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Article

On New Matrix Version Extension of the Incomplete Wright Hypergeometric Functions and Their Fractional Calculus

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Abstract: Through this article, we will discuss a new extension of the incomplete Wright hypergeometric matrix function by using the extended incomplete Pochhammer matrix symbol. First, we give a generalization of the extended incomplete Wright hypergeometric matrix function and state some integral equations and differential formulas about it. Next, we obtain some results about fractional calculus of these extended incomplete Wright hypergeometric matrix functions. Finally, we discuss an application of the extended incomplete Wright hypergeometric matrix function in the kinetic equations.

Keywords: incomplete wright hypergeometric function; integral representation; fractional calculus; kinetic equation; pochhammer matrix symbol

MSC: 33C05; 26A33; 11S23

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1. Introduction and Preliminaries

In this century, special functions have an important place in many branches of mathematics because some sciences such as mathematical physics, probability theory, computer science, engineering and others consider the special functions as an essential tool for it (see [1–5]).

The recent advances in fractional order calculus are dominated by its multidisciplinary applications. Undoubtedly, fractional calculus has become an exciting new mathematical approach to solving various problems in mathematics, model physical, engineering, and many branches of science (see, for example [6–9] and the references therein).

Special matrix functions have an important place in solving some physics problems, and their applications are increasing and becoming an active area in recent literature including statistics, lie group and differential equations. New extensions of matrix special functions such as beta, gamma matrix functions and Gaussian hypergeometric matrix function are studied independently. In this article, $\mathbb{C}^{h\times h}$ is the vector space of h square matrices with complex entries, and we will denote the null matrix and identity matrix in $\mathbb{C}^{h\times h}$ by $\mathbf{0}$ and $\mathbf{I}$, respectively. If a matrix $\mathrm{E} \in \mathbb{C}^{h\times h}$, then, the spectrum of E is the set of all eigenvalues of E and is denoted by σ (E). A matrix $E \in \mathbb{C}^{h\times h}$ is a positive stable if $Re(\mu) > 0$ for all $\mu \in \sigma(E)$.

If $w(z)$ and $s(z)$ are holomorphic functions defined on an open set $D \subseteq \mathbb{C}$ and if E is a matrix in $\mathbb{C}^{h\times h}$ such that $\sigma(E) \subset D$, then $w(E)s(E) = s(E)w(E)$ (see [10]). Furthermore, if F is a matrix in $\mathbb{C}^{h\times h}$ such that $\sigma(E) \subset D$ and $EF = FE$, then $w(E)s(F) = s(F)w(E)$.

If E is a positive stable matrix in $\mathbb{C}^{h\times h}$, then the matrix gamma function $\Gamma(E)$ as defined by (see [10–12])

$$\Gamma(E) = \int_0^\infty t^{E-I}e^{-t} \quad dt, \quad \text{where} \quad t^{E-I} = e^{(E-I)\ln t}. \tag{1}$$

If E in $\mathbb{C}^{h\times h}$, such that

$$E + mI \text{ is invertible for all } m \geq 0\,, \tag{2}$$

then, the version Pochhammer matrix symbol is defined by (see [10,13]):

$$(E)_m = E(E+I)(E+2I)\dots(E+(m-1)I) \quad \text{where} \quad m \geq 1 \text{ and } (E)_0 = I. \tag{3}$$

From [14], if E and P are positive stable matrices in $\mathbb{C}^{h\times h}$ and E satisfy condition (2), then the extended Gamma matrix function is defined by:

$$\Gamma(E,P) = \begin{cases} \int_0^\infty t^{E-I}e^{-It-\frac{P}{t}}\mathrm{d}t & if \quad P \neq \mathbf{0}\,, \\ \Gamma(E) & if \quad P = \mathbf{0}\,. \end{cases} \tag{4}$$

and the new extended Pochhammer matrix symbol is given by:

$$(E,P)_m = \begin{cases} \Gamma^{-1}(E)\Gamma(E+mI,P) & if \quad P \neq \mathbf{0}\,, \\ (E)_m & if \quad P = \mathbf{0}\,. \end{cases} \tag{5}$$

If E is a matrix positive stable in $\mathbb{C}^{h\times h}$ and $y \in R_+$, then the incomplete and complement Gamma matrix functions as follows (see [15,16])

$$\gamma(E,y) = \int_0^y t^{E-I}e^{-t}\,dt \tag{6}$$

and

$$\Gamma(E,y) = \int_y^\infty t^{E-I}e^{-t}\,dt\,, \tag{7}$$

respectively, and they satisfy the following decomposition

$$\gamma(E,y) + \Gamma(E,y) = \Gamma(E)\,. \tag{8}$$

In [17], if E is a positive stable matrix in $\mathbb{C}^{h\times h}$ and $y \in R_+$, then we have the incomplete Pochhammer matrix symbol $(E,y)_m$ and its complement $[E,y]_m$ are defined by

$$(E,y)_m = \gamma(E+mI,y)\Gamma^{-1}(E) \tag{9}$$

and

$$[E,y]_m = \Gamma(E+mI,y)\Gamma^{-1}(E)\,, \tag{10}$$

respectively, and they hold the decomposition formula

$$(E,y)_m + [E,y]_m = (E)_m\,. \tag{11}$$

Let E and P be positive stable matrices in $\mathbb{C}^{h\times h}$ and $y \in R_+$, then the extended incomplete Gamma matrix function $\gamma(E,P;y)$ matrix function and its complement $\Gamma(E,P;y)$ are defined in [18] as follows

$$\gamma(E,P;y) = \int_0^y t^{E-I} e^{-t-\frac{P}{t}} \quad dt \tag{12}$$

and

$$\Gamma(E,P;y) = \int_y^\infty t^{E-I} e^{-t-\frac{P}{t}} \quad dt\,, \tag{13}$$

respectively, and they achieve the following decomposition

$$\gamma(E,P;y) + \Gamma(E,P;y) = \Gamma(E,P)\,. \tag{14}$$

The Laplace transform of a function ϕ(t) is defined as follows (see [19])

$$\bar{\phi}(h) = L\big[\phi(t)\big](h) = \int_0^\infty e^{-ht}\,\phi(t)\;dt, \qquad\qquad Re(h) > 0, \tag{15}$$

where $\bar{\phi}(h)$ denotes the Laplace transform of $\phi(t)$.

