta f dV < \int_{M^n}\Big(3l_2\epsilon - \frac{\|h_\mu\|^2}{l_2}\Big)f^2 dV. \tag{27}$$

From the divergence theorem $\int_{M^n}(\Delta f)dV = 0$ in [34] without boundary. Using this fact, we can compute the following as

$$\begin{aligned} 0 &= \int_{M^n}\Delta\Big(\frac{f^2}{2}\Big)dV = -\int_{M^n} div\Big(\nabla\Big(\frac{f^2}{2}\Big)\Big)dV \\ &= -\int_{M^n} div(f\nabla f)dV = -\int_{M^n} g(\nabla f,\nabla f)dV + \int_{M^n} f\Delta f dV \end{aligned}$$

which implies that

$$\int_{M^n} f\Delta f dV = \int_{M^n}\|\nabla f\|^2 dV. \tag{28}$$

The inequality (27) takes its new form by virtue of (28), that is,

$$2\int_{M^n}\big(\csc^2\theta||\nabla f||^2\big)dV + l_2\int_{M^n}||\nabla f||^2 dV < \int_{M^n}\Big(3l_1\epsilon - \frac{\|h_\mu\|^2}{l_2}\Big)f^2 dV. \tag{29}$$

Using the Dirichlet energy from Formula (5) in the above equation, we have

$$2\big(2\csc^2\theta + l_2\big)\mathbb{E}(f) < \int_{M^n}\Big(3l_1\epsilon - \frac{\|h_\mu\|^2}{l_2}\Big)f^2 dV.$$

Thus, we obtain the required result (6). This completes the proof of the theorem.

5.3. Proof of Corollary 1 and 2

The proof of Corollary 1 and Corollary 3 arises directly from Theorems 2 and 5 by substituting $\theta = \frac{\pi}{2}$ to point out a totally real submanifold from a pointwise slant submanifold, which then provides the promised results.

5.4. Proof of Theorem 5

From Theorem 2, we can find that there do not exist stable integral l_1-currents in a warped product pointwise semi-slant submanifold M^n and that the homology groups are zero for all positive integers l_1, l_2 such that $n = l_1 + l_2 \neq 3$; that is, $\mathbb{H}_{l_1}(M^n,\mathbb{G}) = \mathbb{H}_{l_2}(M^n,\mathbb{G}) = 0$. Therefore, M^n is a homology sphere, and in addition is a homotopic sphere following the same arguments as in [19].

Therefore, applying the generalized *Poincarẽ conjecture* (Smale $n \geq 5$ [4], Freedman $n = 4$ [8]), we know that M^n is homotopic to the sphere $\mathbb{S}^n$ as an immediate consequence of Sjerve[10], implying that the fundamental group $\pi_1(M^n) = 0$ on M^n when applying the same arguments as above. This implies that $M^{l_1+l_2}$ is homeomorphic to the sphere $\mathbb{S}^{l_1+l_2}$. Similarly, it is not hard to check that M^3 is homotopic to a sphere $\mathbb{S}^3$ when $n = 3$ from [9,16]. This completes the proof of Theorem 5.

5.5. Proof of Theorem 6

From the minimum principle on the first eigenvalue λ_1, we can obtain the outcome from [32], p. 186. Let us assume that f is a non-constant warping function

$$\lambda_1 \int_{M^n} f^2 dV \leq \int_{M^n} \|\nabla f\|^2 dV. \tag{30}$$

where the equality holds if and only if $\Delta f = \lambda_1 f$. Integrating Equation (29) and Green's lemma, we have

$$\big(2\csc^2\theta + q\big)\int_{M^n} \|\nabla f\|^2 dV < \int_{M^n} \Big(3l_1\epsilon - \frac{\|h_\mu\|^2}{l_2}\Big) f^2 dV,$$

which implies that

$$\int_{M^n} \|\nabla f\|^2 dV < \frac{1}{\big(2\csc^2\theta + l_2\big)} \int_{M^n} \Big(3l_1\epsilon - \frac{\|h_\mu\|^2}{l_2}\Big) f^2 dV. \tag{31}$$

By virtue of (30) in (31), we can find that

$$\int_{M^n} \left\{\lambda_1 - \frac{\big(3l_1 l_2\epsilon - \|h_\mu\|\big)}{l_2\big(2\csc^2\theta + l_2\big)}\right\} f^2 dV < 0.$$

From this, we arrive at our result (7) by combining Theorems 2 and 5, which completes the proof.

