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# A Class of Semilinear Parabolic Problems and Analytic Semigroups †

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† Dedicated to Professor Emeritus Francesco Altomare of University of Bari (Italy).

**Abstract:** (1) Background: This paper is devoted to the study of a class of semilinear initial boundary value problems of parabolic type. (2) Methods: We make use of fractional powers of analytic semigroups and the interpolation theory of compact linear operators due to Lions–Peetre. (3) Results: We give a functional analytic proof of the $C^2$ compactness of a bounded regular solution orbit for semilinear parabolic problems with Dirichlet, Neumann and Robin boundary conditions. (4) Conclusions: As an application, we study the dynamics of a population inhabiting a strongly heterogeneous environment that is modeled by a class of diffusive logistic equations with Dirichlet and Neumann boundary conditions.

**Keywords:** semilinear initial boundary value problem; bounded regular solution orbit; compactness; analytic semigroup; diffusive logistic equation

**MSC:** 35J65; 35P30; 35J25; 47D07; 92D25

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## 1. Introduction and Main Theorem

Let $\Omega$ be a bounded domain of Euclidean space $\mathbf{R}^n$, with $C^\infty$ boundary $\Gamma = \partial\Omega$; its closure $\overline{\Omega} = \Omega \cup \Gamma$ is an $n$-dimensional, compact $C^\infty$ manifold with boundary. We let

$$Au = \sum_{j,k=1}^{n} a^{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b^j(x)\frac{\partial u}{\partial x_j} + c(x)u$$

be a second-order, strictly elliptic differential operator with real coefficients on $\overline{\Omega} = \Omega \cup \Gamma$ such that:

(1) $a^{jk} \in C^\infty(\overline{\Omega})$ and $a^{jk}(x) = a^{kj}(x)$ for all $x \in \overline{\Omega}$ and all $1 \leq j,k \leq n$, and there exists a constant $a_0 > 0$ such that

$$\sum_{j,k=1}^{n} a^{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2 \quad \text{for all } (x,\xi) \in T^*(\overline{\Omega}) = \overline{\Omega} \times \mathbf{R}^n,$$

where $T^*(\overline{\Omega})$ is the cotangent bundle of $\overline{\Omega}$.

(2) $b^j \in C^\infty(\overline{\Omega})$ for all $1 \leq j \leq n$.

(3) $c \in C^\infty(\overline{\Omega})$ and $c(x) \leq 0$ in $\Omega$.

For simplicity, we suppose that:

The function $c(x)$ does not vanish *identically* in $\Omega$.

First, we study the following linear elliptic boundary value problem: Given functions $f(x)$ and $\varphi(x')$ defined in $\Omega$ and on $\Gamma$, respectively, find a function $u(x)$ in $\Omega$ such that

$$\begin{cases} Au = f & \text{in } \Omega, \\ Bu := a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = \varphi & \text{on } \Gamma. \end{cases} \tag{1}$$

Here:

(1) $a(x')$ and $b(x')$ are real-valued, $C^\infty$ functions on $\Gamma$.
(2) $\partial/\partial\boldsymbol{\nu}$ is the *conormal* derivative associated with the differential operator $A$:

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \sum_{j,k=1}^{n} a^{jk}(x')n_k \frac{\partial}{\partial x_j},$$

$\mathbf{n} = (n_1, n_2, \ldots, n_n)$ being the unit outward normal to the boundary $\Gamma$ (see Figure 1).

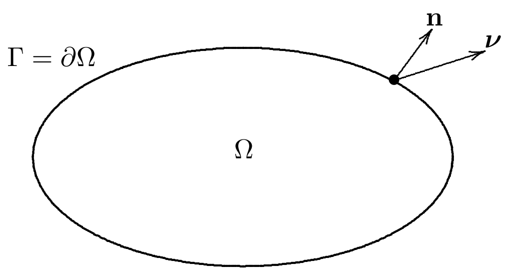


**Figure 1.** The unit outward normal $\mathbf{n}$ and the conormal $\boldsymbol{\nu}$ to $\Gamma$.

We associate with the linear problem (1) an unbounded linear operator $\mathfrak{A}_p$ from the Banach space $L^p(\Omega)$ into itself as follows (see [1], Theorem 1.2):

(a) The domain $D(\mathfrak{A}_p)$ of definition of $\mathfrak{A}_p$ is the set

$$D(\mathfrak{A}_p) = \left\{ u \in W^{2,p}(\Omega) : Bu = a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 \text{ on } \Gamma \right\}. \tag{2}$$

(b) $\mathfrak{A}_p\, u = Au$ for every $u \in D(\mathfrak{A}_p)$.

Then we have the following proposition (see [1], Lemma 8.1):

**Proposition 1.** *Let $1 < p < \infty$. Suppose that the following conditions (H.1) and (H.2) are satisfied:*
*(H.1) $a(x') \geq 0$ and $b(x') \geq 0$ on $\Gamma$.*
*(H.2) $a(x') + b(x') > 0$ on $\Gamma$.*
*Then we have the global a priori estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|Au\|_{L^p(\Omega)} \quad \text{for all } u \in D(\mathfrak{A}_p). \tag{3}$$

Moreover, we have the following generation theorem for analytic semigroups in the framework of $L^p$ Sobolev spaces (see [1], Theorem 1.2):

**Theorem 1** (the generation theorem for analytic semigroups)**.** *Let $1 < p < \infty$. If the conditions (H.1) and (H.2) are satisfied, then we have the following two assertions (i) and (ii):*

*(i) For every $0 < \varepsilon < \pi/2$, there exists a constant $r(\varepsilon) > 0$ such that the resolvent set of $\mathfrak{A}_p$ contains the set*

$$\Sigma(\varepsilon) = \left\{ \lambda = r^2 e^{i\theta} : r \geq r(\varepsilon),\ |\theta| \leq \pi - \varepsilon \right\},$$

*and that the resolvent* $(\mathfrak{A}_p - \lambda I)^{-1}$ *satisfies the estimate*

$$\left\| (\mathfrak{A}_p - \lambda I)^{-1} \right\| \leq \frac{c_p(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma(\varepsilon), \tag{4}$$

*where* $c_p(\varepsilon) > 0$ *is a constant depending on* $\varepsilon$.

*(ii)* *The operator* $\mathfrak{A}_p$ *generates a semigroup* $U(z) = e^{z\mathfrak{A}_p}$ *on the space* $L^p(\Omega)$ *which is analytic in the sector*

$$\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$$

*for any* $0 < \varepsilon < \pi/2$ *(see Figure 2).*

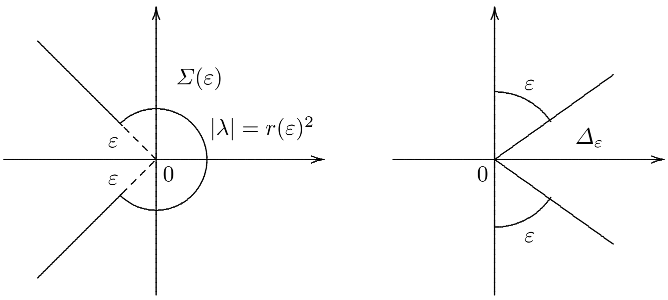


**Figure 2.** The set $\Sigma(\varepsilon)$ and the sector $\Delta_\varepsilon$.

*1.1. Statement of Main Theorem*

Now let $g(t)$ be a real-valued, *locally Lipschitz continuous* function on $\mathbf{R}$. In this section we consider the following semilinear initial boundary value problem of parabolic type:

$$\begin{cases} \frac{\partial u}{\partial t} - Au = g(u) & \text{in } \Omega \times (0, \infty), \\ Bu = a(x')\frac{\partial u}{\partial \nu} + b(x')u = 0 & \text{on } \Gamma \times (0, \infty), \\ u(x, 0) = u_0 & \text{in } \overline{\Omega}. \end{cases} \tag{5}$$

A function $u(x, t)$ is called a *regular solution* of the semilinear problem (5) if it belongs to the space $C^{2+\beta, 1+\beta/2}(\overline{\Omega} \times [0, \infty))$ for $0 < \beta < 1$:

$C^{2+\beta,1+\beta/2}(\overline{\Omega} \times [0, \infty))$
:= the space of continuously differentiable functions $u(x, t) \in C(\overline{\Omega} \times [0, \infty))$ twice with respect to $x$ and once with respect to $t$ such that $\partial_x u(x, t)$ are $(1 + \beta)/2$-Hölder continuous with respect to $t$, that $\partial_t u(x, t)$ is $\beta$-Hölder continuous with respect to $x$ and $\beta/2$-Hölder continuous with respect to $t$ and further that $\partial_x^2 u(x, t)$ are $\beta$-Hölder continuous with respect to $x$ and $\beta/2$-Hölder continuous with respect to $t$.

