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Boundary Coupling for Consensus of Nonlinear Leaderless Stochastic Multi-Agent Systems Based on PDE-ODEs

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Abstract: This paper studies the leaderless consensus of the stochastic multi-agent systems based on partial differential equations–ordinary differential equations (PDE-ODEs). Compared with the traditional state coupling, the most significant difference between this paper is that the space state coupling is designed. Two boundary couplings are investigated in this article, respectively, collocated boundary measurement and distributed boundary measurement. Using the Lyapunov directed method, sufficient conditions for the stochastic multi-agent system to achieve consensus can be obtained. Finally, two simulation examples show the feasibility of the proposed spatial boundary couplings.

Keywords: stochastic; consensus; boundary coupling; partial differential equations–ordinary differential equations

MSC: 93B70

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1. Introduction

Multi-agent system (MAS) is a group system composed of several agents, which has attracted a lot of researchers in recent years [1]. Although many works have been performed on MAS, most of the work assumes that each agent in the system receives accurate information about its neighbors without any system overhead. However, in fact, this kind of situation is actually very difficult to realize in the real application, the reason is that agents are often affected by uncertain factors such as information delay, information loss, and electromagnetic interference in the process of transmitting and receiving information. If we ignore the influence of these factors on the system, there will be greater errors, and even affect the stability of the system. Therefore, considering the consensus of stochastic multi-agent systems (MASs) is a realistic and urgent problem. Stochastic MASs have attracted extensive attention and research from various scholars [2–4]. MASs have many applications such as robotics [5,6], distributed control [7–9], and telecommunications and economic [10]. Consequently, it is of theoretical significance and practical value for the research team cooperating with MASs.

Consensus is the most fundamental problem in multi-agent system, has important applications in many fields, such as in robotic systems, the consensus problem is studied to make all robots cooperate with each other to complete the same task (arrive a certain position, speed reaches a certain value, etc.). In the aerial refueling system, the aerial vehicles can cooperate with each other to complete the oil transportation task. Therefore, consensus is also the most important dynamic behaviors of MASs, which has been widely used in engineering fields such as image encryption [11,12], automatic control [13,14], and communication security [15,16]. Over the past few decades, many important controllers have been designed, for instance, cluster control [17,18], adaptive control [19–22], event-triggered control [23–25], pining control [26], quantized control [27], global asymptotic consensus control [28], etc. At this stage, most literature on the consensus of MASs is based on ODEs, which leads to the neglect of the spatial dynamic behavior of the system. In

fact, there are various spatio-temporal behaviors in nature, such as ecosystems, chemical reactions, food webs, etc., all of which depend on time and space and need to be modeled with the help of PDEs. Consequently, it is very important to study the consensus of stochastic MASs modeled by PDEs.

It should be noticed that, MASs are mostly full of randomness [29,30]. Consequently, research on stochastic systems is very important. In the cooperative control of linear Gaussian MASs, the mean and variance are usually used as the objectives to achieve the purpose of consensus. In addition, the control of stochastic multi-agent systems can be divided into internal control and boundary control, the former is easy to obtain in theory, but difficult to achieve in industrial production and with high production costs. The boundary control only acts on the space boundary, which has the vital significance in the reality, when the point in the space domain cannot be controlled, the boundary control can solve this problem very well [31]. Traditional state coupling does not consider the position information of agent. However, many systems in reality not only change with time, but also have relation with the displacement of agent [32]. For example, the ecosystem, the aerial refueling system, the robot cooperation system, etc. These systems are often associated with the spatial state of the system, so it is necessary to design the spatial state coupling to depict them. Ref. [33] studies the boundary consensus of continuous-time linear MASs. These results provide an effective method for consensus of MASs based on PDE. Boundary control only acts on the spatial boundary, which is of great significance in practice. However, there are few research achievements on boundary coupling for consensus of MASs based on PDEs.

Over the past few decades, nonlinear systems have been further studied. Most meaningful works have been successfully achieved, such as consensus or synchronization of switched nonlinear systems [34–36]. In [37], the adaptive fuzzy finite-time tracking control problem of switched nonlinear systems is addressed, they used a finite-time command filter to handle the drawback in the recursive design method. By introducing TP and MDADT to the switching signal, ref. [38] studied global exponential synchronization of CNs. In addition, the study of nonlinear systems is not limited to this, a great deal with control protocols. These references provide the basis for our work.

Motivated by the results above and drawing on research [39,40], in this article, boundary coupling for the consensus of nonlinear leaderless stochastic MASs based on PDE-ODEs is proposed.

Notations: The following is represented by some of the symbols used in this article: $P > 0$ regards a positive definite matrix P; $\|\cdot\|$ be used to regard the Euclidean norm for vector; $\lambda_{\max(\min)}(\cdot)$ is the maximum (minimum) eigenvalue; $\mathbb{E}$ denotes the mathematical expectation.

Consider the stochastic MAS based on PDE-ODEs in an one-dimensional spatial domain as

$$\mathrm{d}x_i(t) = \left[v(x_i(t)) + \int_0^L w(y_i(\eta,t))\mathrm{d}\eta + u_i(t)\right]\mathrm{d}t + C_1 x_i(t)\mathrm{d}\omega_1(t), \tag{1}$$

$$\mathrm{d}y_i(\eta,t) = \left[\Theta y_{i,\eta\eta}(\eta,t) + p(y_i(\eta,t)) + q(x_i(t))\right]\mathrm{d}t + C_2 y_i(\eta,t)\mathrm{d}\omega_2(t), \tag{2}$$

in which $(\eta,t) \in [0,L] \times [0,\infty)$. Here, $y_i(\eta,t) \in \mathbb{R}^n$ is the state; $u_i(t)$ is the control inputs; $C_1, C_2 \in \mathbb{R}^{n\times n}$ are known constant matrices, where n stands for n-dimensional space, $\mathbb{R}$ represents the field of real numbers; $0 < L \in \mathbb{R}$; $\Theta \in \mathbb{R}^{n\times n}$ is symmetric positive definite the subscript η means the partial derivative with respect to η; $\omega_1(t), \omega_2(t) \in \mathbb{R}$ is stochastic disturbance.

