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Z-Symmetric Manifolds Admitting Schouten Tensor

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Abstract: The paper deals with the study of Z-symmetric manifolds $(ZS)_n$ admitting certain cases of Schouten tensor (specifically: Ricci-recurrent, cyclic parallel, Codazzi type and covariantly constant), and investigate some geometric and physical properties of the manifold. Moreover, we also study $(ZS)_4$ spacetimes admitting Codazzi type Schouten tensor. Finally, we construct an example of $(ZS)_4$ to verify our result.

Keywords: schouten tensor; Z-symmetric tensor; codazzi type tensor; Z symmetric spacetimes

MSC: 53C21; 53Z05

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1. Introduction

Let the manifold (M^n, g) (dim $M = n \geq 3$) be connected and semi-Riemannian, and the endowed metric g is of signature $(s, n-s)$, $0 \leq s \leq n$. If $s = n$ or 0 (resp., $s = n-1$ or 1), then (M^n, g) is a Riemannian (resp., Lorentzian) manifold. A Riemannian manifold is called locally symmetric [1] if $\nabla K = 0$, where K and ∇ appear for the Riemannian curvature tensor and the Levi-Civita connection, resectively. The class of Riemannian symmetric manifold is very natural generalization of the class of manifold of constant curvature. The notion of locally symmetric manifolds have been studied by many authors in several ways to a different extent such as conformally symmetric manifolds by Chaki and Gupta [2], recurrent manifolds by Walker [3], conformally recurrent manifolds by Adati and Miyazawa [4], pseudo symmetric manifolds by Chaki [5], weakly symmetric manifolds by Tamassy and Binh [6] etc.

The general relativity is the geometric theory of gravitation developed by Einstein. In this theory, the Einstein equations relate the geometry of spacetime to the distribution of matter within it [7]. As a consequence of the Einstein equations, the divergence of energy-momentum tensor $\mathcal{T}$ vanishes [8]. In 1966, the authors Chaki and Ray [9] proved that for a covariantly constant energy-momentum tensor, the general relativistic spacetime is Ricci-symmetric, that is, $\nabla \mathrm{Ric} = 0$, where Ric denotes the Ricci tensor of the spacetime.

Approximately a decade ago, the notion of weakly Z-symmetric manifolds was introduced by Mantica and Molinari [10], this notion generalizes weakly Ricci-symmetric manifolds. During the last decade, Z-symmetric manifolds have been studied by various authors, for example, weakly cyclic Z-symmetric manifolds by De, Mantica and Suh [11], pseudo Z-symmetric Riemannian manifolds by Mantica and Suh [12], almost pseudo Z-symmetric manifolds by De and Pal [13], concircularly flat Z-symmetric manifold and Z-symmetric manifold with the projective curvature tensor by Zengin and Yavuz Tasci [14,15].

The symmetric endomorphism R corresponding to the Ricci tensor Ric of type $(0,2)$ is defined through the relation

$$\mathrm{Ric}(Y,V) = g(R(Y),V), \tag{1}$$

for all vector fields Y and V.

In an (M^n, g), $(n > 2)$, a symmetric tensor of type $(0,2)$ is a generalized Z tensor if [12,16]

$$Z(Y,V) = \mathrm{Ric}(Y,V) + \phi g(Y,V), \tag{2}$$

where ϕ is an arbitrary scalar function.

From (2), we have

$$Z(Y,V) = Z(V,Y),$$

and

$$Z(Y,Q) = \mathrm{Ric}(Y,Q) + \phi g(Y,Q).$$

Here the vector field Q is called the basic vector field of the manifold corresponding to the 1-form ϕ.

Contraction of (2) over Y and V gives the scalar $\tilde{Z}$ as follows:

$$\tilde{Z} = r + n\phi, \tag{3}$$

where r is the scalar curvature. With the choice of $\phi = -\frac{1}{n}r$, we obtain the classical Z tensor. Here, the generalized Z tensor is referred as a Z tensor.