Here, we remember the lemma below.

Lemma 2 ([12]). *Assume that $\widetilde{M}^{2m}$ is a Kaehler manifold and $M^{l_1+l_2} = N_T^{l_1} \times_f N_\theta^{l_2}$ is a warped product pointwise semi-slant submanifold of $\widetilde{M}^{2m}$. Then, we have*

$$g\big(h(X2, Y_2), FZ_2\big) = 0. \tag{32}$$

for any $X_2, Y_2 \in \mathfrak{X}(TN_T)$ and $Z_2, W_2 \in \mathfrak{X}(TN_\theta)$.

In view of Lemma 2, we can find our next result.

5.6. Proof of Theorem 3

We can write the following from (12) as follows:

$$\begin{aligned}\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\Big\{2||h(e_i,e_j)||^2 - g\big(h(e_j,e_j),h(e_i,e_i)\big)\Big\} =& 2\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}||h(e_i,e_j)||^2 \\ &- \sum_{i=1}^{l_1}\sum_{j=1}^{l_2} g\big(h(e_i,e_i),e_j\big)^2,\end{aligned}$$

or equivalently as

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\Big\{2||h(e_i,e_j)||^2 - g\big(h(e_j,e_j),h(e_i,e_i)\big)\Big\}$$
$$= 2\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}||h(e_i,e_j)||^2 - \sum_{i=1}^{l_1}\sum_{j=1}^{l_2} g\big(h(e_i,e_i).Fe_j^*\big)^2.$$

By virtue of (32), we have

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\Big\{2||h(e_i,e_j)||^2 - g\big(h(e_j,e_j),h(e_i,e_i)\big)\Big\} = 2\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}||h(e_i,e_j)||^2.$$

Using Equation (22) on the right hand side of the above equation, we have

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\Big\{2||h(e_i,e_j)||^2 - g\big(h(e_j,e_j),h(e_i,e_i)\big)\Big\} = \frac{2l_2}{f^2}\Big(\csc^2\theta + \cot^2\theta\Big)\|\nabla f\|^2 + \|h_\mu\|^2. \quad (33)$$

If assumption (3) is satisfied, then the following inequality is implied by (33):

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\Big\{2||h(e_i,e_j)||^2 - g\big(h(e_j,e_j),h(e_i,e_i)\big)\Big\} < 4l_1l_2\epsilon. \quad (34)$$

Thus, the proof is complete from Theorem 1 and from (34).

Based on Theorem 3 and the similar proof of Theorem 5, we reach the following result.

Corollary 5. *Assume that $M^{l_1+l_2} = N_T^{l_1} \times_f N_\theta^{l_2}$ is a compact warped product pointwise semi-slant submanifold of a complex space form $\widetilde{M}^{2m}(4\epsilon)$ satisfying the following:*

$$\|\nabla f\|^2 < \left\{\frac{(4l_1l_2\epsilon - \|h_\mu\|^2)f^2}{2l_2\big(\csc^2\theta + \cot^2\theta\big)}\right\}.$$

Then, M^{p+q} is homeomorphic to a sphere $\mathbb{S}^{p+q}$ when $p+q \neq 3$, while M^3 is homotopic to a sphere $\mathbb{S}^3$.

5.7. Proof of Theorem 7

In this theorem, we replace our pinching condition (2) with the Hessian of the warping function and Ricci curvature by using the concept of the eigenvalue of the warped function. If f is a first eigenfunction of the Laplacian of M^n associated with the first eigenvalue λ_1, that is, $\Delta f = \lambda_1 f$, then we an recall Bochner's formula (see, e.g., [32]), which states that for a differentiable function f defined on a Riemannian manifold, the following relation holds:

$$\frac{1}{2}\Delta\|\nabla f\|^2 = \|\nabla^2 f\|^2 + Ric(\nabla f,\nabla f) + g\big(\nabla f,\nabla(\Delta f)\big).$$

By integrating the above equation with the aid of Stokes' theorem, we obtain

$$\int \|\nabla^2 f\|^2 dV + \int Ric(\nabla f,\nabla f)dV + \int g\big(\nabla f,\nabla(\Delta f)\big)dV = 0.$$

Now, by using $\Delta f = \lambda_1 f$ and slightly rearranging the above equation, we derive

$$\int \|\nabla f\|^2 dV = -\frac{1}{\lambda_1}\Big(\int \|\nabla^2 f\|^2 dV + \int Ric(\nabla f,\nabla f)dV\Big). \quad (35)$$