For systems with N agents, the coupling in Equation (1) is initialized as

$$u_i(t) = -k\sum_{i=1}^{N} h_{ij}(x_i(t) - x_j(t)), \tag{3}$$

the Neumann boundary condition under the collocated boundary measurement form is designed as

$$\begin{cases} y_{i,\eta}(0,t) = 0, \\ y_{i,\eta}(L,t) = -d\sum_{j=1}^{N} g_{ij}(y_i(L,t) - y_j(L,t)), \end{cases} \tag{4}$$

another Neumann boundary condition under the distributed measurement form is constructed as

$$\begin{cases} y_{i,\eta}(0,t) = 0, \\ y_{i,\eta}(L,t) = -d\int_0^L \sum_{j=1}^{N} g_{ij}(y_i(\eta,t) - y_j(\eta,t))\mathrm{d}\eta, \end{cases} \tag{5}$$

where k, d are positive feedback gains that will be determined later, and the coupling matrix $G = [g_{ij}]$ is constructed as : $g_{ij} > 0 (i \neq j)$, if i is connected to j, otherwise $g_{ij} = 0 (i \neq j)$; and $g_{ii} = 0$, $i, j \in \{1, 2, \cdots, N\}$. $H = [h_{ij}]$ is defined same to G.

Define consensus error $e_i(t) \triangleq x_i(\cdot) - \bar{x}(\cdot)$, $\zeta_i(\eta,t) \triangleq y_i(\cdot) - \bar{y}(\cdot)$, one has

$$\begin{aligned} \mathrm{d}e_i(t) \quad = \quad & \Bigg[v(x_i(t)) - \frac{1}{N}\sum_{j=1}^{N} v(x_j(t)) + \int_0^L w(y_i(\eta,t))\mathrm{d}\eta \\ & - \frac{1}{N}\sum_{j=1}^{N}\int_0^L w(y_i(\eta,t))\mathrm{d}\eta + u_i(t)\Bigg]\mathrm{d}t + C_1 e_i(t)\mathrm{d}\omega_1(t), \end{aligned} \tag{6}$$

$$\begin{aligned} \mathrm{d}\zeta_i(\eta,t) \quad = \quad & \Bigg[\Theta\zeta_{i,\eta\eta}(\eta,t) + p(y(\eta,t)) - \frac{1}{N}\sum_{j=1}^{N} p(y_i(\eta,t)) \\ & + q(x_i(t)) - \frac{1}{N}\sum_{j=1}^{N} q(x_i(t))\Bigg]\mathrm{d}t + C_2\zeta_i(\eta,t)\mathrm{d}\omega_2(t), \end{aligned} \tag{7}$$

where $\bar{x}(\cdot) = \frac{1}{N}\sum_{j=1}^{N} x_j(t)$ and $\bar{y}(\cdot) = \frac{1}{N}\sum_{j=1}^{N} y_j(\eta,t)$. With the Neumann boundary condition under the collocated boundary measurement form is shown as

$$\begin{cases} \zeta_{i,\eta}(0,t) = 0, \\ \zeta_{i,\eta}(L,t) = -d\sum_{j=1}^{N} g_{ij}(\zeta_i(L,t) - \zeta_j(L,t)), \end{cases} \tag{8}$$

the other Neumann boundary condition under the distributed boundary measurement form is displayed as

$$\begin{cases} \zeta_{i,\eta}(0,t) = 0, \\ \zeta_{i,\eta}(L,t) = -d\int_0^L \sum_{j=1}^{N} g_{ij}(\zeta_i(\eta,t) - \zeta_j(\eta,t))d\eta, \end{cases} \tag{9}$$

and the initial condition is defined as

$$\zeta_i(\eta,0) = \varphi_i(\eta), \tag{10}$$

where $\varphi_i(\eta) \triangleq \varphi_i(\eta) - \frac{1}{N}\sum_{j=1}^{N}\varphi_j(\eta)$ and $v(e_i(\eta,t)) \triangleq v(y_i(\eta,t)) - \sum_{j=1}^{N} v(y_j(\eta,t))$.

Definition 1 ([39]). *MAS (1)–(3) achieves consensus if there exist scalar $M > 0$, $\eta > 0$, for instance,*

$$\mathbb{E}||\zeta(\eta,t)||^2 \leq M||\varphi(\eta)||^2 exp^{-\eta t}, \tag{11}$$

for all $\varphi_i(\eta) \in L^2(0,1)$, $\zeta \triangleq [\zeta_1^T(\eta,t), \zeta_2^T(\eta,t), \cdots, \zeta_N^T(\eta,t)]^T$ and $\varphi \triangleq [\varphi_1(\eta)^T, \varphi_2^T(\eta), \cdots, \varphi_N^T(\eta)]^T$.

Lemma 1 ([41]). *Define $\zeta(\eta)$ as a vector function with $\zeta(0) = 0$ or $\zeta(L) = 0$, then the following inequality holds for any matrix $\Phi > 0$, then*

$$\int_0^L \zeta^T(s)\Phi\zeta(s)\mathrm{d}s \leq \frac{4L^2}{\pi^2}\int_0^L \dot{\zeta}^T(s)\Phi\dot{\zeta}(s)\mathrm{d}s. \tag{12}$$

Suppose $\mathcal{W} \in \mathbb{R}^{N\times N}$ is a symmetric Laplacian matrix, then $0 = \lambda_1(\mathcal{W}) < \lambda_2(\mathcal{W}) \leq \lambda_i(\mathcal{W}) \leq \lambda_N(\mathcal{W}), \lambda_i \in \{\lambda_1, \lambda_2, \cdots, \lambda_N\}$.

