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A New Fractal-Fractional Version of Giving up Smoking Model: Application of Lagrangian Piece-Wise Interpolation along with Asymptotical Stability

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Abstract: In this paper, a new kind of mathematical modeling is studied by providing a five-compartmental system of differential equations with respect to new hybrid generalized fractal-fractional derivatives. For the first time, we design a model of giving up smoking to analyze its dynamical behaviors by considering two parameters of such generalized operators; i.e., fractal dimension and fractional order. We apply a special sub-category of increasing functions to investigate the existence of solutions. Uniqueness property is derived by a standard method based on the Lipschitz rule. After proving stability property, the equilibrium points are obtained and asymptotically stable solutions are studied. Finally, we illustrate all analytical results and findings via numerical algorithms and graphs obtained by Lagrangian piece-wise interpolation, and discuss all behaviors of the relevant solutions in the fractal-fractional system.

Keywords: hybrid fractal-fractional derivative; smoking model; approximate solution; stability; sensitivity analysis; Lagrangian piece-wise interpolation

MSC: 34A08; 65P99; 49J15

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Citation: Etemad, S.; Shikongo, A.; Owolabi, K.M.; Tellab, B.; Avcı, İ.; Rezapour, S.; Agarwal, R.P. A New Fractal-Fractional Version of Giving up Smoking Model: Application of Lagrangian Piece-Wise Interpolation along with Asymptotical Stability. *Mathematics* **2022**, *10*, 4369. https://doi.org/10.3390/math10224369

Academic Editor: Chunrui Zhang

Received: 22 September 2022
Accepted: 15 November 2022
Published: 20 November 2022

1. Introduction

Smoking has always been one of the known causes of many human diseases, which threatens the physical health of a large part of the world's population (both smokers and non-smokers). The impact of tobacco abuse, especially cigarettes, on different parts of the human body can be seen so clearly that one of its primary effects is the death of more than 5,000,000 people per year. If we want to make a comparison between smokers and non-smokers, we can refer to the results of medical reports in hospitals around the world, in which the rate of heart attacks and the prevalence of lung cancer in smokers compared to non-smokers are more than 70% and 10%, respectively. Even based on the reports of WHO, the lifespan of non-smokers has been reported to be 10 to 13 years longer than that of smokers. Smoking in the short term can cause bad breath, yellowing of teeth, wrinkled skin, persistent cough, and high blood pressure. In the long term, this bad habit causes dangerous diseases such as stomach ulcers, heart diseases and cancers such as lung, mouth and gums, and throat.

Due to the widespread use of tobacco among teenagers and even children, every year the World Health Organization requests researchers and doctors to study its effects on people's health and present the results in the form of detailed reports and charts and tables. Recently, using the data and numerical results of such studies, researchers have done mathematical modeling of the process of smoking in various statistical societies, and using mathematical algorithms and computer calculations, they are trying to provide methods to control and optimize the outputs. In this direction, researchers turned to mathematical tools and models, and tried to simulate the dynamics of processes and phenomena with the help of differential equations (ordinary and partial) and to get accurate solutions by solving them. Zaman [1] in 2011 designed a model of giving up smoking and analyzed the system qualitatively. In 2008, Sharomi et al. [2] proposed a system of curtailing smoking in the form of four differential equations and investigated stable and unstable solutions. After that, Alkhudhari et al. [3] suggested a four-compartmental system of smoking and checked the effect of smokers on temporary quitters based on the equlibrium criteria. Rahman et al. [4] studied an age-structured model of giving up smoking and conducted an optimal analysis. In addition, in another work, Rahman et al. [5] extended their model of giving up smoking by considering the harmonic mean incidence rate. Other full studies can be found in this regard such as Refs. [6,7].

Due to the weakness of classical operators, generalized fractional operators (Caputo–Fabrizio [8] and Atangana–Baleanu [9]) quickly attracted the attention of many researchers. The non-locality property of the new operators along with their memory property narrowed the field to classical operators such as Riemann–Liouville and Caputo [10]. Of course, there were still those who used classical operators for their modeling to study the dynamics of smoking. For instances, Erturk et al. [11] constructed a five-compartmental fractional giving up smoking model (based on the standard model [12]) via the singular Caputo derivative, and by using the MSGDT method, derived the approximate solutions and lastly, compared their results with the data obtained by the Runge–Kutta algorithm. Zeb et al. [13] gave another fractional model of such a phenomena and analyzed it via the HAM technique. Finally, the giving up smoking and smoking cessation models have been evaluated with various parameters and control tools in different mathematical models with new nonsingular operators, among which we can refer to the research articles published in Refs. [14–16].

More recently, another class of hybrid two-parametric operators was given by Atangaga, for which we can derive more accurate numerical outputs in comparison to both fractional and integer-order operators [17,18]. Due to the effect of fractal dimension and fractional order in the final result, these operators are called "Fractal-Fractional Operators". In the structure of these operators, the role of fractal derivative is essential, and by considering the kernels, these operators divide into three types called the Power law, exponential decay law, and generalized Mittag–Leffler law-type fractal-fractional operators. For more information, one can refer to [17,18]. The effectiveness and efficiency of new operators in obtaining accurate results can be seen in a large number of relevant studies [19–29].

To state the contribution of our work, as we said above, we know that the classical standard time-derivatives are local operators and have some weaknesses in the prediction of the dynamics of a phenomenon. Even a well-known fractional derivative such as the Caputo–Liouville has its own limitations. Since its kernel has a weak memory effect in comparison to the newly-defined fractal-fractional derivatives, this type of derivative cannot precisely describe the full effect of the memory. Hence, due to the strong memory effect, complex dynamics, and non-locality of the generalized hybrid fractal-fractional operators, our main objective in the present research is to use the novel two-parametric power-law type $(\varkappa_1, \varkappa_2)$-fractal-fractional derivative to model the giving up smoking efficiently. In addition, dynamics of the supposed fractal-fractional model is predicted by a numerical scheme with respect to two fractional and fractal parameters continuously for which we can analyze some behaviors of the system accurately.

The rest of the paper is organized as follows. Section 2 deals with preliminaries and Section 3 describes the extended model. Mathematical analysis is carried out in Section 4, whereas in Section 6, the numerical simulations are done based on the algorithms derived in Section 5. Section 7 concludes the paper.

2. Preliminaries

In this section, we recall some basic definitions and theorems about fixed point theory and fractal-fractional calculus that are needed in the sequel.

Let Φ denotes a family of non-decreasing functions $\phi : [0,\infty) \to [0,\infty)$ such that

$$\sum_{m=1}^{\infty} \phi^m(\mathrm{t}) < \infty, \ \forall\, \mathrm{t} > 0,$$

and

$$\phi(\mathrm{t}) < \mathrm{t}, \ \forall\, \mathrm{t} > 0.$$

Definition 1 ([30])**.** *Let $\mathcal{X}$ be a metric space, $\psi : \mathcal{X}^2 \to \mathbb{R}^+ \cup \{0\}$, and $\mathcal{V} : \mathcal{X} \to \mathcal{X}$ be a selfmap.*

(1) $\mathcal{V}$ is called ψ-ϕ-contraction if for each $z_1, z_2 \in \mathcal{X}$,

$$\psi(z_1, z_2)\mathbf{d}(\mathcal{V}z_1, \mathcal{V}z_2) \le \phi(\mathbf{d}(z_1, z_2)),$$

where $\mathbf{d}$ denotes the metric function.

(2) $\mathcal{V}$ is called ψ-admissible if $\psi(z_1, z_2) \ge 1$ gives $\psi(\mathcal{V}z_1, \mathcal{V}z_2) \ge 1$.

Now, we will state two theorems in relation to the existence of a fixed point for such special contractions, which is used in the following sections.

Theorem 1 ([30])**.** *Assume that $(\mathcal{X}, \mathbf{d})$ is a complete metric space, $\psi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, $\phi \in \Phi$, and $\mathcal{V} : \mathcal{X} \to \mathcal{X}$ is an ψ-ϕ-contraction such that*

(1) $\mathcal{V}$ is ψ-admissible;

(2) There is $z_0 \in \mathcal{X}$ such that $\psi(z_0, \mathcal{V}z_0) \ge 1$;

(3) For every sequence $\{z_n\}$ in $\mathcal{X}$ with $z_n \to z$ and $\psi(z_n, z_{n+1}) \ge 1$ for all $n \ge 1$, we have $\psi(z_n, z) \ge 1$ for all $n \ge 1$.

Then, $\mathcal{V}$ has at least a fixed point.

In addition, the following theorem is another theorem that is used for existence results in the sequel.

Theorem 2 (Leray–Schauder [31])**.** *Assume that $\mathcal{X}$ is a Banach space, $\mathbb{A}$ is a convex, bounded and closed set in $\mathcal{X}$, $\mathbb{G}$ is an open subset of $\mathbb{A}$ such that $0 \in \mathbb{G}$, and $\mathcal{Y} : \bar{\mathbb{G}} \to \mathbb{A}$ is a compact and continuous map. Then either:*

(i) There is $z \in \bar{\mathbb{G}}$ such that $\mathcal{Y}(z) = z$, or;

(ii) There are $z \in \partial\mathbb{G}$ and $\alpha \in (0,1)$ so that $z = \alpha\mathcal{Y}(z)$.

Now, we recall fractal-fractional operators.

Definition 2 ([17])**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Assume that a continuous real-valued function $\mathcal{V}$ is a fractal differentiable on (a, b) from the dimension $\varkappa_2$. Then the power-law type $(\varkappa_1, \varkappa_2)$-fractal-fractional derivative of $\mathcal{V}$ in the Riemann–Liouville sense is defined by*

$${}^{\mathbb{FFP}}\mathfrak{D}_{a,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{V}(\mathrm{t}) = \frac{1}{\Gamma(n-\varkappa_1)} \frac{\mathrm{d}}{\mathrm{d}\mathrm{t}^{\varkappa_2}} \int_a^{\mathrm{t}} (\mathrm{t}-\mathrm{u})^{n-\varkappa_1-1}\mathcal{V}(\mathrm{u})\,\mathrm{d}\mathrm{u}, \quad (n-1 < \varkappa_1, \varkappa_2 \le n \in \mathbb{N}),$$

where $\mathrm{t} \in (a, b)$ and $\dfrac{\mathrm{d}\mathcal{V}(\mathrm{u})}{\mathrm{d}\mathrm{u}^{\varkappa_2}} = \lim_{\mathrm{t}\to\mathrm{u}} \dfrac{\mathcal{V}(\mathrm{t}) - \mathcal{V}(\mathrm{u})}{\mathrm{t}^{\varkappa_2} - \mathrm{u}^{\varkappa_2}}$ is the fractal derivative.

If $\varkappa_2 = 1$, then ${}^{\mathbb{FFP}}\mathfrak{D}_{a,\mathrm{t}}^{\varkappa_1,\varkappa_2}$ reduces to the Riemann–Liouville fractional derivative ${}^{\mathbb{RLV}}\mathfrak{D}_{a,\mathrm{t}}^{\varkappa_1}$ of order $\varkappa_1$.

Definition 3 ([17])**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Assume that the real-valued function $\mathcal{V}$ is continuous on (a, b). The power-law type $(\varkappa_1, \varkappa_2)$-fractal-fractional integral of $\mathcal{V}$ is defined by*

$$
{}^{\mathbb{FFP}}\mathfrak{I}_{a,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{V}(\mathrm{t}) = \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_a^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\mathcal{V}(\mathrm{u})\,\mathrm{du}, \tag{1}
$$

where $\mathrm{t} \in (a, b)$.

3. Description of the Giving up Smoking Model

This section is devoted to introducing a new generalized version of the giving up smoking model conducted by Singh, Kumar, Al Qurashi, and Baleanu in [14]. The total population in this model is illustrated by $\mathcal{N}(\mathrm{t})$ at every time $\mathrm{t} \in [0, T]$. This general class $\mathcal{N}(\mathrm{t})$ is divided into five subclasses; i.e., we have three types of smokers, such as potential, occasional, and heavy smokers denoted by $\mathcal{P}(\mathrm{t})$, $\mathcal{O}(\mathrm{t})$ and $\mathcal{H}(\mathrm{t})$, respectively. In addition, we have two other groups such as temporary quitters denoted by $\mathcal{Q}(\mathrm{t})$ and those smokers who quit permanently denoted by $\mathcal{R}(\mathrm{t})$. Therefore, $\mathcal{N}(\mathrm{t}) = \mathcal{P}(\mathrm{t}) + \mathcal{O}(\mathrm{t}) + \mathcal{H}(\mathrm{t}) + \mathcal{Q}(\mathrm{t}) + \mathcal{R}(\mathrm{t})$. By the above assumptions, the mentioned model is designed by:

$$
\begin{cases}
\dfrac{\mathrm{d}\mathcal{P}(\mathrm{t})}{\mathrm{dt}} = v - \vartheta\mathcal{P}(\mathrm{t}) - \omega\mathcal{P}(\mathrm{t})\mathcal{O}(\mathrm{t}), \\
\dfrac{\mathrm{d}\mathcal{O}(\mathrm{t})}{\mathrm{dt}} = -\vartheta\mathcal{O}(\mathrm{t}) + \omega\mathcal{P}(\mathrm{t})\mathcal{O}(\mathrm{t}) - \gamma\mathcal{O}(\mathrm{t})\mathcal{H}(\mathrm{t}), \\
\dfrac{\mathrm{d}\mathcal{H}(\mathrm{t})}{\mathrm{dt}} = (-(\vartheta+\theta) + \gamma\mathcal{O}(\mathrm{t}))\mathcal{H}(\mathrm{t}) + \zeta\mathcal{Q}(\mathrm{t}), \\
\dfrac{\mathrm{d}\mathcal{Q}(\mathrm{t})}{\mathrm{dt}} = -(\vartheta+\zeta)\mathcal{Q}(\mathrm{t}) + \theta(1-q)\mathcal{H}(\mathrm{t}), \\
\dfrac{\mathrm{d}\mathcal{R}(\mathrm{t})}{\mathrm{dt}} = -\vartheta\mathcal{R}(\mathrm{t}) + q\theta\mathcal{H}(\mathrm{t}).
\end{cases} \tag{2}
$$

In view of the widespread use of tobacco among teenagers and children coupled with the weakness of classical and generalized fractional operators in description of such phenomena, by considering the effect of fractal dimension and fractional order in the final result on modeling dynamical systems, the above model is extended by replacing the classical time-derivative with the generalized new hybrid $(\varkappa_1, \varkappa_2)$-fractal-fractional derivative as follows

$$
\begin{cases}
{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{P}(\mathrm{t}) = v - \vartheta\mathcal{P}(\mathrm{t}) - \omega\mathcal{P}(\mathrm{t})\mathcal{O}(\mathrm{t}), \\
{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{O}(\mathrm{t}) = -\vartheta\mathcal{O}(\mathrm{t}) + \omega\mathcal{P}(\mathrm{t})\mathcal{O}(\mathrm{t}) - \gamma\mathcal{O}(\mathrm{t})\mathcal{H}(\mathrm{t}), \\
{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{H}(\mathrm{t}) = (-(\vartheta+\theta) + \gamma\mathcal{O}(\mathrm{t}))\mathcal{H}(\mathrm{t}) + \zeta\mathcal{Q}(\mathrm{t}), \\
{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{Q}(\mathrm{t}) = -(\vartheta+\zeta)\mathcal{Q}(\mathrm{t}) + \theta(1-q)\mathcal{H}(\mathrm{t}), \\
{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{R}(\mathrm{t}) = -\vartheta\mathcal{R}(\mathrm{t}) + q\theta\mathcal{H}(\mathrm{t}),
\end{cases} \tag{3}
$$

subject to initial conditions

$$
\mathcal{P}(0) = \mathcal{P}_0 \geq 0, \;\; \mathcal{O}(0) = \mathcal{O}_0 \geq 0, \;\; \mathcal{H}(0) = \mathcal{H}_0 \geq 0,
$$

$$
\mathcal{Q}(0) = \mathcal{Q}_0 \geq 0, \;\; \mathcal{R}(0) = \mathcal{R}_0 \geq 0,
$$

where ${}^{\mathbb{FFP}}\mathfrak{D}_{0,t}^{\varkappa_1,\varkappa_2}$ is the power-law type $(\varkappa_!, \varkappa_2)$-fractal-fractional derivative with $\varkappa_1, \varkappa_2 \in (0,1]$.

In (3), we have some non-negative parameters that we here aim to introduce:

(1) ω: the contact rate between the potential smokers and smokers who smoke occasionally;
(2) γ: the rate of contact between occasional smokers and heavy smokers;
(3) ζ: the rate at which temporary quitters return back to smoking;
(4) ϑ: the rate of natural death;
(5) θ: the rate of giving up smoking;
(6) $(1-q)$: (at a rate θ) the fraction of smokers who temporarily give up smoking;
(7) q: (at a rate θ) the remaining fraction of smokers who give up smoking forever;
(8) υ: the rate of becoming a potential smoker.

The main point of difference of our contribution about the model derived in Ref. [14] is that the first equation in Ref. [14] is somehow confusing and ineffective. Therefore, one of our major contributions is to modify it with the constant influx of potential smokers.

4. Mathematical Analysis

In this section, the existence of unique solution, stability analysis for the fractal-fractional operator, equilibrium point, sensitivity analysis and asymptotic stability analysis are carried out.