By using the operator $\mathfrak{A}_p$ defined by Formula (2), we can rewrite the semilinear initial boundary value problem (5) in the following abstract semilinear Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \mathfrak{A}_p u + g(u) & \text{in } (0, \infty), \\ u(0) = u_0. \end{cases} \tag{6}$$

In this paper, in the light of the interpolation theory of compact linear operators due to Lions–Peetre [2] we give a *functional analytic proof* of the following $C^2$ compactness of a bounded regular solution orbit of the semilinear Cauchy problem (6) (cf. [3], Satz):

**Theorem 2.** *Suppose that the following conditions (H.1) and (H.3) are satisfied:*

*(H.1) $a(x') \geq 0$ and $b(x') \geq 0$ on $\Gamma$.*

*(H.3) Either $a(x') > 0$ on $\Gamma$ (regular Robin and Neumann cases) or $a(x') \equiv 0$ and $b(x') > 0$ on $\Gamma$ (Dirichlet case).*

*Let $n < p < \infty$ and*

$$\frac{1}{2} + \frac{n}{2p} < \alpha < 1. \tag{7}$$

*If $u_0 \in C^{2+\alpha}(\overline{\Omega})$ such that $Bu = 0$ on $\Gamma$ and if $0 < \beta < 2\alpha - 1 - \frac{n}{p}$, then the Hölder norm $\|u(t)\|_{C^{2+\beta}(\overline{\Omega})}$ of a bounded regular solution $u(\cdot, t)$ of the semilinear Cauchy problem (6) is* uniformly bounded *for all $t \geq 0$. In particular, the orbit of a bounded regular solution $u(\cdot, t)$ is* relatively compact *in the space $C^2(\overline{\Omega})$.*

### *1.2. Outline of the Paper*

The rest of this paper is organized as follows.

Section 2 is devoted to the Hille–Yosida theory of analytic semigroups which forms a functional analytic background for the proof of Theorem 2. We consider fractional powers $(-\mathfrak{A}_p)^\alpha$ of the infinitesimal generator $\mathfrak{A}_p$ for $0 < \alpha \leq 1$ (see Formulas (9) and (10)), and summarize some basic facts about the fractional powers $(-\mathfrak{A}_p)^\alpha$ and the analytic semigroup $U(t) = e^{t\mathfrak{A}_p}$. In particular, we study the imbedding characteristics of the spaces $D\left((-\mathfrak{A}_p)^\alpha\right)$, which make these spaces so useful in the study of the solution $u(t)$ of the semilinear Cauchy problem (6) (Lemmas 1–3).

In Section 3, we formulate the *interpolation theory* of compact linear operators of Lions–Peetre [2] (Theorem 3) in order to give a functional analytic proof of Theorem 2 (Theorem 4 and Corollary 1). This section is the heart of the subject.

In Section 4, we give the proof of Theorem 2. In view of the Ascoli–Arzelà theorem, we have only to show that, for some $0 < \beta < 1$ the Hölder norm $\|u(t)\|_{C^{2+\beta}(\overline{\Omega})}$ of a bounded regular solution orbit $u(\cdot, t)$ of the semilinear Cauchy problem (6) is *uniformly bounded* for all $t \geq 0$. The proof is given by a series of claims (Claims 2–5). We make use of the classical elliptic Schauder theory for $t \geq 1$ and the classical linear parabolic theory for $0 \leq t \leq 1$.

In Section 5, we study of the existence of positive solutions of semilinear Dirichlet and Neumann problems for diffusive logistic equations, which models population dynamics in environments with spatial heterogeneity (Theorems 5 and 6 for the Dirichlet case and Theorems 8 and 9 for the Neumann case). Moreover, as an application of Theorem 2 (see [4], Section 6), we discuss the stability properties for positive steady states (Theorem 7 for the Dirichlet case and Theorem 10 for the Neumann case). A biological interpretation of main theorems is that an initial population will grow exponentially until limited by a lack of available resources if the diffusion rate is below some critical value; this idea is generally credited to the English economist Thomas Robert Malthus (1766–1834). On the other hand, if the diffusion rate is above this critical value, then the model obeys the logistic equation introduced by the Belgian mathematical biologist Pierre François Verhulst (1804–1849). We remark that this critical value tends to be smaller in situations where favorable and unfavorable habits are closely intermingled, and larger when the favorable region consists of a relatively small number of relatively large isolated components (see Formula (69)).

In Appendix A we study linear initial boundary value problems of parabolic type in the framework of Hölder spaces, following Ladyzhenskaya et al. [5] and Friedman [6]. This makes the paper fairly self-contained.

## 2. Fractional Powers for Analytic Semigroups

By virtue of Theorem 1, we may suppose that the operator $\mathfrak{A}_p$, defined by Formula (2), satisfies the following two conditions:

(R.1) The resolvent set of $\mathfrak{A}_p$ contains the region $\Sigma$ shown in Figure 3.

(R.2) There exists a constant $M > 0$ such that the resolvent $\left(\lambda I - \mathfrak{A}_p\right)^{-1}$ satisfies the estimate (see estimate (4))

$$\left\| \left(\lambda I - \mathfrak{A}_p\right)^{-1} \right\| \leq \frac{M}{1 + |\lambda|} \quad \text{for all } \lambda \in \Sigma. \tag{8}$$

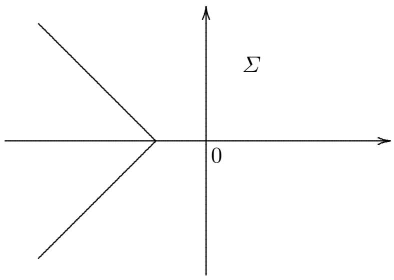


**Figure 3.** The region $\Sigma$ in condition (R.1).

Thus, we can define the fractional powers $(-\mathfrak{A}_p)^{-\alpha}$ and $(-\mathfrak{A}_p)^{\alpha}$ for $0 < \alpha < 1$ as follows:

$$\begin{aligned} \left(-\mathfrak{A}_p\right)^{-\alpha} u &= -\frac{\sin \alpha\pi}{\pi} \int_0^\infty s^{-\alpha} \left(\mathfrak{A}_p - sI\right)^{-1} u \, ds \\ &= \frac{\sin \alpha\pi}{\pi} \int_0^\infty s^{-\alpha} \left(sI - \mathfrak{A}_p\right)^{-1} u \, ds \quad \text{for all } u \in L^p(\Omega), \end{aligned} \tag{9}$$

and

$$\left(-\mathfrak{A}_p\right)^{\alpha} = \text{the inverse of } (-\mathfrak{A}_p)^{-\alpha}.$$

Remark that the operator $(-\mathfrak{A}_p)^{\alpha}$ is a closed linear operator with domain

$$D((-\mathfrak{A}_p)^{\alpha}) \supset D(-\mathfrak{A}_p) = D(\mathfrak{A}_p),$$

and further that

$$\begin{aligned} \left(-\mathfrak{A}_p\right)^{\alpha} u &= \frac{\sin \alpha\pi}{\pi} \int_0^\infty s^{\alpha-1} \left(\mathfrak{A}_p - sI\right)^{-1} \mathfrak{A}_p u \, ds \\ &= \frac{\sin \alpha\pi}{\pi} \int_0^\infty s^{\alpha-1} \left(sI - \mathfrak{A}_p\right)^{-1} \left(-\mathfrak{A}_p u\right) ds \quad \text{for all } u \in D\left(\mathfrak{A}_p\right). \end{aligned} \tag{10}$$

In this section, we study the imbedding characteristics of the spaces $D\left(\left(-\mathfrak{A}_p\right)^{\alpha}\right)$, which will make these spaces so useful in the study of semilinear parabolic differential equations. For detailed studies of this subject, the reader might be referred to Henry [7], Pazy [8], Lunardi [9] and Amann [10].