The essential contribution of this study is to provide a new extension of the incomplete Wright hypergeometric matrix function (EIWHMF). We generalize the definition of incomplete Pochhammer matrix function and its complement. Consequently, we produce a generalization of the incomplete hypergeometric and the incomplete Wright hypergeometric matrix functions and prove some theorem about them. In a fractional view, we discuss the Riemann–Liouville fractional integral of (EIWHMF). Further, an application of the (EIWHMF) for the fractional kinetic equations is implemented.

The rest of this paper is organized as follows. In Section 2, we will give a new extension of the incomplete Wright hypergeometric matrix function (EIWHMF) and state some theorems about integral and derivative formula of the (EIWHMF). In Section 3, we apply some theories of fractional calculus to the (EIWHMF). In the last section, we state some applications of (EIWHMF) in fractional kinetic equations.

2. Extended Incomplete Wright Hypergeometric Matrix Function EIWHMF

In this section, in terms of the general definition of the incomplete Pochhammer matrix function and its complement, also we will give a generalization of the incomplete hypergeometric matrix and the incomplete Wright hypergeometric matrix function and state some theorem about them.

Definition 1. *Let E and P be positive stable matrices in $\mathbb{C}^{h\times h}$ and $y \in R_+$; then, the extended incomplete Pochhammer matrix symbols $(E,P;y)_m$ and $[E,P;y]_m$ are defined as follows:*

$$(E,P;y)_m = \gamma(E+mI,P;y)\Gamma^{-1}(E) \tag{16}$$

and

$$[E,P;y]_m = \Gamma(E+mI,P;y)\Gamma^{-1}(E). \tag{17}$$

If we add (16) to (17), then we obtain

$$(E,P;y)_m + [E,P;y]_m = (E,P)_m. \tag{18}$$

Remark 1. *If $P = \mathbf{0}$ in (16) and (17), then we have the incomplete Pochhammer matrix symbols $(E;y)_m$ and $[E;y]_m$ as defined in (9) and (10).*

Definition 2. *The new extended incomplete Gauss hypergeometric matrix function and its complement are defined by:*

$$ {}_2\gamma_1\Big[(E,P;y),F;G;z\Big] = \sum_{m=0}^{\infty}(E,P;y)_m(F)_m(G)_m^{-1}\frac{z^m}{m!} \tag{19}$$

and

$$ {}_2\Gamma_1\Big[[E,P;y],F;G;z\Big] = \sum_{m=0}^{\infty}[E,P;y]_m(F)_m(G)_m^{-1}\frac{z^m}{m!}, \tag{20}$$

where E,F,G and P are positive stable matrices in the space $\mathbb{C}^{h\times h}$ such that G satisfies condition (2) and $y \in R_+$.

Definition 3. *Let E, F and G be positive stable matrices in $\mathbb{C}^{h\times h}$ such that G satisfies condition (2), in terms of the extended incomplete Pochhammer matrix function $\gamma(E,P,y)$ and $\Gamma(E,P,y)$ defined by (12) and (13), we defined EIWHMF as follows:*

$$ {}_2\gamma_1^{(\zeta)}\Big[(E,P,y),F;G;z\Big] = \Gamma^{-1}(F)\,\Gamma(G)\sum_{m=0}^{\infty}(E,P;y)_m\,\Gamma^{-1}(G+\zeta mI)\,\Gamma(F+\zeta mI)\frac{z^m}{m!} \tag{21}$$

and

$$ {}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;z\Big] = \Gamma^{-1}(F)\,\Gamma(G)\sum_{n=m}^{\infty}[E,P;y]_m\,\Gamma^{-1}(G+\zeta mI)\,\Gamma(F+\zeta mI)\frac{z^m}{m!}, \tag{22}$$

where $\zeta \in R_+ = (0,\infty)$.

One can notice that

$$ {}_2\gamma_1^{(\zeta)}\Big[(E,P,y),F;G;z\Big] + {}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;z\Big] = {}_2R_1^{(\zeta)}\Big[(E,P),F;G;z\Big], \tag{23}$$

where the extended Wright hypergeometric matrix function as

$$ {}_2R_1^{(\zeta)}\Big[(E,P),F;G;z\Big] = \Gamma^{-1}(F)\Gamma(G)\sum_{m=0}^{\infty}(E,P)_m\Gamma^{-1}(G+\zeta mI)\Gamma(F+\zeta mI)\frac{z^m}{m!}, \tag{24}$$

where $\zeta \in (0,\infty)$.

In view of the composition Formula (18), it is sufficient to discuss the properties of ${}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;z\Big]$.

Remark 2.