4.1. Existence of Solutions

In real cases, the existence of such dynamical systems is an important question before every analysis and simulation. To answer such a question, we apply fixed point theory. We guarantee this existence in this section. For conducting a qualitative analysis, we consider the Banach space $\mathcal{X} = \mathbb{U}^5$, where $\mathbb{U} = C(\mathbb{J}, \mathbb{R})$, and the norm

$$\|\Lambda\|_{\mathcal{X}} = \|(\mathcal{P}, \mathcal{O}, \mathcal{H}, \mathcal{Q}, \mathcal{R})\|_{\mathcal{X}} = \max\left\{|\mathcal{P}(t)| + |\mathcal{O}(t)| + |\mathcal{H}(t)| + |\mathcal{Q}(t)| + |\mathcal{R}(t)| : \ t \in \mathbb{J}\right\}.$$

At first, the model (3) can be rewritten by follows

$$\begin{cases} \mathcal{V}_1\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big) = \upsilon - \vartheta\mathcal{P}(t) - \omega\mathcal{P}(t)\mathcal{O}(t), \\ \mathcal{V}_2\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big) = -\vartheta\mathcal{O}(t) + \omega\mathcal{P}(t)\mathcal{O}(t) - \gamma\mathcal{O}(t)\mathcal{H}(t), \\ \mathcal{V}_3\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big) = (-(\vartheta+\theta) + \gamma\mathcal{O}(t))\mathcal{H}(t) + \zeta\mathcal{Q}(t), \\ \mathcal{V}_4\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big) = -(\vartheta+\zeta)\mathcal{Q}(t) + \theta(1-q)\mathcal{H}(t), \\ \mathcal{V}_5\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big) = -\vartheta\mathcal{R}(t) + q\theta\mathcal{H}(t). \end{cases} \tag{4}$$

Hence, it becomes

$$\begin{cases} {}^{\mathbb{RLV}}\mathfrak{D}_{0,t}^{\varkappa_1}\mathcal{P}(t) = \varkappa_2 t^{\varkappa_2-1}\mathcal{V}_1\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big), \\ {}^{\mathbb{RLV}}\mathfrak{D}_{0,t}^{\varkappa_1}\mathcal{O}(t) = \varkappa_2 t^{\varkappa_2-1}\mathcal{V}_2\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big), \\ {}^{\mathbb{RLV}}\mathfrak{D}_{0,t}^{\varkappa_1}\mathcal{H}(t) = \varkappa_2 t^{\varkappa_2-1}\mathcal{V}_3\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big), \\ {}^{\mathbb{RLV}}\mathfrak{D}_{0,t}^{\varkappa_1}\mathcal{Q}(t) = \varkappa_2 t^{\varkappa_2-1}\mathcal{V}_4\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big), \\ {}^{\mathbb{RLV}}\mathfrak{D}_{0,t}^{\varkappa_1}\mathcal{R}(t) = \varkappa_2 t^{\varkappa_2-1}\mathcal{V}_5\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big). \end{cases} \tag{5}$$

By (5), we can write a mini-compact system of IVPs (3) as

$$\begin{cases} {}^{\mathbb{RLV}}\mathfrak{D}_{0,t}^{\varkappa_1}\Lambda(t) = \varkappa_2 t^{\varkappa_2 - 1}\mathcal{V}\big(t, \Lambda(t)\big), \quad \varkappa_1, \varkappa_2 \in (0,1], \\ \Lambda(0) = \Lambda_0, \end{cases} \tag{6}$$

where

$$\Lambda(t) = \big(\mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big)^T, \qquad \Lambda_0 = \big(\mathcal{P}_0, \mathcal{O}_0, \mathcal{H}_0, \mathcal{Q}_0, \mathcal{R}_0\big)^T, \tag{7}$$

and

$$\mathcal{V}\big(t, \Lambda(t)\big) = \begin{cases} \mathcal{V}_1\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big), \\ \mathcal{V}_2\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big), \\ \mathcal{V}_3\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big), \\ \mathcal{V}_4\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big), \\ \mathcal{V}_5\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big). \end{cases} \tag{8}$$

By properties of the hybrid $(\varkappa_1, \varkappa_2)$-fractal-fractional integral, the solution of the mini-compact system of IVP (6) is given by

$$\Lambda(t) = \Lambda(0) + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t u^{\varkappa_2 - 1}(t-u)^{\varkappa_1 - 1}\mathcal{V}\big(u, \Lambda(u)\big)\, du. \tag{9}$$

Now, we extend the above compact $(\varkappa_1, \varkappa_2)$-fractal-fractional integral equation to a system of $(\varkappa_1, \varkappa_2)$-fractal-fractional integral equations as

$$\begin{cases} \mathcal{P}(t) = \mathcal{P}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t u^{\varkappa_2 - 1}(t-u)^{\varkappa_1 - 1}\mathcal{V}_1\big(u, \mathcal{P}(u), \mathcal{O}(u), \mathcal{H}(u), \mathcal{Q}(u), \mathcal{R}(u)\big)\, du, \\ \mathcal{O}(t) = \mathcal{O}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t u^{\varkappa_2 - 1}(t-u)^{\varkappa_1 - 1}\mathcal{V}_2\big(u, \mathcal{P}(u), \mathcal{O}(u), \mathcal{H}(u), \mathcal{Q}(u), \mathcal{R}(u)\big)\, du, \\ \mathcal{H}(t) = \mathcal{H}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t u^{\varkappa_2 - 1}(t-u)^{\varkappa_1 - 1}\mathcal{V}_3\big(u, \mathcal{P}(u), \mathcal{O}(u), \mathcal{H}(u), \mathcal{Q}(u), \mathcal{R}(u)\big)\, du, \\ \mathcal{Q}(t) = \mathcal{Q}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t u^{\varkappa_2 - 1}(t-u)^{\varkappa_1 - 1}\mathcal{V}_4\big(u, \mathcal{P}(u), \mathcal{O}(u), \mathcal{H}(u), \mathcal{Q}(u), \mathcal{R}(u)\big)\, du, \\ \mathcal{R}(t) = \mathcal{R}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t u^{\varkappa_2 - 1}(t-u)^{\varkappa_1 - 1}\mathcal{V}_5\big(u, \mathcal{P}(u), \mathcal{O}(u), \mathcal{H}(u), \mathcal{Q}(u), \mathcal{R}(u)\big)\, du. \end{cases} \tag{10}$$

Our aim in this step is to transform the problem (3) into a fixed point problem. Define $\mathcal{Y} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{Y}(\Lambda(t)) = \Lambda(0) + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t u^{\varkappa_2 - 1}(t-u)^{\varkappa_1 - 1}\mathcal{V}\big(u, \Lambda(u)\big)\, du, \tag{11}$$

for each $t \in \mathbb{J}$ and $\Lambda \in \mathcal{X}$.

Theorem 3. *There are $\kappa : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\mathcal{V} \in C(\mathbb{J} \times \mathcal{X}, \mathcal{X})$ and an increasing function $\phi \in \Phi$ such that*
$(\mathfrak{H}_1)$ *for any $\Lambda_1, \Lambda_2 \in \mathcal{X}$ and $t \in \mathbb{J}$,*

$$\big|\mathcal{V}(t, \Lambda_1(t)) - \mathcal{V}(t, \Lambda_2(t))\big| \le \delta\phi\big(|\Lambda_1(t) - \Lambda_2(t)|\big),$$

where $\delta = \dfrac{\Gamma(\varkappa_2 + \varkappa_1)}{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}$, *and* $\kappa\big(\Lambda_1(\mathrm{t}), \Lambda_2(\mathrm{t})\big) \geq 0$.
$(\mathfrak{H}_2)$ *There is* $\Lambda_0 \in \mathcal{X}$ *such that for all* $\mathrm{t} \in \mathbb{J}$,

$$\kappa\big(\Lambda_0(\mathrm{t}), \mathcal{Y}\big(\Lambda_0(\mathrm{t})\big)\big) \geq 0,$$

and the inequality

$$\kappa\big(\Lambda_1(\mathrm{t}), \Lambda_2(\mathrm{t})\big) \geq 0,$$

implies that

$$\kappa\big(\mathcal{Y}\big(\Lambda_1(\mathrm{t})\big), \mathcal{Y}\big(\Lambda_2(\mathrm{t})\big)\big) \geq 0,$$

for any $\Lambda_1, \Lambda_2 \in \mathcal{X}$ *and* $\mathrm{t} \in \mathbb{J}$.
$(\mathfrak{H}_3)$ *For every sequence* $\{\Lambda_n\}_{n\geq 1}$ *in* $\mathcal{X}$ *converging to* Λ *and for each* $\mathrm{t} \in \mathbb{J}$,

$$\kappa\big(\Lambda_n(\mathrm{t}), \Lambda_{n+1}(\mathrm{t})\big) \geq 0,$$

gives

$$\kappa\big(\Lambda_n(\mathrm{t}), \Lambda(\mathrm{t})\big) \geq 0.$$

Then, there is at least a solution for the fractal-fractional hybrid model of giving up smoking (3).

Proof. Let Λ_1 and Λ_2 belong to $\mathcal{X}$ with $\kappa\big(\Lambda_1(\mathrm{t}), \Lambda_2(\mathrm{t})\big) \geq 0$ for each $\mathrm{t} \in \mathbb{J}$. In this case, the Euler Beta function gives

$$\begin{aligned}
\big|\mathcal{Y}\big(\Lambda_1(\mathrm{t})\big) - \mathcal{Y}\big(\Lambda_2(\mathrm{t})\big)\big| &\leq \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\big|\mathcal{V}\big(\mathrm{u}, \Lambda_1(\mathrm{u})\big) - \mathcal{V}\big(\mathrm{u}, \Lambda_2(\mathrm{u})\big)\big|\mathrm{du} \\
&\leq \frac{\varkappa_2\delta}{\Gamma(\varkappa_1)}\int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\phi\big(\big|\Lambda_1(\mathrm{u}) - \Lambda_2(\mathrm{u})\big|\big)\mathrm{du} \\
&\leq \frac{\varkappa_2\delta\phi\big(\|\Lambda_1 - \Lambda_2\|_{\mathcal{X}}\big)}{\Gamma(\varkappa_1)}\int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\mathrm{du} \\
&\leq \frac{\varkappa_2\delta T^{\varkappa_2+\varkappa_1-1}\mathcal{B}(\varkappa_2, \varkappa_1)}{\Gamma(\varkappa_1)}\phi\big(\|\Lambda_1 - \Lambda_2\|_{\mathcal{X}}\big) \\
&= \phi\big(\|\Lambda_1 - \Lambda_2\|_{\mathcal{X}}\big).
\end{aligned}$$

Thus,

$$\big\|\mathcal{Y}(\Lambda_1) - \mathcal{Y}(\Lambda_2)\big\|_{\mathcal{X}} \leq \phi\big(\|\Lambda_1 - \Lambda_2\|_{\mathcal{X}}\big).$$

Now, for each $\Lambda_1, \Lambda_2 \in \mathcal{X}$, we define a function $\psi : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ as

$$\psi(\Lambda_1, \Lambda_2) = \begin{cases} 1 & \text{if } \kappa\big(\Lambda_1(\mathrm{t}), \Lambda_2(\mathrm{t})\big) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

Then, for every $\Lambda_1, \Lambda_2 \in \mathcal{X}$, we will obtain

$$\psi(\Lambda_1, \Lambda_2)\mathbf{d}\big(\mathcal{Y}(\Lambda_1), \mathcal{Y}(\Lambda_2)\big) \leq \phi\big(\mathbf{d}(\Lambda_1, \Lambda_2)\big).$$

Hence, $\mathcal{Y}$ is an $\psi - \phi-$contraction. To show that $\mathcal{Y}$ is $\psi-$admissible, let $\Lambda_1, \Lambda_2 \in \mathcal{X}$ be arbitrary with $\psi(\Lambda_1, \Lambda_2) \geq 1$. From property of ψ, it yields

$$\kappa\big(\Lambda_1(\mathrm{t}), \Lambda_2(\mathrm{t})\big) \geq 0.$$

Then, the condition $(\mathfrak{H}_2)$ gives

$$\kappa\big(\mathcal{Y}\big(\Lambda_1(\mathrm{t})\big), \mathcal{Y}\big(\Lambda_2(\mathrm{t})\big)\big) \geq 0.$$

Once again, by property of ψ, we follow that $\psi\big(\mathcal{Y}(\Lambda_1), \mathcal{Y}(\Lambda_2)\big) \geq 1$. Therefore, $\mathcal{Y}$ is $\psi-$admissible on $\mathcal{X}$.

The condition $(\mathfrak{H}_2)$ ensures the existence of $\Lambda_0 \in \mathcal{X}$, which satisfies

$$\kappa\big(\Lambda_0(\mathrm{t}), \mathcal{Y}(\Lambda_0(\mathrm{t}))\big) \geq 0,$$

for each $\mathrm{t} \in \mathbb{J}$. Evidently, $\psi\big(\Lambda_0, \mathcal{Y}(\Lambda_0)\big) \geq 1$,

Now, suppose that $\{\Lambda_n\}_{n\geq 1}$ is a sequence defined in $\mathcal{X}$ converging to Λ and for all $n \geq 1$, $\psi\big(\Lambda_n, \Lambda_{n+1}\big) \geq 1$. From the property of ψ, we obtain

$$\kappa\big(\Lambda_n(\mathrm{t}), \Lambda_{n+1}(\mathrm{t})\big) \geq 0.$$

Thus, the condition $(\mathfrak{C}_3)$ gives us that

$$\kappa\big(\Lambda_n(\mathrm{t}), \Lambda(\mathrm{t})\big) \geq 0.$$

This implies $\psi\big(\Lambda_n, \Lambda\big) \geq 1$ for all $n \geq 1$. Thus the item (3) of Theorem 1 is valid. Therefore, Theorem 1 is valid. In consequence, $\mathcal{Y}$ has a fixed point $\Lambda^* \in \mathcal{X}$. Hence $\Lambda^* = \big(\mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^*\big)^T$ is a solution of the fractal-fractional model of giving up smoking (3). □

Theorem 4. *Let $\mathcal{V} \in C(\mathbb{J} \times \mathcal{X}, \mathcal{X})$.*

$(\mathcal{D}_1)$ There are $K \in L^1(\mathbb{J}, [0, +\infty))$ and an increasing function $B \in C([0, +\infty), (0, +\infty))$ provided that

$$\big|B(\mathrm{t}, \Lambda(\mathrm{t}))\big| \leq K(\mathrm{t})B\big(|\Lambda(\mathrm{t})|\big), \ \ \forall \mathrm{t} \in \mathbb{J}, \ \ \textit{and} \ \ \Lambda \in \mathcal{X};$$

$(\mathcal{D}_2)$ There is $b > 0$ such that

$$b > \Lambda_0 + \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)} K_0^* B(b), \tag{12}$$

where $K_0^* = \sup\limits_{\mathrm{t}\in\mathbb{J}} |K(\mathrm{t})|$.

Then, there is a solution for the fractal-fractional model of giving up smoking (3).

Proof. To complete the proof, we consider $\mathcal{Y}$ defined in (11), and the closed ball

$$\mathbf{N}_L = \{\Lambda \in \mathcal{X} : \ \ \|\Lambda\|_{\mathcal{X}} \leq L\}.$$

The continuity of $\mathcal{V}$ implies that of $\mathcal{Y}$. Now, by $(\mathcal{D}_1)$ and for $\Lambda \in \mathbf{N}_L$, we estimate

$$\begin{aligned}
\big|\mathcal{Y}(\Lambda(\mathrm{t}))\big| &\leq |\Lambda(0)| + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\big|\mathcal{V}(\mathrm{u}, \Lambda(\mathrm{u}))\big| d\mathrm{u} \\
&\leq \Lambda_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1} K(\mathrm{u}) B\big(|\Lambda(\mathrm{u})|\big) d\mathrm{u} \\
&\leq \Lambda_0 + \frac{\varkappa_2 T^{\varkappa_2+\varkappa_1-1}\mathfrak{B}(\varkappa_2, \varkappa_1)}{\Gamma(\varkappa_1)} K_0^* B\big(\|\Lambda\|_{\mathcal{X}}\big) \\
&\leq \Lambda_0 + \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)} K_0^* B(L).
\end{aligned}$$

Consequently, we get

$$\|\mathcal{Y}\Lambda\|_{\mathcal{X}} \leq \Lambda_0 + \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)} K_0^* B(L) < +\infty. \tag{13}$$

Thus, $\mathcal{Y}$ is uniformly bounded on $\mathcal{X}$. Next, we choose arbitrarily $\mathrm{t}, \tau \in [0, T]$ with $\mathrm{t} < \tau$ and $\Lambda \in \mathbf{N}_L$. By

$$\mathcal{V}^* = \sup_{(\mathrm{t},\Lambda)\in\mathbb{J}\times\mathbf{N}_L} \left|\mathcal{V}(\mathrm{t}, \Lambda(\mathrm{t}))\right| < +\infty,$$

we find

$$\begin{aligned}
\left|\mathcal{Y}(\Lambda(\tau)) - \mathcal{Y}(\Lambda(\mathrm{t}))\right| &= \bigg|\frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^\tau \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\mathcal{V}(\mathrm{u}, \Lambda(\mathrm{u}))\mathrm{du} \\
&\quad - \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\mathcal{V}(\mathrm{u}, \Lambda(\mathrm{u}))\mathrm{du}\bigg| \\
&\leq \frac{\varkappa_2\mathcal{V}^*}{\Gamma(\varkappa_1)}\left|\int_0^\tau \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\mathrm{du} - \int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\mathrm{du}\right| \\
&\leq \frac{\varkappa_2\mathcal{V}^* B(\varkappa_2, \varkappa_1)}{\Gamma(\varkappa_1)}\left(\tau^{\varkappa_2+\varkappa_1-1} - \mathrm{t}^{\varkappa_2+\varkappa_1-1}\right) \\
&= \frac{\mathcal{V}^*\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}\left(\tau^{\varkappa_2+\varkappa_1-1} - \mathrm{t}^{\varkappa_2+\varkappa_1-1}\right).
\end{aligned} \tag{14}$$

Note that from the above computations, the right-hand side of (14) is not dependent on Λ and also converges to 0 as $\mathrm{t} \to \tau$. So

$$\|\mathcal{Y}(\Lambda(\tau)) - \mathcal{Y}(\Lambda(\mathrm{t}))\|_{\mathcal{X}} \to 0,$$

as $\mathrm{t} \to \tau$, which shows the equicontinuity of $\mathcal{Y}$. By referring to the Arzelà–Ascoli theorem, $\mathcal{Y}$ is compact on $\mathbf{N}_L$. Now, Theorem 2 is valid on $\mathcal{Y}$. We have one of the consequences (i) or (ii). We know that from $(\mathcal{D}_2)$, there exists $b > 0$ such that

$$\Lambda_0 + \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}K_0^* B(b) < b. \tag{15}$$

Then, we consider

$$\mathbb{G} = \{\Lambda \in \mathcal{X} : \ \|\Lambda\|_{\mathcal{X}} < b\}.$$

By assuming the existence of $\Lambda \in \partial\mathbb{G}$ and $\alpha \in (0,1)$ such that $\Lambda = \alpha\mathcal{Y}(\Lambda)$, we can write

$$\begin{aligned}
b = \|\Lambda\|_{\mathcal{X}} = \alpha\|\mathcal{Y}\Lambda\|_{\mathcal{X}} &< \Lambda_0 + \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}K_0^* B(\|\Lambda\|_{\mathcal{X}}) \\
&< \Lambda_0 + \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}K_0^* B(b) < b,
\end{aligned}$$

by (15). However, this is impossible. Thus (ii) does not hold and by Theorem 2, $\mathcal{Y}$ has a fixed point in $\overline{\mathbb{G}}$ which is considered as a solution of the fractal-fractional model of giving up smoking (3). □

4.2. Unique Solution

To prove the uniqueness of the solution in the model of giving up smoking (3), we use the Lipschitz property under the functions $\mathcal{V}_i, (i = 1, \ldots 5)$ defined by (4).