We let

$$\begin{aligned} \mathcal{X}_\alpha &:= \text{the space } D\left(\left(-\mathfrak{A}_p\right)^{\alpha}\right) \text{ endowed with the } \textit{graph norm } \|\cdot\|_\alpha \\ &\quad \text{of the fractional power } \left(-\mathfrak{A}_p\right)^{\alpha} \text{ for } 0 < \alpha < 1, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \mathcal{X}_0 &:= L^p(\Omega), \\ \mathcal{X}_1 &:= D\left(-\mathfrak{A}_p\right) = D\left(\mathfrak{A}_p\right). \end{aligned}$$

Here

$$\|u\|_\alpha = \left( \|u\|^2_{L^p(\Omega)} + \left\| \left(-\mathfrak{A}_p\right)^{\alpha} u \right\|^2_{L^p(\Omega)} \right)^{1/2} \quad \text{for all } u \in D\left(\left(-\mathfrak{A}_p\right)^{\alpha}\right). \tag{12}$$

Then we have the following three assertions (1), (2) and (3) (see Henry [7], Theorem 1.4.8; [1], Proposition 3.16):

(1) The space $\mathcal{X}_\alpha$ is a Banach space.
(2) By the *a priori* estimate (3), it follows that the graph norm $\|u\|_\alpha$ is equivalent to the norm $\|(-\mathfrak{A}_p)^\alpha u\|_{L^p(\Omega)}$:

$$\|u\|_\alpha \approx \left\|\left(-\mathfrak{A}_p\right)^\alpha u\right\|_{L^p(\Omega)} \quad \text{for all } u \in D\Big(\left(-\mathfrak{A}_p\right)^\alpha\Big). \tag{13}$$

(3) If $0 \leq \alpha < \beta \leq 1$, then we have $\mathcal{X}_\beta \subset \mathcal{X}_\alpha$ with continuous injection.

Moreover, we recall the following fundamental inequlaities for analytic semigroups ([7], Theorems 1.3.4 and 1.4.3; [1], Remark 3.1):

$$\left\|e^{t\mathfrak{A}_p}\right\| \leq C_1\, e^{-\delta t} \quad \text{for all } t \geq 0. \tag{14}$$

$$\left\|\left(-\mathfrak{A}_p\right)^\alpha e^{t\mathfrak{A}_p}\right\| \leq \frac{C(\alpha)}{t^\alpha}\, e^{-\delta t} \quad \text{for all } t > 0, \tag{15}$$

$$\left\|\left(e^{t\mathfrak{A}_p} - I\right)\left(-\mathfrak{A}_p\right)^{-1}\right\| \leq C_2\, t \quad \text{for all } t \geq 0. \tag{16}$$

It is easy to verify that every solution $u(t)$ of the abstract semilinear Cauchy problem (6) is given by the following integral formula

$$u(t) = e^{t\mathfrak{A}_p}\, u_0 + \int_0^t e^{(t-s)\mathfrak{A}_p}\, f(s)\, ds \quad \text{for all } t \geq 0, \tag{17}$$

with

$$f(t) := g(u(t)) \quad \text{for } t \geq 0.$$

If the orbit of a regular solution $u(t)$ of the semilinear problem (5) is *bounded*, then we can find constants $K > 0$ and $L > 0$ such that

$$\|f(t)\|_{L^p(\Omega)} \leq K \quad \text{for all } t \geq 0, \tag{18}$$

$$\|f(s) - f(t)\|_{L^p(\Omega)} \leq L\|u(s) - u(t)\|_{L^p(\Omega)} \quad \text{for all } s, t \geq 0. \tag{19}$$

Indeed, it suffices to note that the function

$$\mathbf{R} \ni s \longmapsto f(s) = g(u(s))$$

is bounded and Lipschitz continuous if the uniform norm $\|u(\cdot, s)\|_{C(\overline{\Omega})}$ is bounded for all $s \geq 0$.

First, we prove the $\mathcal{X}_\alpha$-boundedness of the solution $u(t)$ of the abstract semilinear Cauchy problem (6) (cf. [1], Theorem 3.18):

**Lemma 1.** *Let $u_0 \in D(\mathfrak{A}_p)$ and $0 < \alpha < 1$. If the conditions (R.1) and (R.2) are satisfied, then we have, for all $t \geq 0$,*

$$u(t) \in \mathcal{X}_\alpha = D\Big(\left(-\mathfrak{A}_p\right)^\alpha\Big), \tag{20}$$

$$\|u(t)\|_\alpha \leq C_1(\alpha), \tag{21}$$

*with a constant $C_1(\alpha) > 0$. For example, we may take*

$$C_1(\alpha) := C_1\|u_0\|_\alpha + C(\alpha)\, K\, \frac{\Gamma(1-\alpha)}{\delta^{1-\alpha}}.$$

**Proof.** Since the fractional power $\left(-\mathfrak{A}_p\right)^\alpha$ is a closed operator and since

$$u_0 \in D\left(\mathfrak{A}_p\right) \subset D\left(\left(-\mathfrak{A}_p\right)^\alpha\right),$$

it follows from the integral Formula (17) that

$$\begin{aligned}\left(-\mathfrak{A}_p\right)^\alpha u(t) &= \left(-\mathfrak{A}_p\right)^\alpha e^{t\mathfrak{A}_p} u_0 + \left(-\mathfrak{A}_p\right)^\alpha \left(\int_0^t e^{(t-s)\mathfrak{A}_p} f(s)\, ds\right) \\ &= \left(-\mathfrak{A}_p\right)^\alpha e^{t\mathfrak{A}_p}\, u_0 + \int_0^t \left(-\mathfrak{A}_p\right)^\alpha e^{(t-s)\mathfrak{A}_p} f(s)\, ds \\ &= e^{t\mathfrak{A}_p} \left(-\mathfrak{A}_p\right)^\alpha u_0 + \int_0^t \left(-\mathfrak{A}_p\right)^\alpha e^{(t-s)\mathfrak{A}_p} f(s)\, ds \quad \text{for all } t \geq 0.\end{aligned}$$

Hence, by using inequalities (15) and (18) we obtain that

$$\begin{aligned}\|u(t)\|_\alpha &= \left\|\left(-\mathfrak{A}_p\right)^\alpha u(t)\right\|_{L^p(\Omega)} \\ &= \left\|e^{t\mathfrak{A}_p} \left(-\mathfrak{A}_p\right)^\alpha u_0 + \int_0^t \left(-\mathfrak{A}_p\right)^\alpha e^{(t-s)\mathfrak{A}_p} f(s)\, ds\right\|_{L^p(\Omega)} \\ &\leq \left\|e^{t\mathfrak{A}_p} \cdot \left(-\mathfrak{A}_p\right)^\alpha u_0\right\|_{L^p(\Omega)} + \int_0^t \left\|\left(-\mathfrak{A}_p\right)^\alpha e^{(t-s)\mathfrak{A}_p} f(s)\right\|_{L^p(\Omega)} ds \\ &\leq \left\|e^{t\mathfrak{A}_p}\right\| \cdot \left\|\left(-\mathfrak{A}_p\right)^\alpha u_0\right\|_{L^p(\Omega)} + \int_0^t \left\|\left(-\mathfrak{A}_p\right)^\alpha e^{(t-s)\mathfrak{A}_p} f(s)\right\|_{L^p(\Omega)} ds \\ &\leq C_1\, e^{-\delta t} \|u_0\|_\alpha + C(\alpha) \int_0^t \frac{e^{-\delta(t-s)}}{(t-s)^\alpha} \|f(s)\|_{L^p(\Omega)} ds \\ &\leq C_1\, e^{-\delta t} \|u_0\|_\alpha + C(\alpha)\, K \int_0^t \frac{e^{-\delta(t-s)}}{(t-s)^\alpha}\, ds \\ &\leq C_1 \|u_0\|_\alpha + C(\alpha)\, K\, \frac{\Gamma(1-\alpha)}{\delta^{1-\alpha}} \\ &:= C_1(\alpha) \quad \text{for all } t \geq 0.\end{aligned}$$

The proof of Lemma 1 is complete. □

Secondly, we prove the boundedness of the difference

$$\left\|\frac{u(t+h) - u(t)}{h}\right\|_{L^p(\Omega)}$$

of the solution $u(t)$ of the abstract semilinear Cauchy problem (6):

**Lemma 2.** *Let $u_0 \in D\left(-\mathfrak{A}_p\right) = D\left(\mathfrak{A}_p\right)$. If the conditions (R.1) and (R.2) are satisfied, then there exists a constant $C > 0$ such that*

$$\|u(t+h) - u(t)\|_{L^p(\Omega)} \leq C\, h \quad \textit{for all } t \geq 0 \textit{ and } h > 0. \tag{22}$$