A Riemannian or a semi-Riemannian manifold (M^n, g), $(n > 2)$ is said to be weakly Z-symmetric [10] and denoted by $(WZS)_n$, if the generalized Z tensor satisfies the condition

$$(\nabla_U Z)(Y,V) = A(U)Z(Y,V) + B(Y)Z(U,V) + D(V)Z(Y,U), \tag{4}$$

where A, B, D are 1-forms which are non-zero simultaneously. If $A = B = D = 0$, then the manifold reduces to a Z symmetric ($\nabla Z = 0$) manifold.

On an n-dimensional Riemannian (semi-Riemannian) manifold $(M^n, g), n \geq 3$, the Schouten tensor is defined by [17]

$$P(Y,V) = \frac{1}{n-2}\Big[\mathrm{Ric}(Y,V) - \frac{r}{2(n-1)}g(Y,V)\Big]. \tag{5}$$

By combining (2) and (5), we have

$$P(Y,V) = \frac{1}{n-2}\Big[Z(Y,V) - \Big\{\frac{r}{2(n-1)} + \phi\Big\}g(Y,V)\Big], \tag{6}$$

where r is the scalar curvature and ϕ is a non-zero 1-form such that $g(Y,Q) = \phi(Y)$ for every vector field Y.

The Riemannian curvature tensor K decomposes as [18]

$$K = P \odot g + \mathcal{C},$$

where $\mathcal{C}$, $\odot$ and P represent the Weyl tensor of g, the Kulkarni-Nomizu product and the Schouten tensor, respectively. Since $\mathcal{C}$ is conformally invariant, therefore, to study the deformation of the conformal metric, we need a good understanding of the Schouten tensor [17,19].

The scalar $\bar{P}$ is obtained by contracting (5) over Y and V as follows:

$$\bar{P} = \frac{r}{2(n-1)}. \tag{7}$$

An (M^n, g) is said to have Codazzi type Ricci tenor if its $Ric(\neq 0)$ of type $(0,2)$ satisfies [20,21]

$$(\nabla_U \mathrm{Ric})(Y,V) = (\nabla_Y \mathrm{Ric})(U,V). \tag{8}$$

An (M^n, g) is said to have cyclic parallel Ricci tensor if its $Ric(\neq 0)$ of type $(0,2)$ satisfies [20,22]

$$(\nabla_U \mathrm{Ric})(Y,V) + (\nabla_Y \mathrm{Ric})(U,V) + (\nabla_V \mathrm{Ric})(Y,U) = 0. \tag{9}$$

An (M^n, g) is said to be Ricci-recurrent if its $Ric(\neq 0)$ of type $(0,2)$ satisfies the following relation [23]

$$(\nabla_U \mathrm{Ric})(Y,V) = \lambda(U)\mathrm{Ric}(Y,V), \tag{10}$$

where λ is non-zero 1-form.

An (M^n, g) is said to be generalized Ricci-recurrent if the following relation holds [24,25]

$$(\nabla_U \mathrm{Ric})(Y,V) = \lambda(U)\mathrm{Ric}(Y,V) + \beta(U)g(Y,V), \tag{11}$$

where λ and β are two non-zero 1-forms of the manifold. If $\beta = 0$, then the generalized Ricci-recurrent manifold reduces to a Ricci-recurrent manifold.

The paper is presented as follows: After introduction in Section 2, we investigate the $(ZS)_n$ admitting certain cases of Schouten tensor and find out some interesting results on corresponding Z tensor. In Section 3, we study the $(ZS)_4$ spacetime admitting Schouten tensor and proved some remarkable results. In Section 4, we give an example to illustrate our result.

2. $(ZS)_n$ Admitting Schouten Tensor

In the current section, by using the concepts and definitions given in previous section, we will prove some results on $(ZS)_n$ admitting Schouten tensor satisfying certain curvature conditions. First we prove the following result:

Theorem 1. *If the Schouten tensor in a $(ZS)_n$ is Ricci-recurrent, then the corresponding Z tensor is generalized Ricci-recurrent.*

Proof. By taking the covariant derivative of (6) along U, we find

$$(\nabla_U P)(Y,V) = \frac{1}{n-2}\Big[(\nabla_U Z)(Y,V) - \Big\{\frac{(\nabla_U r)}{2(n-1)} + (\nabla_U \phi)\Big\}g(Y,V)\Big]. \tag{12}$$