Lemma 1. *Let the functions* $\mathcal{P}, \mathcal{O}, \mathcal{H}, \mathcal{Q}, \mathcal{R}, \mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^* \in \mathbb{U} = C(\mathbb{J}, \mathbb{R})$ *and assume that* $(\mathcal{P}_1)$ $\|\mathcal{P}\| \leq \gamma_1, \|\mathcal{O}\| \leq \gamma_2, \|\mathcal{H}\| \leq \gamma_3, \|\mathcal{Q}\| \leq \gamma_4, \|\mathcal{R}\| \leq \gamma_5$ *for some positive constants* $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$.

Then, the functions $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5$ *defined by* (4) *satisfy the Lipschitz property with respect to the corresponding components if*

$$\delta_1 = \vartheta + \omega\gamma_2, \quad \delta_2 = \vartheta + \omega\gamma_1 + \gamma\gamma_3,$$

$$\delta_3 = \vartheta + \theta + \gamma\gamma_2, \quad \delta_4 = \vartheta + \zeta, \quad \delta_5 = \vartheta. \tag{16}$$

Proof. We begin with function $\mathcal{V}_1$. For other solution functions, the proof is similar. For any functions $\mathcal{P}, \mathcal{P}^* \in \mathbb{U} = C(\mathbb{J}, \mathbb{R})$, we get

$$\begin{aligned}
& \left\| \mathcal{V}_1\big(t, \mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)\big) - \mathcal{V}_1\big(t, \mathcal{P}^*(t), \mathcal{O}^*(t), \mathcal{H}^*(t), \mathcal{Q}^*(t), \mathcal{R}^*(t)\big) \right\| \\
= & \left\| \big(\vartheta - \vartheta\mathcal{P}(t) - \omega\mathcal{P}(t)\mathcal{O}(t)\big) - \big(\vartheta - \vartheta\mathcal{P}^*(t) - \omega\mathcal{P}^*(t)\mathcal{O}(t)\big) \right\| \\
\leq & \big(\vartheta + \omega\|\mathcal{O}(t)\|\big)\left\|\mathcal{P}(t) - \mathcal{P}^*(t)\right\| \\
\leq & \big(\vartheta + \omega\gamma_2\|\big)\left\|\mathcal{P}(t) - \mathcal{P}^*(t)\right\| \\
= & \delta_1\left\|\mathcal{P}(t) - \mathcal{P}^*(t)\right\|.
\end{aligned}$$

This shows that $\mathcal{V}_1$ is a Lipschitz function with respect to $\mathcal{P}$ with the Lipschitz constant $\delta_1 > 0$. By continuing similar proofs, we see that the functions $\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5$ are Lipschitiz with respect to the corresponding components with the Lipschitz constants $\delta_2, \delta_3, \delta_4, \delta_5 > 0$, respectively. □

Theorem 5. *By considering the condition $(\mathcal{P}_1)$, the fractal-fractional model of giving up smoking* (3) *has a unique solution if*

$$\frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}\delta_i < 1, \quad i \in \{1, \dots, 5\}. \tag{17}$$

Proof. Let us consider the fact that the conclusion is not to be held. That is, there exists another solution. Assume that $\big(\mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^*\big)$ is another solution with initial condition $\big(\mathcal{P}_0, \mathcal{O}_0, \mathcal{H}_0, \mathcal{Q}_0, \mathcal{R}_0\big)$ such that by (10), we have

$$\begin{aligned}
\mathcal{P}^*(t) &= \mathcal{P}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^t u^{\varkappa_2-1}(t-u)^{\varkappa_1-1}\mathcal{V}_1\big(u, \mathcal{P}^*(u), \mathcal{O}^*(u), \mathcal{H}^*(u), \mathcal{Q}^*(u), \mathcal{R}^*(u)\big)\,du, \\
\mathcal{O}^*(t) &= \mathcal{O}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^t u^{\varkappa_2-1}(t-u)^{\varkappa_1-1}\mathcal{V}_2\big(u, \mathcal{P}^*(u), \mathcal{O}^*(u), \mathcal{H}^*(u), \mathcal{Q}^*(u), \mathcal{R}^*(u)\big)\,du, \\
\mathcal{H}^*(t) &= \mathcal{H}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^t u^{\varkappa_2-1}(t-u)^{\varkappa_1-1}\mathcal{V}_3\big(u, \mathcal{P}^*(u), \mathcal{O}^*(u), \mathcal{H}^*(u), \mathcal{Q}^*(u), \mathcal{R}^*(u)\big)\,du, \\
\mathcal{Q}^*(t) &= \mathcal{Q}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^t u^{\varkappa_2-1}(t-u)^{\varkappa_1-1}\mathcal{V}_4\big(u, \mathcal{P}^*(u), \mathcal{O}^*(u), \mathcal{H}^*(u), \mathcal{Q}^*(u), \mathcal{R}^*(u)\big)\,du, \\
\mathcal{R}^*(t) &= \mathcal{R}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^t u^{\varkappa_2-1}(t-u)^{\varkappa_1-1}\mathcal{V}_5\big(u, \mathcal{P}^*(u), \mathcal{O}^*(u), \mathcal{H}^*(u), \mathcal{Q}^*(u), \mathcal{R}^*(u)\big)\,du.
\end{aligned}$$

Now, we can estimate

$$\begin{aligned}
\left|\mathcal{P}(t) - \mathcal{P}^*(t)\right| \leq\; & \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^t u^{\varkappa_2-1}(t-u)^{\varkappa_1-1}\Big|\mathcal{V}_1\big(u, \mathcal{P}(u), \mathcal{O}(u), \mathcal{H}(u), \mathcal{Q}(u), \mathcal{R}(u)\big) \\
& -\mathcal{V}_1\big(u, \mathcal{P}^*(u), \mathcal{O}^*(u), \mathcal{H}^*(u), \mathcal{Q}^*(u), \mathcal{R}^*(u)\big)\Big|\,du \\
\leq\; & \frac{\varkappa_2}{\Gamma(\varkappa_1)}\delta_1\left\|\mathcal{P} - \mathcal{P}^*\right\|\int_0^t u^{\varkappa_2-1}(t-u)^{\varkappa_1-1}\,du \\
\leq\; & \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}\delta_1\left\|\mathcal{P} - \mathcal{P}^*\right\|.
\end{aligned}$$

This gives

$$\left[1 - \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}\delta_1\right]\left\|\mathcal{P} - \mathcal{P}^*\right\| \leq 0.$$

Then from (17), it follows that $\|\mathcal{P} - \mathcal{P}^*\| = 0$, and accordingly $\mathcal{P} = \mathcal{P}^*$. Similarly, we get

$$\left[1 - \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}\delta_2\right]\|\mathcal{O} - \mathcal{O}^*\| \leq 0,$$

which gives that $\|\mathcal{O} - \mathcal{O}^*\| = 0$, and so $\mathcal{O} = \mathcal{O}^*$. By the same arguments, we obtain

$$\left[1 - \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}\delta_3\right]\|\mathcal{H}_-\mathcal{H}^*\| \leq 0.$$

Therefore, $\|\mathcal{H} - \mathcal{H}^*\| = 0$, and so $\mathcal{H} = \mathcal{H}^*$. In a similar way, we immediately get

$$\left[1 - \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}\delta_4\right]\|\mathcal{Q} - \mathcal{Q}^*\| \leq 0,$$

and

$$\left[1 - \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}\delta_5\right]\|\mathcal{R} - \mathcal{R}^*\| \leq 0,$$

The last two inequalities give $\|\mathcal{Q} - \mathcal{Q}^*\| = 0$ and $\|\mathcal{R} - \mathcal{R}^*\| = 0$, respectively. Thus, $\mathcal{Q} = \mathcal{Q}^*$ and $\mathcal{R} = \mathcal{R}^*$. Consequently, we find that

$$(\mathcal{P}, \mathcal{O}, \mathcal{H}, \mathcal{Q}, \mathcal{R}) = (\mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^*).$$

This shows that the fractal-fractional model of giving up smoking (3) has a unique solution. □

4.3. Stability Criterion

In this section, we aim to study the stability property based on the definition of Ulam–Hyers. This definition has applicable significance since it states that if we are studying an Ulam–Hyers stable system then we do not have to obtain the exact solution. Therefore, by proving the stability of the solutions of the given system, we can confidently focus on its approximate solutions in the next sections. More precisely, we are here to study the stability property for solutions of the fractal-fractional model of giving up smoking (3). The main focus is on the Ulam–Hyers and Ulam–Hyers–Rassias stability. For more information, we refer to Refs. [32,33].

Definition 4. *The fractal-fractional model of of giving up smoking* (3) *is Ulam–Hyers stable if there are real constants $\mathbb{M}_{\mathcal{V}_i} > 0$, $i \in \{1, \ldots, 5\}$ such that for all $L_i > 0$ and for all $(\mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^*) \in \mathcal{X}$ satisfying*

$$\begin{cases} \left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{P}^*(\mathrm{t}) - \mathcal{V}_1(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t}))\right| < L_1, \\ \left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{O}^*(\mathrm{t}) - \mathcal{V}_2(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t}))\right| < L_2, \\ \left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{H}^*(\mathrm{t}) - \mathcal{V}_3(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t}))\right| < L_3, \\ \left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{Q}^*(\mathrm{t}) - \mathcal{V}_4(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t}))\right| < L_4, \\ \left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{R}^*(\mathrm{t}) - \mathcal{V}_5(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t}))\right| < L_5, \end{cases} \tag{18}$$

there is $\big(\mathcal{P}, \mathcal{O}, \mathcal{H}, \mathcal{Q}, \mathcal{R}\big) \in \mathcal{X}$ *satisfying the fractal-fractional hybrid model of giving up smoking* (3) *such that*

$$\begin{cases} \Big|\mathcal{P}^*(\mathrm{t}) - \mathcal{P}(\mathrm{t})\Big| \leq \mathbb{M}_{\mathcal{V}_1} L_1, \\ \Big|\mathcal{O}^*(\mathrm{t}) - \mathcal{O}(\mathrm{t})\Big| \leq \mathbb{M}_{\mathcal{V}_2} L_2, \\ \Big|\mathcal{H}^*(\mathrm{t}) - \mathcal{H}(\mathrm{t})\Big| \leq \mathbb{M}_{\mathcal{V}_3} L_3, \\ \Big|\mathcal{Q}^*(\mathrm{t}) - \mathcal{Q}(\mathrm{t})\Big| \leq \mathbb{M}_{\mathcal{V}_4} L_4, \\ \Big|\mathcal{R}^*(\mathrm{t}) - \mathcal{R}(\mathrm{t})\Big| \leq \mathbb{M}_{\mathcal{V}_5} L_5, \quad \forall \mathrm{t} \in \mathbb{J}. \end{cases} \tag{19}$$

Definition 5. *The fractal-fractional model of giving up smoking* (3) *is generalized Ulam–Hyers stable if there are real constants* $\mathbb{M}_{\mathcal{V}_i} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $(i \in \{1, \ldots, 5\})$ *with* $\mathbb{M}_{\mathcal{V}_i}(0) = 0$ *such that for all* $L_i > 0$ *and for all* $\big(\mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^*\big) \in \mathcal{X}$ *satisfying* (18), *there is* $\big(\mathcal{P}, \mathcal{O}, \mathcal{H}, \mathcal{Q}, \mathcal{R}\big) \in \mathcal{X}$ *as a solution of the fractal-fractional hybrid model of giving up smoking* (3) *such that*

$$\begin{cases} \Big|\mathcal{P}^*(\mathrm{t}) - \mathcal{P}(\mathrm{t})\Big| \leq \mathbb{M}_{\mathcal{V}_1}(L_1), & \Big|\mathcal{O}^*(\mathrm{t}) - \mathcal{O}(\mathrm{t})\Big| \leq \mathbb{M}_{\mathcal{V}_2}(L_2), \\ \Big|\mathcal{H}^*(\mathrm{t}) - \mathcal{H}(\mathrm{t})\Big| \leq \mathbb{M}_{\mathcal{V}_3}(L_3), & \Big|\mathcal{Q}^*(\mathrm{t}) - \mathcal{Q}(\mathrm{t})\Big| \leq \mathbb{M}_{\mathcal{V}_4}(L_4), \\ \Big|\mathcal{R}^*(\mathrm{t}) - \mathcal{R}(\mathrm{t})\Big| \leq \mathbb{M}_{\mathcal{V}_5}(L_5), \quad \forall \mathrm{t} \in \mathbb{J}. \end{cases}$$

Note that Definition 4 is obtained from Definition 5.