**Proof.** The proof of Lemma 2 is divided into three steps.

**Step 1**: By using the integral Formula (17), we have, for $0 \leq \tau \leq t$ and $h > 0$,

$$\begin{aligned}&u(t+h) - u(t) \\ &= \left(e^{(t+h)\mathfrak{A}_p} - e^{t\mathfrak{A}_p}\right) u_0 + \int_\tau^{\tau+h} e^{(t+h-s)\mathfrak{A}_p} f(s)\, ds \\ &\quad + \int_0^\tau \left(e^{(t+h-s)\mathfrak{A}_p} - e^{(t-s)\mathfrak{A}_p}\right) f(s)\, ds + \int_\tau^t e^{(t-s)\mathfrak{A}_p} (f(s+h) - f(s)) ds.\end{aligned} \tag{23}$$

However, we obtain from inequalities (14)–(16) that

$$
\begin{aligned}
&\bullet \left\| \left(e^{(t+h)\mathfrak{A}_p} - e^{t\mathfrak{A}_p}\right) u_0 \right\|_{L^p(\Omega)} \qquad (24)\\
&= \left\| \left(e^{h\mathfrak{A}_p} - I\right)(-\mathfrak{A}_p)^{-1} \cdot e^{t\mathfrak{A}_p}((-\mathfrak{A}_p)u_0) \right\|_{L^p(\Omega)}\\
&\leq C_2\, h \left\| e^{t\mathfrak{A}_p}((-\mathfrak{A}_p u_0)) \right\|_{L^p(\Omega)}\\
&\leq C_2\, h\, C_1\, e^{-\delta t} \| ((-\mathfrak{A}_p)u_0) \| = C_1\, C_2\, h\, e^{-\delta t} \|u_0\|_1\\
&\leq (C_1\, C_2 \|u_0\|_1) h\\
&= C_3\, h \quad \text{for all } t \geq 0,
\end{aligned}
$$

where

$$
C_3 := C_1\, C_2 \|u_0\|_1,
$$

and further from inequalities (18) and (19) that

$$
\begin{aligned}
&\bullet \left\| \int_\tau^{\tau+h} e^{(t+h-s)\mathfrak{A}_p} f(s)\, ds \right\|_{L^p(\Omega)} \qquad (25)\\
&\leq \int_\tau^{\tau+h} \left\| e^{(h-s)\mathfrak{A}_p} f(s) \right\|_{L^p(\Omega)} ds \cdot \left\| e^{t\mathfrak{A}_p} \right\|\\
&\leq K \int_\tau^{\tau+h} C_1\, e^{-\delta(h-s)}\, ds \cdot C_1\, e^{-\delta t}\\
&\leq K\, C_1^2 \int_{\tau-h}^{\tau} e^{\delta s}\, ds \cdot e^{-\delta t}\\
&\leq K\, C_1^2\, e^{-\delta t} \cdot \frac{1}{\delta}\, e^{\tau\delta}\left(1 - e^{-h\delta}\right)\\
&\leq K\, C_1^2\, e^{-(t-\tau)\delta}\, \frac{1}{\delta} \cdot h\delta = K\, C_1^2\, e^{-(t-\tau)\delta}\, h\\
&\leq K\, C_1^2\, h \quad \text{for } 0 \leq \tau \leq t,\\
&\bullet \left\| \int_0^\tau \left(e^{(t+h-s)\mathfrak{A}_p} - e^{(t-s)\mathfrak{A}_p}\right) f(s)\, ds \right\|_{L^p(\Omega)} \qquad (26)\\
&\leq \int_0^\tau \left\| (-\mathfrak{A}_p) e^{(t-s)\mathfrak{A}_p} \cdot \left(e^{h\mathfrak{A}_p} - I\right)(-\mathfrak{A}_p)^{-1} \right\| \cdot \|f(s)\|_{L^p(\Omega)} ds\\
&\leq K\, C_2\, h \int_0^\tau \left\| (-\mathfrak{A}_p) e^{(t-s)\mathfrak{A}_p} \right\| ds\\
&\leq K\, C_2\, h\, C(1) \int_0^\tau \frac{1}{t-s}\, e^{-\delta(t-s)}\, ds\\
&\leq K\, C_2\, h\, C(1)\, \frac{e^{-\delta t}}{t-\tau} \int_0^\tau e^{\delta s}\, ds = K\, C_2\, h\, \frac{C(1)}{\delta}\, \frac{e^{-\delta t}}{t-\tau}\left(e^{\delta\tau} - 1\right)\\
&\leq K\, C_2\, h\, \frac{C(1)}{\delta}\, \frac{1}{t-\tau} \quad \text{for } 0 \leq \tau < t,\\
&\bullet \left\| \int_\tau^t e^{(t-s)\mathfrak{A}_p} (f(s+h) - f(s)) ds \right\|_{L^p(\Omega)} \qquad (27)\\
&\leq \int_\tau^t \left\| e^{(t-s)\mathfrak{A}_p} \right\| \cdot \|f(s+h) - f(s)\|_{L^p(\Omega)} ds\\
&\leq C_1\, L \int_\tau^t e^{-(t-s)\delta} \|u(s+h) - u(s)\|_{L^p(\Omega)} ds \quad \text{for } 0 \leq \tau \leq t.
\end{aligned}
$$

By combining inequalities (24) through (27), we have proved that

$$\begin{aligned}
&\|u(t+h)-u(t)\|_{L^p(\Omega)} \\
&\leq \left(C_3 + C_1^2\, K + \frac{C_2\, C(1)\, K}{\delta}\, \frac{1}{t-\tau}\right) h \\
&\quad + C_1 L \int_\tau^t e^{-(t-s)\delta} \|u(s+h)-u(s)\|_{L^p(\Omega)} ds \quad \text{for } 0 \leq \tau < t. \\
&\leq C_4 \left(1 + \frac{1}{t-\tau}\right) h + C_1 L \int_\tau^t e^{-(t-s)\delta} \|u(s+h)-u(s)\|_{L^p(\Omega)} ds \quad \text{for } 0 \leq \tau < t.
\end{aligned} \tag{28}$$

Here

$$C_4 := \max\left\{C_1\, C_2 \|u_0\|_1 + C_1^2\, K,\ \frac{C_2\, C(1)\, K}{\delta}\right\}.$$

**Step 2**: If we let

$$\phi(t) := e^{\delta t} \|u(t+h)-u(t)\|_{L^p(\Omega)} \quad \text{for } t \geq 0,$$

then we obtain from inequality (28) that

$$\begin{aligned}
e^{-\delta t}\, \phi(t) &= \|u(t+h)-u(t)\|_{L^p(\Omega)} \\
&\leq C_4 \left(1 + \frac{1}{t-\tau}\right) h + C_1\, L \left(\int_\tau^t \phi(s)\, ds\right) e^{-\delta t} \quad \text{for } 0 \leq \tau < t,
\end{aligned}$$

so that

$$\begin{aligned}
&e^{\delta t} \|u(t+h)-u(t)\|_{L^p(\Omega)} = \phi(t) \\
&\leq C_4\, h \left(1 + \frac{1}{t-\tau}\right) e^{\delta t} + C_1\, L \int_\tau^t \phi(s)\, ds \quad \text{for } 0 \leq \tau < t.
\end{aligned} \tag{29}$$

Now we need the following form of Gronwall's inequality ([11], Lemma 29.2):

**Claim 1** (Gronwall)**.** *Let $g(t)$, $v(t)$ be continuous functions in $C[0,T]$, and let $h(t)$ be a non-negative, integrable function in $(0,T)$. Suppose that the following inequality holds true:*

$$v(t) \leq g(t) + \int_0^t h(s)\, v(s)\, ds \quad \textit{for all } 0 \leq t \leq T.$$

*Then we have the inequality*

$$v(t) \leq g(t) + \int_0^t g(\tau)\, h(\tau)\, e^{\int_\tau^t h(s)\, ds}\, d\tau \quad \textit{for all } 0 \leq t \leq T.$$

