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New Hille Type and Ohriska Type Criteria for Nonlinear Third-Order Dynamic Equations

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Abstract: The objective of this paper is to derive new Hille type and Ohriska type criteria for third-order nonlinear dynamic functional equations in the form of $\left\{a_2(\zeta)\varphi_{\alpha_2}\left(\left[a_1(\zeta)\varphi_{\alpha_1}(x^\Delta(\zeta))\right]^\Delta\right)\right\}^\Delta + q(\zeta)\varphi_\alpha(x(g(\zeta))) = 0$, on a time scale $\mathbb{T}$, where Δ is the forward operator on $\mathbb{T}$, $\alpha_1, \alpha_2, \alpha > 0$, and g, q, a_i, $i = 1,2$, are positive rd-continuous functions on $\mathbb{T}$, and $\varphi_\theta(u) := |u|^{\theta-1}u$. Our results in this paper are new and substantial for dynamic equations of the third order on arbitrary time scales. An example is included to illustrate the results.

Keywords: Hille type; Ohriska type; third order; time scales; dynamic equations

MSC: 34K11; 39A10; 39A99; 34N05

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1. Introduction

This paper deals with new Hille type and Ohriska type criteria for the oscillation of third-order functional dynamic equations in the form of

$$\left\{a_2(\zeta)\varphi_{\alpha_2}\left(\left[a_1(\zeta)\varphi_{\alpha_1}\left(x^\Delta(\zeta)\right)\right]^\Delta\right)\right\}^\Delta + q(\zeta)\varphi_\alpha(x(g(\zeta))) = 0, \tag{1}$$

on an arbitrary time scale $\mathbb{T}$ with $\sup\mathbb{T} = \infty$, where Δ is the forward operator on $\mathbb{T}$; $\varphi_\theta(u) := |u|^{\theta-1}u$, $\theta > 0$; $\alpha_1, \alpha_2, \alpha > 0$; q, a_i, $i = 1,2$, are positive rd-continuous functions on $\mathbb{T}$ such that

$$\int_{\zeta_0}^\infty \frac{\Delta t}{a_i^{1/\alpha_i}(t)} = \infty, \; i = 1,2, \tag{2}$$

and $g : \mathbb{T} \to \mathbb{T}$ is an rd-continuous nondecreasing function such that $\lim_{t\to\infty} g(t) = \infty$. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals. A forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is given by

$$\sigma(\zeta) = \inf\{s \in \mathbb{T} : s > \zeta\},$$

where $\inf\phi = \sup\mathbb{T}$, and it is said that $f : \mathbb{T} \to \mathbb{R}$ is differentiable at $\zeta \in \mathbb{T}$ provided

$$f^\Delta(\zeta) := \lim_{s\to\zeta} \frac{f(\zeta) - f(s)}{\zeta - s}$$

exists when $\sigma(\zeta) = \zeta$ and when f is continuous at ζ and $\sigma(\zeta) > \zeta$,

$$f^{\Delta}(\zeta) := \frac{f(\sigma(\zeta)) - f(\zeta)}{\sigma(\zeta) - \zeta}.$$

We say that the point $\zeta \in \mathbb{T}$ is right-dense if $\zeta < \sup \mathbb{T}$ and $\sigma(\zeta) = \zeta$ and f is an *rd*-continuous function if for any right-dense point $\zeta \in \mathbb{T}$, $f(\zeta) = \lim_{s \to \zeta^+} f(s)$; for a fantastic introduction to time-scale calculus, see [1–4].

We shall not investigate solutions which vanish in the neighbourhood of infinity. A solution x of (1) is said to be oscillatory if it is neither eventually positive nor negative; otherwise, it is said to be nonoscillatory. By a solution of Equation (1), we mean a nontrivial real-valued function $x \in \mathrm{C}^1_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}$ for some T_x in $[\zeta_0, \infty)_{\mathbb{T}}$ for a positive constant $\zeta_0 \in \mathbb{T}$, such that $x(\zeta)$ satisfies Equation (1) on $[T_x, \infty)_{\mathbb{T}}$ and $a_1(\zeta)\varphi_{\alpha_1}(x^{\Delta}(\zeta))$, $a_2(\zeta)\varphi_{\alpha_2}\Big(\big[a_1(\zeta)\varphi_{\alpha_1}(x^{\Delta}(\zeta))\big]^{\Delta}\Big) \in \mathrm{C}^1_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions.