Lemma 2 ([42]). *For Laplacian matrix $\mathcal{W}$, symmetric positive definite P and $x \in \mathbb{R}^{N\times n}$ such that $1_{Nn}^T x = 0$, the following inequality is satisfied:*

$$\lambda_2(\mathcal{W})x^T(I_N \otimes P)x \leq x^T(\mathcal{W} \otimes P)x.$$

Assumption 1. *Assume for any real constants ξ_1, ξ_2, there exist positive numbers $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ satisfying Lipschitz condition:*

$$\begin{aligned}
&|v(\xi_1) - v(\xi_2)| \leq \gamma_1|\xi_1 - \xi_2|,\\
&|p(\xi_1) - p(\xi_2)| \leq \gamma_2|\xi_1 - \xi_2|,\\
&|w(\xi_1) - w(\xi_2)| \leq \gamma_3|\xi_1 - \xi_2|,\\
&|q(\xi_1) - q(\xi_2)| \leq \gamma_4|\xi_1 - \xi_2|.
\end{aligned}$$

2. Consensus of PDE-ODEs Based MASs under the Collocated Boundary Measurement Form

Choose the Lyapunov functional candidate for MASs (1)–(3) as

$$V(t) = \frac{1}{2}\sum_{i=1}^N e_i^T(t)e_i(t) + \frac{1}{2}\sum_{i=1}^N \int_0^L \zeta_i^T(\eta,t)\zeta_i(\eta,t)\mathrm{d}\eta. \tag{13}$$

The differential of $V(t)$ along MASs (1)–(3) is obtained as

$$\begin{aligned}
\mathrm{d}V \quad &= \sum_{i=1}^N e_i^T(t)\mathrm{d}e_i(t) + \sum_{i=1}^N \int_0^L \zeta_i^T(\eta,t)\mathrm{d}\zeta_i(\eta,t)\mathrm{d}\eta\\
&= \Bigg[\sum_{i=1}^N e_i^T(t)(v(x_i(t)) - \frac{1}{N}\sum_{j=1}^N v(x_j(t))\\
&+ \int_0^L w(y_i(\eta,t))\mathrm{d}\eta - \frac{1}{N}\sum_{i=1}^N \int_0^L w(y_i(\eta,t))\mathrm{d}\eta)\Bigg]\mathrm{d}t\\
&+ \sum_{j=1}^N e_i^T(t)u_i(t)\mathrm{d}t + \sum_{i=1}^N e_i^T(t)C_1e_i(t)\mathrm{d}\omega_1(t)\\
&+ \Bigg[\sum_{i=1}^N \int_0^L \zeta_i^T(\eta,t)[\Theta\zeta_{i,\eta\eta}(\eta,t) + p(y_i(\eta,t)) - \frac{1}{N}\sum_{j=1}^N p(y_i(\eta,t))\\
&+ q(x_i(t)) - \frac{1}{N}\sum_{j=1}^N q(x_i(t))]\mathrm{d}\eta\Bigg]\mathrm{d}t\\
&+ \sum_{i=1}^N \int_0^L \zeta_i^T(\eta,t)C_2\zeta_i(\eta,t)\mathrm{d}\eta\mathrm{d}\omega_2(t).
\end{aligned} \tag{14}$$

Owing to $\sum_{i=1}^{N} e_i(t)(v(\frac{1}{N}\sum_{j=1}^{N} x_j(t)) - \frac{1}{N}\sum_{j=1}^{N} v(x_j(t))) = 0$, using Hypothesis 1, we can obtain that

$$\begin{aligned}
& \sum_{i=1}^{N} e_i(t)[v(x_i(t)) - \frac{1}{N}\sum_{j=1}^{N} v(x_j(t))] \\
= & \sum_{i=1}^{N} e_i(t)\left[v(x_i(t)) - v(\frac{1}{N}\sum_{j=1}^{N} x_j(t)) + v(\frac{1}{N}\sum_{j=1}^{N} x_j(t)) - \frac{1}{N}\sum_{j=1}^{N} v(x_j(t))\right] \\
\leq & \gamma_1 \sum_{i=1}^{N} e_i^T(t)e_i(t),
\end{aligned} \tag{15}$$

treating the following formula in the same way, one has

$$\begin{aligned}
& \sum_{i=1}^{N} \int_0^L \zeta_i^T(\eta,t)\left[p(y_i(\eta,t)) - \frac{1}{N}\sum_{j=1}^{N} p(y_i(\eta,t))\right]\mathrm{d}\eta \\
\leq & \gamma_3 \sum_{i=1}^{N} \int_0^L \zeta_i^T(\eta,t)\zeta_i(\eta,t)\mathrm{d}\eta.
\end{aligned} \tag{16}$$

Under Hypothesis 1, the following can be obtained,

$$\begin{aligned}
& \sum_{i=1}^{N} e_i^T(t) \int_0^L \left[w(y_i(\eta,t)) - \frac{1}{N}\sum_{j=1}^{N} w(y_i(\eta,t))\right]\mathrm{d}\eta \\
= & \sum_{i=1}^{N} e_i^T(t) \int_0^L \left[w(y_i(\eta,t)) - w(\frac{1}{N}\sum_{j=1}^{N} y_j(\eta,t))\right]\mathrm{d}\eta \\
\leq & \frac{1}{2}\sum_{i=1}^{N} e_i^T e_i(t) + \frac{1}{2}\sum_{i=1}^{N} \int_0^L [w(y_i(\eta,t)) - w(\frac{1}{N}\sum_{j=1}^{N} y_j(\eta,t))^2]\mathrm{d}\eta \\
\leq & \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) + \frac{1}{2}\sum_{i=1}^{N} \gamma_2^2 \int_0^L \zeta_i^T(\eta,t)\zeta_i(\eta,t)\mathrm{d}\eta.
\end{aligned} \tag{17}$$