**Step 3**: We consider the following two cases (i) and (ii).

• The case (i): $0 \leq t \leq 1/(C_1 L)$. If we put $\tau = 0$ in Formula (23), we obtain from inequalities (24), (25) and (27) that

$$\begin{aligned}
&\|u(t+h) - u(t)\|_{L^p(\Omega)} \\
&\leq \left\| \left( e^{(t+h)\mathfrak{A}_p} - e^{t\mathfrak{A}_p} \right) u_0 \right\|_{L^p(\Omega)} + \left\| \int_0^h e^{(t+h-s)\mathfrak{A}_p} f(s)\, ds \right\|_{L^p(\Omega)} \\
&\quad + \left\| \int_0^t e^{(t-s)\mathfrak{A}_p} (f(s+h) - f(s)) ds \right\|_{L^p(\Omega)} \\
&\leq \left( C_3 + C_1^2 K \right) h + C_1 L \int_0^t e^{-(t-s)\delta} \|u(s+h) - u(s)\|_{L^p(\Omega)} ds \\
&:= C_5\, h + C_1 L \int_0^t e^{-(t-s)\delta} \|u(s+h) - u(s)\|_{L^p(\Omega)} ds,
\end{aligned}$$

so that

$$\begin{aligned}
&\phi(t) = e^{\delta t} \|u(t+h) - u(t)\|_{L^p(\Omega)} \\
&\leq C_5\, h\, e^{\delta t} + C_1 L \int_0^t \phi(s)\, ds \quad \text{for all } 0 \leq t \leq 1/(C_1 L).
\end{aligned}$$

Here

$$C_5 := C_3 + C_1^2 K = C_1\, C_2 \|u_0\|_1 + C_1^2 K.$$

Hence, by applying Gronwall's inequality with

$$\begin{aligned}
v(t) &:= \phi(t), \\
g(t) &:= C_5\, h\, e^{\delta t}, \\
h(t) &:= C_1\, L,
\end{aligned}$$

we have, for some constant $C' > 0$,

$$\begin{aligned}
&e^{\delta t} \|u(t+h) - u(t)\|_{L^p(\Omega)} \\
&= \phi(t) \leq \frac{C_5 \delta}{\delta - C_1 L}\, h \left( 1 - e^{(C_1 L - \delta) t} \right) e^{\delta t} \quad \text{for all } 0 \leq t \leq 1/(C_1 L).
\end{aligned} \tag{30}$$

Therefore, we have, by inequality (30),

$$\|u(t+h) - u(t)\|_{L^p(\Omega)} \leq C'\, h \quad \text{for all } 0 \leq t \leq 1/(C_1 L). \tag{31}$$

Here the positive constant $C'$ is given by the formula

$$C' := \begin{cases} \frac{C_5 \delta}{\delta - C_1 L} & \text{if } \delta > C_1 L, \\ 2\, C_5 & \text{if } \delta = C_1 L, \\ \frac{C_5 \delta}{C_1 L - \delta}\, e^{(1 - \delta/(C_1 L))} & \text{if } \delta < C_1 L. \end{cases}$$

• The case (ii): $t > 1/(C_1 L)$. If we let

$$\tau := t - \frac{1}{C_1 L}$$

and so

$$t - \tau = \frac{1}{C_1 L},$$

then we have, by inequality (29),

$$\begin{aligned}\phi(t) &\le C_4\, h(1+C_1L)e^{\delta t} + C_1\, L \int_{t-1/(C_1L)}^{t} \phi(s)\, ds \\ &= C_4\, h(1+C_1L)e^{\delta t} + C_1\, L \int_0^t \chi(s)\, \phi(s)\, ds \quad \text{for } t > 1/(C_1L),\end{aligned}$$

where

$$\chi_{[t-1/(C_1L),t]}(t) = \text{ the characteristic function of the ineterval } [t-1/(C_1L),t].$$

Hence, by applying Gronwall's inequality (Claim 1) with

$$\begin{aligned} v(t) &:= \phi(t), \\ g(t) &:= C_4\, h(1+C_1\, L)e^{\delta t}, \\ h(t) &:= C_1\, L\, \chi_{[t-1/(C_1L),t]}(t), \end{aligned}$$

we have the inequality

$$e^{\delta t}\|u(t+h)-u(t)\|_{L^p(\Omega)} = \phi(t) \le C''\, h\, e^{\delta t} \quad \text{for all } t > 1/(C_1L),$$

and hence

$$\|u(t+h)-u(t)\|_{L^p(\Omega)} \le C''\, h \quad \text{for all } t > 1/(C_1L). \tag{32}$$

Here the positive constant $C''$ is given by the formula

$$C'' := C_4(1+C_1\, L)(1+e).$$

Therefore, by combining inequalities (31) and (32) we obtain that

$$\|u(t+h)-u(t)\|_{L^p(\Omega)} \le C\, h \quad \text{for all } t \ge 0,$$

with

$$C := \max\{C', C''\}.$$

Now the proof of Lemma 2 is complete. □

Finally, we prove the $\mathcal{X}_\alpha$-boundedness of the derivative

$$\frac{\partial u}{\partial t} = g(u(t)) + \mathfrak{A}_p u(t)$$

of the solution $u(t)$ of the abstract semilinear Cauchy problem (6):

**Lemma 3.** *Let $u_0 \in D(-\mathfrak{A}_p) = D(\mathfrak{A}_p)$ and $0 < \alpha < 1$. If the conditions (R.1) and (R.2) are satisfied, then we have, for all $t \ge 1$,*

$$\frac{\partial u}{\partial t} \in \mathcal{X}_\alpha = D\Big((-\mathfrak{A}_p)^\alpha\Big), \tag{33}$$

$$\left\|\frac{\partial u}{\partial t}\right\|_\alpha \le C_6(\alpha), \tag{34}$$

*with a constant $C_6(\alpha) > 0$. For example, we may take*

$$C_6(\gamma) := \max\left\{C(\gamma)\, C_2\|u_0\|_1,\ K\, C(\gamma),\ L\, C\, C(\gamma)\, \frac{\Gamma(1-\gamma)}{\delta^{1-\gamma}}\right\} \quad \text{for } 0 < \gamma < 1.$$

**Proof.** The proof of Lemma 3 is divided into two steps.

**Step 1**: First, we have, by integral Formula (17),

$$\begin{aligned} u(t+h)-u(t) &= \left(e^{(t+h)\mathfrak{A}_p}-e^{t\mathfrak{A}_p}\right)u_0+\int_0^h e^{(t+h-s)\mathfrak{A}_p} f(s)\,ds \\ &\quad +\int_0^t e^{(t-s)\mathfrak{A}_p}[f(s+h)-f(s)]ds \\ &:= \left(e^{(t+h)\mathfrak{A}_p}-e^{t\mathfrak{A}_p}\right)u_0+J_1(h)+J_2(h), \end{aligned}$$

where

$$\begin{aligned} J_1(h) &:= \int_0^h e^{(t+h-s)\mathfrak{A}_p} f(s)\,ds, \\ J_2(h) &:= \int_0^t e^{(t-s)\mathfrak{A}_p}[f(s+h)-f(s)]ds. \end{aligned}$$

However, by inequalities (16) and (15) it follows that we have, for $0<\gamma<1$,

$$\begin{aligned} \bullet\ & \left\|\left(e^{(t+h)\mathfrak{A}_p}-e^{t\mathfrak{A}_p}\right)u_0\right\|_\gamma \\ &= \left\|e^{t\mathfrak{A}_p}\left(e^{h\mathfrak{A}_p}-I\right)(-\mathfrak{A}_p)^{-1}\cdot(-\mathfrak{A}_p)u_0\right\|_\gamma \\ &\le \left\|(-\mathfrak{A}_p)^\gamma e^{t\mathfrak{A}_p}\right\|\cdot\left\|\left(e^{h\mathfrak{A}_p}-I\right)(-\mathfrak{A}_p)^{-1}\right\|\cdot\|(-\mathfrak{A}_p)u_0\|_{L^p(\Omega)} \\ &\le C(\gamma)\,\frac{e^{-\delta t}}{t^\gamma}\cdot C_2 h\|u_0\|_1 \\ &\le (C(\gamma)\,C_2\|u_0\|_1)h \quad \text{for all } t\ge 1. \end{aligned}$$