Oscillation criteria for solutions to dynamic equations on time scales are receiving more attention as a result of their applications in engineering and the natural sciences. Hille [5] showed that the solutions of the second-order linear differential equation

$$x''(\zeta) + q(\zeta)x(\zeta) = 0, \tag{3}$$

were oscillatory if

$$\liminf_{\zeta \to \infty} \zeta \int_{\zeta}^{\infty} q(t)\mathrm{d}t > \frac{1}{4}. \tag{4}$$

Erbe [6] improved condition (4) and showed that if

$$\liminf_{\zeta \to \infty} \zeta \int_{\zeta}^{\infty} \left(\frac{g(t)}{t}\right) q(t)\mathrm{d}t > \frac{1}{4},$$

then all solutions of the delay second-order linear differential equation

$$x''(\zeta) + q(\zeta)x(g(\zeta)) = 0, \tag{5}$$

were oscillatory, where $g(\zeta) \le \zeta$. Ohriska [7] proved that, if

$$\limsup_{t \to \infty} \zeta \int_{\zeta}^{\infty} \left(\frac{g(t)}{t}\right) q(t)\mathrm{d}t > 1,$$

then all solutions of (5) were oscillatory.

The results in [8–16] generalized the Hille type criterion for different forms of second-order dynamic equations. Regarding third-order dynamic equations, the results in [17–24] established several Hille type oscillation criteria for various dynamic equations of the third order, which ensured that the solutions were either oscillatory or nonoscillatory and converged to a finite limit under various restrictive conditions. See [23] (Discussions and Conclusions Section) for a good comparison among those results. The technique used there was by reducing the third-order dynamic equations to second-order ones. The reader is directed to papers [25–42] and the references therein.

The goal of this study was to utilize a Riccati transformation technique to find new Hille type and Ohriska type criteria for the oscillation of third-order functional dynamic Equation (1) without restricting the conditions on the time scales for both delay and advanced types. As far as the authors know, this approach for investigating has not been used for the Hille type criterion for third-order dynamic Equation (1) before.

We point out that all of the inequalities presented in this paper, if not specifically mentioned, are assumed to hold eventually, that is, for all sufficiently large ζ.

In this paper, we denote that for sufficiently large $T \in [\zeta_0, \infty)_{\mathbb{T}}$,

$$P(\zeta, T) := H_2^{\alpha_2}(\zeta, T) \int_{\zeta}^{\infty} \left(\frac{H_2(\psi(t), T)}{H_2(\sigma(t), T)} \right)^{\alpha_2} q(t) \Delta t,$$

$$H_i(\zeta, T) := \varphi_{\alpha_{i-1}} \left(\int_T^{\zeta} \varphi_{\alpha_{i-1}}^{-1} \left(\frac{H_{i-1}(t, T)}{a_{i-1}(t)} \right) \Delta t \right), \; i = 1, 2, 3,$$

with

$$\psi(\zeta) := \min\{\sigma(\zeta), g(\zeta)\}, \; H_0(t, T) := \frac{1}{a_2^{1/\alpha_2}(t)}, \; a_0 = \alpha_0 = 1,$$

$$l := \liminf_{\zeta \to \infty} \frac{H_2(\zeta, \zeta_0)}{H_2(\sigma(\zeta), \zeta_0)}, \tag{6}$$

and

$$x^{[0]}(\zeta) = x \quad \text{and} \quad x^{[i]}(\zeta) := a_i(\zeta) \varphi_{\alpha_i}([x^{[i-1]}(\zeta)]^{\Delta}), \; i = 1, 2. \tag{7}$$

2. Main Results

Before stating the main results, we offer a preliminary lemma which is used in the proof of the main results.

Lemma 1. *Let $x(\zeta)$ be*

$$\left(x^{[2]}(\zeta) \right)^{\Delta} < 0 \text{ and } x^{[i]}(\zeta) > 0, \; i = 0, 1, 2, \quad \text{on } [\zeta_0, \infty)_{\mathbb{T}},$$

then, for $v \in (u, \infty)_{\mathbb{T}} \subseteq [\zeta_0, \infty)_{\mathbb{T}}$,

$$\left(\frac{x(v)}{H_2^{1/\alpha_1}(v, u)} \right)^{\Delta_v} < 0. \tag{8}$$

Proof. Suppose, without loss of generality, that

$$x(g(\zeta)) > 0 \text{ and } x^{[i]}(\zeta) > 0, \; i = 0, 1, 2, \text{ on } [\zeta_0, \infty)_{\mathbb{T}}.$$

In view of the definition of H_i and the fact that $\left(x^{[2]}(\zeta) \right)^{\Delta} < 0$, we see that

$$\begin{aligned} x^{[1]}(v) &> x^{[1]}(v) - x^{[1]}(u) = \int_u^v \varphi_{\alpha_2}^{-1}\left(x^{[2]}(\omega) \right) H_0(\omega, u) \, \Delta\omega \\ &\geq \varphi_{\alpha_2}^{-1}\left(x^{[2]}(v) \right) \int_u^v H_0(\omega, u) \, \Delta\omega \\ &= \varphi_{\alpha_2}^{-1}\left(x^{[2]}(v) \right) H_1(v, u), \end{aligned} \tag{9}$$

which implies that

$$x^{\Delta}(v) > \varphi_{\alpha}^{-1}\left(x^{[2]}(v) \right) \left(\frac{H_1(v, u)}{a_1(v)} \right)^{1/\alpha_1}. \tag{10}$$