Using the boundary coupling (4) with $\Theta > 0$ and Lemma 1, Lemma 2, we can have that

$$\begin{aligned}
& \sum_{i=1}^{N} \int_0^L \zeta_i^T(\eta,t)\Theta\zeta_{\eta\eta}(\eta,t)\mathrm{d}\eta \\
= & \sum_{i=1}^{N} \zeta_i^T(L,t)\Theta d \sum_{j=1}^{N} g_{ij}(\zeta_i(L,t) - \zeta_j(L,t))\mathrm{d}\eta - \int_0^L \sum_{i=1}^{N} \zeta_{i,\eta}^T(\eta,t)\Theta\zeta_{i,\eta}(\eta,t)\mathrm{d}\eta \\
\leq & -d\sum_{i=1}^{N} \zeta_i^T(L,t)(\mathcal{L}_G \otimes \Theta)\zeta_i(L,t) \\
& -0.25L^{-2}\pi^2 \sum_{i=1}^{N} \int_0^L (\zeta_i^T(\eta,t) - \zeta_i^T(L,t))(I_N \otimes \Theta)(\zeta_i(\eta,t) - \zeta_i(L,t))\mathrm{d}\eta \\
\leq & -d\lambda_2(\mathcal{L}_G)\sum_{i=1}^{N} \zeta_i^T(L,t)(I_N \otimes \Theta)\zeta_i(L,t) \\
& -0.25L^{-2}\pi^2 \int_0^L (\zeta_i^T(\eta,t) - \zeta_i^T(L,t))(I_N \otimes \Theta)(\zeta_i(\eta,t) - \zeta_i(L,t))\mathrm{d}\eta \\
\leq & -d\lambda_2(\mathcal{L}_G)\lambda_{min}(\Theta)\sum_{i=1}^{N} \zeta_i^T(L,t)\zeta_i(L,t) \\
& -0.25L^{-2}\pi^2 \int_0^L (\zeta_i^T(\eta,t) - \zeta_i^T(L,t))(I_N \otimes \Theta)(\zeta_i(\eta,t) - \zeta_i(L,t))\mathrm{d}\eta,
\end{aligned} \tag{18}$$

where $\mathcal{L}_G = D - G, D = diag\{d_1, d_2, \cdots, d_N\}, d_i = \sum_{j=1}^{N} g_{ij}$, therefore, $\mathcal{L}_G$ is a Laplacian matrix and $\lambda_2(\cdot)$ denotes the minimum nonzero eigenvalue of $\cdot$.

In the same way, the following can be derived as

$$\begin{aligned}
& \sum_{i=1}^{N} \int_0^L \zeta_i^T(\eta,t)[q(x_i(t)) - \frac{1}{N}\sum_{j=1}^{N} q(x_j(t))]\mathrm{d}\eta \\
= & \sum_{i=1}^{N} \int_0^L \zeta_i^T(\eta,t)[q(x_i(t)) - q(\frac{1}{N}\sum_{j=1}^{N} x_j(t))]\mathrm{d}\eta \\
\leq & \frac{1}{2}\sum_{i=1}^{N} \int_0^L \zeta_i^T(\eta,t)\zeta_i(\eta,t)\mathrm{d}\eta + \frac{1}{2}\sum_{i=1}^{N} \int_0^L [q(x_i(t)) - q(\frac{1}{N}\sum_{j=1}^{N} x_j(t))]^2\mathrm{d}\eta \\
\leq & \frac{1}{2}\int_0^L \zeta_i^T(\eta,t)\zeta_i(\eta,t)\mathrm{d}\eta + \frac{1}{2}\gamma_4^2\sum_{i=1}^{N} e_i^T(t)e_i(t). \qquad (19)
\end{aligned}$$

Theorem 1. *MASs (1)–(4) is said to be mean-square exponential consensus, if there exists $d > 0$ such that*

$$\Psi \triangleq \begin{bmatrix} \Psi_{11} & 0 & 0 \\ * & \Psi_{22} & \Psi_{23} \\ * & * & \Psi_{33} \end{bmatrix} < 0, \qquad (20)$$

where

$$\begin{aligned}
\Psi_{11} & \triangleq L^{-1}(\gamma_1 + 0.5\gamma_4^2 + 0.5 - k\lambda_2(\mathcal{L}_H))I_{Nn}, \\
\Psi_{22} & \triangleq -d\lambda_2(\mathcal{L}_G)\lambda_{min}(\Theta)I_{Nn} - 0.25L^{-2}\pi^2 I_N \otimes \Theta, \\
\Psi_{23} & \triangleq 0.25L^{-2}\pi^2 I_N \otimes \Theta, \\
\Psi_{33} & \triangleq 0.5\gamma_2^2 + \gamma_3 + 0.5 - 0.25L^{-2}\pi^2 I_N \otimes \Theta.
\end{aligned}$$

Proof. Substituting (15)–(19) into (14), one obtains

$$\begin{aligned}
\mathrm{d}V \leq & \Bigg[(\gamma_1 - k\lambda_2(\mathcal{L}_H) + 0.5\gamma_4^2 + 0.5)\sum_{i=1}^{N} e_i^T(t)e_i(t) \\
& + (0.5\gamma_2^2 + 0.5 + \gamma_3)\sum_{i=1}^{N}\int_0^L \zeta_i^T(\eta,t)\zeta_i(\eta,t)\mathrm{d}\eta\Bigg]\mathrm{d}t \\
& + \sum_{i=1}^{N} e_i^T(t)C_1 e_i^T(t)\mathrm{d}\omega_1(t) \\
& + \Bigg[-d\lambda_2(\mathcal{L}_G)\lambda_{min}(\Theta)\sum_{i=1}^{N}\zeta_i^T(L,t)\zeta_i(L,t) \\
& - 0.25L^{-2}\pi^2\sum_{i=1}^{N}\int_0^L \zeta_i^T(\eta,t) - \zeta_i^T(L,t)(I_N \otimes \Theta)(\zeta_i(\eta,t) - \zeta_i(L,t))\mathrm{d}\eta\Bigg]\mathrm{d}t \\
& + (\sum_{i=1}^{N}\int_0^L \zeta_i^T(\eta,t)C_2\zeta_i(\eta,t))\mathrm{d}\omega_2(t) \\
\leq & \Bigg[\int_0^L \hat{\zeta}^T(\eta,t)\Psi\hat{\zeta}(\eta,t)\mathrm{d}\eta\Bigg]\mathrm{d}t + \sum_{i=1}^{N} e_i^T(t)C_1 e_i(t)\mathrm{d}\omega_1(t) \\
& + (\sum_{i=1}^{N}\int_0^L \zeta_i^T(\eta,t)C_2\zeta_i(\eta,t)\mathrm{d}\eta)\mathrm{d}\omega_2(t), \qquad (21)
\end{aligned}$$

where $\hat{\zeta}(\eta,t) \triangleq [e^T(t), \zeta^T(L,t), \zeta^T(\eta,t)]^T$ and $\zeta \triangleq [\zeta_1^T, \zeta_2^T, \cdots, \zeta_N^T]^T$ and Ψ has defined in Equation (20).