Moreover, we have, by inequalities (15) and (18),

$$\begin{aligned} \bullet\ \|J_1(h)\|_\gamma &= \left\|\int_0^h (-\mathfrak{A}_p)^\gamma e^{(t+h-s)\mathfrak{A}_p} f(s)\,ds\right\|_{L^p(\Omega)} \\ &\le \int_0^h \left\|(-\mathfrak{A}_p)^\gamma e^{(t+h-s)\mathfrak{A}_p}\right\|\cdot\|f(s)\|_{L^p(\Omega)}ds \\ &\le C(\gamma)\,K\int_0^h \frac{1}{(t+h-s)^\gamma}\,ds \\ &\le (K\,C(\gamma))h \quad \text{for all } t\ge 1, \end{aligned}$$

and, by inequalities (19) and (22),

$$\begin{aligned} \bullet\ \|J_2(h)\|_\gamma &= \left\|\int_0^t (-\mathfrak{A}_p)^\gamma e^{(t-s)\mathfrak{A}_p} f(s)\,ds\right\|_{L^p(\Omega)} \\ &\le \int_0^h \left\|(-\mathfrak{A}_p)^\gamma e^{(t-s)\mathfrak{A}_p}\right\|\cdot\|f(s+h)-f(s)\|_{L^p(\Omega)}ds \\ &\le L\int_0^h \left\|(-\mathfrak{A}_p)^\gamma e^{(t-s)\mathfrak{A}_p}\right\|\cdot\|u(s+h)-u(s)\|_{L^p(\Omega)}ds \\ &\le C(\gamma)\,L(C\,h)\int_0^t \frac{e^{-\delta(t-s)}}{(t-s)^\gamma}\,ds \\ &\le \left(L\,C\,C(\gamma)\,\frac{\Gamma(1-\gamma)}{\delta^{1-\gamma}}\right)h \quad \text{for all } t\ge 1. \end{aligned}$$

Summing up, we have the inequality

$$\left\|\frac{u(t+h)-u(t)}{h}\right\|_\gamma \le C_6(\gamma) \quad \text{for all } t\ge 1 \text{ and } h>0. \tag{35}$$

**Step 2**: By inequality (35), it follows that the set

$$\left\| (-\mathfrak{A}_p)^{\gamma} \left( \frac{u(t+h) - u(t)}{h} \right) \right\|_{L^p(\Omega)}$$

is bounded for $h > 0$, for each $t \geq 1$. On the other hand, the negative fractional power

$$(-\mathfrak{A}_p)^{-(\gamma-\alpha)} \colon L^p(\Omega) \longrightarrow L^p(\Omega)$$

is *compact* if $\gamma > \alpha$. The proof of this fact (Corollary 1) will be given in the next section, due to its length.

Hence, we can find a sequence $\{h_n\}$, $h_n \downarrow 0$ as $n \to \infty$, such that the sequqnce

$$(-\mathfrak{A}_p)^{\alpha} \left( \frac{u(t+h_n) - u(t)}{h_n} \right) = (-\mathfrak{A}_p)^{-(\gamma-\alpha)} \left( (-\mathfrak{A}_p)^{\gamma} \left( \frac{u(t+h_n) - u(t)}{h_n} \right) \right)$$

is a Cauchy sequence in $L^p(\Omega)$, for each $t \geq 1$. Namely, there exists a function $v(t) \in L^p(\Omega)$, for each $t \geq 1$, such that

$$(-\mathfrak{A}_p)^{\alpha} \left( \frac{u(t+h_n) - u(t)}{h_n} \right) \longrightarrow v(t) \quad \text{in } L^p(\Omega) \text{ as } n \to \infty, \text{ for each } t \geq 1. \tag{36}$$

By passing to the limit in inequality (35) with $\gamma := \alpha$, we obtain that

$$\begin{aligned} \|v(t)\|_{L^p(\Omega)} &= \lim_{n\to\infty} \left\| \left(-\mathfrak{A}_p^{\alpha}\right) \left( \frac{u(t+h_n) - u(t)}{h_n} \right) \right\|_{L^p(\Omega)} \\ &\leq C_6(\alpha) \quad \text{for each } t \geq 1, \end{aligned} \tag{37}$$

where

$$C_6(\alpha) = \max\left\{ C(\alpha)\, C_2 \|u_0\|_1, K\, C(\alpha), L\, C\, C(\alpha)\, \frac{\Gamma(1-\alpha)}{\delta^{1-\alpha}} \right\}.$$

Moreover, we have the assertions

$$\frac{u(t+h_n) - u(t)}{h_n} \in D(-\mathfrak{A}_p) \subset D\left( (-\mathfrak{A}_p)^{\alpha} \right) = \mathcal{X}_{\alpha} \quad \text{for each } t \geq 1, \tag{38}$$

$$\frac{u(t+h_n) - u(t)}{h_n} \longrightarrow \frac{\partial u}{\partial t} \quad \text{in } L^p(\Omega) \text{ as } n \to \infty, \text{ for each } t \geq 1. \tag{39}$$

Therefore, since the operator

$$(-\mathfrak{A}_p)^{\alpha} \colon L^p(\Omega) \longrightarrow L^p(\Omega)$$

is closed, it follows from assertions (36) and (38,39) and inequality (37) that

$$\begin{gathered} \frac{\partial u}{\partial t} \in D\left( (-\mathfrak{A}_p)^{\alpha} \right) = \mathcal{X}_{\alpha}, \\ (-\mathfrak{A}_p)^{\alpha} \left( \frac{\partial u}{\partial t} \right) = v(t), \end{gathered}$$

and further that

$$\left\| \frac{\partial u}{\partial t} \right\|_{\alpha} = \|v(t)\|_{L^p(\Omega)} \leq C_6(\alpha) \quad \text{for each } t \geq 1.$$

Now the proof of Lemma 3 is complete. □

## 3. Compactness Theorem for Spaces of Class $\overline{\mathcal{K}}_\theta(A_0, A_1)$

In this section, we formulate the *interpolation theory* of compact linear operators of Lions–Peetre [2] (Theorem 3) in order to give a functional analytic proof of Theorem 2 (Theorem 4 and Corollary 1). This section is the heart of the subject.

Let $A_0$, $A_1$ be Banach spaces that are contained in a separable topological vector space $\mathcal{A}$. We suppose that the injections

$$A_0 \longrightarrow \mathcal{A},$$
$$A_1 \longrightarrow \mathcal{A}$$

are both continuous. The norm of $A_i$ $(i = 0, 1)$ will be denoted by $\|\cdot\|_{A_i}$.

We consider the normed linear space $A_0 \cap A_1 \subset \mathcal{A}$ with the norm

$$\|a\|_{A_0 \cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\}, \quad a \in A_0 \cap A_1,$$

and the normed linear space

$$A_0 + A_1 = \{a = a_0 + a_1 : a_0 \in A_0,\ a_1 \in A_1\} \subset \mathcal{A}$$

with the norm

$$\|a\|_{A_0+A_1} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1,\ a_0 \in A_0, a_1 \in A_1\}.$$

Then the spaces $A_0 \cap A_1$ and $A_0 + A_1$ are both Banach spaces, and we have the inclusions

$$A_0 \cap A_1 \subset A_0,\ A_1 \subset A_0 + A_1 \subset \mathcal{A},$$

with continuous injections.

A separable locally convex, topological vector space $A$ is called an *intermediate space* between $A_0$ and $A_1$ if we have the inclusions

$$A_0 \cap A_1 \subset A \subset A_0 + A_1 \subset \mathcal{A},$$

Let $\{A_0, A_1, \mathcal{A}\}$ be a triplet as above, and $0 < \theta < 1$. We say that a Banach space $A \subset A_0 + A_1$ is of class $\overline{\mathcal{K}}_\theta(A_0, A_1)$ if it satisfies the following condition: For every $t > 0$, there exist elements $a_0 \in A_0$ and $a_1 \in A_1$ such that

$$a = a_0 + a_1 \in A, \tag{40}$$

and that we have the inequalities

$$\|a_0\|_{A_0} \leq \frac{C}{t^\theta}\,\|a\|_A, \tag{41}$$
$$\|a_1\|_{A_1} \leq C\,t^{1-\theta}\,\|a\|_A, \tag{42}$$

with a constant $C > 0$ (see [2], Définition (1.2)).