Replacing v by s in (10) and integrating with respect to s from u to v, we have

$$\begin{aligned} x(v) &\geq \int_u^v \varphi_{\alpha}^{-1}\left(x^{[2]}(\omega) \right) \left(\frac{H_1(\omega, u)}{a_1(\omega)} \right)^{1/\alpha_1} \Delta\omega \\ &\geq \varphi_{\alpha}^{-1}\left(x^{[2]}(v) \right) \int_u^v \left(\frac{H_1(\omega, u)}{a_1(\omega)} \right)^{1/\alpha_1} \Delta\omega \\ &= \varphi_{\alpha}^{-1}\left(x^{[2]}(v) \right) H_2^{1/\alpha_1}(v, u). \end{aligned} \tag{11}$$

By virtue of (9), there exists a $v \in (u,\infty)_{\mathbb{T}}$ such that

$$\left(\frac{x^{[1]}(v)}{H_1(v,u)}\right)^{\Delta_v} < 0 \qquad \text{for } v \in (u,\infty)_{\mathbb{T}} \subseteq [\zeta_0,\infty)_{\mathbb{T}}.$$

Hence, for $v \in (u,\infty)_{\mathbb{T}}$,

$$\begin{aligned} x(v) &> \int_u^v \varphi_{\alpha_1}^{-1}\left(\frac{x^{[1]}(\omega)}{H_1(\omega,u)}\right)\left(\frac{H_1(\omega,u)}{a_1(\omega)}\right)^{\frac{1}{\alpha_1}}\Delta\omega \\ &\geq \varphi_{\alpha_1}^{-1}\left(\frac{x^{[1]}(v)}{H_1(v,u)}\right)\int_u^v \left(\frac{H_1(\omega,u)}{a_1(\omega)}\right)^{\frac{1}{\alpha_1}}\Delta\omega \\ &= \varphi_{\alpha_1}^{-1}\left(\frac{x^{[1]}(v)}{H_1(v,u)}\right) H_2^{\frac{1}{\alpha_1}}(v,u). \end{aligned} \tag{12}$$

If follows from (12) that

$$\begin{aligned} &\left(\frac{x(v)}{H_2^{1/\alpha_1}(v,u)}\right)^{\Delta_v} \\ = \;& \frac{1}{\{H_2(v,u)H_2(\sigma(v),u)\}^{1/\alpha_1}}\left\{H_2^{1/\alpha_1}(v,u)x^{\Delta}(v) - \left(\frac{H_1(v,u)}{a_1(v)}\right)^{1/\alpha_1} x(v)\right\} \\ = \;& \frac{\left(\frac{H_1(v,u)}{a_1(v)}\right)^{1/\alpha_1}}{\{H_2(v,u)H_2(\sigma(v),u)\}^{1/\alpha_1}}\left\{\varphi_{\alpha_1}^{-1}\left(\frac{x^{[1]}(v)}{H_1(v,u)}\right)H_2^{1/\alpha_1}(v,u) - x(v)\right\} < 0. \end{aligned}$$

The proof is now complete. □

The Hille type and Ohriska type criteria for Equation (1) are established as follows.

Theorem 1. *Suppose there exists an $l > 0$ such that for sufficiently large $T \in [\zeta_0,\infty)_{\mathbb{T}}$,*

$$\liminf_{\zeta\to\infty} P(\zeta,T) > \frac{\alpha^{\alpha}}{l^{\alpha\alpha_2}(1+\alpha)^{1+\alpha}}. \tag{13}$$

Then, any solution $x(\zeta)$ of Equation (1) is either oscillatory or all functions $x^{[i]}(\zeta)$, $i = 0,1,2,$ converge.

Proof. Assume, without loss of generality, that $x(\zeta)$ and $x(g(\zeta))$ are eventually positive. From (1), we deduce that $x^{[i]}(\zeta)$, $i = 1,2$ are eventually of one sign. Applying (2), we see $x^{[2]}(\zeta)$ is eventually positive, see ([43] Part $(\mathcal{I})$ of the proof of Theorem 2.1). In the following, we consider two cases:

$(\mathcal{I})$ $x^{[1]}(\zeta)$ is eventually positive. In this case, there is a $\zeta_1 \in [\zeta_0,\infty)_{\mathbb{T}}$ such that

$$\left(x^{[2]}(\zeta)\right)^{\Delta} < 0 \text{ and } x^{[i]}(\zeta) > 0,\ i = 0,1,2, \quad \text{on } [\zeta_1,\infty)_{\mathbb{T}}.$$