(i) *When $\zeta = 1$, the Formulas (21) and (22) are reduced to the extended matrix version of the incomplete Gauss hypergeometric functions defined in (19) and (20), respectively.*

(ii) *When $\zeta = 1$ and $y = 0$, Formulas (21) and (22) are reduced to the extended Gauss hypergeometric matrix function as*

$$ {}_2F_1[(E,P),F;G;z] = \sum_{n=0}^{\infty}(E,P)_n(F)_n[(G)_n]^{-1}\frac{z^n}{n!}. \tag{25}$$

(iii) *If we put $\zeta = 1$ and $P = \mathbf{0}$ in (21) and (22), then we obtain the incomplete Gauss hypergeometric matrix function (see [17]) .*

Integral Representation and Differentiation Formulas

Theorem 1. *Suppose that E, F, G and P are positive stable matrices in $\mathbb{C}^{h\times h}$ satisfying the condition (2), then for $|z|<1$, we have*

$$ {}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;z\Big] = \Gamma^{-1}(E)\left(\int_y^\infty t^{E-I}e^{-t-\frac{P}{t}} \quad {}_1R_1^{(\zeta)}(F;G;zt) \quad dt\right), \tag{26} $$

where

$$ {}_1R_1^{(\zeta)}(F;G;zt) = \Gamma^{-1}(F)\,\Gamma(G)\sum_{m=0}^{\infty}\Gamma^{-1}(G+m\zeta I)\Gamma(F+m\zeta I)\frac{(zt)^m}{m!}. $$

Proof. By using (17) and (22), we find that

$$ \begin{aligned} &{}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;z\Big] \\ &= \Gamma^{-1}(E)\,\Gamma^{-1}(F)\,\Gamma(G)\times\sum_{m=0}^{\infty}\ \Gamma^{-1}(G+m\zeta I)\,\Gamma(F+m\zeta I)\frac{z^m}{m!}\int_y^\infty t^{E+(m-1)I}e^{-t-\frac{P}{t}} \quad dt. \end{aligned} $$

Which can be written as

$$ \begin{aligned} &{}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;z\Big] \\ &= \quad \Gamma^{-1}(E)\int_y^\infty t^{E-I}e^{-t-\frac{P}{t}} \quad \left[\Gamma^{-1}(F)\,\Gamma(G)\sum_{m=0}^{\infty}\Gamma^{-1}(G+m\zeta I)\,\Gamma(F+m\zeta I)\frac{(zt)^m}{m!}\right] \quad dt \\ &= \Gamma^{-1}(E)\left(\int_y^\infty t^{E-I}e^{-t-\frac{P}{t}} \quad {}_1R_1^{(\zeta)}(F;G;zt) \quad dt\right), \end{aligned} $$

this completes the proof. □

Remark 3. *Note that when $y=0$ in (26), then the following relation holds true*

$$ {}_2R_1^{(\zeta)}\Big[(E,P),F;G;z\Big] = \Gamma^{-1}(E)\left(\int_0^\infty t^{E-I}e^{-t-\frac{P}{t}} \quad {}_1R_1^{(\zeta)}(F,G;zt) \quad dt\right). \tag{27} $$

Theorem 2. *All E, F, G and P are matrices in $\mathbb{C}^{h\times h}$ such that $GF=FG$ and P, G, F, E satisfy condition (2), then for $|z|<1$, we find:*

$$ \begin{aligned} &{}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;z\Big] \\ &= \Gamma^{-1}(F)\,\Gamma^{-1}(G-F)\,\Gamma(G)\times\left(\int_0^1 {}_1\Gamma_0\Big[[E,P;y];-,-;zt^\zeta\Big]\ t^{F-I}\ (1-t)^{G-F-I} \quad dt\right), \end{aligned} \tag{28} $$

where ${}_1\Gamma_0\Big[[E,P;y];-,-;zt^\zeta\Big] = \sum_{m=0}^{\infty}[E,P;y]_m\dfrac{\left(zt^\zeta\right)^m}{m!}.$

Proof. First, we notice that

$$ \begin{aligned} &\Gamma^{-1}(F)\,\Gamma^{-1}(G+m\zeta I)\,\Gamma(G)\Gamma(F+m\zeta I) \\ =&\Gamma^{-1}(F)\,\Gamma^{-1}(G-F)\,\Gamma(G)\Gamma(F+m\zeta I)\,\Gamma(G-F)\Gamma^{-1}(G+m\zeta I) \\ =&\Gamma^{-1}(F)\,\Gamma^{-1}(G-F)\,\Gamma(G)\ \int_0^1 t^{F+(m\zeta-1)I}\ (1-t)^{G-F-I} \quad dt\,, \end{aligned} $$

Now, we can write

$$
\begin{aligned}
&{}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;z\Big]\\
=&\Gamma^{-1}(F)\,\Gamma(G)\,\sum_{m=0}^{\infty}[E,P;y]_m\,\Gamma^{-1}(G+\zeta mI)\,\Gamma(F+\zeta mI)\frac{z^m}{m!}\\
=&\Gamma^{-1}(F)\,\Gamma^{-1}(G-F)\,\Gamma(G)\,\sum_{m=0}^{\infty}\int_0^1 t^{F+(m\zeta-1)I}(1-t)^{G-F-I}\,[E,P;y]_m\,\frac{z^m}{m!}\quad dt\\
=&\Gamma^{-1}(F)\,\Gamma^{-1}(G-F)\,\Gamma(G)\,\int_0^1 t^{F-I}(1-t)^{G-F-I}\,\sum_{m=0}^{\infty}[E,P;y]_m\,\frac{(zt^{\zeta})^m}{m!}\quad dt\\
=&\Gamma^{-1}(F)\,\Gamma^{-1}(G-F)\,\Gamma(G)\times\left(\int_0^1 {}_1\Gamma_0\Big[[E,P;y];-,-;zt^{\zeta}\Big]t^{F-I}\,(1-t)^{G-F-I}\quad dt\right).
\end{aligned}
$$