Remark 1. *Notice that* $\big(\mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^*\big) \in \mathcal{X}$ *is called a solution for inequalities* (4) *if and only if there are* $\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5 \in C([0, T], \mathbb{R})$ *(depending on* $\mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^*$*, respectively) so that for each* $\mathrm{t} \in \mathbb{J}$,
(i) $\big|\hbar_i(\mathrm{t})\big| < L_i, \quad (i \in \{1, \ldots, 5\})$,
(ii) We have

$$\begin{cases} {}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1, \varkappa_2} \mathcal{P}^*(\mathrm{t}) = \mathcal{V}_1\big(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})\big) + \hbar_1(\mathrm{t}), \\ {}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1, \varkappa_2} \mathcal{O}^*(\mathrm{t}) = \mathcal{V}_2\big(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})\big) + \hbar_2(\mathrm{t}), \\ {}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1, \varkappa_2} \mathcal{H}^*(\mathrm{t}) = \mathcal{V}_3\big(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})\big) + \hbar_3(\mathrm{t}), \\ {}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1, \varkappa_2} \mathcal{Q}^*(\mathrm{t}) = \mathcal{V}_4\big(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})\big) + \hbar_4(\mathrm{t}), \\ {}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1, \varkappa_2} \mathcal{R}^*(\mathrm{t}) = \mathcal{V}_5\big(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})\big) + \hbar_5(\mathrm{t}). \end{cases}$$

Definition 6. *The fractal-fractional model of giving up smoking* (3) *is Ulam–Hyers–Rassias stable with respect to the functions* β_i, $(i \in \{1, \ldots, 5\})$, *if there are constants* $0 < \mathbb{M}_{(\mathcal{V}_i, \beta_i)} \in \mathbb{R}$ *so that for each* $L_i > 0$ *and for each* $\big(\mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^*\big) \in \mathcal{X}$ *satisfying*

$$\begin{cases} \Big|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1, \varkappa_2} \mathcal{P}^*(\mathrm{t}) - \mathcal{V}_1\big(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})\big)\Big| < L_1 \beta_1(\mathrm{t}), \\ \Big|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1, \varkappa_2} \mathcal{O}^*(\mathrm{t}) - \mathcal{V}_2\big(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})\big)\Big| < L_2 \beta_2(\mathrm{t}), \\ \Big|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1, \varkappa_2} \mathcal{H}^*(\mathrm{t}) - \mathcal{V}_3\big(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})\big)\Big| < L_3 \beta_3(\mathrm{t}), \\ \Big|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1, \varkappa_2} \mathcal{Q}^*(\mathrm{t}) - \mathcal{V}_4\big(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})\big)\Big| < L_4 \beta_4(\mathrm{t}), \\ \Big|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1, \varkappa_2} \mathcal{R}^*(\mathrm{t}) - \mathcal{V}_5\big(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})\big)\Big| < L_5 \beta_5(\mathrm{t}), \end{cases} \tag{20}$$

there is $(\mathcal{P}, \mathcal{O}, \mathcal{H}, \mathcal{Q}, \mathcal{R}) \in \mathcal{X}$ *satisfying the fractaional-fractal hybrid model of giving up smoking* (3) *such that*

$$\begin{cases} \left|\mathcal{P}^*(\mathrm{t}) - \mathcal{P}(\mathrm{t})\right| \le L_1 \mathbb{M}_{(\mathcal{V}_1,\beta_1)} \beta_1(\mathrm{t}), \\ \left|\mathcal{O}^*(\mathrm{t}) - \mathcal{O}(\mathrm{t})\right| \le L_2 \mathbb{M}_{(\mathcal{V}_2,\beta_2)} \beta_2(\mathrm{t}), \\ \left|\mathcal{H}^*(\mathrm{t}) - \mathcal{H}(\mathrm{t})\right| \le L_3 \mathbb{M}_{(\mathcal{V}_3,\beta_3)} \beta_3(\mathrm{t}), \\ \left|\mathcal{Q}^*(\mathrm{t}) - \mathcal{Q}(\mathrm{t})\right| \le L_4 \mathbb{M}_{(\mathcal{V}_4,\beta_4)} \beta_4(\mathrm{t}), \\ \left|\mathcal{R}^*(\mathrm{t}) - \mathcal{R}(\mathrm{t})\right| \le L_5 \mathbb{M}_{(\mathcal{V}_5,\beta_5)} \beta_5(\mathrm{t}), \quad \forall \mathrm{t} \in \mathbb{J}. \end{cases}$$

Definition 7. *The fractal-fractional model of giving up smoking* (3) *is generalized Ulam–Hyers–Rassias stable with respect to* β_i, $(i \in \{1, \dots, 5\})$, *if there are constants* $0 < \mathbb{M}_{(\mathcal{V}_i,\beta_i)} \in \mathbb{R}$ *so that for each* $(\mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^*) \in \mathcal{X}$ *satisfying*

$$\begin{cases} \left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2} \mathcal{P}^*(\mathrm{t}) - \mathcal{V}_1(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t}))\right| < \beta_1(\mathrm{t}), \\ \left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2} \mathcal{O}^*(\mathrm{t}) - \mathcal{V}_2(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t}))\right| < \beta_2(\mathrm{t}), \\ \left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2} \mathcal{H}^*(\mathrm{t}) - \mathcal{V}_3(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t}))\right| < \beta_3(\mathrm{t}), \\ \left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2} \mathcal{Q}^*(\mathrm{t}) - \mathcal{V}_4(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t}))\right| < \beta_4(\mathrm{t}), \\ \left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2} \mathcal{R}^*(\mathrm{t}) - \mathcal{V}_5(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t}))\right| < \beta_5(\mathrm{t}), \end{cases}$$

there is $(\mathcal{P}, \mathcal{O}, \mathcal{H}, \mathcal{Q}, \mathcal{R}) \in \mathcal{X}$ *satisfying the fractal-fractional hybrid model of giving up smoking* (3) *such that*

$$\begin{cases} \left|\mathcal{P}^*(\mathrm{t}) - \mathcal{P}(\mathrm{t})\right| \le \mathbb{M}_{(\mathcal{V}_1,\beta_1)} \beta_1(\mathrm{t}), \\ \left|\mathcal{O}^*(\mathrm{t}) - \mathcal{O}(\mathrm{t})\right| \le \mathbb{M}_{(\mathcal{V}_2,\beta_2)} \beta_2(\mathrm{t}), \\ \left|\mathcal{H}^*(\mathrm{t}) - \mathcal{H}(\mathrm{t})\right| \le \mathbb{M}_{(\mathcal{V}_3,\beta_3)} \beta_3(\mathrm{t}), \\ \left|\mathcal{Q}^*(\mathrm{t}) - \mathcal{Q}(\mathrm{t})\right| \le \mathbb{M}_{(\mathcal{V}_4,\beta_4)} \beta_4(\mathrm{t}), \\ \left|\mathcal{R}^*(\mathrm{t}) - \mathcal{R}(\mathrm{t})\right| \le \mathbb{M}_{(\mathcal{V}_5,\beta_5)} \beta_5(\mathrm{t}), \quad \forall \mathrm{t} \in \mathbb{J}. \end{cases}$$

Remark 2. *Notice that* $(\mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^*) \in \mathbb{X}$ *is called a solution for inequalities* (5) *if and only if there are* $\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5 \in C([0,T], \mathbb{R})$ *(depending on* $\mathcal{P}^*, \mathcal{O}^*, \mathcal{H}^*, \mathcal{Q}^*, \mathcal{R}^*$, *respectively) such that for each* $\mathrm{t} \in \mathbb{J}$,
(i) $\left|\hbar_i(\mathrm{t})\right| < L_i \beta_i(\mathrm{t}), \quad (i \in \{1, \dots, 5\})$,
(ii) *We have*

$$\begin{cases} {}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2} \mathcal{P}^*(\mathrm{t}) = \mathcal{V}_1(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})) + \hbar_1(\mathrm{t}), \\ {}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2} \mathcal{O}^*(\mathrm{t}) = \mathcal{V}_2(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})) + \hbar_2(\mathrm{t}), \\ {}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2} \mathcal{H}^*(\mathrm{t}) = \mathcal{V}_3(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})) + \hbar_3(\mathrm{t}), \\ {}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2} \mathcal{Q}^*(\mathrm{t}) = \mathcal{V}_4(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})) + \hbar_4(\mathrm{t}), \\ {}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2} \mathcal{R}^*(\mathrm{t}) = \mathcal{V}_5(\mathrm{t}, \mathcal{P}^*(\mathrm{t}), \mathcal{O}^*(\mathrm{t}), \mathcal{H}^*(\mathrm{t}), \mathcal{Q}^*(\mathrm{t}), \mathcal{R}^*(\mathrm{t})) + \hbar_5(\mathrm{t}). \end{cases}$$

Theorem 6. *If the condition* $(\mathcal{P}_1)$ *holds, then the fractal-fractional model of giving up smoking* (3) *is Ulam–Hyers and generalized Ulam–Hyers stable such that*

$$\frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}\delta_i < 1, \quad i \in \{1,\dots,5\},$$

where δ_i*'s are defined by* (16).

Proof. Let $L_1 > 0$ and $\mathcal{P}^* \in \mathbb{U}$ be arbitrary so that

$$\left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathfrak{t}}^{\varkappa_1,\varkappa_2}\mathcal{P}^*(\mathfrak{t}) - \mathcal{V}_1\big(\mathfrak{t},\mathcal{P}^*(\mathfrak{t}),\mathcal{O}^*(\mathfrak{t}),\mathcal{H}^*(\mathfrak{t}),\mathcal{Q}^*(\mathfrak{t}),\mathcal{R}^*(\mathfrak{t})\big)\right| < L_1.$$

Then, in view of Remark 1, we can find a function $\hbar_1(\mathfrak{t})$ satisfying

$${}^{\mathbb{FFP}}\mathfrak{D}_{0,(\mathfrak{t})}^{\varkappa_1,\varkappa_2}\mathcal{P}^*(\mathfrak{t}) = \mathcal{V}_1\big(\mathfrak{t},\mathcal{P}^*(\mathfrak{t}),\mathcal{O}^*(\mathfrak{t}),\mathcal{H}^*(\mathfrak{t}),\mathcal{Q}^*(\mathfrak{t}),\mathcal{R}^*(\mathfrak{t})\big) + \hbar_1(\mathfrak{t}),$$

with $|\hbar_1(\mathfrak{t})| \le L_1$. It follows that

$$\begin{aligned}\mathcal{P}^*(\mathfrak{t}) \;=\; & \mathcal{P}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathfrak{t}} \mathfrak{u}^{\varkappa_2-1}(\mathfrak{t}-\mathfrak{u})^{\varkappa_1-1}\mathcal{V}_1\big(\mathfrak{u},\mathcal{P}^*(\mathfrak{u}),\mathcal{O}^*(\mathfrak{u}),\mathcal{H}^*(\mathfrak{u}),\mathcal{Q}^*(\mathfrak{u}),\mathcal{R}^*(\mathfrak{u})\big)\,d\mathfrak{u} \\ & + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathfrak{t}} \mathfrak{u}^{\varkappa_2-1}(\mathfrak{t}-\mathfrak{u})^{\varkappa_1-1}\hbar_1(\mathfrak{u})\,d\mathfrak{u}.\end{aligned}$$

By using Theorem 5, let $\mathcal{P} \in \mathbb{U}$ be a unique solution of the fractal-fractional model of giving up smoking (3). Then $\mathcal{P}(\mathfrak{t})$ is given by

$$\mathcal{P}(\mathfrak{t}) = \mathcal{P}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathfrak{t}} \mathfrak{u}^{\varkappa_2-1}(\mathfrak{t}-\mathfrak{u})^{\varkappa_1-1}\mathcal{V}_1\big(\mathfrak{u},\mathcal{P}(\mathfrak{u}),\mathcal{O}(\mathfrak{u}),\mathcal{H}(\mathfrak{u}),\mathcal{Q}(\mathfrak{u}),\mathcal{R}(\mathfrak{u})\big)\,d\mathfrak{u}.$$

Then

$$\begin{aligned}\big|\mathcal{P}^*(\mathfrak{t}) - \mathcal{P}(\mathfrak{t})\big| \;\le\; & \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathfrak{t}} \mathfrak{u}^{\varkappa_2-1}(\mathfrak{t}-\mathfrak{u})^{\varkappa_1-1} \\ \times\; & \big|\mathcal{V}_1\big(\mathfrak{u},\mathcal{P}^*(\mathfrak{u}),\mathcal{O}^*(\mathfrak{u}),\mathcal{H}^*(\mathfrak{u}),\mathcal{Q}^*(\mathfrak{u}),\mathcal{R}^*(\mathfrak{u})\big) \\ & -\mathcal{V}_1\big(\mathfrak{u},\mathcal{P}(\mathfrak{u}),\mathcal{O}(\mathfrak{u}),\mathcal{H}(\mathfrak{u}),\mathcal{Q}(\mathfrak{u}),\mathcal{R}(\mathfrak{u})\big)\big|\,d\mathfrak{u} \\ & + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathfrak{t}} \mathfrak{u}^{\varkappa_2-1}(\mathfrak{t}-\mathfrak{u})^{\varkappa_1-1}|\hbar_1(\mathfrak{u})|\,d\mathfrak{u} \\ \le\; & \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}\delta_1\big\|\mathcal{P}^*-\mathcal{P}\big\| + \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}L_1.\end{aligned}$$

Hence, we get

$$\big\|\mathcal{P}^*-\mathcal{P}\big\| \le \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1) - T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)\delta_1}L_1.$$

If we let $\mathbb{M}_{\mathcal{V}_1} = \dfrac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1) - T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)\delta_1}$, then we obtain $\big\|\mathcal{P}^*-\mathcal{P}\big\| \le \mathbb{M}_{\mathcal{V}_1}L_1$. Again, we find that

$$\begin{aligned}&\big\|\mathcal{O}^*-\mathcal{O}\big\| \le \mathbb{M}_{\mathcal{V}_2}L_2, \quad \big\|\mathcal{H}^*-\mathcal{H}\big\| \le \mathbb{M}_{\mathcal{V}_3}L_3, \quad \big\|\mathcal{Q}^*-\mathcal{Q}\big\| \le \mathbb{M}_{\mathcal{V}_4}L_4, \\ &\big\|\mathcal{R}^*-\mathcal{R}\big\| \le \mathbb{M}_{\mathcal{V}_5}L_5,\end{aligned}$$

where
$$\mathbb{M}_{\mathcal{V}_i} = \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1) - T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)\delta_i}, \quad (i \in \{2,\ldots,5\}).$$

Thus, the Ulam–Hyers stability of the fractal-fractional model of giving up smoking (3) is fulfilled. Now, set

$$\mathbb{M}_{\mathcal{V}_i}(L_i) = \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)L_i}{\Gamma(\varkappa_2+\varkappa_1) - T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)\delta_i}, \quad (i \in \{1,\ldots,5\}).$$

Thus, $\mathbb{M}_{\mathcal{V}_i}(0) = 0$. Hence, the generalized Ulam–Hyers stability is satisfied for the mentioned model (3). $\square$

Theorem 7. *Assume that the condition $(\mathcal{P}_1)$ holds and,*

$(\mathcal{P}_2)$ *There are increasing functions $\beta_i \in C([0,T],\mathbb{R})$ and $\Omega_{\beta_i} > 0$ $(i \in \{1,\ldots,5\})$ such that for each $\mathrm{t} \in \mathbb{J}$,*

$$^{\mathbb{FFP}}\mathfrak{I}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\beta_i(\mathrm{t}) < \Omega_{\beta_i}\beta_i(\mathrm{t}), \quad (i \in \{1,\ldots,5\}). \tag{21}$$

Then, the fractal-fractional model of giving up smoking (3) is Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stable.

Proof. For each constant $L_1 > 0$ and for each $\mathcal{P}^* \in \mathbb{U}$ satisfying

$$\left|{}^{\mathbb{FFP}}\mathfrak{D}_{0,\mathrm{t}}^{\varkappa_1,\varkappa_2}\mathcal{P}^*(\mathrm{t}) - \mathcal{V}_1\big(\mathrm{t},\mathcal{P}^*(\mathrm{t}),\mathcal{O}^*(\mathrm{t}),\mathcal{H}^*(\mathrm{t}),\mathcal{Q}^*(\mathrm{t}),\mathcal{R}^*(\mathrm{t})\big)\right| < L_1\beta_1(\mathrm{t}),$$

we can find the function $\hbar_1(\mathrm{t})$ such that

$$^{\mathbb{FFP}}\mathfrak{D}_{0,(\mathrm{t})}^{\varkappa_1,\varkappa_2}\mathcal{P}^*(\mathrm{t}) = \mathcal{V}_1\big(\mathrm{t},\mathcal{P}^*(\mathrm{t}),\mathcal{O}^*(\mathrm{t}),\mathcal{H}^*(\mathrm{t}),\mathcal{Q}^*(\mathrm{t}),\mathcal{R}^*(\mathrm{t})\big) + \hbar_1(\mathrm{t}),$$

with $|\hbar_1(\mathrm{t})| < L_1\beta_1(\mathrm{t})$. It follows that

$$\begin{aligned}\mathcal{P}^*(\mathrm{t}) \quad = \quad & \mathcal{P}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\mathcal{V}_1\big(\mathrm{u},\mathcal{P}^*(\mathrm{u}),\mathcal{O}^*(\mathrm{u}),\mathcal{H}^*(\mathrm{u}),\mathcal{Q}^*(\mathrm{u}),\mathcal{R}^*(\mathrm{u})\big)\,\mathrm{du} \\ & + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\hbar_1(\mathrm{u})\,\mathrm{du}.\end{aligned}$$

By using Theorem 5, let $\mathcal{P} \in \mathbb{U}$ be a unique solution of the fractal-fractional model of giving up smoking (3). Then $\mathcal{P}(\mathrm{t})$ is formulated as

$$\mathcal{P}(\mathrm{t}) = \mathcal{P}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\mathcal{V}_1\big(\mathrm{u},\mathcal{P}(\mathrm{u}),\mathcal{O}(\mathrm{u}),\mathcal{H}(\mathrm{u}),\mathcal{Q}(\mathrm{u}),\mathcal{R}(\mathrm{u})\big)\,\mathrm{du}.$$

Then, by (21), we get

$$\begin{aligned}\big|\mathcal{P}^*(\mathrm{t}) - \mathcal{P}(\mathrm{t})\big| \quad \leq \quad & \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1} \\ \times \quad & \big|\mathcal{V}_1\big(\mathrm{u},\mathcal{P}^*(\mathrm{u}),\mathcal{O}^*(\mathrm{u}),\mathcal{H}^*(\mathrm{u}),\mathcal{Q}^*(\mathrm{u}),\mathcal{R}^*(\mathrm{u})\big) \\ & -\mathcal{V}_1\big(\mathrm{u},\mathcal{P}(\mathrm{u}),\mathcal{O}(\mathrm{u}),\mathcal{H}(\mathrm{u}),\mathcal{Q}(\mathrm{u}),\mathcal{R}(\mathrm{u})\big)\big|\,\mathrm{du} \\ & + \frac{\varkappa_2}{\Gamma(\varkappa_1)}\int_0^{\mathrm{t}} \mathrm{u}^{\varkappa_2-1}(\mathrm{t}-\mathrm{u})^{\varkappa_1-1}\beta_1(\mathrm{u})\,\mathrm{du} \\ \leq \quad & L_1\Omega_{\beta_1}\beta_1(\mathrm{t}) + \frac{T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2+1)}{\Gamma(\varkappa_2+\varkappa_1)}\delta_1\big\|\mathcal{P}^* - \mathcal{P}\big\|.\end{aligned}$$

Accordingly, it gives

$$\left\|\mathcal{P}^* - \mathcal{P}\right\| \leq \frac{L_1 \Gamma(\varkappa_2 + \varkappa_1)\Omega_{\beta_1}}{\Gamma(\varkappa_2 + \varkappa_1) - T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2 + 1)\delta_1}\beta_1(\mathfrak{t}).$$