Consider the Riccati substitution

$$w(\zeta) := \frac{x^{[2]}(\zeta)}{x^{\alpha}(\zeta)}. \tag{14}$$

Then,

$$\begin{aligned} w^{\Delta}(\zeta) &= \frac{x^{\alpha}(\zeta)(x^{[2]}(\zeta))^{\Delta} - x^{[2]}(\zeta)(x^{\alpha}(\zeta))^{\Delta}}{x^{\alpha}(\zeta)x^{\alpha}(\sigma(\zeta))} \\ &= \frac{(x^{[2]}(\zeta))^{\Delta}}{x^{\alpha}(\sigma(\zeta))} - \frac{(x^{\alpha}(\zeta))^{\Delta}}{x^{\alpha}(\zeta)} \frac{x^{[2]}(\zeta)}{x^{\alpha}(\sigma(\zeta))}. \end{aligned}$$

It follows from (1) that

$$w^{\Delta}(\zeta) = -\left(\frac{x(g(\zeta))}{x(\sigma(\zeta))}\right)^{\alpha} q(\zeta) - \frac{(x^{\alpha}(\zeta))^{\Delta}}{x^{\alpha}(\zeta)} \frac{x^{[2]}(\zeta)}{x^{\alpha}(\sigma(\zeta))}.$$

We first consider the case when $g(\zeta) \leq \sigma(\zeta)$ on $[\zeta_1, \infty)_{\mathbb{T}}$. Using the fact that $\left(\frac{x(\zeta)}{H_2^{1/\alpha_1}(\zeta, \zeta_1)}\right)^{\Delta} < 0$, we get

$$\frac{x(h(\zeta))}{H_2^{1/\alpha_1}(g(\zeta), \zeta_1)} \geq \frac{x(\sigma(\zeta))}{H_2^{1/\alpha_1}(\sigma(\zeta), \zeta_1)} \quad \text{for } \zeta \in [\zeta_2, \infty)_{\mathbb{T}} \subseteq (\zeta_1, \infty)_{\mathbb{T}}. \tag{15}$$

Next, we consider the case when $h(\zeta) \geq \sigma(\zeta)$ on $[\zeta_2, \infty)_{\mathbb{T}}$. Since $x^{\Delta}(\zeta) > 0$, we obtain

$$x(g(\zeta)) \geq x(\sigma(\zeta)) \quad \text{for } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}. \tag{16}$$

It follows from (15) and (16) that

$$\frac{x(g(\zeta))}{x(\sigma(\zeta))} \geq \left(\frac{H_2(\psi(\zeta), \zeta_1)}{H_2(\sigma(\zeta), \zeta_1)}\right)^{1/\alpha_1} \quad \text{for } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}.$$

Hence, we deduce that for $\zeta \in [\zeta_2, \infty)_{\mathbb{T}}$,

$$w^{\Delta}(\zeta) \leq -\left(\frac{H_2(\psi(\zeta), \zeta_1)}{H_2(\sigma(\zeta), \zeta_1)}\right)^{\alpha_2} q(\zeta) - \frac{(x^{\alpha}(\zeta))^{\Delta} x^{[2]}(\zeta)}{x^{\alpha}(\zeta) x^{\alpha}(\sigma(\zeta))}.$$

By the definition of $w(\zeta)$ and $\left(x^{[2]}(\zeta)\right)^{\Delta} < 0$, we see that for $\zeta \geq \zeta_2$,

$$\begin{aligned} w^{\Delta}(\zeta) &\leq -\left(\frac{H_2(\psi(\zeta), \zeta_1)}{H_2(\sigma(\zeta), \zeta_1)}\right)^{\alpha_2} q(\zeta) - \frac{(x^{\alpha}(\zeta))^{\Delta}}{x^{\alpha}(\zeta)} \frac{x^{[2]}(\zeta)}{x^{\alpha}(\sigma(\zeta))} \\ &\leq -\left(\frac{H_2(\psi(\zeta), \zeta_1)}{H_2(\sigma(\zeta), \zeta_1)}\right)^{\alpha_2} q(\zeta) - \frac{(x^{\alpha}(\zeta))^{\Delta}}{x^{\alpha}(\zeta)} \frac{x^{[2]}(\sigma(\zeta))}{x^{\alpha}(\sigma(\zeta))} \\ &= -\left(\frac{H_2(\psi(\zeta), \zeta_1)}{H_2(\sigma(\zeta), \zeta_1)}\right)^{\alpha_2} q(\zeta) - \frac{(x^{\alpha}(\zeta))^{\Delta}}{x^{\alpha}(\zeta)} w(\sigma(\zeta)). \end{aligned} \tag{17}$$

Using the Pötzsche chain rule ([2] Theorem 1.90) and the fact that $x^{[1]}(\zeta) > 0$, we conclude that

$$\begin{aligned} \frac{(x^{\alpha}(\zeta))^{\Delta}}{x^{\alpha}(\zeta)} &\geq \begin{cases} \alpha \left(\dfrac{x(\sigma(\zeta))}{x(\zeta)}\right)^{\alpha} \dfrac{x^{\Delta}(\zeta)}{x(\sigma(\zeta))}, & 0 < \alpha \leq 1 \\[2ex] \alpha \dfrac{x(\sigma(\zeta))}{x(\zeta)} \dfrac{x^{\Delta}(\zeta)}{x(\sigma(\zeta))}, & \alpha \geq 1 \end{cases} \\ &\geq \alpha \frac{x^{\Delta}(\zeta)}{x(\sigma(\zeta))} \quad \text{for } \alpha > 0. \end{aligned} \tag{18}$$