Now we are in a position to state the main result due to Lions–Petre [2], Théorème (2.2):

**Theorem 3** (Lions–Peetre). *Let $\{A_0, A_1, \mathcal{A}\}$ be a triplet as above. Let $\pi$ be a linear operator from $\mathcal{A}$ into a Banach space $B$. We suppose that*

$$\pi\colon A_0 \longrightarrow B$$

*is compact (or completely continuous) and further that*

$$\pi\colon A_1 \longrightarrow B$$

*is continuous.*

*If an intermediate Banach space $A$ is of class $\overline{\mathcal{K}}_\theta$ for $0 < \theta < 1$, then the operator*

$$\pi\colon A \longrightarrow B$$

*is compact.*

The purpose of this section is to prove the following theorem:

**Theorem 4.** *Suppose that conditions (R.1) and (R.2) are satisfied. Then the injection*

$$\iota\colon D\Big(\big(-\mathfrak{A}_p\big)^\alpha\Big) \longrightarrow L^p(\Omega)$$

*is compact for every $0 < \alpha < 1$.*

**Proof.** Our proof is based on Theorem 3 due to Lions–Peetre [2]. The proof is divided into three steps.

**Step 1**: First, we show that the domain $D\Big(\big(-\mathfrak{A}_p\big)^\alpha\Big)$ of the fractional power is of class

$$\overline{\mathcal{K}}_{1-\alpha}\big(D\big(-\mathfrak{A}_p\big), L^p(\Omega)\big),$$

if we take

$$\begin{aligned}
&A_0 := D\big(-\mathfrak{A}_p\big) = D\big(\mathfrak{A}_p\big),\\
&A_1 := L^p(\Omega),\\
&A := D\Big(\big(-\mathfrak{A}_p\big)^\alpha\Big),\\
&\theta := 1-\alpha.
\end{aligned}$$

By inequality (8) with $\lambda := s$, it follows that

$$\left\|\big(sI-\mathfrak{A}_p\big)^{-1}\right\| \leq \frac{M}{1+s} \quad \text{for all } s \geq 0. \tag{43}$$

Then, by using the integral representation Formula (9) we find that every function

$$u := \big(-\mathfrak{A}_p\big)^{-\alpha} f \in D\Big(\big(-\mathfrak{A}_p\big)^\alpha\Big) \quad \text{with } f \in L^p(\Omega),$$

can be expressed in the form

$$\begin{aligned}
u &= \big(-\mathfrak{A}_p\big)^{-\alpha} f = \frac{\sin \pi\alpha}{\pi} \int_0^\infty s^{-\alpha}\big(sI-\mathfrak{A}_p\big)^{-1} f\, ds \qquad (44)\\
&= \frac{\sin \pi\alpha}{\pi} \int_0^{t^{-1}} s^{-\alpha}\big(sI-\mathfrak{A}_p\big)^{-1} f\, ds + \frac{\sin \pi\alpha}{\pi} \int_{t^{-1}}^\infty s^{-\alpha}\big(sI-\mathfrak{A}_p\big)^{-1} f\, ds\\
&:= u_0 + u_1 \quad \text{for every } t > 0.
\end{aligned}$$

However, we have, by resolvent estimate (43),

$$\begin{aligned}
\bullet\ \|u_0\|_1 &= \frac{\sin \pi\alpha}{\pi} \left\| \int_0^{t^{-1}} s^{-\alpha} \left(sI - \mathfrak{A}_p\right)^{-1} f\, ds \right\|_{D(-\mathfrak{A}_p)} \qquad (45)\\
&= \frac{\sin \pi\alpha}{\pi} \left\| \int_0^{t^{-1}} s^{-\alpha} \left(-\mathfrak{A}_p\right)\left(sI - \mathfrak{A}_p\right)^{-1} f\, ds \right\|_{L^p(\Omega)}\\
&= \frac{\sin \pi\alpha}{\pi} \left\| \int_0^{t^{-1}} s^{-\alpha} \left(-s\left(sI - \mathfrak{A}_p\right)^{-1} + I\right) f\, ds \right\|_{L^p(\Omega)}\\
&\leq \frac{\sin \pi\alpha}{\pi} \left( \int_0^{t^{-1}} s^{-\alpha+1} \left\| \left(sI - \mathfrak{A}_p\right)^{-1} \right\| ds + \int_0^{t^{-1}} s^{-\alpha}\, ds \right) \|f\|_{L^p(\Omega)}\\
&\leq \frac{\sin \pi\alpha}{\pi} \left( M \int_0^{t^{-1}} \frac{s^{1-\alpha}}{1+s}\, ds + \int_0^{t^{-1}} s^{-\alpha}\, ds \right) \|f\|_{L^p(\Omega)}\\
&\leq \frac{\sin \pi\alpha}{\pi} \left( M \int_0^{t^{-1}} s^{-\alpha}\, ds + \int_0^{t^{-1}} s^{-\alpha}\, ds \right) \|f\|_{L^p(\Omega)}\\
&= \frac{\sin \pi\alpha}{\pi} (M+1) \frac{t^{\alpha-1}}{1-\alpha} \|f\|_{L^p(\Omega)}\\
&\approx \frac{\sin \pi\alpha}{\pi} (M+1) \frac{1}{1-\alpha}\, \frac{1}{t^{1-\alpha}} \|u\|_\alpha,
\end{aligned}$$

and

$$\begin{aligned}
\bullet\ \|u_1\|_{L^p(\Omega)} &= \frac{\sin \pi\alpha}{\pi} \left\| \int_{t^{-1}}^{\infty} s^{-\alpha} \left(sI - \mathfrak{A}_p\right)^{-1} f\, ds \right\|_{L^p(\Omega)} \qquad (46)\\
&\leq \frac{\sin \pi\alpha}{\pi} \left( \int_{t^{-1}}^{\infty} s^{-\alpha} \left\| \left(sI - \mathfrak{A}_p\right)^{-1} \right\| ds \right) \|f\|_{L^p(\Omega)}\\
&\leq \frac{\sin \pi\alpha}{\pi} \left( \int_{t^{-1}}^{\infty} s^{-\alpha} \frac{M}{1+s}\, ds \right) \|f\|_{L^p(\Omega)}\\
&\leq \frac{\sin \pi\alpha}{\pi} M \left( \int_{t^{-1}}^{\infty} s^{-\alpha-1}\, ds \right) \|f\|_{L^p(\Omega)} = \frac{\sin \pi\alpha}{\pi} M \frac{t^\alpha}{\alpha} \|f\|_{L^p(\Omega)}\\
&\approx \frac{\sin \pi\alpha}{\pi} M \frac{t^\alpha}{\alpha} \|u\|_\alpha,
\end{aligned}$$

where (see assertions (12) and (13))

$$\|f\|_{L^p(\Omega)} = \left\| \left(-\mathfrak{A}_p\right)^\alpha u \right\|_{L^p(\Omega)} \approx \|u\|_\alpha.$$

By virtue of inequalities (45) and (46), we obtain from Formula (44) that

$$u = u_0 + u_1, \quad u_0 \in D\left(-\mathfrak{A}_p\right),\ u_1 \in L^p(\Omega), \qquad (47)$$

and further that

$$\|u_0\|_{D(-\mathfrak{A}_p)} = \|u_0\|_1 \leq \frac{C}{t^{1-\alpha}} \left\| \left(-\mathfrak{A}_p\right)^\alpha u \right\|_{L^p(\Omega)} = \frac{C}{t^{1-\alpha}} \|u\|_\alpha, \qquad (48)$$

$$\|u_1\|_{L^p(\Omega)} \leq C\, t^\alpha \left\| \left(-\mathfrak{A}_p\right)^\alpha u \right\|_{L^p(\Omega)} = C\, t^\alpha \|u\|_\alpha, \qquad (49)$$

with the constant

$$C := \frac{\sin \alpha\pi}{\pi} \max\left\{ \frac{M+1}{1-\alpha}, \frac{M}{\alpha} \right\}.$$

We remark that inequalities (48) and (49) correspond to inequalities (41) and (42), respectively, and further that Formula (47) corresponds to Formula (40).

In this way, we have proved that $D\Big((-\mathfrak{A}_p)^\alpha\Big)$ is of class $\overline{\mathcal{K}}_{1-\alpha}\big(D(-\mathfrak{A}_p), L^p(\Omega)\big)$.