Let the Schouten tensor be Ricci-recurrent, then by virtue of (10) we have

$$(\nabla_U P)(Y,V) = \lambda(U)P(Y,V), \tag{13}$$

which in view of (6) and (12) takes the form

$$\begin{aligned}(\nabla_U Z)(Y,V) - &\Big\{\frac{(\nabla_U r)}{2(n-1)} + (\nabla_U \phi)\Big\}g(Y,V) \\ &= \lambda(U)\Big[Z(Y,V) - \Big\{\frac{r}{2(n-1)} + \phi\Big\}g(Y,V)\Big].\end{aligned} \tag{14}$$

By using (7), (14) can be written as

$$\begin{aligned}&(\nabla_U Z)(Y,V) \\ &= \lambda(U)Z(Y,V) + \Big[(\nabla_U \bar{P}) - \lambda(U)\bar{P} + (\nabla_U \phi) - \lambda(U)\phi\Big]g(Y,V).\end{aligned} \tag{15}$$

If we take $\Big[(\nabla_U \bar{P}) - \lambda(U)\bar{P} + (\nabla_U \phi) - \lambda(U)\phi\Big] = \beta(U)$, then (15) transforms to

$$(\nabla_U Z)(Y,V) = \lambda(U)Z(Y,V) + \beta(U)g(Y,V),$$

which shows that the Z-tensor is generalized Ricci-recurrent. □

Next we prove the following result:

Theorem 2. *If the Schouten tensor in a $(ZS)_n$ is cyclic parallel, then the scalar curvature of $(ZS)_n$ is constant.*

Proof. If the Schouten tensor in a $(ZS)_n$ is cyclic parallel. Then by virtue of (9), we have

$$(\nabla_U P)(Y,V) + (\nabla_Y P)(U,V) + (\nabla_V P)(U,Y) = 0. \tag{16}$$

The covariant differentiation of (5) over U leads to

$$(\nabla_U P)(Y,V) = \frac{1}{n-2}\Big[(\nabla_U \mathrm{Ric})(Y,V) - \frac{(\nabla_U r)}{2(n-1)} g(Y,V)\Big]. \tag{17}$$

In view of (17), (16) takes the form

$$\begin{aligned} &(\nabla_U \mathrm{Ric})(Y,V) - \frac{(\nabla_U r)}{2(n-1)} g(Y,V) + (\nabla_Y \mathrm{Ric})(U,V) \\ &- \frac{(\nabla_Y r)}{2(n-1)} g(U,V) + (\nabla_V \mathrm{Ric})(U,Y) - \frac{(\nabla_V r)}{2(n-1)} g(U,Y) = 0. \end{aligned} \tag{18}$$

By contracting (18) over U and Y, we get

$$\nabla_V r = 0 \implies r = constant. \tag{19}$$

This implies that the scalar curvature r is constant. This completes the proof □

Now we prove the following:

Theorem 3. *If the Schouten tensor in a $(ZS)_n$ of constant scalar curvature is of Codazzi type, then the necessary and sufficient condition for the corresponding Z tensor to be Codazzi type is that the associated 1-form of the manifold is constant.*

Proof. We consider that the Schouten tensor in a $(ZS)_n$ is of Codazzi type. By interchanging V and U in (12) we have

$$(\nabla_V P)(Y,U) = \frac{1}{n-2}\Big[(\nabla_V Z)(Y,U) - \Big\{\frac{(\nabla_V r)}{2(n-1)} + (\nabla_V \phi)\Big\} g(Y,U)\Big]. \tag{20}$$

By using (12) and (20) in (8), we find

$$\begin{aligned} &(\nabla_U Z)(Y,V) - (\nabla_V Z)(Y,U) \\ &\quad - (\nabla_U \phi) g(Y,V) + (\nabla_V \phi) g(Y,U) = 0, \end{aligned} \tag{21}$$

r being constant.