Using (10) and setting $u = \zeta_1$ and $v = \zeta$, and by the fact that $\left(x^{[2]}(\zeta)\right)^{\Delta} < 0$, we obtain

$$
\begin{aligned}
x^{\Delta}(\zeta) &> \left(\frac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1}\left(x^{[2]}(\zeta)\right)^{1/\alpha} \\
&= \left(\frac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1}\left(x^{[2]}(\zeta)\right)^{1/\alpha} \qquad (19)\\
&\geq \left(\frac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1}\left(x^{[2]}(\sigma(\zeta))\right)^{1/\alpha}. \qquad (20)
\end{aligned}
$$

Substituting (20) into (18), we get

$$
\begin{aligned}
\frac{(x^{\alpha}(\zeta))^{\Delta}}{x^{\alpha}(\zeta)} &> \alpha\left(\frac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1}\frac{\left(x^{[2]}(\sigma(\zeta))\right)^{1/\alpha}}{x(\sigma(\zeta))} \\
&= \alpha\left(\frac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1} w^{1/\alpha}(\sigma(\zeta)). \qquad (21)
\end{aligned}
$$

Therefore, (17) becomes

$$
\begin{aligned}
w^{\Delta}(\zeta) &\leq -\left(\frac{H_2(\psi(\zeta),\zeta_1)}{H_2(\sigma(\zeta),\zeta_1)}\right)^{\alpha_2} q(\zeta) \\
&\quad -\alpha\left(\frac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1} w^{1+1/\alpha}(\sigma(\zeta)) \qquad (22)\\
&\leq -\left(\frac{H_2(\psi(\zeta),\zeta_1)}{H_2(\sigma(\zeta),\zeta_1)}\right)^{\alpha_2} q(\zeta) \\
&\quad -\alpha\frac{(H-\varepsilon)^{1+1/\alpha}}{H_2^{\alpha_2+1/\alpha_1}(\zeta,\zeta_1)}\left(\frac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1}. \qquad (23)
\end{aligned}
$$

Hence, for any $\varepsilon > 0$, there exists a $\zeta_3 \in [\zeta_2,\infty)_{\mathbb{T}}$ such that for $\zeta \in [\zeta_3,\infty)_{\mathbb{T}}$,

$$
H_2^{\alpha_2}(\zeta,\zeta_1)w(\sigma(\zeta)) \geq H-\varepsilon \quad \text{and} \quad \frac{H_2(\zeta,\zeta_1)}{H_2(\sigma(\zeta),\zeta_1)} \geq l-\varepsilon, \qquad (24)
$$

with

$$
H := \liminf_{\zeta\to\infty} H_2^{\alpha_2}(\zeta,\zeta_1)w(\sigma(\zeta)), \quad 0 \leq H \leq 1. \qquad (25)
$$

Note that

$$
\begin{aligned}
\left(\frac{-1}{H_2^{\alpha_2}(\zeta,\zeta_1)}\right)^{\Delta} &= \left(\frac{-1}{\left(H_2^{1/\alpha_1}(\zeta,\zeta_1)\right)^{\alpha}}\right)^{\Delta} \\
&\leq \begin{cases} \alpha\left(\dfrac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1} \dfrac{1}{H_2^{1/\alpha_1}(\zeta,\zeta_1)H_2^{\alpha_2}(\sigma(\zeta),\zeta_1)}, & 0<\alpha\leq 1 \\ \alpha\left(\dfrac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1} \dfrac{1}{H_2^{\alpha_2}(\zeta,\zeta_1)H_2^{1/\alpha_1}(\sigma(\zeta),\zeta_1)}, & \alpha\geq 1 \end{cases} \qquad (26) \\
&= \begin{cases} \alpha\left(\dfrac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1} \dfrac{\left(\dfrac{H_2(\zeta,\zeta_1)}{H_2(\sigma(\zeta),\zeta_1)}\right)^{\alpha_2}}{H_2^{\alpha_2+1/\alpha_1}(\zeta,\zeta_1)}, & 0<\alpha\leq 1 \\ \alpha\left(\dfrac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1} \dfrac{\left(\dfrac{H_2(\zeta,\zeta_1)}{H_2(\sigma(\zeta),\zeta_1)}\right)^{1/\alpha_1}}{H_2^{\alpha_2+1/\alpha_1}(\zeta,\zeta_1)}, & \alpha\geq 1 \end{cases} \\
&\leq \alpha\left(\frac{H_1(\zeta,\zeta_1)}{a_1(\zeta)}\right)^{1/\alpha_1} \frac{1}{H_2^{\alpha_2+1/\alpha_1}(\zeta,\zeta_1)}, \quad \alpha>0.
\end{aligned}
$$