If we let

$$\mathbb{M}_{(\mathcal{V}_1,\beta_1)} = \frac{\Gamma(\varkappa_2 + \varkappa_1)\Omega_{\beta_1}}{\Gamma(\varkappa_2 + \varkappa_1) - T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2 + 1)\delta_1},$$

then, we obtain $\left\|\mathcal{P}^* - \mathcal{P}\right\| \leq L_1\mathbb{M}_{(\mathcal{V}_1,\beta_1)}\beta_1(\mathfrak{t})$. In a similar way, we also have

$$\begin{aligned} &\left\|\mathcal{O}^* - \mathcal{O}\right\| \leq L_2\mathbb{M}_{(\mathcal{V}_2,\beta_2)}\beta_2(\mathfrak{t}), \quad \left\|\mathcal{H}^* - \mathcal{H}\right\| \leq L_3\mathbb{M}_{(\mathcal{V}_3,\beta_3)}\beta_3(\mathfrak{t}), \\ &\left\|\mathcal{Q}^* - \mathcal{Q}\right\| \leq L_4\mathbb{M}_{(\mathcal{V}_4,\beta_4)}\beta_4(\mathfrak{t}), \quad \left\|\mathcal{R}^* - \mathcal{R}\right\| \leq L_5\mathbb{M}_{(\mathcal{V}_5,\beta_5)}\beta_5(\mathfrak{t}), \end{aligned}$$

where

$$\mathbb{M}_{(\mathcal{V}_i,\beta_i)} = \frac{\Gamma(\varkappa_2 + \varkappa_1)\Omega_{\beta_i}}{\Gamma(\varkappa_2 + \varkappa_1) - T^{\varkappa_2+\varkappa_1-1}\Gamma(\varkappa_2 + 1)\delta_i}, \quad (i \in \{2, \ldots, 5\}).$$

Therefore, the fractal-fractional model of giving up smoking (3) is Ulam–Hyers–Rassias stable. If $L_i = 1, \ (i \in \{1, \ldots, 5\})$, then the fractal-fractional model of giving up smoking (3) is generalized Ulam–Hyers–Rassias stable. □

4.4. Equilibrium Points

When

$${}^{\mathbb{FFP}}\mathfrak{D}^{\varkappa_1,\varkappa_2}_{0,\mathfrak{t}}\mathcal{P}(\mathfrak{t}) = {}^{\mathbb{FFP}}\mathfrak{D}^{\varkappa_1,\varkappa_2}_{0,\mathfrak{t}}\mathcal{O}(\mathfrak{t}) = {}^{\mathbb{FFP}}\mathfrak{D}^{\varkappa_1,\varkappa_2}_{0,\mathfrak{t}}\mathcal{H}(\mathfrak{t}) = {}^{\mathbb{FFP}}\mathfrak{D}^{\varkappa_1,\varkappa_2}_{0,\mathfrak{t}}\mathcal{Q}(\mathfrak{t}) = {}^{\mathbb{FFP}}\mathfrak{D}^{\varkappa_1,\varkappa_2}_{0,\mathfrak{t}}\mathcal{R}(\mathfrak{t}) = 0,$$

we can find the following results from the fractal-fractional model of giving up smoking (3).

Theorem 8. *The fractal-fractional model of giving up smoking* (3) *has at most three equilibrium points, namely the smoking-free equilibrium point* $\left(\frac{v}{\vartheta}, 0, 0, 0, 0\right)$ *and smoking equilibrium points*

$$\begin{cases} \left(\frac{v}{\vartheta+\omega\bar{\mathcal{O}}}, \frac{v}{\vartheta+\omega\bar{\mathcal{O}}}, \frac{\omega v-\vartheta^2-\vartheta\omega\bar{\mathcal{O}}}{\gamma\vartheta+\gamma\omega\bar{\mathcal{O}}}, \frac{(\vartheta+\theta)}{\zeta}\bar{\mathcal{H}}, \frac{q\theta}{\vartheta}\bar{\mathcal{H}}\right), \\ \textit{and} \\ \left(\frac{v}{\vartheta}, \frac{v}{\vartheta+\omega\bar{\mathcal{O}}}, \frac{\omega v-\vartheta^2-\vartheta\omega\bar{\mathcal{O}}}{\gamma\vartheta+\gamma\omega\bar{\mathcal{O}}}, \frac{\theta(1-q)}{(\vartheta+\zeta)}\bar{\mathcal{H}}, \frac{q\theta}{\vartheta}\bar{\mathcal{H}}\right). \end{cases}$$

Proof. Let $(\bar{\mathcal{P}}, \bar{\mathcal{O}}, \bar{\mathcal{H}}, \bar{\mathcal{Q}}, \bar{\mathcal{R}})$ denote the equilibrium point for the fractal-fractional model of giving up smoking (3). When $\bar{\mathcal{O}} = \bar{\mathcal{H}} = \bar{\mathcal{Q}} = \bar{\mathcal{R}} = 0$, then from the first equation in (3), we find that

$$\bar{\mathcal{P}} = \frac{v}{\vartheta},$$

whereas, if $\bar{\mathcal{O}} \neq 0, \ \bar{\mathcal{H}} \neq 0, \ \bar{\mathcal{Q}} \neq 0, \ \bar{\mathcal{R}} \neq 0,$ then, we obtain

$$\bar{\mathcal{P}} = \frac{v}{\vartheta + \omega\bar{\mathcal{O}}},$$

which is substituted into the second equation in (3) to give

$$\bar{\mathcal{H}} = \frac{\omega v - \vartheta^2 - \vartheta\omega\bar{\mathcal{O}}}{\gamma\vartheta + \gamma\omega\bar{\mathcal{O}}}.$$

Consequently, it follows trivially from the remaining subsequent equations that

$$\bar{\mathcal{Q}} = \frac{(\vartheta + \theta)}{\zeta}\bar{\mathcal{H}} \text{ or } \bar{\mathcal{Q}} = \frac{\theta(1 - q)}{(\vartheta + \zeta)}\bar{\mathcal{H}} \ \& \ \bar{\mathcal{R}} = \frac{q\theta}{\vartheta}\bar{\mathcal{H}},$$

and it completes the proof. □

4.5. Time-Dependent Basic Reproduction Number

In view of the derivation of the model in Equation (3), the basic reproduction number, henceforth denoted as R_0 is expected to define the expected number of secondary cases produced, in a completely potential population, by a typical smoking individual [34]. Therefore, the progression from $\mathcal{O}$ to $\mathcal{H}$ and failure to quit smoking are not considered to be new cases, but rather the progression of a smoking individual through various compartments. Hence, the following results are stated. Moreover, time-dependent variations in the transmission potential of infectious diseases are of practical importance. Consequently, in Ref. [35] it is reported that time-dependent reproduction number $R(\mathrm{t})$ measures the disease transmissibility, which can be estimated over the course of disease progression. Thus, $R(\mathrm{t})$ has been particularly useful for monitoring epidemic trends by measuring the progress of interventions over time and for providing parameters for mathematical phenomena. Hence, by following [36], one can find the following results.

Theorem 9. *The time-dependent basic reproduction number for the fractal-fractional model of giving up smoking* (3) *is*

$$R(\mathrm{t}) = \frac{\mathcal{S}(\mathrm{t})}{\mathcal{S}(0)} R_0, \text{ where, } R_0 = \frac{\omega v}{\vartheta^2}.$$

Proof. It suffices to derive the R_0. Then, the remaining part of the proof is followed easily. The respective vectors for the rate of appearance of new smokers and transfer of individuals in the model (3) are

$$\mathcal{F} = \begin{pmatrix} \omega \mathcal{P}\mathcal{O} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ \& \ \mathcal{V} = \begin{pmatrix} \vartheta\mathcal{P} + \omega\mathcal{P}\mathcal{O} - v \\ \vartheta\mathcal{O} + \gamma\mathcal{O}\mathcal{H} \\ (\vartheta+\theta)\mathcal{H} - \gamma\mathcal{O}(\mathrm{t})\mathcal{H}(\mathrm{t}) - \zeta\mathcal{Q}(\mathrm{t}) \\ (\vartheta+\zeta)\mathcal{Q}(\mathrm{t}) - \theta(1-q)\mathcal{H}(\mathrm{t}) \\ \vartheta\mathcal{R}(\mathrm{t}) - q\theta\mathcal{H}(\mathrm{t}). \end{pmatrix}.$$

Based on the smoking compartments, i.e., $\mathcal{O}, \mathcal{H}, \mathcal{Q}$ and free-smoking equilibrium, we find that

$$F = \begin{pmatrix} \frac{\omega v}{\vartheta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ \& \ V = \begin{pmatrix} \vartheta & 0 & 0 \\ 0 & (\vartheta+\theta) & -\zeta \\ 0 & -\theta(1-q) & (\vartheta+\zeta) \end{pmatrix}.$$

It gives

$$V^{-1} = \frac{1}{\vartheta[(\vartheta+\theta)(\vartheta+\zeta) - \zeta\theta(1-q)]} \begin{pmatrix} (\vartheta+\theta)(\vartheta+\zeta) - \zeta\theta(1-q) & 0 & 0 \\ 0 & \vartheta(\vartheta+\zeta) & \vartheta\theta(1-q) \\ 0 & \vartheta\zeta & \vartheta(\vartheta+\zeta) \end{pmatrix}.$$

Thus, by Ref. [37], the basic reproduction number is followed easily. □

4.6. Sensitivity Analysis

In view of theorem 9, one can see that the sensitivity analysis of $R(t)$ depends mainly on R_0. Thus, in what follows, the analysis is therefore curtailed to the sensitivity of R_0. By recalling that the sensitivity analysis enables us to predict which parameters have a high impact on the basic reproduction number [38], one of the main objectives is therefore to suggest strategies to ensure that the necessary control measures are taken to stop smoking and prevent a possible increase in the number of smokers in the future. Such attempts are, of course, attained in the direction of supporting the efforts of lowering the value of the basic reproduction number. Considering that there are many negative conditions brought about by smoking, together with the challenge of completely eliminating the smoking epidemic in a population in a short time, attempts to reduce the spread of smoking are therefore very important. Thus, lowering the value R_0 is one of the most fundamental issues, as it possesses a major influence on the effect of parameters on the change of R_0. To this end, we will evaluate the influence aspects of the parameters that affect R_0 by determining the normalized forward sensitivity index of it [38]. Starting with the first to the last parameter listed under model in Equation (2), the normalized forward sensitivity index of the variable R_0 yields the following results.

Theorem 10. *The parameters $\gamma, \zeta, \vartheta, q$ are likely to bring about the decrease in the time-dependent basic reproduction number.*

Proof. It follows trivially through R_0 that

$$\begin{cases} \frac{\partial R_0}{\partial \omega} \times \frac{\omega}{R_0} = \frac{\partial \left[\frac{\omega v}{\vartheta^2}\right]}{\partial \omega} \times \frac{\omega}{\frac{\omega v}{\vartheta^2}} = \frac{v}{\vartheta^2} \frac{\vartheta^2}{\omega v} = \omega^{-1} > 0, \\ \frac{\partial R_0}{\partial \gamma} \times \frac{\gamma}{R_0} = \frac{\partial \left[\frac{\omega v}{\vartheta^2}\right]}{\partial \gamma} \times \frac{\gamma}{\frac{\omega v}{\vartheta^2}} = 0, \\ \frac{\partial R_0}{\partial \zeta} \times \frac{\zeta}{R_0} = \frac{\partial \left[\frac{\omega v}{\vartheta^2}\right]}{\partial \zeta} \times \frac{\zeta}{\frac{\omega v}{\vartheta^2}} = 0, \\ \frac{\partial R_0}{\partial \vartheta} \times \frac{\vartheta}{R_0} = \frac{\partial \left[\frac{\omega v}{\vartheta^2}\right]}{\partial \vartheta} \times \frac{\vartheta}{\frac{\omega v}{\vartheta^2}} = -2 < 0, \\ \frac{\partial R_0}{\partial \theta} \times \frac{\theta}{R_0} = \frac{\partial \left[\frac{\omega v}{\vartheta^2}\right]}{\partial \theta} \times \frac{\theta}{\frac{\omega v}{\vartheta^2}} = 0, \\ \frac{\partial R_0}{\partial q} \times \frac{q}{R_0} = \frac{\partial \left[\frac{\omega v}{\vartheta^2}\right]}{\partial q} \times \frac{q}{\frac{\omega v}{\vartheta^2}} = 0, \\ \frac{\partial R_0}{\partial v} \times \frac{v}{R_0} = \frac{\partial \left[\frac{\omega v}{\vartheta^2}\right]}{\partial v} \times \frac{v}{\frac{\omega v}{\vartheta^2}} = \vartheta^{-1} > 0, \end{cases}$$

which concludes the proof. □

4.7. Asymptotically Stability Analysis

To investigate the local asymptotic stability for the fractal-fractional model of giving up smoking (3), one requires the Jacobian matrix [39] computed at the equilibrium points and associated characteristic equation. Let $\mathcal{E} := (\bar{\mathcal{P}}, \bar{\mathcal{O}}, \bar{\mathcal{H}}, \bar{\mathcal{Q}}, \bar{\mathcal{R}})$. Thus, the non-zero entries of the Jacobian matrix are

$$J(\mathcal{E})_{(1,1)} = -(\vartheta + \omega\bar{\mathcal{O}}),\ J(\mathcal{E})_{(1,2)} = -\omega\bar{\mathcal{P}},\ J(\mathcal{E})_{(2,2)} = \omega\bar{\mathcal{P}} - \vartheta - \gamma\bar{\mathcal{H}},$$
$$J(\mathcal{E})_{(3,2)} = \gamma\bar{\mathcal{H}},\ J(\mathcal{E})_{(3,3)} = -(\theta + \vartheta),\ J(\mathcal{E})_{(3,4)} = \zeta,\ J(\mathcal{E})_{(4,3)} = -\theta(q-1),$$
$$J(\mathcal{E})_{(4,4)} = -(\vartheta + \zeta),\ J(\mathcal{E})_{(5,3)} = q\theta,\ J(\mathcal{E})_{(5,5)} = -\vartheta,$$

and the associated characteristic equation [40] is

$$\lambda^5 - a_4\lambda^4 - a_3\lambda^3 - a_2\lambda^2 - a_1\lambda - a_0 = 0,$$

where,

$$\begin{cases} a_4 = \gamma\bar{\mathcal{O}} - 5\vartheta - \zeta - \gamma\bar{\mathcal{H}} - \theta - \omega\bar{\mathcal{O}} + \omega\bar{\mathcal{P}}, \\ a_3 = 4\bar{\mathcal{O}}\gamma\vartheta - 4\vartheta\zeta - 10\vartheta^2 - \bar{\mathcal{H}}\gamma\theta - 4\bar{\mathcal{H}}\gamma\vartheta - \bar{\mathcal{H}}\gamma\zeta - 4\theta\vartheta + \bar{\mathcal{O}}\gamma\zeta - \bar{\mathcal{O}}\omega\theta + \bar{\mathcal{P}}\omega\theta \\ \quad -4\bar{\mathcal{O}}\omega\vartheta + 4\bar{\mathcal{P}}\omega\vartheta - \bar{\mathcal{O}}\omega\zeta + \bar{\mathcal{P}}\omega\zeta - q\theta\zeta + \bar{\mathcal{O}}^2\gamma\omega - \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega - \bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega, \\ a_2 = 6\bar{\mathcal{O}}\gamma\vartheta^2 - 6\vartheta^2\zeta - 10\vartheta^3 - 6\bar{\mathcal{H}}\gamma\vartheta^2 - 6\theta\vartheta^2 - 6\bar{\mathcal{O}}\omega\vartheta^2 + 6\bar{\mathcal{P}}\omega\vartheta^2 + 3\bar{\mathcal{O}}\gamma\vartheta\zeta - 3\bar{\mathcal{O}}\omega\theta\vartheta \\ \quad +3\bar{\mathcal{P}}\omega\theta\vartheta - 3\bar{\mathcal{O}}\omega\vartheta\zeta + 3\bar{\mathcal{P}}\omega\vartheta\zeta - 3q\theta\vartheta\zeta + 3\bar{\mathcal{O}}^2\gamma\omega\vartheta + \bar{\mathcal{O}}^2\gamma\omega\zeta - 3\bar{\mathcal{H}}\gamma\theta\vartheta - 3\bar{\mathcal{H}}\gamma\vartheta\zeta \\ \quad -\bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\theta - 3\bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\vartheta - \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\zeta - 3\bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\vartheta - \bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\zeta - \bar{\mathcal{H}}\gamma q\theta\zeta - \bar{\mathcal{O}}\omega q\theta\zeta + \bar{\mathcal{P}}\omega q\theta\zeta, \\ a_1 = 4\bar{\mathcal{O}}\gamma\vartheta^3 - 4\vartheta^3\zeta - 5\vartheta^4 - 4\bar{\mathcal{H}}\gamma\vartheta^3 - 4\theta\vartheta^3 - 4\bar{\mathcal{O}}\omega\vartheta^3 + 4\bar{\mathcal{P}}\omega\vartheta^3 - 3\bar{\mathcal{H}}\gamma\theta\vartheta^2 - 3\bar{\mathcal{H}}\gamma\vartheta^2\zeta \\ \quad +3\bar{\mathcal{O}}\gamma\vartheta^2\zeta - 3\bar{\mathcal{O}}\omega\theta\vartheta^2 + 3\bar{\mathcal{P}}\omega\theta\vartheta^2 - 3\bar{\mathcal{O}}\omega\vartheta^2\zeta + 3\bar{\mathcal{P}}\omega\vartheta^2\zeta - 3q\theta\vartheta^2\zeta + 3\bar{\mathcal{O}}^2\gamma\omega\vartheta^2 \\ \quad -3\bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\vartheta^2 - 3\bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\vartheta^2 + 2\bar{\mathcal{O}}^2\gamma\omega\vartheta\zeta - 2\bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\theta\vartheta - 2\bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\vartheta\zeta - 2\bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\vartheta\zeta - 2\bar{\mathcal{H}}\gamma q\theta\vartheta\zeta \\ \quad -2\bar{\mathcal{O}}\omega q\theta\vartheta\zeta + 2\bar{\mathcal{P}}\omega q\theta\vartheta\zeta - \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega q\theta\zeta, \\ a_0 = \bar{\mathcal{O}}\gamma\vartheta^4 - \vartheta^4\zeta - \vartheta^5 - \bar{\mathcal{H}}\gamma\vartheta^4 - \theta\vartheta^4 - \bar{\mathcal{O}}\omega\vartheta^4 + \bar{\mathcal{P}}\omega\vartheta^4 - \bar{\mathcal{H}}\gamma\theta\vartheta^3 - \bar{\mathcal{H}}\gamma\vartheta^3\zeta + \bar{\mathcal{O}}\gamma\vartheta^3\zeta \\ \quad -\bar{\mathcal{O}}\omega\theta\vartheta^3 + \bar{\mathcal{P}}\omega\theta\vartheta^3 - \bar{\mathcal{O}}\omega\vartheta^3\zeta + \bar{\mathcal{P}}\omega\vartheta^3\zeta - q\theta\vartheta^3\zeta + \bar{\mathcal{O}}^2\gamma\omega\vartheta^3 + \bar{\mathcal{O}}^2\gamma\omega\vartheta^2\zeta - \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\vartheta^3 \\ \quad -\bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\vartheta^3 - \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\theta\vartheta^2 - \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\vartheta^2\zeta - \bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\vartheta^2\zeta - \bar{\mathcal{H}}\gamma q\theta\vartheta^2\zeta - \bar{\mathcal{O}}\omega q\theta\vartheta^2\zeta \\ \quad +\bar{\mathcal{P}}\omega q\theta\vartheta^2\zeta - \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega q\theta\vartheta\zeta. \end{cases}$$