Again, we assume that the Z tensor is of Codazzi type, then (21) reduces to

$$(\nabla_U \phi) g(Y,V) - (\nabla_V \phi) g(Y,U) = 0.$$

By contracting the foregoing equation over Y and V we infer

$$(\nabla_U \phi) = 0. \tag{22}$$

This implies that the 1-form ϕ is constant.

Conversely, if we assume that the 1-form ϕ is constant then from (21) it follows that the Z tensor is of Codazzi type. This completes the proof. □

Thus we have

Corollary 1. *In a $(ZS)_n$ of constant scalar curvature, if both the Z-symmetric and Schouten tensors are of Codazzi type, then the Z-symmetric tensor is of constant trace.*

Proof. By taking the covariant derivative of (3) along U, we find

$$(\nabla_U \tilde{Z}) = (\nabla_U r) + n(\nabla_U \phi). \tag{23}$$

Now we suppose that both the Z-symmetric and the Schouten tensors in a $(ZS)_n$ of constant scalar curvtaure tensor are of Codazzi type. Then in view of Theorem 3, (23) reduces to

$$(\nabla_U \tilde{Z}) = 0.$$

This implies that $\tilde{Z} = constant$. This completes the proof. □

Further, we prove the following:

Theorem 4. *If the Schouten tensor in a $(ZS)_n$ is covariantly constant, then the necessary and sufficient condition for the Z-symmetric tensor to be (i) covariantly constant, or (ii) Codazzi type is that ϕ is constant.*

Proof. We consider that the Schouten tensor in a $(ZS)_n$ is covariantly constant. Then from (5) it can be easily seen that r is constant. Thus (20) leads to

$$(\nabla_U Z)(Y, V) = (\nabla_U \phi) g(Y, V). \tag{24}$$

Since Z-symmetric tensor in $(ZS)_n$ is covariantly constant, then (24) reduces to

$$(\nabla_U \phi) = 0. \tag{25}$$

This implies that the 1-form ϕ is constant.

Conversely, if the 1-form ϕ of the manifold is constant then from (24), it follows that

$$(\nabla_U Z)(Y, V) = 0.$$

This implies that the Z tensor is covariantly constant.

By interchanging U and V in (24), we have

$$(\nabla_V Z)(Y, U) = (\nabla_V \phi) g(Y, U). \tag{26}$$

Now subtracting (26) from (25), we have

$$(\nabla_U Z)(Y, V) - (\nabla_V Z)(Y, U) = (\nabla_U \phi) g(Y, V) - (\nabla_V \phi) g(Y, U). \tag{27}$$

If the Z tensor is of Codazzi type, then from (27) it follows that

$$(\nabla_U \phi) g(Y, V) - (\nabla_V \phi) g(Y, U) = 0,$$

which on contracting over Y and V gives

$$(\nabla_U \phi) = 0. \tag{28}$$

Conversely, if the relation (28) holds, then from (27), we obtain

$$(\nabla_U Z)(Y, V) - (\nabla_V Z)(Y, U) = 0.$$

This shows that the Z tensor is of Codazzi type. □

3. $(ZS)_4$ Spacetimes Admitting Schouten Tensor

The energy-momentum tensor $\mathcal{T}$ of a perfect fluid spacetime is given by [8]:

$$T(Y,V) = (p+\sigma)\phi(Y)\phi(V) + pg(V,Y), \tag{29}$$

where p is the isotropic pressure of the fluid, σ is the energy denisty and ϕ is the non-zero 1-form such that

$$g(Y,\xi) = \phi(Y), \qquad g(\xi,\xi) = -1. \tag{30}$$

Here ξ being the unit timelike velocity vector field. For a perfect fluid spacetime the Einstein's field equation without cosmological constant is given by

$$\mathrm{Ric}(Y,V) - \frac{r}{2}g(Y,V) = kT(Y,V), \tag{31}$$

where k as the gravitational constant.