From (23), we see that

$$
w^{\Delta}(\zeta) \leq -\left(\frac{H_2(\psi(\zeta),\zeta_1)}{H_2(\sigma(\zeta),\zeta_1)}\right)^{\alpha_2} q(\zeta) - (H-\varepsilon)^{1+1/\alpha}\left(\frac{-1}{H_2^{\alpha_2}(\zeta,\zeta_1)}\right)^{\Delta}. \qquad (27)
$$

Integrating (27) from $\sigma(\zeta)$ to v, we have

$$
\begin{aligned}
&\int_{\sigma(\zeta)}^{v} \left(\frac{H_2(\psi(t),\zeta_1)}{H_2(\sigma(t),\zeta_1)}\right)^{\alpha_2} q(t)\Delta t \\
\leq\;& w(\sigma(\zeta)) - w(v) \\
&-(H-\varepsilon)^{1+1/\alpha}\left(\frac{1}{H_2^{\alpha_2}(\sigma(\zeta),\zeta_1)} - \frac{1}{H_2^{\alpha_2}(v,\zeta_1)}\right). \qquad (28)
\end{aligned}
$$

Taking into consideration that $w>0$ and passing to the limit of (28) as $v\to\infty$, we obtain

$$
\begin{aligned}
P(\zeta,\zeta_1) &= \int_{\sigma(\zeta)}^{\infty} \left(\frac{H_2(\psi(t),\zeta_1)}{H_2(\sigma(t),\zeta_1)}\right)^{\alpha_2} q(t)\Delta t \\
&\leq w(\sigma(\zeta)) - (H-\varepsilon)^{1+1/\alpha}\left(\frac{1}{H_2^{\alpha_2}(\sigma(\zeta),\zeta_1)}\right). \qquad (29)
\end{aligned}
$$

Multiplying both sides of (29) by $H_2^{\alpha_2}(\zeta,\zeta_1)$, we deduce that for $\zeta\in[\zeta_3,\infty)_{\mathbb{T}}$,

$$
\begin{aligned}
P(\zeta,\zeta_1) &\leq w(\sigma(\zeta)) - (H-\varepsilon)^{1+1/\alpha}\left(\frac{H_2(\zeta,\zeta_1)}{H_2(\sigma(\zeta),\zeta_1)}\right)^{\alpha_2} \\
&\leq H_2^{\alpha_2}(\zeta,\zeta_1)w(\sigma(\zeta)) - (H-\varepsilon)^{1+1/\alpha}(l-\varepsilon)^{\alpha_2}. \qquad (30)
\end{aligned}
$$

We obtain by taking the liminf on both sides of (30) as $\zeta\to\infty$

$$
\liminf_{\zeta\to\infty} P(\zeta,\zeta_1) \leq H - (H-\varepsilon)^{1+1/\alpha}(l-\varepsilon)^{\alpha_2}.
$$

Since $\varepsilon > 0$ is arbitrary, we deduce that

$$\liminf_{\zeta\to\infty} P(\zeta,\zeta_1) \le H - H^{1+1/\alpha}\, l^{\alpha_2}. \tag{31}$$

Set

$$X := H\, l^{\alpha\alpha_2/(1+\alpha)} \qquad \text{and} \qquad x := \left(\frac{\alpha}{1+\alpha}\right)^{\alpha} l^{-\alpha^2\alpha_2/(1+\alpha)},$$

and $\lambda := 1 + 1/\alpha$. By using the inequality (see [44])

$$\lambda X x^{\lambda-1} - X^{\lambda} \le (\lambda-1)x^{\lambda}, \tag{32}$$

we deduce that

$$H - H^{1+1/\alpha}\, l^{\alpha_2} \le \frac{\alpha^{\alpha}}{l^{\alpha\alpha_2}(1+\alpha)^{1+\alpha}}.$$

Thus, (31) becomes

$$\liminf_{\zeta\to\infty} P(\zeta,\zeta_1) \le \frac{\alpha^{\alpha}}{l^{\alpha\alpha_2}(1+\alpha)^{1+\alpha}}.$$

As a result, we reach a contradiction to (13).

$(\mathcal{II})$ $x^{[1]}(\zeta)$ is eventually negative. In this case, there is a $\zeta_1 \in [\zeta_0,\infty)_{\mathbb{T}}$ such that

$$\left(x^{[2]}(\zeta)\right)^{\Delta} < 0 \text{ and } (-1)^i x^{[i]}(\zeta) > 0,\ i = 0,1,2, \quad \text{on } [\zeta_1,\infty)_{\mathbb{T}}. \tag{33}$$

By dint of (33), it is easy to show that $x^{[i]}(\zeta)$, $i = 0,1,2$, converge. This completes the proof. □

Theorem 2. *Suppose that for a sufficiently large $T \in [\zeta_0,\infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty} P(\zeta,T) > 1, \tag{34}$$

If $x(\zeta)$ is a solution of Equation (1), then $x(\zeta)$ is either oscillatory or all functions $x^{[i]}(\zeta)$, $i = 0,1,2$, converge.