These end the proof. □

Theorem 3. *The derivative formula for* ${}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;z\Big]$ *holds true*

$$
\begin{aligned}
&\frac{d^m}{dz^m}\left\{{}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;z\Big]\right\}\\
&=(E)_m\,\Gamma^{-1}(G+m\zeta I)\,\Gamma^{-1}(F)\,\Gamma(G)\,\Gamma(F+m\zeta I)\,{}_2\Gamma_1^{(\zeta)}\Big[[E+mI,P;y],F+m\zeta I,G+m\zeta I;z\Big].
\end{aligned}
\tag{29}
$$

Proof. By differentiating both sides of (22), we find that

$$
\begin{aligned}
&\frac{d}{dz}\left\{{}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;z\Big]\right\}\\
&=\Gamma^{-1}(F)\,\Gamma(G)\,\sum_{m=1}^{\infty}[E,P;y]_m\,\Gamma^{-1}(G+m\zeta I)\,\Gamma(F+m\zeta I)\,\frac{z^{m-1}}{(m-1)!}\\
&=\Gamma^{-1}(F)\,\Gamma(G)\,\sum_{m=0}^{\infty}[E,P;y]_{m+1}\Gamma^{-1}(G+(m+1)\zeta I)\Gamma(F+(m+1)\zeta I)\frac{z^m}{m!}\\
&=E\quad\Gamma^{-1}(F)\,\Gamma^{-1}(G+\zeta I)\,\Gamma^{-1}(F+\zeta I)\,\Gamma(G)\,\Gamma(F+\zeta I)\,\Gamma(G+\zeta I)\\
&\times\sum_{m=0}^{\infty}[E+I,P;y]_m\;\Gamma^{-1}(G+(m+1)\zeta I)\,\Gamma(F+(m+1)\zeta I)\,\frac{z^m}{m!}\\
&=E\,\Gamma^{-1}(F)\,\Gamma^{-1}(G+\zeta I)\Gamma(G)\,\Gamma(F+\zeta I)\,{}_2\Gamma_1^{(\zeta)}\Big[[E+I,P;y],F+\zeta I,G+\zeta I;z\Big].
\end{aligned}
$$

By using the mathematical induction on m, we obtain the required result (29). This finishes the proof. □

Theorem 4. *Assume that E, F, G and P are positive stable matrices in* $\mathbb{C}^{h\times h}$*. Then, we have the following derivative formula,*

$$
\begin{aligned}
&\left(\frac{d}{dz}\right)^n\left\{z^{G-I}\,{}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G;\alpha z^{\zeta}\Big]\right\}\\
&=\Gamma^{-1}(G-nI)\,\Gamma(G)\,z^{G-(n+1)I}\,{}_2\Gamma_1^{(\zeta)}\Big[[E,P;y],F;G-nI;\alpha z^{\zeta}\Big]
\end{aligned}
\tag{30}
$$

Proof. From using Definition 3 and differentiating term by term, we obtain

$$\left(\frac{d}{dz}\right)^n \left\{ z^{G-I}\, {}_2\Gamma_1^{(\zeta)}\left[[E,P;y],F;G;\alpha z^{\zeta}\right] \right\}$$
$$= \Gamma^{-1}(F)\,\Gamma(G)\,\sum_{m=0}^{\infty}[E,P;y]_m\,\Gamma^{-1}(G+\zeta mI)\,\Gamma(F+\zeta mI)\,\frac{\alpha^m}{m!}\left(\frac{d}{dz}\right)^n z^{G+(\zeta m-1)I}$$
$$= \Gamma^{-1}(F)\,\Gamma(G)\,\sum_{m=0}^{\infty}[E,P;y]_m\Gamma^{-1}(G+(\zeta m-n)I)\Gamma(F+\zeta mI)\,\frac{\alpha^m}{m!}\,z^{G+(\zeta m-n-1)I}$$
$$= z^{G-(n+1)I}\,\Gamma^{-1}(F)\,\Gamma(G)\,\sum_{m=0}^{\infty}[E,P;y]_m\,\Gamma^{-1}(G+\zeta m-n)I)\,\Gamma(F+\zeta mI)\quad\frac{(\alpha z^{\zeta})^m}{m!}$$
$$= z^{G-(n+1)I}\,\Gamma^{-1}(G-nI)\,\Gamma(G)\,{}_2\Gamma_1^{(\zeta)}\left[[E,P;y],F;G-nI;\alpha z^{\zeta}\right].$$

This finishes the proof. □

Theorem 5. *The extended incomplete gamma matrix function achieves the following relation:*

$${}_2\Gamma_1^{(\zeta)}\left[[E,P;y],F;F;z\right] = \Gamma^{-1}(E)(1-z)^{-E}\,\Gamma\Big(E,P(1-z);y(1-z)\Big)\quad,(|z|<1,y\geq 0) \tag{31}$$

Proof. If we put $G=F$ in (26), then we find that

$$\begin{aligned}{}_2\Gamma_1^{(\zeta)}\left[[E,P;y],F;F;z\right] &= \Gamma^{-1}(E)\times\sum_{n=0}^{\infty}\int_y^{\infty}\frac{(zt)^n}{n!}t^{E-I}e^{-t-\frac{P}{t}}\quad dt\\ &= \Gamma^{-1}(E)\times\int_y^{\infty}t^{E-I}e^{-t-\frac{P}{t}}\sum_{n=0}^{\infty}\frac{(zt)^n}{n!}\quad dt\\ &= \Gamma^{-1}(E)\times\int_y^{\infty}t^{E-I}e^{-t(1-z)-\frac{P}{t}}\quad dt.\end{aligned}$$