By using (29), (31) turns to

$$\mathrm{Ric}(Y,V) = k(\sigma+p)\phi(Y)\phi(V) + \left(\frac{r}{2}+kp\right)g(Y,V). \tag{32}$$

By contracting (32) over Y and V, we obtain

$$r = k(\sigma - 3p). \tag{33}$$

By using the relation (33) in (32), we have

$$\mathrm{Ric}(Y,V) = k(\sigma+p)\phi(Y)\phi(V) + \frac{k}{2}(\sigma-p)g(Y,V). \tag{34}$$

Now combining (2) and (34), we finally obtain

$$Z(Y,V) = k(\sigma+p)\phi(Y)\phi(V) + \left(\phi + \frac{k}{2}(\sigma-p)\right). \tag{35}$$

Recently, spacetimes and its properties have been studied in several ways by various authors such as [16,26–34] and many others.

Now we prove the following:

Theorem 5. *Let a $(ZS)_4$ of constant scalar curvature tensor admit Codazzi type Schouten tensor. If the velocity vector $(\nabla_U\phi)$ associated with the 1-form ϕ is recurrent and the matter content is a perfect fluid whose velocity vector field is the basic vector field of $(ZS)_4$, then the matter contents of $(ZS)_4$ satisfy the vacuum-like equation of state.*

Proof. We consider that the Schouten tensor in $(ZS)_4$ of constant scalar curvature is of Codazzi type. Now by taking the covariant derivative of (33), we find

$$k[(\nabla_U\sigma) - 3(\nabla_U p)] = 0. \tag{36}$$

This implies that

$$(\nabla_U\sigma) = 3(\nabla_U p), \;\; k \neq 0. \tag{37}$$

By the covariant differentiation of (35) along U, we arrive at

$$\begin{aligned}(\nabla_U Z)(Y,V) = {} & k\Big[(\nabla_U\sigma) + (\nabla_U p)\Big]\phi(Y)\phi(V) \\ & + k(p+\sigma)\Big[(\nabla_U\phi)(Y)\phi(V) + \phi(Y)(\nabla_U\phi)(V)\Big] \\ & + \Big[(\nabla_U\phi) - \frac{k}{2}\Big\{(\nabla_U p) - (\nabla_U\sigma)\Big\}\Big]g(Y,V).\end{aligned} \tag{38}$$

Now we consider that the velocity vector $\nabla_U \phi$ is recurrent, then from (37) and (38), we obtain

$$
\begin{aligned}
(\nabla_U Z)(Y,V) &= 4k(\nabla_U p)\phi(Y)\phi(V) + 2k\lambda(U)(\sigma+p)\phi(Y)\phi(V) \\
&\quad + \Big\{(\nabla_U \phi) + k(\nabla_U p)\Big\} g(Y,V).
\end{aligned} \tag{39}
$$

On contracting (39) over Y and V, we lead to

$$(\nabla_U \tilde{Z}) = 4(\nabla_U \phi) - 2k\lambda(U)(\sigma+p). \tag{40}$$

Now taking the covariant derivative of (3) along U and comparing it with (40), we get

$$\lambda(U)(\sigma+p) = 0.$$

Since $\lambda(U)$ is non-vanishing, therefore, we find $\sigma + p = 0$, which leads to the statement of our theorem. □

Now we prove the following theorem:

Theorem 6. *Let $(ZS)_4$ of constant scalar curvature admit Codazzi type Schouten tensor. If the Z-symmetric tensor is covariantly constant and the matter content is a perfect fluid whose velocity vector field is the basic vector field, then both the isotropic pressure and the energy density are constant.*

Proof. We suppose that $(ZS)_4$ admits Codazzi type Schouten tensor and the Z tensor is covariantly constant. Then from (2) we find

$$(\nabla_U \mathrm{Ric})(Y,V) + (\nabla_U \phi) g(Y,V) = 0,$$

which by contracting over Y and V, and considering r as constant gives

$$(\nabla_U \phi) = 0. \tag{41}$$

This shows that the 1-form ϕ is constant.