Thus, if

(a) $5\vartheta + \zeta + \theta > \frac{\omega v}{\vartheta}$,
(b) $q\theta\zeta + 4\vartheta\zeta + 10\vartheta^2 + 4\theta\vartheta > (\theta + 4\vartheta + \zeta)\frac{\omega v}{\vartheta}$,
(c) $6\vartheta^2\zeta + 10\vartheta^3 + 6\theta\vartheta^2 + 3q\theta\vartheta\zeta > (6\vartheta^2 + 3\theta\vartheta + 3\vartheta\zeta + q\theta\zeta)\frac{\omega v}{\vartheta}$,
(d) $4\vartheta^2\zeta + 5\vartheta^3 + 4\theta\vartheta^2 + 3q\theta\vartheta\zeta > (4\vartheta^2 + 3\theta\vartheta + 3\vartheta\zeta + 2q\theta\zeta)\frac{\omega v}{\vartheta}$,
(e) $\vartheta^2\zeta + \vartheta^3 + \theta\vartheta^2 + q\theta\vartheta\zeta > (\vartheta^2 + \theta\vartheta + \vartheta\zeta + q\theta\zeta)\frac{\omega v}{\vartheta}$,

in that case, the smoking-free equilibrium point is locally stable if the equilibrium points are positive [41].

Similarly, one finds that if

(a)

$$5\vartheta + \zeta + \gamma\bar{\mathcal{H}} + \theta + \omega\bar{\mathcal{O}} > \gamma\bar{\mathcal{O}} + \omega\bar{\mathcal{P}},$$

(b)

$$\begin{cases} 4\bar{\mathcal{O}}\omega\vartheta + \bar{\mathcal{O}}\omega\zeta + q\theta\zeta + \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega + \bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega + 4\vartheta\zeta + 10\vartheta^2 + \bar{\mathcal{H}}\gamma\theta \\ +4\bar{\mathcal{H}}\gamma\vartheta + \bar{\mathcal{H}}\gamma\zeta + 4\theta\vartheta + \bar{\mathcal{O}}\omega\theta > 4\bar{\mathcal{O}}\gamma\vartheta + \bar{\mathcal{O}}\gamma\zeta + \bar{\mathcal{P}}\omega\theta + 4\bar{\mathcal{P}}\omega\vartheta + \bar{\mathcal{P}}\omega\zeta + \bar{\mathcal{O}}^2\gamma\omega, \end{cases}$$

(c)

$$\begin{cases} 6\vartheta^2\zeta + 10\vartheta^3 + 6\bar{\mathcal{H}}\gamma\vartheta^2 + 6\theta\vartheta^2 + 6\bar{\mathcal{O}}\omega\vartheta^2 + 3\bar{\mathcal{O}}\omega\theta\vartheta + 3\bar{\mathcal{O}}\omega\vartheta\zeta + 3q\theta\vartheta\zeta \\ +3\bar{\mathcal{H}}\gamma\theta\vartheta + 3\bar{\mathcal{H}}\gamma\vartheta\zeta + \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\theta + 3\bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\vartheta + \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\zeta + 3\bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\vartheta + \bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\zeta \\ +\bar{\mathcal{H}}\gamma q\theta\zeta + \bar{\mathcal{O}}\omega q\theta\zeta > 6\bar{\mathcal{O}}\gamma\vartheta^2 + 6\bar{\mathcal{P}}\omega\vartheta^2 + 3\bar{\mathcal{O}}\gamma\vartheta\zeta + 3\bar{\mathcal{P}}\omega\theta\vartheta + 3\bar{\mathcal{P}}\omega\vartheta\zeta + 3\bar{\mathcal{O}}^2\gamma\omega\vartheta, \\ +\bar{\mathcal{O}}^2\gamma\omega\zeta + \bar{\mathcal{P}}\omega q\theta\zeta, \end{cases}$$

(d)

$$\begin{cases} 4\vartheta^3\zeta + 5\vartheta^4 + 4\bar{\mathcal{H}}\gamma\vartheta^3 + 4\theta\vartheta^3 + 4\bar{\mathcal{O}}\omega\vartheta^3 + 3\bar{\mathcal{H}}\gamma\theta\vartheta^2 + 3\bar{\mathcal{H}}\gamma\vartheta^2\zeta \\ +3\bar{\mathcal{O}}\omega\theta\vartheta^2 + 3\bar{\mathcal{O}}\omega\vartheta^2\zeta + 3q\theta\vartheta^2\zeta + 3\bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\vartheta^2 + 3\bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\vartheta^2 \\ +2\bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\theta\vartheta + 2\bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\vartheta\zeta + 2\bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\vartheta\zeta - 2\bar{\mathcal{H}}\gamma q\theta\vartheta\zeta \\ +2\bar{\mathcal{O}}\omega q\theta\vartheta\zeta + \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega q\theta\zeta > 4\bar{\mathcal{O}}\gamma\vartheta^3 + 4\bar{\mathcal{P}}\omega\vartheta^3 \\ +3\bar{\mathcal{O}}\gamma\vartheta^2\zeta + 3\bar{\mathcal{P}}\omega\theta\vartheta^2 + 3\bar{\mathcal{P}}\omega\vartheta^2\zeta + 3\bar{\mathcal{O}}^2\gamma\omega\vartheta^2 \\ +2\bar{\mathcal{O}}^2\gamma\omega\vartheta\zeta + 2\bar{\mathcal{P}}\omega q\theta\vartheta\zeta, \end{cases}$$

(e)

$$\begin{cases} \vartheta^4\zeta + \vartheta^5 + \bar{\mathcal{H}}\gamma\vartheta^4 + \theta\vartheta^4 + \bar{\mathcal{O}}\omega\vartheta^4 + \bar{\mathcal{H}}\gamma\theta\vartheta^3 + \bar{\mathcal{H}}\gamma\vartheta^3\zeta \\ +\bar{\mathcal{O}}\omega\theta\vartheta^3 + \bar{\mathcal{O}}\omega\vartheta^3\zeta + q\theta\vartheta^3\zeta + \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\vartheta^3 + \bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\vartheta^3 + \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\theta\vartheta^2 \\ +\bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega\vartheta^2\zeta + \bar{\mathcal{O}}\bar{\mathcal{P}}\gamma\omega\vartheta^2\zeta + \bar{\mathcal{H}}\gamma q\theta\vartheta^2\zeta + \bar{\mathcal{O}}\omega q\theta\vartheta^2\zeta + \bar{\mathcal{H}}\bar{\mathcal{O}}\gamma\omega q\theta\vartheta\zeta \\ > \bar{\mathcal{O}}\gamma\vartheta^4 + \bar{\mathcal{P}}\omega\vartheta^4 + \bar{\mathcal{O}}\gamma\vartheta^3\zeta + \bar{\mathcal{P}}\omega\theta\vartheta^3 + \bar{\mathcal{P}}\omega\vartheta^3\zeta + \bar{\mathcal{O}}^2\gamma\omega\vartheta^3 \\ +\bar{\mathcal{O}}^2\gamma\omega\vartheta^2\zeta + \bar{\mathcal{P}}\omega q\theta\vartheta^2\zeta, \end{cases}$$

then the smoking equilibrium point is locally stable if the equilibrium points are positive [41].

Lemma 2. *The time-dependent basic reproduction number $R(t) < 1$ is globally stable in $\mathcal{X}$, whereas, if $R(t) > 1$, the unique smoking equilibrium point is globally asymptotically stable in the interior of $\mathcal{X}$.*

Proof. The proof of lemma 2 is similar to the proof established in [42]. □

5. Numerical Algorithm

In this section, we describe the numerical algorithm for the fractal-fractional model of giving up smoking (3). To do this, we apply the technique based on the fractal-fractional derivative operator [18]. To begin this process, we note that the system of fractal-fractional derivatives in the Riemann–Liouville sense in Equation (3) can be converted to

$$\begin{cases} {}^{RL}\mathfrak{D}_{0,t}^{\varkappa_1}\mathcal{P}(t) = \varkappa_2\tau^{\varkappa_2-1}[v - \vartheta\mathcal{P}(t) - \omega\mathcal{P}(t)\mathcal{O}(t)], \\ {}^{RL}\mathfrak{D}_{0,t}^{\varkappa_1}\mathcal{O}(t) = \varkappa_2\tau^{\varkappa_2-1}[-\vartheta\mathcal{O}(t) + \omega\mathcal{P}(t)\mathcal{O}(t) - \gamma\mathcal{O}(t)\mathcal{H}(t)], \\ {}^{RL}\mathfrak{D}_{0,t}^{\varkappa_1}\mathcal{H}(t) = \varkappa_2\tau^{\varkappa_2-1}[(-(\vartheta+\theta) + \gamma\mathcal{O}(t))\mathcal{H}(t) + \zeta\mathcal{Q}(t)], \\ {}^{RL}\mathfrak{D}_{0,t}^{\varkappa_1}\mathcal{Q}(t) = \varkappa_2\tau^{\varkappa_2-1}[-(\vartheta+\zeta)\mathcal{Q}(t) + \theta(1-q)\mathcal{H}(t)], \\ {}^{RL}\mathfrak{D}_{0,t}^{\varkappa_1}\mathcal{R}(t) = \varkappa_2\tau^{\varkappa_2-1}[-\vartheta\mathcal{R}(t) + q\theta\mathcal{H}(t)]. \end{cases} \tag{22}$$

By applying the Riemann–Liouville fractional integral on both sides of equation in (22) one obtains

$$\begin{cases} \mathcal{P}(\mathrm{t}) = \mathcal{P}(0) + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t \iota^{\varkappa_2-1}(t-\iota)^{\varkappa_1-1}[\upsilon - \vartheta\mathcal{P}(\iota) - \omega\mathcal{P}(\iota)\mathcal{O}(\iota)]d\iota, \\ \mathcal{O}(\mathrm{t}) = \mathcal{O}(0) + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t \iota^{\varkappa_2-1}(t-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{O}(\iota) + \omega\mathcal{P}(\iota)\mathcal{O}(\iota) - \gamma\mathcal{O}(\iota)\mathcal{H}(\iota)]d\iota, \\ \mathcal{H}(\mathrm{t}) = \mathcal{H}(0) + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t \iota^{\varkappa_2-1}(t-\iota)^{\varkappa_1-1}[(-(\vartheta+\theta) + \gamma\mathcal{O}(\iota))\mathcal{H}(\iota) + \zeta\mathcal{Q}(\iota)]d\iota, \\ \mathcal{Q}(\mathrm{t}) = \mathcal{Q}(0) + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t \iota^{\varkappa_2-1}(t-\iota)^{\varkappa_1-1}[-(\vartheta+\zeta)\mathcal{Q}(\iota) + \theta(1-q)\mathcal{H}(\iota)]d\iota, \\ \mathcal{R}(\mathrm{t}) = \mathcal{R}(0) + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^t \iota^{\varkappa_2-1}(t-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{R}(\iota) + q\theta\mathcal{H}(\iota)]d\iota. \end{cases} \tag{23}$$

Using a new approach at t_{n+1}, (where n denotes the denotes the number of sub-intervals) we discretize the mentioned Equation (23) for $\mathrm{t} = \mathrm{t}_{n+1}$, and we get

$$\begin{cases} \mathcal{P}(\mathrm{t}_{n+1}) = \mathcal{P}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^{t_{n+1}} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[\upsilon - \vartheta\mathcal{P}(\iota) - \omega\mathcal{P}(\iota)\mathcal{O}(\iota)]d\iota, \\ \mathcal{O}(\mathrm{t}_{n+1}) = \mathcal{O}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^{t_{n+1}} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{O}(\iota) + \omega\mathcal{P}(\iota)\mathcal{O}(\iota) - \gamma\mathcal{O}(\iota)\mathcal{H}(\iota)]d\iota, \\ \mathcal{H}(\mathrm{t}_{n+1}) = \mathcal{H}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^{t_{n+1}} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[(-(\vartheta+\theta) + \gamma\mathcal{O}(\iota))\mathcal{H}(\iota) + \zeta\mathcal{Q}(\iota)]d\iota, \\ \mathcal{Q}(\mathrm{t}_{n+1}) = \mathcal{Q}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^{t_{n+1}} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-(\vartheta+\zeta)\mathcal{Q}(\iota) + \theta(1-q)\mathcal{H}(\iota)]d\iota, \\ \mathcal{R}(\mathrm{t}_{n+1}) = \mathcal{R}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \int_0^{t_{n+1}} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{R}(\iota) + q\theta\mathcal{H}(\iota)]d\iota. \end{cases} \tag{24}$$

Approximating the obtained integrals in Equation (24), we obtain

$$\begin{cases} \mathcal{P}(\mathrm{t}_{n+1}) = \mathcal{P}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \sum_{i=0}^{n} \int_{t_i}^{t_{i-1}} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[\upsilon - \vartheta\mathcal{P}(\iota) - \omega\mathcal{P}(\iota)\mathcal{O}(\iota)]d\iota, \\ \mathcal{O}(\mathrm{t}_{n+1}) = \mathcal{O}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \sum_{i=0}^{n} \int_{t_i}^{t_{i-1}} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{O}(\iota) + \omega\mathcal{P}(\iota)\mathcal{O}(\iota) - \gamma\mathcal{O}(\iota)\mathcal{H}(\iota)]d\iota, \\ \mathcal{H}(\mathrm{t}_{n+1}) = \mathcal{H}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \sum_{i=0}^{n} \int_{t_i}^{t_{i-1}} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[(-(\vartheta+\theta) + \gamma\mathcal{O}(\iota))\mathcal{H}(\iota) + \zeta\mathcal{Q}(\iota)]d\iota, \\ \mathcal{Q}(\mathrm{t}_{n+1}) = \mathcal{Q}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \sum_{i=0}^{n} \int_{t_i}^{t_{i-1}} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-(\vartheta+\zeta)\mathcal{Q}(\iota) + \theta(1-q)\mathcal{H}(\iota)]d\iota, \\ \mathcal{R}(\mathrm{t}_{n+1}) = \mathcal{R}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \sum_{i=0}^{n} \int_{t_i}^{t_{i-1}} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{R}(\iota) + q\theta\mathcal{H}(\iota)]d\iota. \end{cases} \tag{25}$$