Substitute $t(1-z)=u$, we have

$$\begin{aligned}&= \Gamma^{-1}(E)(1-z)^{-E}\times\int_{y(1-z)}^{\infty}u^{E-I}e^{-u-\frac{P(1-z)}{u}}\quad du\\ &= \Gamma^{-1}(E)(1-z)^{-E}\,\Gamma\Big(E,P(1-z);y(1-z)\Big).\end{aligned}$$

This completes the proof. □

3. Fractional Calculus of the EIWHMF

In this section, we will discuss some theorems about the fractional Riemann–Liouville integral of the EIWHMF.

The fractional integral and derivative of Riemann–Liouville of order μ and $y>0$ are given, respectively, as follows (see [1,12]):

$${}_0D_y^{-\mu}[f(y)] = I^{\mu}[f(y)] = \frac{1}{\Gamma(\mu)}\int_0^y (y-t)^{\mu-1}f(t)\quad dt \tag{32}$$

and

$$D^{\mu}f(y) = D^n\left[I^{n-\mu}f(y)\right]\quad,D=\frac{d}{dy}. \tag{33}$$

In [19], the Laplace transform for Riemann–Liouville fractional integral is given as follows

$$L\left[{}_0D_t^{-\mu}\phi(t)\right](h) = h^{-\mu}\bar{\phi}(h), \tag{34}$$

where $\bar{\phi}(h)$ denotes the Laplace transform of $\phi(t)$.

If E is a positive stable matrix in $\mathbb{C}^{h\times h}$, and $Re(\mu) > 0$, then the following relation holds true (see [20,21]):

$$I^{\mu}\left(y^{E-I}\right) = \Gamma(E)\,\Gamma^{-1}(E+\mu I)\,y^{E+(\mu-1)I}. \tag{35}$$

Theorem 6. *Assume that E, F, G are positive stable matrices in $\mathbb{C}^{h\times h}$ and $\zeta > 0$, $\mu \in \mathbb{C}$ such that $Re(\mu) > 0$ and $|wz^{\zeta}| < 1$, then we obtain*

$$\begin{aligned} &I^{\mu}\left\{z^{G-I}\,{}_2\Gamma_1^{(\zeta)}\left[[E,P;y],F;G;wz^{\zeta}\right]\right\} \\ &= \Gamma(G)\,\Gamma^{-1}(G+\mu I)\,z^{G+(\mu-1)I} \times {}_2\Gamma_1^{(\zeta)}\left[[E,P;y],F;G+\mu I;wz^{\zeta}\right]. \end{aligned} \tag{36}$$

Proof. Substituting (22) in the left-hand side of (36), we find that

$$\begin{aligned} &I^{\mu}\left\{z^{G-I}\,{}_2\Gamma_1^{(\zeta)}\left[[E,P;y],F;G;wz^{\zeta}\right]\right\} \\ &= I^{\mu}\left\{\Gamma^{-1}(F)\,\Gamma(G) \times \sum_{m=0}^{\infty}[E,p;y]_m\Gamma^{-1}(G+\zeta mI)\,\Gamma(F+\zeta mI)\,\frac{w^m}{m!}\,z^{G+(\zeta m-1)I}\right\} \\ &= \Gamma^{-1}(F)\,\Gamma(G) \times \sum_{m=0}^{\infty}[E,p;y]_m\,\Gamma^{-1}(G+\zeta mI)\,\Gamma(F+\zeta mI)\,\frac{w^m}{m!}\,I^{\mu}\left\{z^{G+(\zeta m-1)I}\right\} \\ &= \Gamma^{-1}(F)\,\Gamma(G) \times \sum_{m=0}^{\infty}\left[[E,p;y]_m\Gamma^{-1}(G+\zeta mI)\Gamma(F+\zeta mI)\,\frac{w^m}{m!}\right. \\ &\left.\times\,\Gamma(G+m\zeta I)\Gamma^{-1}(G+(\zeta m+\mu)I)\,z^{G+(\mu+\zeta m-1)I}\right] \\ &= \Gamma^{-1}(F)\,\Gamma(G) \times \sum_{m=0}^{\infty}\left[[E,p;y]_m\Gamma(F+\zeta mI) \times \Gamma^{-1}(G+(\zeta m+\mu)I)\,\frac{(wz^{\zeta})^m}{m!}\,z^{G+(\mu-1)I}\right] \\ &= z^{G+(\mu-1)I}\,\Gamma(G)\,\Gamma^{-1}(G+\mu I) \\ &\times\left[\Gamma(G+\mu I)\,\Gamma^{-1}(F) \times \sum_{m=0}^{\infty}[E,p;y]_m\Gamma^{-1}(G+\mu I+\zeta mI)\,\Gamma(F+\zeta mI)\,\frac{w^m}{m!}\right] \\ &= \Gamma(G)\,\Gamma^{-1}(G+\mu I)\,z^{G+(\mu-1)I}\,{}_2\Gamma_1^{(\zeta)}\left[[E,P;y],F;G+\mu I;wz^{\zeta}\right]. \end{aligned}$$