By virtue of (37) and (41) and the fact that Z-symmetric tensor is covariantly constant, the relation (38) reduces to

$$4k(\nabla_U p)\phi(Y)\phi(V) + k(\nabla_U p) g(Y,V) = 0,$$

which by taking $Y = V = \xi$ and using (30) leads to

$$\nabla_U p = 0,$$

i.e., the isotropic pressure p is constant. Thus from (37), we lead to $\nabla_U \sigma = 0$, i.e., the energy density is constant. This completes the proof. □

Now we have the following result:

Corollary 2. *Let a perfect fluid $(ZS)_4$ with constant scalar curvature admit Codazzi type Schouten tensor. If the Z tensor is covariantly constant and the velocity vector $(\nabla_U \phi)$ associated with the 1-form ϕ is recurrent, then the spacetime reduces to an Einstein space.*

Proof. We suppose that in $(ZS)_4$, the Z tensor is covariantly constant and the Schouten tensor is of Codazzi type. Then in view of Theorems 5 and 6, from (34), we get

$$\mathrm{Ric}(Y,V) = \frac{k}{2}(\sigma - p) g(Y,V), \tag{42}$$

which by contracting over Y and V gives

$$r = 2k(\sigma - p). \tag{43}$$

On comparing (42) and (43), we obtain

$$\mathrm{Ric}(Y,V) = \frac{r}{4} g(Y,V). \tag{44}$$

This shows that our spacetime is an Einstein space. □

4. Example

In this section, we construct an example of Z-symmetric manifold admitting Schouten tensor on the real number space $\mathbb{R}^4$. First we calculate the components of K, Ric, Z-symmetric tensor and P. Then we verify Theorem 4(i).
Define a semi-Riemannian metric on $\mathbb{R}^4$ by

$$ds^2 = -(dx^1)^2 + e^{x^1}[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]. \tag{45}$$

The non-vanishing components of the Christoffel symbols are given by

$$\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = \frac{1}{2}e^{x^1}, \quad \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{1}{2}. \tag{46}$$

The curvature tensor and the Ricci tensor are obtained as follows:

$$K_{1221} = K_{1331} = K_{1441} = -\frac{1}{4}e^{x^1}, \quad K_{2332} = K_{2442} = K_{3443} = -\frac{1}{4}e^{2x^1}, \tag{47}$$
$$R_{11} = -\frac{1}{2}, \quad R_{22} = R_{33} = R_{44} = -\frac{1}{4}e^{x^1},$$

and the components enlisted by the symmetry properties. Thus we can easily show that $r = -\frac{1}{4}$. The non-vanishing components of the Z tensor and the Schouten tensor are as follows:

$$Z_{11} = -\frac{1}{2} - \phi, \;\; Z_{22} = Z_{33} = Z_{44} = e^{x^1}(-\frac{1}{4} + \phi), \tag{48}$$
$$P_{11} = -\frac{13}{48}, \;\; P_{22} = P_{33} = P_{44} = -\frac{5}{48}e^{x^1}.$$

In view of the above relations, the non-zero components of the covariant derivatives of the Z tensor are obatined as follows:

$$Z_{11,i} = -\phi_i, \quad Z_{22,i} = Z_{33,i} = Z_{44,i} = \phi_i e^{x^1}, \;\; for \quad i = 1,2,3,4, \tag{49}$$

and the components can be easily obtained from (49) by the symmetric properties where “,” denotes for the covariant differentiation with respect to the metric tensor g. Hence the manifold under the consideration has covariantly constant Schouten tensor.

If Z-symmetric tensor is covariantly constant, then

$$Z_{11,i} = Z_{22,i} = Z_{33,i} = Z_{44,i} = 0. \tag{50}$$

Thus from (49), we obtain

$$\phi_{,i} = 0 \quad for \quad i = 1,2,3,4. \tag{51}$$

This verifies Theorem 4 (i).

5. Discussion

The importance of spaces with constant curvature is well understood in cosmology. The simplest cosmological model of the universe is obtained by assuming that the universe is isotropic and homogeneous. This is called the cosmological principle. Isotropy means that all spatial directions are equivalent, whereas homogeneity means that no place in the universe can be distinguished from another. In terms of Riemannian geometry it asserts that the three dimensional position space is a space of maximal symmetry [35], that is, a space of constant curvature whose curvature depends upon time. The cosmological solution of Einstein's equations which contain a three dimensional spacelike surface of a constant curvature are the Robertson-Walker metrics, while four dimensional space of constant curvature is the de Sitter model of the universe [35,36].