Applying the Lagrangian piece-wise interpolation [43] to each functions

$$\begin{cases} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[\upsilon - \vartheta\mathcal{P}(\iota) - \omega\mathcal{P}(\iota)\mathcal{O}(\iota)], \\ \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{O}(\iota) + \omega\mathcal{P}(\iota)\mathcal{O}(\iota) - \gamma\mathcal{O}(\iota)\mathcal{H}(\iota)], \\ \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[(-(\vartheta+\theta) + \gamma\mathcal{O}(\iota))\mathcal{H}(\iota) + \zeta\mathcal{Q}(\iota)], \\ \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-(\vartheta+\zeta)\mathcal{Q}(\iota) + \theta(1-q)\mathcal{H}(\iota)], \\ \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{R}(\iota) + q\theta\mathcal{H}(\iota)], \end{cases} \tag{26}$$

we find

$$\begin{cases} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[v-\vartheta\mathcal{P}(\iota)-\omega\mathcal{P}(\iota)\mathcal{O}(\iota)] \\ =\frac{\iota-t_{i-1}}{t_i-t_{i-1}}t_i^{\varkappa_2-1}[v-\vartheta\mathcal{P}(\mathrm{t_i})-\omega\mathcal{P}(\mathrm{t_i})\mathcal{O}(\mathrm{t_i})] \\ \quad -\frac{\iota-t_i}{t_i-t_{i-1}}t_{i-1}^{\varkappa_2-1}[v-\vartheta\mathcal{P}(\mathrm{t_{i-1}})-\omega\mathcal{P}(\mathrm{t_{i-1}})\mathcal{O}(\mathrm{t_{i-1}})], \\ \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{O}(\iota)+\omega\mathcal{P}(\iota)\mathcal{O}(\iota)-\gamma\mathcal{O}(\iota)\mathcal{H}(\iota)] \\ =\frac{\iota-t_{i-1}}{t_i-t_{i-1}}t_i^{\varkappa_2-1}[-\vartheta\mathcal{O}(\mathrm{t_i})+\omega\mathcal{P}(\mathrm{t_i})\mathcal{O}(\mathrm{t_i})-\gamma\mathcal{O}(\mathrm{t_i})\mathcal{H}(\mathrm{t_i})] \\ \quad -\frac{\iota-t_i}{t_i-t_{i-1}}t_{i-1}^{\varkappa_2-1}[-\vartheta\mathcal{O}(\mathrm{t_{i-1}})+\omega\mathcal{P}(\mathrm{t_{i-1}})\mathcal{O}(\mathrm{t_{i-1}})-\gamma\mathcal{O}(\mathrm{t_{i-1}})\mathcal{H}(\mathrm{t_{i-1}})], \\ \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[(-(\vartheta+\theta)+\gamma\mathcal{O}(\iota))\mathcal{H}(\iota)+\zeta\mathcal{Q}(\iota)] \\ =\frac{\iota-t_{i-1}}{t_i-t_{i-1}}t_i^{\varkappa_2-1}[(-(\vartheta+\theta)+\gamma\mathcal{O}(\mathrm{t_i}))\mathcal{H}(\mathrm{t_i})+\zeta\mathcal{Q}(\mathrm{t_i})] \\ \quad -\frac{\iota-t_i}{t_i-t_{i-1}}t_{i-1}^{\varkappa_2-1}[(-(\vartheta+\theta)+\gamma\mathcal{O}(\mathrm{t_{i-1}}))\mathcal{H}(\mathrm{t_{i-1}})+\zeta\mathcal{Q}(\mathrm{t_{i-1}})], \\ \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-(\vartheta+\zeta)\mathcal{Q}(\iota)+\theta(1-q)\mathcal{H}(\iota)] \\ =\frac{\iota-t_{i-1}}{t_i-t_{i-1}}t_i^{\varkappa_2-1}[-(\vartheta+\zeta)\mathcal{Q}(\mathrm{t_i})+\theta(1-q)\mathcal{H}(\mathrm{t_i})] \\ \quad -\frac{\iota-t_i}{t_i-t_{i-1}}t_{i-1}^{\varkappa_2-1}[-(\vartheta+\zeta)\mathcal{Q}(\mathrm{t_{i-1}})+\theta(1-q)\mathcal{H}(\mathrm{t_{i-1}})], \\ \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{R}(\iota)+q\theta\mathcal{H}(\iota)]=\frac{\iota-t_{i-1}}{t_i-t_{i-1}}t_i^{\varkappa_2-1}[-\vartheta\mathcal{R}(\mathrm{t_i})+q\theta\mathcal{H}(\mathrm{t_i})] \\ \quad -\frac{\iota-t_i}{t_i-t_{i-1}}t_{i-1}^{\varkappa_2-1}[-\vartheta\mathcal{R}(\mathrm{t_{i-1}})+q\theta\mathcal{H}(\mathrm{t_{i-1}})]. \end{cases} \tag{27}$$

Consequently,

$$\begin{aligned} \mathcal{P}(\mathrm{t}_{n+1}) &= \mathcal{P}_0+\frac{\varkappa_2}{\Gamma(\varkappa_1)}\sum_{i=0}^{n}\int_{t_i}^{t_{i-1}}\iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[\frac{\iota-t_{i-1}}{t_i-t_{i-1}}t_i^{\varkappa_2-1}[v-\vartheta\mathcal{P}(\mathrm{t_i})-\omega\mathcal{P}(\mathrm{t_i})\mathcal{O}(\mathrm{t_i})] \\ &\quad -\frac{\iota-t_i}{t_i-t_{i-1}}t_{i-1}^{\varkappa_2-1}[v-\vartheta\mathcal{P}(\mathrm{t_{i-1}})-\omega\mathcal{P}(\mathrm{t_{i-1}})\mathcal{O}(\mathrm{t_{i-1}})]]d\iota, \\ \mathcal{O}(\mathrm{t}_{n+1}) &= \mathcal{O}_0+\frac{\varkappa_2}{\Gamma(\varkappa_1)}\sum_{i=0}^{n}\int_{t_i}^{t_{i-1}}\iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{O}(\iota)+\omega\mathcal{P}(\iota)\mathcal{O}(\iota)-\gamma\mathcal{O}(\iota)\mathcal{H}(\iota)]d\iota \\ &= \frac{\iota-t_{i-1}}{t_i-t_{i-1}}t_i^{\varkappa_2-1}[-\vartheta\mathcal{O}(\mathrm{t_i})+\omega\mathcal{P}(\mathrm{t_i})\mathcal{O}(\mathrm{t_i})-\gamma\mathcal{O}(\mathrm{t_i})\mathcal{H}(\mathrm{t_i})] \\ &\quad -\frac{\iota-t_i}{t_i-t_{i-1}}t_{i-1}^{\varkappa_2-1}[-\vartheta\mathcal{O}(\mathrm{t_{i-1}})+\omega\mathcal{P}(\mathrm{t_{i-1}})\mathcal{O}(\mathrm{t_{i-1}})-\gamma\mathcal{O}(\mathrm{t_{i-1}})\mathcal{H}(\mathrm{t_{i-1}})]d\iota, \\ \mathcal{H}(\mathrm{t}_{n+1}) &= \mathcal{H}_0+\frac{\varkappa_2}{\Gamma(\varkappa_1)}\sum_{i=0}^{n}\int_{t_i}^{t_{i-1}}\iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[(-(\vartheta+\theta)+\gamma\mathcal{O}(\iota))\mathcal{H}(\iota)+\zeta\mathcal{Q}(\iota)]d\iota \\ &= \frac{\iota-t_{i-1}}{t_i-t_{i-1}}t_i^{\varkappa_2-1}[(-(\vartheta+\theta)+\gamma\mathcal{O}(\mathrm{t_i}))\mathcal{H}(\mathrm{t_i})+\zeta\mathcal{Q}(\mathrm{t_i})] \\ &\quad -\frac{\iota-t_i}{t_i-t_{i-1}}t_{i-1}^{\varkappa_2-1}[(-(\vartheta+\theta)+\gamma\mathcal{O}(\mathrm{t_{i-1}}))\mathcal{H}(\mathrm{t_{i-1}})+\zeta\mathcal{Q}(\mathrm{t_{i-1}})]d\iota, \\ \mathcal{Q}(\mathrm{t}_{n+1}) &= \mathcal{Q}_0+\frac{\varkappa_2}{\Gamma(\varkappa_1)}\sum_{i=0}^{n}\int_{t_i}^{t_{i-1}}\iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-(\vartheta+\zeta)\mathcal{Q}(\iota)+\theta(1-q)\mathcal{H}(\iota)]d\iota \end{aligned} \tag{28}$$

$$= \frac{\iota - t_{i-1}}{t_i - t_{i-1}} t_i^{\varkappa_2-1}[-(\vartheta+\zeta)\mathcal{Q}(\mathrm{t_i}) + \theta(1-q)\mathcal{H}(\mathrm{t_i})]$$

$$- \frac{\iota - t_i}{t_i - t_{i-1}} t_{i-1}^{\varkappa_2-1}[-(\vartheta+\zeta)\mathcal{Q}(\mathrm{t_{i-1}}) + \theta(1-q)\mathcal{H}(\mathrm{t_{i-1}})]d\iota,$$

$$\mathcal{R}(\mathrm{t}_{n+1}) = \mathcal{R}_0 + \frac{\varkappa_2}{\Gamma(\varkappa_1)} \sum_{i=0}^{n} \int_{t_i}^{t_{i-1}} \iota^{\varkappa_2-1}(t_{n+1}-\iota)^{\varkappa_1-1}[-\vartheta\mathcal{R}(\mathrm{t_i}) + q\theta\mathcal{H}(\mathrm{t_i})]d\iota$$

$$- \frac{\iota - t_i}{t_i - t_{i-1}} t_{i-1}^{\varkappa_2-1}[-\vartheta\mathcal{R}(\mathrm{t_{i-1}}) + q\theta\mathcal{H}(\mathrm{t_{i-1}})]d\iota.$$

The equations in (28) are equivalent to

$$\begin{cases} \mathcal{P}(\mathrm{t}_{n+1}) = \mathcal{P}_0 + \frac{\varkappa_2(\Delta t)^{\varkappa_1}}{\Gamma(\varkappa_1+2)} \sum_{i=0}^{n} t_i^{\varkappa_2-1}[\upsilon - \vartheta\mathcal{P}(\mathrm{t_i}) - \omega\mathcal{P}(\mathrm{t_i})\mathcal{O}(\mathrm{t_i})] \\ \quad \times [(n+1-i)^{\varkappa_1}(n-i+2+\varkappa_1) - (n-i)^{\varkappa_1}(n-i+2+2\varkappa_1)] \\ \quad - t_{i-1}^{\varkappa_2-1}[\upsilon - \vartheta\mathcal{P}(\mathrm{t_{i-1}}) - \omega\mathcal{P}(\mathrm{t_{i-1}})\mathcal{O}(\mathrm{t_{i-1}})] \\ \quad \times [(n+1-i)^{\varkappa_1+1} - (n-i)^{\varkappa_1}(n-i+1+\varkappa_1)], \end{cases} \tag{29}$$

and

$$\begin{cases} \mathcal{O}(\mathrm{t}_{n+1}) = \mathcal{O}_0 + \frac{\varkappa_2(\Delta t)^{\varkappa_1}}{\Gamma(\varkappa_1+2)} \sum_{i=0}^{n} t_i^{\varkappa_2-1}[-\vartheta\mathcal{O}(\mathrm{t_i}) + \omega\mathcal{P}(\mathrm{t_i})\mathcal{O}(\mathrm{t_i}) - \gamma\mathcal{O}(\mathrm{t_i})\mathcal{H}(\mathrm{t_i})] \\ \quad \times [(n+1-i)^{\varkappa_1}(n-i+2+\varkappa_1) - (n-i)^{\varkappa_1}(n-i+2+2\varkappa_1)] \\ \quad - t_{i-1}^{\varkappa_2-1}[-\vartheta\mathcal{O}(\mathrm{t_{i-1}}) + \omega\mathcal{P}(\mathrm{t_{i-1}})\mathcal{O}(\mathrm{t_{i-1}}) - \gamma\mathcal{O}(\mathrm{t_{i-1}})\mathcal{H}(\mathrm{t_{i-1}})] \\ \quad \times [(n+1-i)^{\varkappa_1+1} - (n-i)^{\varkappa_1}(n-i+1+\varkappa_1)], \end{cases} \tag{30}$$

and

$$\begin{cases} \mathcal{H}(\mathrm{t}_{n+1}) = \mathcal{H}_0 + \frac{\varkappa_2(\Delta t)^{\varkappa_1}}{\Gamma(\varkappa_1+2)} \sum_{i=0}^{n} t_i^{\varkappa_2-1}[(-(\vartheta+\theta) + \gamma\mathcal{O}(\mathrm{t_i}))\mathcal{H}(\mathrm{t_i}) + \zeta\mathcal{Q}(\mathrm{t_i})] \\ \quad \times [(n+1-i)^{\varkappa_1}(n-i+2+\varkappa_1) - (n-i)^{\varkappa_1}(n-i+2+2\varkappa_1)] \\ \quad - t_{i-1}^{\varkappa_2-1}[(-(\vartheta+\theta) + \gamma\mathcal{O}(\mathrm{t_{i-1}}))\mathcal{H}(\mathrm{t_{i-1}}) + \zeta\mathcal{Q}(\mathrm{t_{i-1}})] \\ \quad \times [(n+1-i)^{\varkappa_1+1} - (n-i)^{\varkappa_1}(n-i+1+\varkappa_1)], \end{cases} \tag{31}$$

and

$$\begin{cases} \mathcal{Q}(\mathrm{t}_{n+1}) = \mathcal{Q}_0 + \frac{\varkappa_2(\Delta t)^{\varkappa_1}}{\Gamma(\varkappa_1+2)} \sum_{i=0}^{n} t_i^{\varkappa_2-1}[-(\vartheta+\zeta)\mathcal{Q}(\mathrm{t_i}) + \theta(1-q)\mathcal{H}(\mathrm{t_i})] \\ \quad \times [(n+1-i)^{\varkappa_1}(n-i+2+\varkappa_1) - (n-i)^{\varkappa_1}(n-i+2+2\varkappa_1)] \\ \quad - t_{i-1}^{\varkappa_2-1}[-(\vartheta+\zeta)\mathcal{Q}(\mathrm{t_{i-1}}) + \theta(1-q)\mathcal{H}(\mathrm{t_{i-1}})]] \\ \quad \times [(n+1-i)^{\varkappa_1+1} - (n-i)^{\varkappa_1}(n-i+1+\varkappa_1)], \end{cases} \tag{32}$$

and

$$\begin{cases} \mathcal{R}(\mathrm{t}_{n+1}) = \mathcal{R}_0 + \frac{\varkappa_2(\Delta t)^{\varkappa_1}}{\Gamma(\varkappa_1+2)} \sum_{i=0}^{n} t_i^{\varkappa_2-1}[-\vartheta\mathcal{R}(\mathrm{t_i}) + q\theta\mathcal{H}(\mathrm{t_i})] \\ \quad \times [(n+1-i)^{\varkappa_1}(n-i+2+\varkappa_1) - (n-i)^{\varkappa_1}(n-i+2+2\varkappa_1)] \\ \quad - t_{i-1}^{\varkappa_2-1}[-\vartheta\mathcal{R}(\mathrm{t_{i-1}}) + q\theta\mathcal{H}(\mathrm{t_{i-1}})]] \\ \quad \times [(n+1-i)^{\varkappa_1+1} - (n-i)^{\varkappa_1}(n-i+1+\varkappa_1)]. \end{cases} \tag{33}$$

We refer to equations in (29)–(33) as the numerical scheme for the solutions of the fractal-fractional model of giving up smoking (3).

6. Simulations and Discussion

Simulation and discussion on the behavior of the fractal-fractional model of giving up smoking (3) are implemented in this section according to the parameters computed in Ref. [14]. Based on this source, we assume $v = 0.2$, $\vartheta = 0.04$, $\gamma = 0.3$, $\zeta = 0.25$, $\omega = 0.23$, $\theta = 2$, $q = 0.4$. The initial values are:

$$\mathcal{P}_0 = 0.60301, \quad \mathcal{O}_0 = 0.24000, \quad \mathcal{H}_0 = 0.10628, \quad \mathcal{Q}_0 = 0.03260, \quad \mathcal{R}_0 = 0.01811.$$

As a first step, to compare the best fitting parameters with our assumption parameters [14], we regenerate the total population ($\mathcal{N} = \mathcal{P} + \mathcal{O} + \mathcal{H} + \mathcal{Q} + \mathcal{R}$) by adding white Gaussian noise. Then, we apply the well-known least square technique for the regenerated total population and find the curve of best fit for the new data. The comparative results including the approximate $\mathcal{N}(t)$ by the Adams–Bashforth technique (blue dashed line), regenerated $\mathcal{N}(t)$ with noise (blue dots), and the curve of best fit for the new data (red line) are graphically represented in Figure 1. From this graphical illustration, we can observe the great agreement between the Adams–Bashforth solution of $\mathcal{N}(t)$ and the curve of best fit created from the regenerated data with white Gaussian noise. In addition, obtained root mean square error for the best fit, which is a criterion to see the goodness of the fit, is produced as 0.3198.

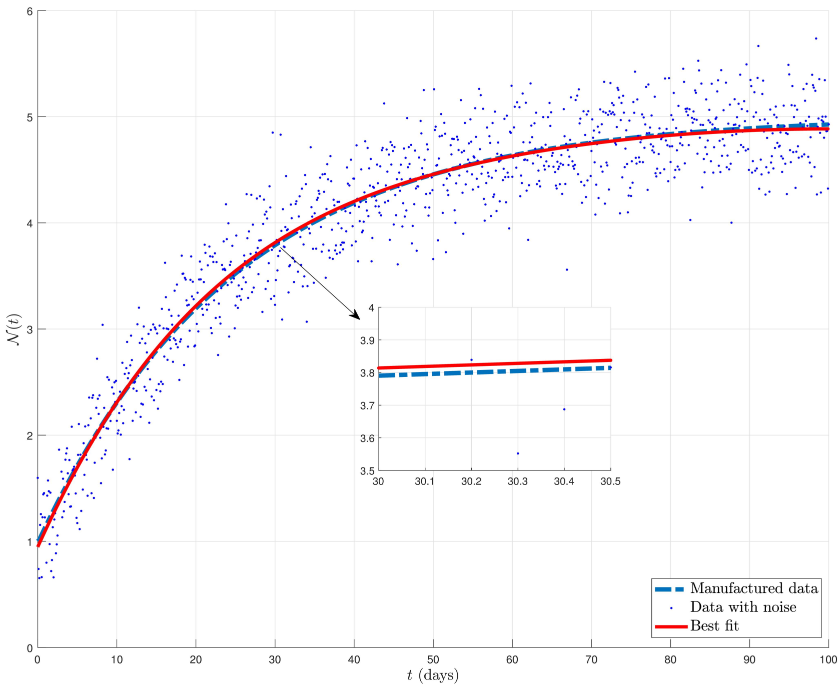

Figure 1. Comparison between the approximate total population $\mathcal{N}(t)$ with noise data and curve of best fit.