By Itô's formula [43], one has

$$\begin{aligned} & \mathbb{E}[\exp(\lambda_{\min}(\Psi)t)V(t)] - \mathbb{E}[\exp(\lambda_{\min}(\Psi)0)V(0)] \\ = \; & \mathbb{E}\int_0^t d[\exp(\lambda_{\min}(\Psi)s)V(s)]\mathrm{d}s \\ = \; & \mathbb{E}\int_0^t \lambda_{\min}(\Psi)\exp(\lambda_{\min}(\Psi)s)V(s)\mathrm{d}s + \mathbb{E}\int_0^t \exp(\lambda_{\min}(\Psi)s)\mathrm{d}V(s). \end{aligned} \tag{22}$$

Substituting (20) and (21) into (22), we can obtain

$$\begin{aligned} & \mathbb{E}[\exp(\lambda_{\min}(\Psi)t)V(t)] \\ \leq \; & V(0) + \mathbb{E}\int_0^t [\lambda_{\min}(\Psi)\exp(\lambda_{\min}(\Psi)s)V(s) \\ & -\lambda_{\min}(\Psi)\exp(\lambda_{\min}(\Psi)s)\int_0^L e^T(\eta,s)e(\eta,s)\mathrm{d}\eta]\mathrm{d}s \\ & +\mathbb{E}\int_0^t \exp(\lambda_{\min}(\Psi)s)e_i^T(s)C_1e_i(s)\mathrm{d}\omega_1(s) \\ & +\mathbb{E}\int_0^t \exp(\lambda_{\min}(\Psi)s)\int_0^L \zeta_i^T(\eta,s)C_2\zeta_i(\eta,s)\mathrm{d}\eta\mathrm{d}\omega_2(s) \\ = \; & V(0), \end{aligned} \tag{23}$$

that is

$$\mathbb{E}V(t) \leq V(0)\exp(-\lambda_{\min}(\Psi)t). \tag{24}$$

The proof is completed. □

3. Consensus of PDE-ODEs Based MASs under the Distributed Boundary Measurement Form

Theorem 2. *MASs (1)–(3) and (5) is said to be mean-square exponential consensus, if there exists $d > 0$ such that*

$$\Xi \triangleq \begin{bmatrix} \Xi_{11} & 0 & 0 \\ * & \Xi_{22} & \Xi_{23} \\ * & * & \Xi_{33} \end{bmatrix} < 0, \tag{25}$$

where

$$\begin{aligned} \Xi_{11} &\triangleq L^{-1}(\gamma_1 - k\lambda_2(\mathcal{L}_H) + 0.5\gamma_4^2 + 0.5)I_{Nn}, \\ \Xi_{22} &\triangleq -0.25L^{-2}\pi^2 I_N \otimes \Theta, \\ \Xi_{23} &\triangleq 0.5d\lambda_2(\mathcal{L}_G)\lambda_{\min}(\Theta)I_{Nn}, \\ \Xi_{33} &\triangleq (0.5\gamma_2^2 + 0.5 + \gamma_3 - d\lambda_2(\mathcal{L}_G)\lambda_{\min}(\Theta))I_{Nn}. \end{aligned}$$

Proof. To prevent repetitions, only the difference with Theorem 1 is shown here.

Choose the same Lyapunov function as in Equation (13).

Using boundary coupling (5) to recalculate (18),

$$
\begin{aligned}
& \int_0^L \zeta_i^T(\eta,t)\Theta\zeta_{i,\eta\eta}(\eta,t)\mathrm{d}\eta \\
= & -\int_0^L \sum_{i=1}^N \zeta_i^T(L,t)\Theta d\sum_{j=1}^N g_{ij}(\zeta_i(\eta,t)-\zeta_j(\eta,t))\mathrm{d}\eta - \int_0^L \zeta_{i,\eta}^T(\eta,t)\Theta\zeta_{i,\eta}(\eta,t)\mathrm{d}\eta \\
= & -\int_0^L d\zeta_i^T(L,t)(\mathcal{L}_G\otimes\Theta)\zeta_i(\eta,t)\mathrm{d}\eta - \int_0^L \zeta_{i,\eta}^T(\eta,t)\Theta\zeta_{i,\eta}(\eta,t)\mathrm{d}\eta \\
\leq & -d\int_0^L \zeta^T(\eta,t)(\mathcal{L}_G\otimes\Theta)\zeta_i(\eta,t)\mathrm{d}\eta \\
- & 0.25L^{-2}\pi^2\int_0^L \zeta_i^T(\eta,t)-\zeta_i^T(L,t)(I_N\otimes\Theta)(\zeta_i(\eta,t)-\zeta_i(L,t))\mathrm{d}\eta \\
\leq & \int_0^L d\bar{\zeta}^T(\eta,t)(\mathcal{L}_G\otimes\Theta)\zeta_i(\eta,t)\mathrm{d}\eta - d\lambda_2(\mathcal{L}_G)\lambda_{min}(\Theta)\int_0^L \zeta_i^T(\eta,t)(I_N\otimes\Theta)\zeta_i(\eta,t)\mathrm{d}\eta \\
- & 0.25L^{-2}\pi^2\int_0^L \bar{\zeta}^T(\eta,t)(I_N\otimes\Theta)\bar{\zeta}(\eta,t)\mathrm{d}\eta,
\end{aligned} \tag{26}
$$

for brevity, we take $\bar{\zeta}(\eta,t) \triangleq \zeta(\eta,t)-\zeta(L,t)$.