The current research is focused on Z-symmetric manifold admitting Schouten tensor with certain investigations in general relativity by the coordinate free method of differential geometry. In this way the spacetime of general relativity is treated as a connected four-dimensional semi-Riemannian manifold $(ZS)_4$ with Lorentz metric g with signature $(-,+,+,+)$. The geometry of the Lorentz manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentz manifold becomes a convenient choice for the study of general relativity. The general theory of relativity, which is a field theory of gravitation, is described by the Einstein's field equations. The Einstein's equations [8] imply that the energy-momentum tensor is of vanishing divergence; and in this direction the authors in [9] showed that for a covariantly constant energy-momentum tensor, the general relativistic spacetime is Ricci symmetric ($\nabla \mathrm{Ric} = 0$).

As a generalization of Ricci symmetric manifold many authors such as [10–13,16,37] studied Z-symmetric manifolds in several ways to a different extent. Motivated by above studies and concepts, we tried to study Z-symmetric manifolds admitting certain types of Schouten tensors, namely, Ricci-recurrent, cyclic parallel, Codazzi type and covariantly constant; and also Z-symmetric spacetime admitting Codazzi type Schouten tensor to prove the some results. In the future, we plan to focus on studying different kinds of curvature tensors on the generalized cases of Z-symmetric manifold. Many problems related to this study are still unresolved, and we hope that the readers of the present paper can do a good amount of work on the subject.

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References

1. Cartan, E. Sur une classe remarquable d'espaces de Riemannian. *Bull. Soc. Math. Fr.* **1926**, *54*, 214–264. [CrossRef]
2. Chaki, M.C.; Gupta, B. On conformally symmetric spaces. *Indian J. Math.* **1963**, *5*, 113–122.