Proof. Assume, without loss of generality, that $x(\zeta)$ and $x(h(\zeta))$ are eventually positive. From (1), we deduce that $x^{[i]}(\zeta)$, $i = 1,2$ are eventually of one sign. Applying (2), we see $x^{[2]}(\zeta)$ is eventually positive, see ([43] Part $(\mathcal{I})$ of the proof of Theorem 2.1). In the following we consider two cases:

$(\mathcal{I})$ $x^{[1]}(\zeta)$ is eventually positive. In this case, there is a $\zeta_1 \in [\zeta_0,\infty)_{\mathbb{T}}$ such that

$$\left(x^{[2]}(\zeta)\right)^{\Delta} < 0 \text{ and } x^{[i]}(\zeta) > 0,\ i = 0,1,2, \quad \text{on } [\zeta_1,\infty)_{\mathbb{T}}. \tag{35}$$

In view of (8), (11), and (35) it follows that

$$\begin{aligned}
\varphi_\alpha(x(g(t))) &\ge \left(\frac{H_2(\psi(t),\zeta_1)}{H_2(\sigma(t),\zeta_1)}\right)^{\alpha_2} \varphi_\alpha(x(\sigma(t))) \\
&\ge \left(\frac{H_2(\psi(t),\zeta_1)}{H_2(\sigma(t),\zeta_1)}\right)^{\alpha_2} \varphi_\alpha(x(\zeta)) \\
&\ge H_2^{\alpha_2}(\zeta,\zeta_1)\left(\frac{H_2(\psi(t),\zeta_1)}{H_2(\sigma(t),\zeta_1)}\right)^{\alpha_2} x^{[2]}(\zeta),
\end{aligned} \tag{36}$$

for $t \in [\zeta,\infty)_{\mathbb{T}}$ and $\zeta \in [\zeta_1,\infty)_{\mathbb{T}}$. Integrating (1) from $\sigma(\zeta)$ to u, we get

$$\int_{\sigma(\zeta)}^{u} q(t)\varphi_\alpha(x(g(t)))\Delta t = x^{[2]}(\sigma(\zeta)) - x^{[2]}(u) \le x^{[2]}(\sigma(\zeta)) \le x^{[2]}(\zeta). \tag{37}$$

Substituting (36) into Inequality (37), we deduce that

$$H_2^{\alpha_2}(\zeta,\zeta_1)\int_{\sigma(\zeta)}^{u}\left(\frac{H_2(\psi(t),\zeta_1)}{H_2(\sigma(t),\zeta_1)}\right)^{\alpha_2}q(t)\Delta t\leq 1.$$

Letting $u\to\infty$, we have

$$P(\zeta,\zeta_1)=H_2^{\alpha_2}(\zeta,\zeta_1)\int_{\sigma(\zeta)}^{\infty}\left(\frac{H_2(\psi(t),\zeta_1)}{H_2(\sigma(t),\zeta_1)}\right)^{\alpha_2}q(t)\Delta t\leq 1,$$

which implies

$$\limsup_{t\to\infty}P(\zeta,\zeta_1)\leq 1,$$

which contradicts (34).

$(\mathcal{II})$ $x^{[1]}(\zeta)$ is eventually negative. The proof in Part $(\mathcal{II})$ is the same as the proof of Theorem 1, hence it is omitted. □

Remark 1.

(1) *The deduction of Theorems 1 and 2 keeps intact if assumptions* (13) *and* (34) *are replaced by*

$$\int^{\infty}\left(\frac{H_2(\psi(t),T)}{H_2(\sigma(t),T)}\right)^{\alpha_2}q(t)\Delta t=\infty. \tag{38}$$

From (22), *we get*

$$w^{\Delta}(\zeta)\leq-\left(\frac{H_2(\psi(\zeta),\zeta_1)}{H_2(\sigma(\zeta),\zeta_1)}\right)^{\alpha_2}q(\zeta). \tag{39}$$

Integrating (39) *from u to v, we obtain*

$$w(v)-w(u)\leq-\int_u^v\left(\frac{H_2(\psi(t),\zeta_1)}{H_2(\sigma(t),\zeta_1)}\right)^{\alpha_2}q(t)\Delta t.$$

Taking into account that $w>0$, we have

$$w(u)\geq\int_u^v\left(\frac{H_2(\psi(t),\zeta_1)}{H_2(\sigma(t),\zeta_1)}\right)^{\alpha_2}q(t)\Delta t,$$

which contradicts (38).