This finishes the proof □

Theorem 7. *Suppose that E, F, G are positive stable matrices in $\mathbb{C}^{h\times h}$ and $\zeta > 0$, $\mu \in \mathbb{C}$ such that $Re(\mu) > 0$ and $|wz^{\zeta}| < 1$, then, the following holds true*

$$\begin{aligned} &D^{\mu}\left\{z^{G-I}\,{}_2\Gamma_1^{(\zeta)}\left[[E,P;y],F;G;wz^{\zeta}\right]\right\} \\ &= \Gamma(G)\,\Gamma^{-1}(G-\mu I)\,z^{G-(\mu+1)I} \times {}_2\Gamma_1^{(\zeta)}\left[[E,P;y],F;G-\mu I;wz^{\zeta}\right]. \end{aligned} \tag{37}$$

Proof. Applying (33), we find that

$$D^{\mu}\left\{z^{G-I}\,{}_{2}\Gamma_{1}^{(\zeta)}\left[[E,P;y],F;G;wz^{\zeta}\right]\right\}$$
$$=\left(\frac{d}{dz}\right)^{n}\left\{I^{n-\mu}\left[z^{G-I}\,{}_{2}\Gamma_{1}^{(\zeta)}\left[[E,P;y],F;G;wz^{\zeta}\right]\right]\right\},$$

by using Theorem (6), we obtain

$$D^{\mu}\left\{z^{G-I}\,{}_{2}\Gamma_{1}^{(\zeta)}\left[[E,P;y],F;G;wz^{\zeta}\right]\right\}$$
$$=\left(\frac{d}{dz}\right)^{n}\left\{z^{G+(n-\mu-1)I}\,\Gamma(G)\,\Gamma^{-1}(G+(n-\mu)I)\,{}_{2}\Gamma_{1}^{(\zeta)}\left[[E,P;y],F;G+(n-\mu)I;wz^{\zeta}\right]\right\}.$$

Applying Theorem (4), we obtain

$$D^{\mu}\left\{z^{G-I}\,{}_{2}\Gamma_{1}^{(\zeta)}\left[[E,P;y],F;G;wz^{\zeta}\right]\right\}$$
$$=\Gamma(G)\,\Gamma^{-1}(G-\mu I)\,z^{G-(\mu+1)I}\times{}_{2}\Gamma_{1}^{(\zeta)}\left[[E,P;y],F;G-\mu I;wz^{\zeta}\right].$$

This completes the proof □

4. Applications: Kinetic Equations

In recent years, the solution of the fractional kinetic equations has attracted the attention many workers due to their importance in the field of applied science, such as physics, dynamical systems, control systems, and engineering, to create the mathematical model of many physical phenomena and mathematical physics. In certain astrophysical problems, the kinetic equations describe the continuity of motion of substance and are the basic equations of mathematical physics and natural science. The extension and generalization of fractional kinetic equations involving various fractional calculus operators were found (for example [22,23]).

Haubold and Mathai in [22] have established a functional differential equation between rate of change of reaction, the destruction rate, and the production rate as follows

$$\frac{dR}{dt}=-d(R_t)+p(R_t), \tag{38}$$

where $R=R(t)$ is the rate of reaction, $d=d(R)$ is the rate of destruction, $p=p(R)$ is the rate of production, and R_t denotes the function defined by $R_t(t^*)=R(t-t^*), t^*>0$.

A special case of (38), when spatial fluctuations or inhomogeneities in the quantity $R(t)$ are neglected, is given by the following differential equation as

$$\frac{dR}{dt}=-m_{\,i}R_i(t), \tag{39}$$

together with the initial condition that $R_i(t=0)=R_0$, is the number of density of species i at time $t=0$, $m_i>0$.

If the index i is dropped, and the typical kinetic Equation (39) is integrated, we receive

$$R(t)-R_0=-m\,{}_0D_t^{-1}R(t), \tag{40}$$

where m is a constant, and ${}_0D_t^{-1}$ is the Riemann–Liouville integral operator of order $\mu = 1$. The fractional kinetic equation (FKE) is redefined by Haubold and Mathai as following (see [22])

$$R(t) - R_0 = -m^{\mu}\, {}_0D_t^{-\mu} R(t), \tag{41}$$

where ${}_0D_t^{-\mu}$ defined in (32).

Then, the solution for $R(t)$ is given by

$$R(t) = R_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(\mu r + 1)} (mt)^{\mu r} = R_0\, E_{\mu}(-m^{\mu} t^{\mu}), \tag{42}$$

where $E_{\mu}(-m^{\mu}t^{\mu})$ denotes the Mittag–Leffler function (see [24,25]).

In addition, Saxena Kalla thought about the subsequent fractional kinetic equation (see [23,26–28])

$$R(t) - R_0 f(t) = -m^{\mu}\, {}_0D_t^{-\mu} R(t), \qquad m > 0,\ Re(\mu) > 0, \tag{43}$$

where $R(t)$ denotes the number density of a given species at time t, $R_0 = R(0)$ is the number density of that species at time $t = 0$, m is a constant, and f is an integrable function on $(0, \infty)$.

Very recently, several different papers appeared to solve the fractional kinetic equations by using different integral transforms, such as Laplace, Fourier, Sumudu and Mellin transforms with special functions and a matrix function, (see [26,29–33]).

Now, in the following section, we derive the solutions of fractional kinetic equations involving the extension of the incomplete Wright Hypergeometric matrix functions. Further, we established various special cases.