Substituting (15)–(18) and (26) into (14),

$$
\begin{aligned}
\mathrm{d}V \leq & \Bigg[(\gamma_1 - k\lambda_2(\mathcal{L}_H)+0.5\gamma_4^2+0.5)\sum_{i=1}^N e_i^T(t)e_i^T(t) \\
& +(0.5\gamma_2^2+0.5+\gamma_3-d\lambda_2(\mathcal{L}_G)\lambda_{min}(\Theta))\sum_{i=1}^N\int_0^L \zeta_i^T(\eta,t)\zeta_i(\eta,t)\mathrm{d}\eta\Bigg]\mathrm{d}t \\
& +\sum_{i=1}^N e_i^T(t)C_1e_i(t)\mathrm{d}\omega_1(t) \\
& +\Bigg[\sum_{i=1}^N\int_0^L d\bar{\zeta}^T(\eta,t)(\mathcal{L}_G\otimes\Theta)\zeta_i(\eta,t)\mathrm{d}\eta \\
& -0.25L^{-2}\pi^2\int_0^L \bar{\zeta}^T(\eta,t)(I_N\otimes\Theta)\bar{\zeta}(\eta,t)\mathrm{d}\eta\Bigg]\mathrm{d}t \\
& +\sum_{i=1}^N\int_0^L \zeta_i^T(\eta,t)C_2\zeta_i(\eta,t)\mathrm{d}\eta\mathrm{d}\omega_2(t) \\
\leq & \Bigg[\lambda_{min}(\Xi)\int_0^L \tilde{\zeta}^T(\eta,t)\tilde{\zeta}(\eta,t)\mathrm{d}\eta\Bigg]\mathrm{d}t \\
& +\sum_{i=1}^N e_i^T(t)C_1e_i(t)\mathrm{d}\omega_1(t)+\sum_{i=1}^N\int_0^L \zeta_i^T(\eta,t)C_2\zeta_i(\eta,t)\mathrm{d}\eta\mathrm{d}\omega_2(t),
\end{aligned} \tag{27}
$$

in which $\tilde{\zeta}(\eta,t) \triangleq [e^T(t),\bar{\zeta}^T(\eta,t),\zeta^T(\eta,t)]^T$.

Most specially, $\mathbb{E}V(t) \leq V(0)\exp(-\lambda_{\min}(\Xi)t)$. This completes the proof. □

Remark 1. *Different from the coupling design in ODE systems for consensus [44,45], this paper solves the consensus of stochastic MASs based on PDE-ODEs by designing boundary coupling.*

Remark 2. *There have been many important results for MASs [46–48]. However, this article studies a class of MASs when only boundary nodes can receive messages, and the boundary coupling is designed to solve this issue.*

4. Simulation Examples

Example 1. *Consider the following nonlinear leaderless MAS (1)–(4) with random initial conditions:*

$$
\mathrm{d}x_i(t) = \Bigg[v(x_i(t))+\int_0^L w(y_i(\eta,t))d\eta+u_i(t)\Bigg]\mathrm{d}t + C_1x_i(t)\mathrm{d}\omega_1(t), \tag{28}
$$

$$\mathrm{d}y_i(\eta,t) = \left[\Theta y_{i,\eta\eta}(\eta,t) + p(y_i(\eta,t)) + q(x_i(t))\right]\mathrm{d}t + C_2 y_i(\eta,t)\mathrm{d}\omega_2(t), \tag{29}$$

where

$$\Theta = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, C1 = C2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, L = 1, \tag{30}$$

$$v(\cdot) = p(\cdot) = w(\cdot) = q(\cdot) = tanh(\cdot). \tag{31}$$

Obviously, $v(\cdot), p(\cdot), w(\cdot)$ and $q(\cdot)$ satisfy Lipschitz condition with $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1$, and the stochastic factors $\omega_1(t), \omega_2(t)$ can be written by matrix, one has

$$\Omega_1 = \Omega_2 = \begin{bmatrix} sin(2x)cos(\pi t) \\ sin(2x)cos(2\pi t) \end{bmatrix}. \tag{32}$$

The coupling matrix represents communication between agents, and it can be designed as:

$$H = G = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \tag{33}$$

and the initial conditions

$$\begin{aligned} y_1^0(\eta,t) =& [-0.8 + 0.5\cos(2\pi\eta - \frac{\pi}{3}), -1.2\sin(\pi\eta + \frac{\pi}{4})]^T, \\ y_2^0(\eta,t) =& [-0.5 + 0.8\sin(0.2\pi\eta), -0.6 - 0.3\exp(0.5\pi\eta + \frac{\pi}{6})]^T, \\ y_3^0(\eta,t) =& [-0.7 + 0.2\cos(0.3\pi\eta), 0.5 + 0.3\sin(\pi\eta + \frac{\pi}{5})]^T, \\ y_4^0(\eta,t) =& [0.5\exp(0.4\pi\eta), -0.8 + 0.2\cos(\pi\eta + \frac{\pi}{3})]^T, \end{aligned} \tag{34}$$

According to Theorem 1, solving LMI (20) by Matlab, the control gains $d = 0.7288$, $k = 4.4242$ is obtained. It can be shown in Figures 1 and 2 that the MAS (1)–(4) achieve consensus, the states of the MAS eventually converge, and the error between agents tends to 0, which means that the agents agree on speed or reach a certain point in time in engineering field. The boundary coupling (3) and (4) with the feedback gain are shown in Figures 3 and 4.