In Figure 2, we illustrate the obtained dynamics of all five state functions $\mathcal{P}, \mathcal{O}, \mathcal{H}, \mathcal{Q}, \mathcal{R}$ via the numerical technique introduced in Section 5.

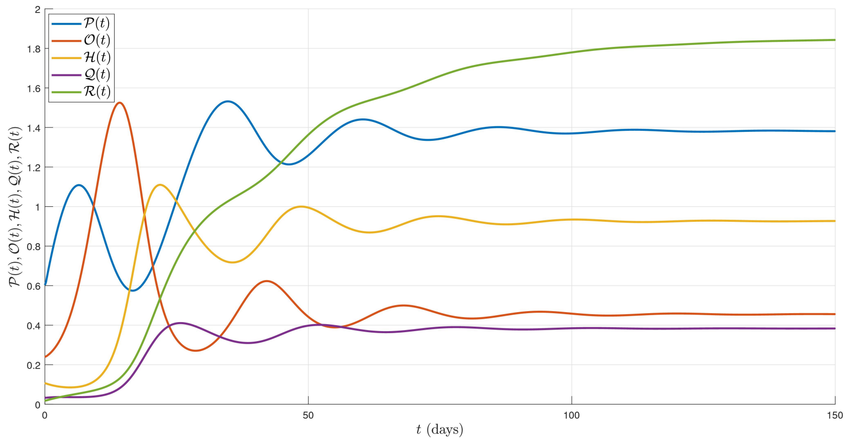

Figure 2. Behaviors of five sub-classes under fractal-fractional order $\varkappa_1 = \varkappa_2 = 1.00$.

In Figures 3–7, we illustrate the behaviors of five state functions $\mathcal{P}(t), \mathcal{O}(t), \mathcal{H}(t), \mathcal{Q}(t), \mathcal{R}(t)$, respectively, when the Adams–Bashforth technique is applied under the fractal-fractional orders $\varkappa_1 = \varkappa_2 = 0.95, 0.96, 0.97, 0.98, 1.00$. From these illustrations, we can observe that while the fractal-fractional order gets closer to the integer case, the density of each state function is increasing at about the same rate. In addition, it can be said that the fractal-fractional orders have an effect on the trajectories regarding converging to a more stable case.

In Figures 8–11, the behaviors of approximate solutions of some pairs of the state functions such as a) $\mathcal{P}(t) - \mathcal{O}(t)$, b) $\mathcal{P}(t) - \mathcal{R}(t)$, c) $\mathcal{O}(t) - \mathcal{H}(t)$, and d) $\mathcal{H}(t) - \mathcal{Q}(t)$ under the integer-order are graphically illustrated where the time $t \in [0, 150]$ and step size $h = 0.1$.

In Figures 12–14, to observe the effects of contact rates on the sub-classes, we illustrate the behaviors of approximate solutions of state functions $\mathcal{P}(t)$, $\mathcal{H}(t)$ and $\mathcal{Q}(t)$ versus the different values of contact rates γ, ω, ζ.

From Figure 12, we can observe that decreasing the contact rate between occasionally smokers and heavy smokers (γ) has a positive effect on the population of potential smokers $\mathcal{P}$; that is, the density of the potential smokers is decreasing at about the same rate. Similarly, when the contact rate between the potential smokers and occasional smokers (ω) decreases, from Figure 13, we can see that the population of heavy smokers $\mathcal{H}$ also decreases. Figure 14 shows us that increasing the contact rate between heavy smokers and temporary quitters who return back to smoking (ζ), has an effect on decreasing the population of temporary quitters who return back to smoking $\mathcal{Q}$.

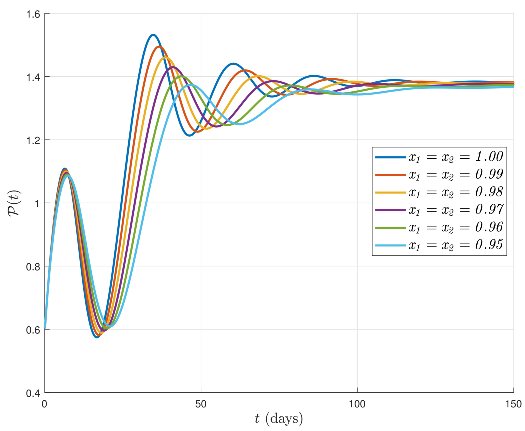

Figure 3. Behaviors of $\mathcal{P}(t)$.

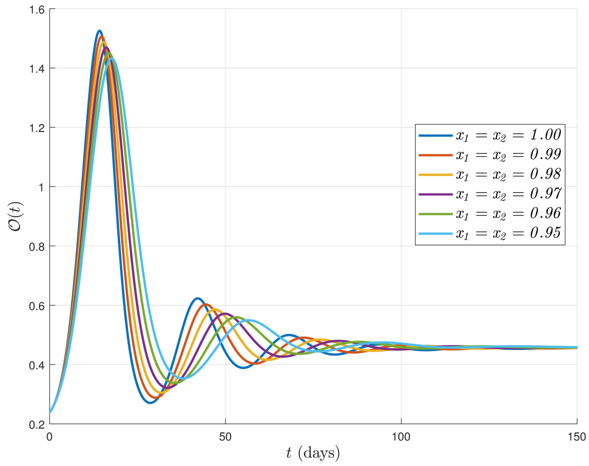

Figure 4. Behaviors of $\mathcal{O}(t)$.

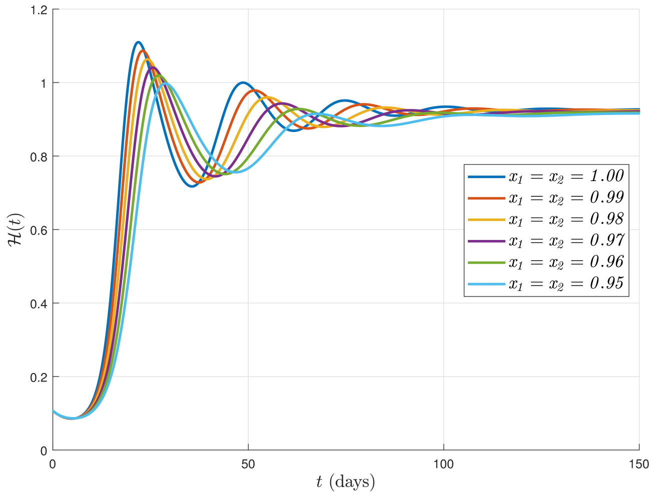

Figure 5. Behaviors of $\mathcal{H}(t)$.

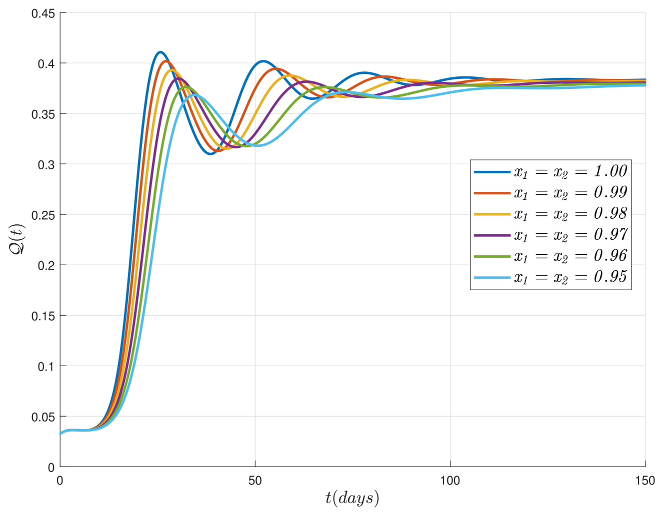

Figure 6. Behaviors of $\mathcal{Q}(t)$.

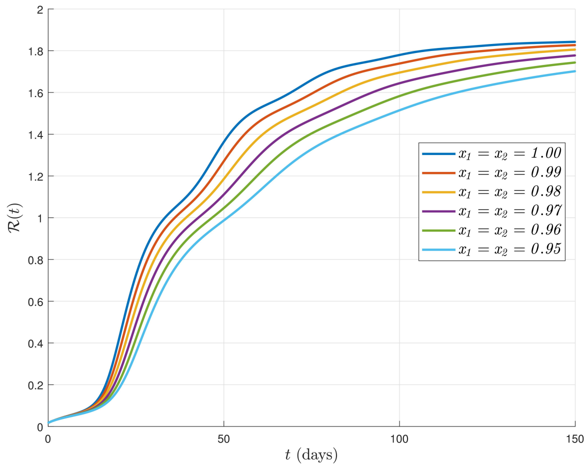

Figure 7. Behaviors of $\mathcal{R}(t)$.

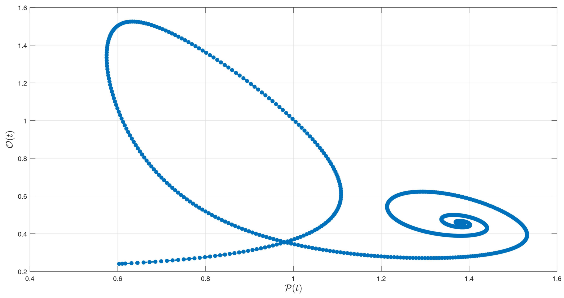

Figure 8. Behaviors of pair of sub-classes $\mathcal{P}(t) - \mathcal{O}(t)$.

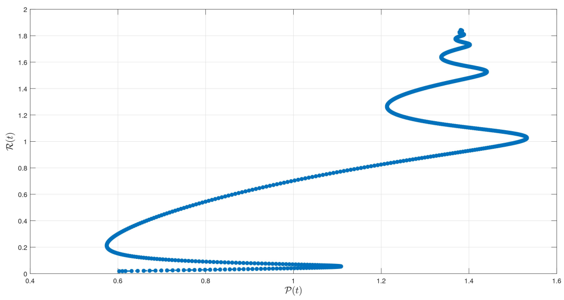

Figure 9. Behaviors of pair of sub-classes $\mathcal{P}(t) - \mathcal{R}(t)$.

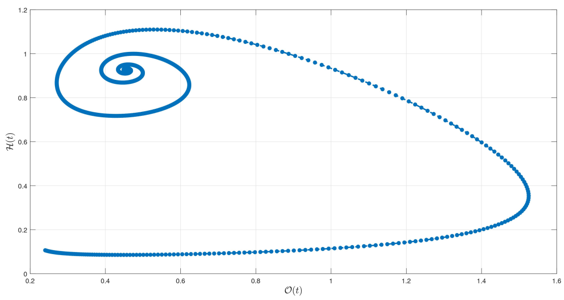

Figure 10. Behaviors of pair of sub-classes $\mathcal{O}(t) - \mathcal{H}(t)$.

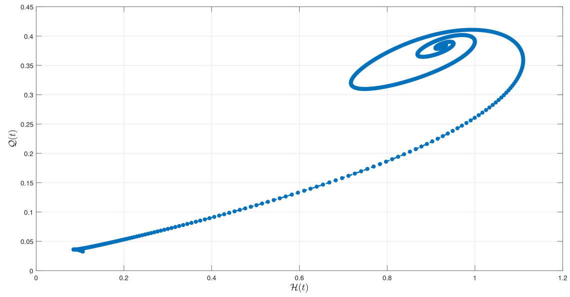

Figure 11. Behaviors of pair of sub-classes $\mathcal{H}(t) - \mathcal{Q}(t)$.

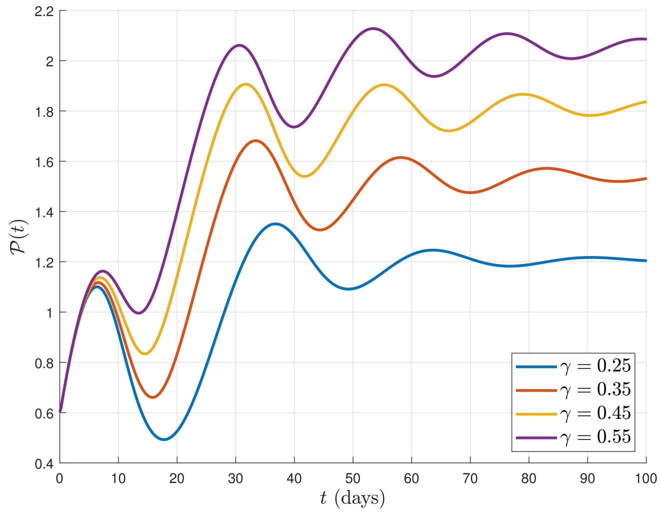

Figure 12. Effects of contact rates on state functions: $\mathcal{P}(t)$ versus γ.

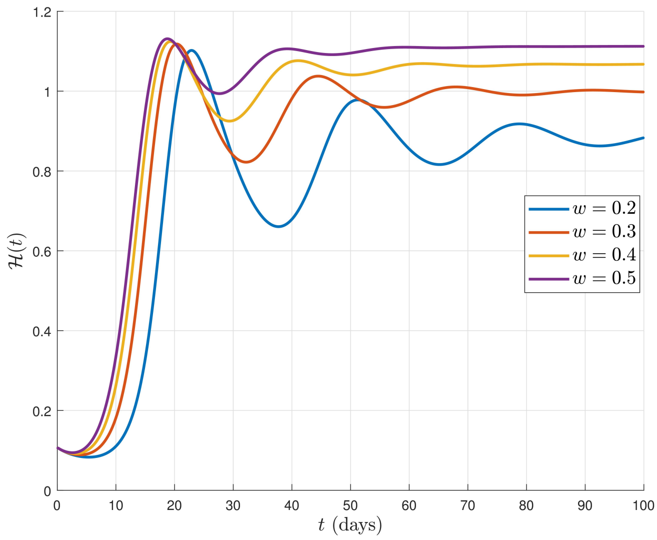

Figure 13. Effects of contact rates on state functions: $\mathcal{H}(t)$ versus ω.

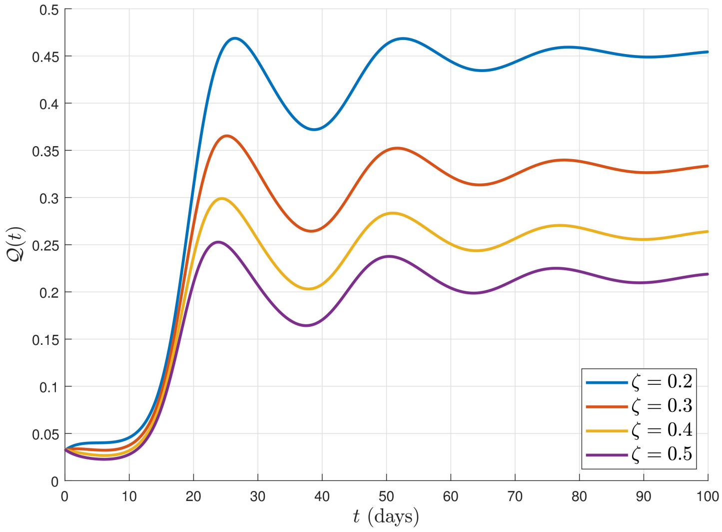

Figure 14. Effects of contact rates on state functions: $\mathcal{Q}(t)$ versus ζ.

7. Conclusions

In this research, a new mathematical model of giving up smoking was designed by defining a five-compartmental system of differential equations based on the new hybrid generalized fractal-fractional derivatives. The properties of solutions to this fractal-fractional model of giving up smoking were discussed from several points of view. A special sub-class of increasing functions along with a special kind of contractions was used to complete the existing section about the solutions.Steady-state analysis was conducted for this model and we derived a numerical scheme for the fractal-fractional model of giving up smoking by terms of fractal and fractional parameters. In other words, we derived approximate solutions of the system (3) via the Adams–Bashforth method and simulated the behaviors of each sub-classes from several aspects such as variations of fractal-fractional dimension-orders. From the illustrated results, we can see that by increasing the fractal-fractional orders, the density of each sub-population also increases. We also observed and discussed the effects of contact rates γ, ω, ζ on the behaviors of sub-classes in Section 6. All the approximate results and calculations are obtained with the help of MATLAB version R2019A. These simulations and graphs show that if we control the contact rate in each sub-class, then we can obtain significant results in reducing the number of people who quit smoking. New directions can be extended by considering other generalized kernels in the fractal-fractional operators in future research projects.

Author Contributions: Conceptualization, S.E., S.R. and R.P.A.; formal analysis, S.E., A.S., K.M.O., B.T., İ.A., S.R. and R.P.A.; methodology, S.E., A.S., K.M.O., B.T., İ.A., S.R. and R.P.A.; software, S.E., K.M.O. and İ.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated nor analyzed during the current study.

Acknowledgments: The first and sixth authors would like to thank Azarbaijan Shahid Madani University. The second author would like to thank University of Namibia. In addition, the third author author would like to thank Federal University of Technology Akure.

Conflicts of Interest: The authors declare no conflict of interest.

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