Theorem 8. *Assume that E, F, G and M are positive stable matrices in $\mathbb{C}^{h\times h}$ such that F, G and M satisfy condition (2), and $\zeta \in R_+$. Then, for $Re(\mu) > 0$ and $t \in \mathbb{C}$, the generalized fractional kinetic matrix equation*

$$R(t)I - R_0\, {}_2\Gamma_1^{(\zeta)}\Big[\big[E, P; y\big], F; G; t\Big] = -M^{\mu}\, {}_0D_t^{-\mu} R(t), \tag{44}$$

has a solution

$$\begin{aligned} R(t)I &= R_0\, \Gamma^{-1}(F)\, \Gamma(G) \\ &\times \sum_{m=0}^{\infty} \big[E, P; y\big]_m\, \Gamma^{-1}(G + \zeta m I)\, \Gamma(F + \ mI) \\ &\times\ t^m\, E_{\mu, m+1}(-M^{\mu} t^{\mu}), \end{aligned} \tag{45}$$

where $E_{\mu,m+1}(-M^{\mu}t^{\mu}) = \sum_{r=0}^{\infty} (-1)^r M^{\mu r} \frac{t^{\mu r}}{\Gamma(\mu r + m + 1)}$ and called the generalized the Mittag–Leffler function (see [25]).

Proof. From (15), (34) and by using Laplace transform in Equation (44), we obtain

$$L\Big[R(t)I\Big](h) = R_0\, L\Bigg[{}_2\Gamma_1^{(\zeta)}\Big[\big[E, P; y\big], F; G; t\Big]\Bigg](h) - M^{\mu}\, L\Big[{}_0D_t^{-\mu} R(t)\Big](h)$$

$$\bar{R}(h)I = R_0 \int_0^{\infty} e^{-ht}\, \Gamma^{-1}(F)\, \Gamma(G) \sum_{m=0}^{\infty} \big[E, P; y\big]_m \Gamma^{-1}(G + \zeta m I)\Gamma(F + \zeta m I) \frac{t^m}{m!}\, dt - M^{\mu} h^{-\mu} R(h),$$

and we can write

$$\begin{aligned}
&\left[I + M^{\mu} h^{-\mu}\right] \bar{R}(h) \\
&= R_0\, \Gamma^{-1}(F)\, \Gamma(G) \sum_{m=0}^{\infty} \left[E, P; y\right]_m \Gamma^{-1}(G + \zeta m I)\Gamma(F + \zeta m I) \frac{1}{m!} \int_0^{\infty} e^{-ht} t^m \quad dt \\
&= R_0\, \Gamma^{-1}(F)\, \Gamma(G) \sum_{m=0}^{\infty} \left[E, P; y\right]_m \Gamma^{-1}(G + \zeta m I)\, \Gamma(F + \zeta m I)\, \frac{1}{h^{m+1}} \\
&= R_0\, \Gamma^{-1}(F)\, \Gamma(G) \sum_{m=0}^{\infty} \left[E, P; y\right]_m \Gamma^{-1}(G + \zeta m I)\Gamma(F + \zeta m I) \left[h^{-(m+1)}\right],
\end{aligned}$$

this can be writen as

$$\begin{aligned}
&\bar{R}(h) I \\
&= R_0\, \Gamma^{-1}(F)\, \Gamma(G) \sum_{m=0}^{\infty} \left[E, P; y\right]_m \Gamma^{-1}(G + \zeta m I)\, \Gamma(F + \zeta m I) \left[h^{-(m+1)}\right] \\
&\times \left[I + M^{\mu} h^{-\mu}\right]^{-1} \\
&= R_0\, \Gamma^{-1}(F)\, \Gamma(G) \sum_{m=0}^{\infty} \left[E, P; y\right]_m \Gamma^{-1}(G + \zeta m I)\, \Gamma(F + \zeta m I) \left[h^{-(m+1)}\right] \\
&\times \sum_{r=0}^{\infty} (-1)^r \left[\left(\frac{M}{h}\right)^{\mu}\right]^r.
\end{aligned}$$

Taking the inverse Laplace transform to the above result, we obtain

$$\begin{aligned}
L^{-1}\left\{R(h) I\right\} &= R_0\, \Gamma^{-1}(F)\, \Gamma(G) \sum_{m=0}^{\infty} \left[E, P; y\right]_m \Gamma^{-1}(G + \zeta m I)\, \Gamma(F + \zeta m I) \\
&\quad \times L^{-1}\left\{\sum_{r=0}^{\infty} (-1)^r M^{\mu r} h^{-(\mu r + m + 1)}\right\} \\
R(t) I &= R_0\, \Gamma^{-1}(F)\, \Gamma(G) \sum_{m=0}^{\infty} \left[E, P; y\right]_m \Gamma^{-1}(G + \zeta m I)\, \Gamma(F + \zeta m I) \\
&\quad \times \left\{\sum_{r=0}^{\infty} (-1)^r M^{\mu r} \frac{t^{\mu r + m}}{\Gamma(\mu r + m + 1)}\right\} \\
&= R_0\, \Gamma^{-1}(F)\, \Gamma(G) \sum_{m=0}^{\infty} \left[E, P; y\right]_m \Gamma^{-1}(G + \zeta m I)\, \Gamma(F + \zeta m I)\, t^m \\
&\quad \times \left\{\sum_{r=0}^{\infty} (-1)^r M^{\mu r} \frac{t^{\mu r}}{\Gamma(\mu r + m + 1)}\right\} \\
&= R_0\, \Gamma^{-1}(F)\, \Gamma(G) \sum_{m=0}^{\infty} \left[E, P; y\right]_m \Gamma^{-1}(G + \zeta m I)\, \Gamma(F + \zeta m I)\, t^m \\
&\times E_{\mu, m+1}(-M^{\mu} t^{\mu}),
\end{aligned}$$

this finishes the proof. □

Remark 4.