(2) *If either*

$$\int^{\infty}q(t)\,\Delta t=\infty;$$

$$\int^{\infty}\left(\frac{1}{a_2(t)}\int_t^{\infty}q(\omega)\,\Delta\omega\right)^{1/\alpha_2}\Delta t=\infty;$$

or

$$\int^{\infty}\left[\frac{1}{a_1(t)}\int_t^{\infty}\left(\frac{1}{a_2(\omega)}\int_{\omega}^{\infty}q(s)\,\Delta s\right)^{1/\alpha_2}\Delta\omega\right]^{1/\alpha_1}\Delta t=\infty, \tag{40}$$

then nonoscillatory solutions of the investigated Equation (1) *are convergent to zero, see ([43] Theorem 2.1).*

3. Illustrative Example

An illustrative example is presented to show the significance of the obtained results.

Example 1. *Consider the dynamic equation of the third order*

$$\left\{\left(\frac{1}{2\zeta}\right)^{2/3}\varphi_{\frac{2}{3}}\left(\left[\left(\frac{1}{3\zeta}\right)^{2}\varphi_{2}\left(x^{\Delta}(\zeta)\right)\right]^{\Delta}\right)\right\}^{\Delta}+\frac{\beta}{\zeta^{5}}\varphi_{\frac{4}{3}}(x(\gamma\zeta))=0,\quad \zeta\in[\zeta_0,\infty), \tag{41}$$

where β and γ are positive constants. Condition (2) is obviously satisfied. Now,

$$\int_{\zeta_0}^{\infty}\left[\frac{1}{a_1(t)}\int_t^{\infty}\left(\frac{1}{a_2(\omega)}\int_{\omega}^{\infty}q(s)\,\Delta s\right)^{1/\alpha_2}\Delta\omega\right]^{1/\alpha_1}\Delta\, t$$
$$=\frac{3\beta^{\frac{3}{4}}}{2}\int_{\zeta_0}^{\infty}t\left[\int_t^{\infty}\frac{\mathrm{d}!}{\omega^{5}}\right]^{1/2}\mathrm{dt}=\frac{3\beta^{\frac{3}{4}}}{4}\int_{\zeta_0}^{\infty}\frac{\mathrm{dt}}{t}=\infty.$$

Therefore, (41) is satisfied. There are two different types of Equation (41):

(i) *Delay type, i.e., $0<\gamma\le 1$. Hence,*

$$\begin{aligned}\liminf_{\zeta\to\infty}P(\zeta,T) &= \liminf_{\zeta\to\infty}H_2^{\alpha_2}(\zeta,T)\int_{\zeta}^{\infty}\left(\frac{H_2(\psi(t),T)}{H_2(\sigma(t),T)}\right)^{\alpha_2}q(t)\Delta\, t\\ &= \beta\,\liminf_{\zeta\to\infty}\left(\zeta^2-T^2\right)^2\int_{\zeta}^{\infty}\left(\gamma^2-\frac{T^2}{t^2}\right)^2\frac{1}{t^5}\,\mathrm{dt}=\frac{\beta\gamma^4}{4}.\end{aligned}$$

According to Theorem 1 and Remark 1, Part (2), then every solution of Equation (41) is either oscillatory or convergent to zero if $0<\gamma\le 1$ and $\beta>\frac{3}{\gamma^4}\left(\frac{4}{7}\right)^{7/3}$.

(ii) *Advanced type, i.e., $\gamma\ge 1$. Hence,*

$$\begin{aligned}\liminf_{\zeta\to\infty}P(\zeta,T) &= \liminf_{\zeta\to\infty}H_2^{\alpha_2}(\zeta,T)\int_{\zeta}^{\infty}\left(\frac{H_2(\psi(t),T)}{H_2(\sigma(t),T)}\right)^{\alpha_2}q(t)\Delta\, t\\ &= \beta\,\liminf_{\zeta\to\infty}\left(\zeta^2-T^2\right)^2\int_{\zeta}^{\infty}\frac{1}{t^5}\,\mathrm{dt}=\frac{\beta}{2}.\end{aligned}$$

Furthermore, according to Theorem 1 and Remark 1, Part (2), then every solution of Equation (41) is either oscillatory or convergent to zero if $\gamma\ge 1$ and $\beta>\frac{3}{2}\left(\frac{4}{7}\right)^{7/3}$.

4. Conclusions

(1) In this paper, new Hille type and Ohriska type criteria were established for (1) which can be applied to various types of time scales, e.g., $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$, $\mathbb{T}=h\mathbb{Z}$ with $h>0$, $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$, etc., (see [2]).

(2) This paper did not require additional relations between $g(\zeta)$ and $\sigma(\zeta)$. Therefore, the results apply to both the delay and advanced cases.

(3) It is interesting that the sharp oscillation criterion given in [6] for the third-order Euler differential equation $x'''(\zeta)+\frac{\beta}{\zeta^3}x(\zeta)=0$ with $\beta>\frac{2}{3\sqrt{3}}$ can be extended to third-order dynamic equations.

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