3. Walker, A.G. On Ruse's spaces of recurrent curvature. *Proc. Lond. Math. Soc.* **1951**, *52*, 36–64. [CrossRef]
4. Adati, T.; Miyazawa, T. On a Riemannian space with recurrent conformal curvature. *Tensor* **1967**, *18*, 348–354.
5. Chaki, M.C. On pseudo symmetric manifolds. *Ann. Stiint. Univ. Alexandru Ioan I Cuza Iasi* **1987**, *33*, 53–58.
6. Tamassy, L.; Binh, T.Q. On weak symmetries of Einstein and Sasakian manifolds. *Tensor* **1993**, *53*, 140–148.
7. Einstein, A. The Foundation of the General Theory of Relativity. *Ann. Der Phys.* **1916**, *354*, 769–822. [CrossRef]
8. O'Neill, B. *Semi-Riemannian Geometry with Application to the Relativity*; Academic Press: New York, NY, USA; London, UK, 1983.
9. Chaki, M.C.; Ray, S. Spacetimes with covariant constant energy momentum tensor. *Int. J. Theor. Phys*. **1996**, *35*, 1027–1032. [CrossRef]
10. Mantica, C.A.; Molinari, L.G. Weakly Z-symmetric manifols. *Acta Math. Hung.* **2012**, *135*, 80–96. [CrossRef]
11. De, U.C.; Mantica, C.A.; Suh, Y.J. On weakly cyclic Z-symmetric manifolds. *Acta Math. Hung.* **2015**, *146*, 153–167. [CrossRef]
12. Mantica, C.A.; Suh, Y.J. Pseudo Z-symmetric Riemannian manifolds with harmonic curvature tensors. *Int. J. Geom. Methods Mod. Phys.* **2012**, *9*, 1250004. [CrossRef]
13. De, U.C.; Pal, P. On almost pseudo-Z-symmetric manifolds. *Acta Univ. Palack. Olomuc Fac. Rerum. Natur. Math.* **2014**, *53*, 25–43.
14. Yavuz Tascia, A.; Zengin, F.Ö. Concircularly flat Z-symmetric manifolds. *Ann. Alexandru Ioan Cuza-Univ.-Math.* **2019**, *65*, 241–250.
15. Yavuz Tascia, A.; Zengin, F.Ö. On Z-symmetric manifold admitting projective curvature tensor. *Int. Elec. J. Geom.* **2022**, *15*, 39–46. [CrossRef]
16. Mantica, C.A.; Suh, Y.J. Pseudo Z-symmetric spacetimes. *J. Math. Phys*. **2014**, *55*, 042502. [CrossRef]
17. Guan, P.; Viaclovsky, J.; Wang, G. Some properties of the Schouten tensor and applications to conformal geometry. *Trans. Am. Math. Soc.* **2003**, *355*, 925–933. [CrossRef]
18. Besse, A.L. *Einstein Manifolds*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1987.
19. Ali, M.; Vasiulla, M. Almost pseudo Ricci symmetric manifold admitting Schouten tensor. *J. Dyn. Syst. Geom. Theor.* **2021**, *19*, 217–225. [CrossRef]
20. Gray, A. Einstein-like manifolds which are not Einstein. *Geom. Dedicata* **1978**, *7*, 259–280. [CrossRef]
21. Haseeb, A.; De, U.C. η-Ricci solitons in ϵ-Kenmotsu manifolds. *J. Geom.* **2019**, *110*, 34. [CrossRef]
22. Haseeb, A.; Bilal, M.; Chaubey, S.K.; Khan, M.N.I. Geometry of Indefinite Kenmotsu Manifolds as $*$-η-Ricci-Yamabe Solitons. *Axioms* **2022**, *11*, 461. [CrossRef]
23. Patterson, E.M. Some theorems on Ricci-recurrent spaces. *J. Lond. Math. Soc.* **1952**, *27*, 287–295. [CrossRef]
24. Chaubey, S.K.; Suh, Y.J. Generalized Ricci recurrent spacetimes and GRW spacetimes. *Int. J. Geom. Methods Mod. Phys*. **2021**, *18*, 2150209. [CrossRef]
25. De, U.C.; Guha, N.; Kamilya, D. On generalized Ricci-recurrent manifolds. *Tensor* **1995**, *56*, 312–317.
26. Ahsan, Z.; Siddiqui, S.A. Concircular curvature tensor and fluid spacetimes. *Int. J. Theor. Phys.* **2009**, *48*, 32023212. [CrossRef]
27. Chattopadhyay, K.; Bhunia, N.; Bhattacharyya, A. On Ricci-symmetric mixed generalized quasi Einstein spacetime. *Bull. Cal. Math. Soc.* **2018**, *110*, 513–524.
28. Chaubey, S.K. Characterization of perfect fluid spacetimes admitting gradient η-Ricci and gradient Einstein solitons. *J. Geom. Phys.* **2021**, *162*, 104069. [CrossRef]
29. Chaubey, S.K.; Suh, Y.J. Characterizations of Lorentzian manifolds. *J. Math. Phys.* **2022**, *63*, 062501. [CrossRef]
30. De, U.C.; Ghosh, G.C. On weakly Ricci symmetric spacetime manifolds. *Rad. Mat.* **2004**, *13*, 93–101.
31. De, U.C.; Velimirovic, L. Spacetimes with semi-symmetric energy-momentum tensor. *Int. J. Theor. Phys.* **2015**, *54*, 1779–1783. [CrossRef]
32. Li, Y.; Haseeb, A.; Ali, M. *LP*-Kenmotsu manifolds admitting η-Ricci solitons and spacetime. *J. Math.* **2022**, *2022*, 6605127. [CrossRef]
33. Vasiulla, M.; Haseeb, A.; Mofarreh, F.; Ali, M. Application of mixed generalized quasi-Einstein spacetimes in general relativity. *Mathematics* **2022**, *10*, 3749. [CrossRef]
34. Zengin, F.Ö. *M*-projectively flat spacetimes. *Math. Rep*. **2012**, *4*, 363–370.
35. Stephani, H. *General Relativity: An Introduction to the Theory of Gravitational Field*; Cambridge University Press: New York, NY, USA, 1982.
36. Narlikar, J.V. *General Relativity and Gravitation*; The Macmillan Co. of India: New Delhi, India, 1978.
37. De, U.C.; Mantica, C.A.; Molinari, L.G.; Suh, Y.J. On weakly cyclic Z-symmetric spacetimes. *Acta Math. Hungar.* **2016**, *149*, 462–477. [CrossRef]