(1) *If $P = \mathbf{0}$, then the extended incomplete Wright hypergeometric matrix function* ${}_2\Gamma_1^{(\zeta)}\Big[[E,p;y],F;G;t\Big]$ *is reduced to the incomplete Wright hypergeometric matrix function* ${}_2\Gamma_1^{(\zeta)}\Big[[E;y],F;G;t\Big]$ *(see [20]), and Equations (44) and (45) become as following*

Corollary 1. *Assume that E, F, G and M are positive stable matrices in $\mathbb{C}^{h\times h}$ such that F, G and M satisfy condition (2), and $\zeta \in R_+$, then for $Re(\mu) > 0$ and $t \in \mathbb{C}$, the generalized fractional kinetic matrix equation*

$$R(t)I - R_0\,{}_2\Gamma_1^{(\zeta)}\Big[[E;y],F;G;t\Big] = -M^\mu\,{}_0D_t^{-\mu}R(t), \tag{46}$$

has a solution

$$\begin{aligned} R(t)I = &R_0\Gamma^{-1}(F)\,\Gamma(G)\sum_{m=0}^{\infty}[E;y]_m\,\Gamma^{-1}(G+\zeta mI)\,\Gamma(F+\zeta mI)\,t^m \\ &\times E_{\mu,m+1}(-M^\mu t^\mu). \end{aligned}$$

(2) *If $P = \mathbf{0}$ and $\zeta = 1$, then the extended incomplete Wright hypergeometric matrix function* ${}_2\Gamma_1^{(\zeta)}\Big[[E,p;y],F;G;t\Big]$ *is reduced to the incomplete Gauss hypergeometric matrix function* ${}_2\Gamma_1\Big[[E;y],F;G;t\Big]$ *(see [17]), and Equations (44) and (45) reduce to the following forms*

Corollary 2. *Suppose that E, F, G and M are positive stable matrices in $\mathbb{C}^{h\times h}$ such that F, G and M satisfy condition (2), and $\zeta \in R_+$, then for $Re(\mu) > 0$ and $t \in \mathbb{C}$, the generalized fractional kinetic matrix equation*

$$R(t)I - R_0\,{}_2\Gamma_1\Big[[E;y],F;G;t\Big] = -M^\mu\,{}_0D_t^{-\mu}R(t), \tag{47}$$

has a solution

$$R(t)I = R_0\sum_{m=0}^{\infty}[E;y]_m\,(F)_m\,[(G)_m]^{-1}\,t^m \times E_{\mu,m+1}(-M^\mu t^\mu).$$

(3) *If $\zeta = 1$ and $y = 0$, then the extended incomplete Wright hypergeometric matrix function* ${}_2\Gamma_1^{(\zeta)}\Big[[E,p;y],F;G;t\Big]$ *reduces to the Gauss hypergeometric matrix function* ${}_2F_1\Big[(E,P),F;G;t\Big]$ *defined in (25), and Equations (44) and (45) reduce to the following forms*

Corollary 3. *Suppose that E, F, G and M are positive stable matrices in $\mathbb{C}^{h\times h}$ such that F, G and M satisfy condition (2), then for $Re(\mu) > 0$ and $t \in \mathbb{C}$, the generalized fractional kinetic matrix equation*

$$R(t)I - R_0\,{}_2F_1\Big[E,F;G;t\Big] = -M^\mu\,{}_0D_t^{-\mu}R(t), \tag{48}$$

has a solution

$$R(t)I = R_0\sum_{m=0}^{\infty}(E)_m\,(F)_m\,[(G)_m]^{-1}\,t^m \times E_{\mu,m+1}(-M^\mu t^\mu). \tag{49}$$

Theorem 9. *Suppose that E, F, G and M are positive stable matrices in $\mathbb{C}^{h\times h}$ such that F, G and M satisfycondition (2), and $\zeta \in R_+$. Then for $Re(\mu) > 0$ and $t, \alpha \in \mathbb{C}$ the generalized fractional kinetic matrix equation*

$$R(t)I - R_0\, {}_2\Gamma_1^{(\zeta)}\Big[[E, P; y], F; G; \alpha^\mu t^\mu\Big] = -M^\mu\, {}_0D_t^{-\mu} R(t), \tag{50}$$

has a solution

$$\begin{aligned} R(t)I &= R_0\, \Gamma^{-1}(F)\, \Gamma(G) \\ &\times \sum_{m=0}^{\infty} [E, P; y]_m\, \Gamma^{-1}(G + \zeta m I)\, \Gamma(F + \zeta m I)\, \frac{\Gamma(m\mu + 1)(\alpha^\mu t^\mu)^m}{m!} \times E_{\mu, m\mu+1}(-M^\mu t^\mu), \end{aligned}$$

where $E_{\mu,m\mu+1}(-M^\mu t^\mu) = \sum_{r=0}^{\infty} (-1)^r M^{\mu r} \dfrac{t^{\mu r}}{\Gamma(\mu r + m\mu + 1)}$ and called the generalized the Mittag-Leffler function.

Proof. Applying the same steps of the proof used in theorem (8), we obtain the required. □

5. Conclusions

Recently, matrix functions with their potential applications have a major role in mathematical physics, probability theory and engineering. In this paper, we introduce an extension of incomplete Wright hypergeometric matrix function and we investigate its properties. Also, we present the Riemann-Liouville fractional integral and derivative of the new extension of incomplete Wright hypergeometric matrix function function. Further, many specific cases are considered. We are motivated to obtain apply an application of fractional kinetic matrix equations involving the new function and we also have many special cases for these fractional equations.The results appear in this paper are seemed new to the literature.

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