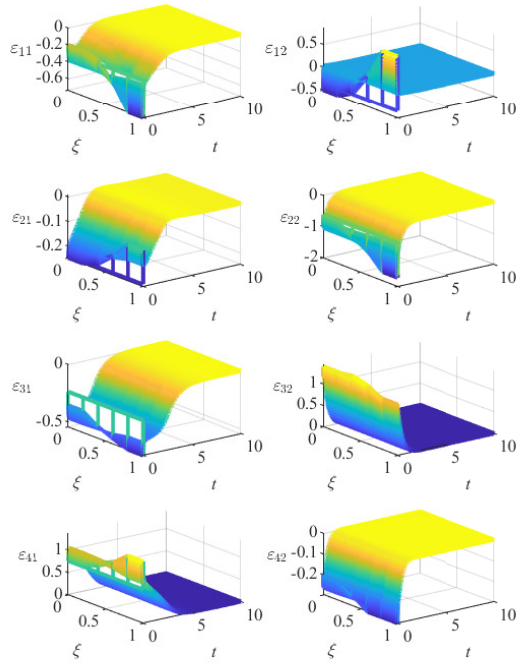

Figure 1. The system error $\zeta_i(\eta, t)$ of MAS with boundary coupling (4) in Example 1.

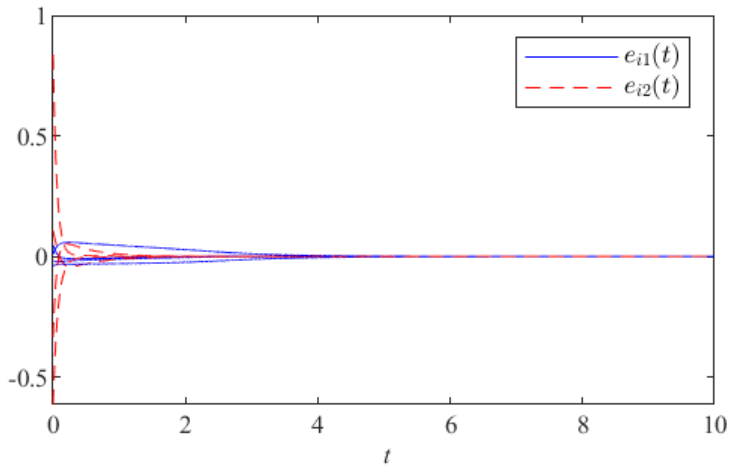

Figure 2. The system error $e_i(t)$ of MAS with boundary coupling (4) in Example 1.

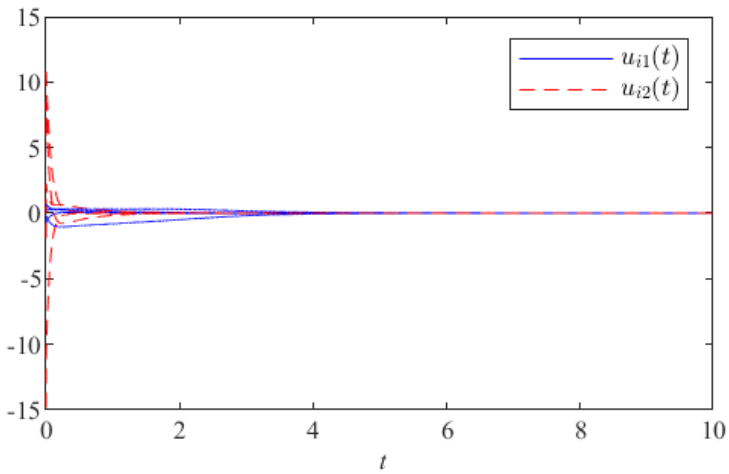

Figure 3. The boundary coupling $u_i(t)$ in Example 1.

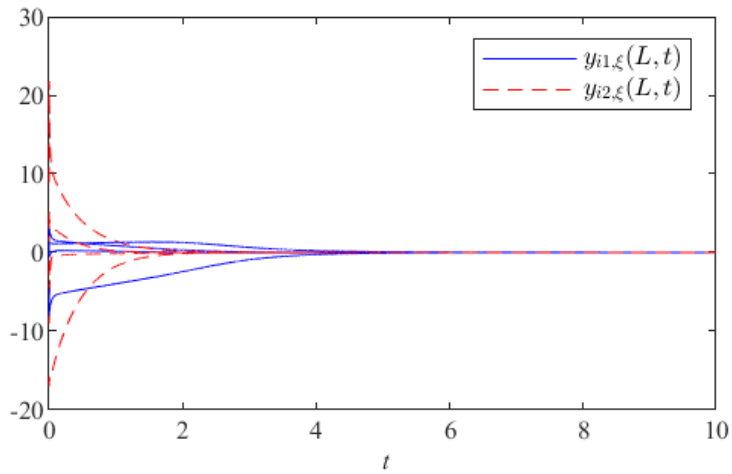

Figure 4. The boundary coupling $y_i(L,t)$ in Example 1.

Example 2. *Consider a stochastic MAS (1), (3), and (5) with the same coefficients as in Example 1. Different from former, the boundary coupling condition is adopted in the form of distributed measurement. According to Theorem 2, solving LMI (25) by Matlab,* $d = 1.7428, k = 3.9784$ *is obtained. Similarly, we can observe that the stochastic MAS in question can eventually be stabilized with the application of a controller. It can be shown in Figures 5 and 6 that the error of the MAS (1), (3), and (5) finally approaches 0 with the change of time. The boundary coupling (3) and (5) with the feedback gain are shown in Figures 7 and 8.*

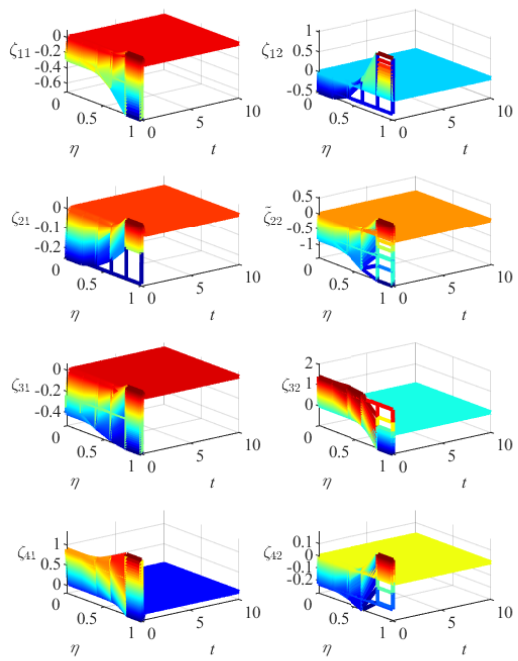

Figure 5. The system error $\zeta_i(\eta,t)$ of MAS with boundary coupling (5) in Example 2.