On combing Equations (28) and (27), we obtain

$$\left(2\csc^2\theta + l_2\right)\int_{M^n}\|\nabla f\|^2 dV + \int_{M^n}\frac{f^2\|h_\mu\|^2}{l_2}dV < 3l_1\epsilon\int_{M^n} f^2 dV. \tag{36}$$

Following from (35) and (36), we find that

$$\int_{M^n}\left(\frac{\|h_\mu\|^2}{l_2} - 3l_1\epsilon\right)f^2 dV < \frac{\left(2\csc^2\theta + l_2\right)}{\lambda_1}\int_{M^n}\Big(\|\nabla^2 f\|^2 + Ric(\nabla f, \nabla f)\Big)dV,$$

which implies that

$$\|\nabla^2 f\|^2 + Ric(\nabla f, \nabla f) > \left\{\frac{\left(\|h_\mu\|^2 - 3l_1 l_2\epsilon\right)\lambda_1 f^2}{\left(2\csc^2\theta + l_2\right)}\right\}. \tag{37}$$

The proof follows from the above Equation (37) along with Theorem 2.

6. Consequences

It is well known that a complete simply-connected complex space form $\widetilde{M}^{2m}(4\epsilon)$ is holomorphicaly isometric to the complex Euclidean space $\mathbb{C}^m$, the complex projective m-space $\mathbb{C}P^m(4)$, and a complex hyperbolic m-space $\mathbb{C}H^m(-4)$ with $\epsilon = 0,\ 1\,\&\ \epsilon = -1$. Therefore, we define the following corollaries in consequence of our Theorem 2 and Theorem 5.

Corollary 6. *Let $M^{l_1+l_2} = N_T^{l_1}\times_f N_\theta^{l_2}$ be a compact warped product pointwise semi-slant submanifold in a complex Euclidean space $\mathbb{C}^m$ satisfying the condition*

$$\left(\csc^2\theta + \cot^2\theta + l_2\right)||\nabla f||^2 + f\Delta f + \frac{f^2}{l_2}\|h_\mu\|^2 < 0.$$

Then, there do not exist stable integral l_1-currents in $M^{l_1+l_2}$ and $\mathbb{H}_{l_1}(M^{l_1+l_2}, \mathbb{G}) = \mathbb{H}_{l_2}(M^{l_1+l_2}, \mathbb{G}) = 0$. Furthermore, $M^{l_1+l_2}$ is homeomorphic to a sphere $\mathbb{S}^{l_1+l_2}$ when $l_1 + l_2 \geq 4$, while M^3 is homotopic to a sphere $\mathbb{S}^3$.

Similarly, for the complex projective m-space $\mathbb{C}P^m(4)$ we have the following.

Corollary 7. *Let $M^{l_1+l_2} = N_T^{l_1}\times_f N_\theta^{l_2}$ be a compact warped product pointwise semi-slant submanifold in a complex projective m-space $\mathbb{C}P^{2m}(4)$ satisfying the condition*

$$\left(\csc^2\theta + \cot^2\theta + l_2\right)||\nabla f||^2 + f\Delta f < \frac{f^2}{l_2}\left(3l_1 l_2 - \|h_\mu\|^2\right).$$

Then, there do not exist stable integral l_1-currents in $M^{l_1+l_2}$ and $\mathbb{H}_{l_1}(M^{l_1+l_2}, \mathbb{G}) = \mathbb{H}_{l_2}(M^{l_1+l_2}, \mathbb{G}) = 0$. In addition, $M^{l_1+l_2}$ is homeomorphic to a sphere $\mathbb{S}^{l_1+l_2}$ when $l_1 + l_2 \geq 4$, while M^3 is homotopic to a sphere $\mathbb{S}^3$.

7. Conclusions

The presented study is significant in light of the extant literature thanks to the new pinching conditions presented in terms of pointwise slant functions and the Laplacian of the warped function. We have discussed the rigidity results and investigated several topological classifications. In addition, we have derived a number of extrinsic conditions involving relevant geometric quantities by analyzing the extent to which the topology of warped product submanifolds is affected by the conditions on the main intrinsic and main extrinsic curvature invariants. A number of topological sphere theorems have been investigated in refeence to the connection between warped product submanifolds and homotopic–

homologic theory. The contents of the present paper can be expected to attract researchers to the prospect of finding possible applications in various research areas of physics.

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