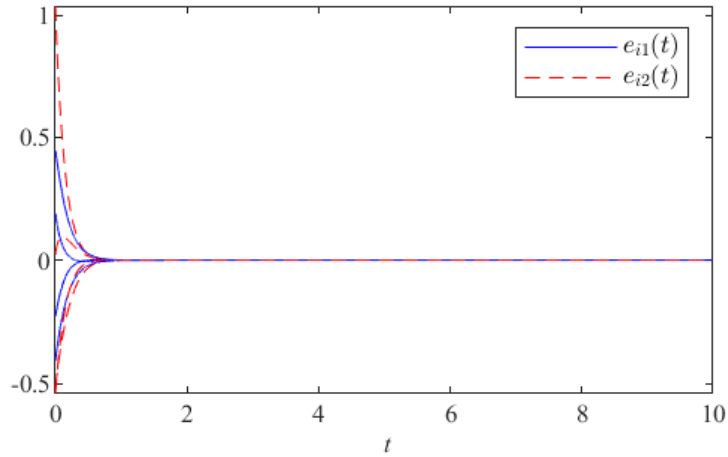

Figure 6. The system error $e_i(t)$ of MAS with boundary coupling (5) in Example 2.

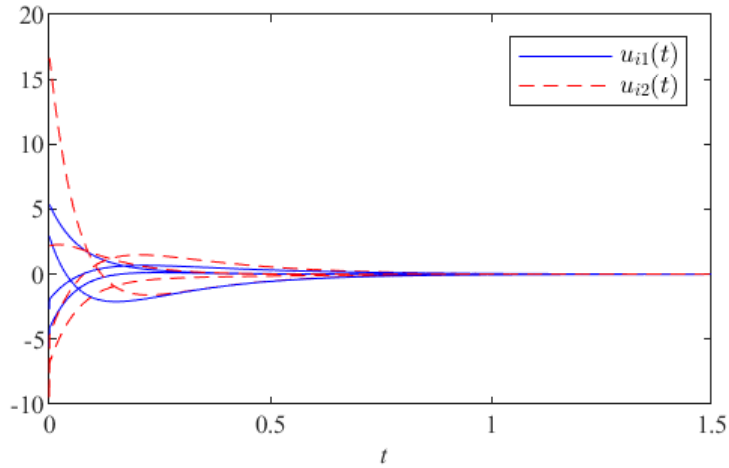

Figure 7. The boundary coupling $u_i(t)$ in Example 2.

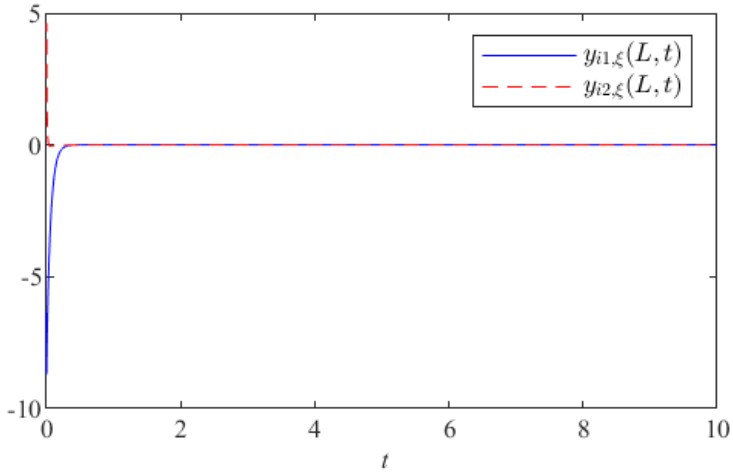

Figure 8. The boundary coupling $y_i(L,t)$ in Example 2.

5. Results

According to the above simulation examples, it is not difficult to obtain the following results. With the collocated boundary coupling and distributed boundary coupling, we designed two boundary controllers, which enable the MAS (1) and (2) with stochastic disturbance to reach agreement. As shown in Examples 1 and 2, by using LMI Matlab tools, the control gains k, d are calculated. Thus, in full consideration of the communication

between agents, all agents can achieve one common goal, so that the stochastic MASs achieve consensus. Different from other references, this paper not only solved the consensus of stochastic MASs based on PDE-ODEs, but also took the boundary control strategy. The experiment proved that the proposed methods can save costs and is very effective and of practical significance for the consensus control of MASs. The reason is that we fully considered the stochastic factors in this paper, the implementation of the proposed controllers only require few actuators located at the boundary of the spatial domain and is thus relatively easy.

6. Conclusions

In this article, we have investigated the leaderless consensus of stochastic MASs. The research model was established by nonlinear PDE-ODEs. Compared with the traditional state coupling, the most significant difference between this paper is that the space state coupling is designed. The spatial position of the system is fully considered. In addition, it is very difficult and costly to apply the traditional control strategy in reality; therefore, this paper adopts the boundary coupling method to design the boundary controller and place the controller only on the boundary of the system. This greatly reduces the cost of resources, but also achieved a good control effect. Thus, two boundary couplings have been investigated in this article, respectively, collocated measurement and distributed measurement. Using Lyapunov directed method, we obtained sufficient conditions for the stochastic multi-agent system to achieve consensus. Finally, two simulation examples are used to be shown the feasibility of the proposed spatial boundary couplings. In future work, backstepping control, and adaptive control for consensus of MASs will be studied.

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