**Step 2**: Secondly, by applying the Rellich–Kondrachov theorem (see [12], Theorem 6.3, Parts I and II, [13], Theorem 7.26) we find that the injection

$$\iota\colon D(-\mathfrak{A}_p) \longrightarrow L^p(\Omega)$$

is compact. Indeed, we have, by *a priori* estimate (3),

$$D(-\mathfrak{A}_p) = D(\mathfrak{A}_p) \underset{\text{continuously}}{\hookrightarrow} W^{2,p}(\Omega) \underset{\text{compactly}}{\hookrightarrow\hookrightarrow} L^p(\Omega).$$

**Step 3**: Finally, by applying Theorem 3 with

$$\begin{aligned}
&A_0 := D(-\mathfrak{A}_p) = D(\mathfrak{A}_p),\\
&A_1 = B := L^p(\Omega),\\
&A := D\Big((-\mathfrak{A}_p)^\alpha\Big),\\
&\pi := \iota,\\
&\theta := 1-\alpha,
\end{aligned}$$

we obtain that the injection

$$\iota\colon D\Big((-\mathfrak{A}_p)^\alpha\Big) \longrightarrow L^p(\Omega)$$

is compact for every $0 < \alpha < 1$.

The proof of Theorem 4 is complete. □

The situation of Theorem 4 can be visualized as follows:

$$\boxed{\begin{array}{ccc}
L^p(\Omega) & \xrightarrow{\ \iota\ } & L^p(\Omega)\\
\uparrow & & \uparrow\\
D\Big((-\mathfrak{A}_p)^\alpha\Big) & \xrightarrow[\iota]{} & L^p(\Omega)\\
\uparrow & & \uparrow\\
D(-\mathfrak{A}_p) & \xrightarrow[\iota]{} & L^p(\Omega)
\end{array}}$$

**Corollary 1.** *If the conditions (R.1) and (R.2) are satisfied, then the negative fractional power*

$$(-\mathfrak{A}_p)^{-\alpha}\colon L^p(\Omega) \longrightarrow L^p(\Omega)$$

*is compact for every* $0 < \alpha < 1$.

Indeed, it suffices to note that

$$L^p(\Omega) \overset{(-\mathfrak{A}_p)^{-\alpha}}{\longrightarrow} D\Big((-\mathfrak{A}_p)^\alpha\Big) \underset{\text{compactly}}{\hookrightarrow\hookrightarrow} L^p(\Omega).$$

## 4. Proof of Theorem 2

In this section, we give the proof of Theorem 2. By virtue of the Ascoli–Arzelà theorem (see [13], Lemma 6.36), we have only to show that, for some $0 < \beta < 1$ the Hölder norm $\|u(t)\|_{C^{2+\beta}(\overline{\Omega})}$ of a bounded regular solution orbit $u(\cdot, t)$ of the semilinear Cauchy problem

(6) is *uniformly bounded* for all $t \geq 0$. The proof is given by a series of claims (Claims 2–5). We make use of the classical elliptic Schauder theory ([13], Chapter 6, Theorem 6.6) for $t \geq 1$ and the classical linear parabolic theory [5,6] for $0 \leq t \leq 1$.

Suppose that conditions (H.1) and (H.3) are satisfied. Let $n < p < \infty$, and choose a constant $\alpha$ satisfying the condition (7)

$$\frac{1}{2} + \frac{n}{2p} < \alpha < 1.$$

Then, by using Sobolev's imbedding theorem we have the following assertion (Henry [7], Theorem 1.6.1; [1], Theorem 9.1):

$$\mathcal{X}_\alpha = D\Big(\big(-\mathfrak{A}_p\big)^\alpha\Big) \subset C^{1+\beta}(\overline{\Omega}) \quad \text{for } 0 < \beta < 2\alpha - 1 - \tfrac{n}{p}.$$

By inequalities (20), (21) and (33), (34), we have the following claim:

**Claim 2.** *If $u(t)$ is a solution of the abstract semilinear Cauchy problem (6) for $t \geq 0$, then it follows that*

$$\begin{aligned} &u(\cdot,t) \in C^{1+\beta}(\overline{\Omega}) \quad \textit{for all } t \geq 0, \\ &\frac{\partial u}{\partial t}(\cdot,t) \in C^{1+\beta}(\overline{\Omega}) \quad \textit{for all } t \geq 1. \end{aligned}$$

*Moreover, their norms in the Hölder space $C^{1+\beta}(\overline{\Omega})$ are uniformly bounded. Namely, we have, for constants $C_1(\beta) > 0$ and $C_2(\beta) > 0$,*

$$\|u(t)\|_{C^{1+\beta}(\overline{\Omega})} \leq C_1(\beta) \quad \textit{for all } t \geq 0, \tag{50}$$

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{C^{1+\beta}(\overline{\Omega})} \leq C_2(\beta) \quad \textit{for all } t \geq 1. \tag{51}$$

For each $t \geq 1$, we consider the following linear *elliptic problem* (see the semilinear parabolic problem (5)):

$$\begin{cases} Au(t) = g(u(t)) - \frac{\partial u}{\partial t} & \text{in } \Omega, \\ Bu(t) = 0 & \text{on } \Gamma, \end{cases} \tag{52}$$

The proof of Theorem 2 is divided into three steps.

**Step 1**: First, we have the following claim:

**Claim 3.** *Let $u(t)$ be a solution of the abstract semilinear Cauchy problem (6) for $t \geq 0$. If $0 < \beta < 2\alpha - 1 - \frac{n}{p}$, then we can find a constant $L_\beta > 0$ such that*

$$\big|g(u(x,t)) - g\big(u(x',t)\big)\big| \leq L_\beta \big|u(x,t) - u(x',t)\big| \quad \textit{for all } x, x' \in \overline{\Omega}. \tag{53}$$

**Proof.** Indeed, we have, by inequality (50),

$$\|u(t)\|_{C^\beta(\overline{\Omega})} \leq \|u(t)\|_{C^{1+\beta}(\overline{\Omega})} \leq C_1(\beta) \quad \text{for all } t \geq 0. \tag{54}$$

Since the function $g(t)$ is locally Lipschitz continuous on $\mathbf{R}$, we can find a constant $L_\beta > 0$ such that

$$\begin{aligned} &\big|g(u(x,t)) - g\big(u(x',t)\big)\big| \\ &\leq L_\beta \big|u(x,t) - u(x',t)\big| \quad \text{for all } x, x' \in \overline{\Omega}. \end{aligned}$$

This proves the desired inequalty (53) for all $t \geq 0$.

The proof of Claim 3 is complete. $\square$

Moreover, we have the following claim:

**Claim 4.** *The function*

$$\overline{\Omega} \ni x \longmapsto g(u(x,t))$$

*is $\beta$-Hölder continuous on $\overline{\Omega}$ for all $t \geq 0$. More precisely, we have, for a constant $C_3(\beta) > 0$,*

$$\|g(u(t))\|_{C^{\beta}(\overline{\Omega})} \leq C_3(\beta) \quad \text{for all } t \geq 0. \tag{55}$$

**Proof.** First, it follows from inequalities (53) and (54) that we have, for all $t \geq 0$,

$$\begin{aligned} \frac{|g(u(x,t)) - g(u(x',t))|}{|x-x'|^{\beta}} &\leq L_{\beta} \frac{|u(x,t) - u(x',t)|}{|x-x'|^{\beta}} \\ &\leq L_{\beta} \|u\|_{C^{\beta}(\overline{\Omega})} \\ &\leq L_{\beta} C_1(\beta) \quad \text{for all } x, x' \in \overline{\Omega} \text{ with } x \neq x'. \end{aligned}$$

This proves the desired inequality (55).

The proof of Claim 4 is complete. □

**Claim 5.** *Let $u(t)$ be a solution of the abstract semilinear Cauchy problem (6) for $t \geq 0$. If $0 < \beta < 2\alpha - 1 - \frac{n}{p}$, then it follows that*

$$u(\cdot,t) \in C^{2+\beta}(\overline{\Omega}) \quad \text{for all } t \geq 1. \tag{56}$$

*Moreover, the Hölder norm $\|u(t)\|_{2+\beta}$ is uniformly bounded for all $t \geq 1$. Namely, we have, for a constant $C_4(\beta) > 0$,*

$$\|u(t)\|_{C^{2+\beta}(\overline{\Omega})} \leq C_4(\beta) \quad \text{for all } t \geq 1. \tag{57}$$

**Proof.** By applying the *elliptic Schauder theory* ([13], Chapter 6, Theorem 6.6) to the linear elliptic problem (52), we obtain that assertion (56) holds true. Moreover, it follows from inequalities (54), (55) and (51) that

$$\begin{aligned} \|u(t)\|_{C^{2+\beta}(\overline{\Omega})} &\leq C\left(\|u(t)\|_{C(\overline{\Omega})} + \left\|g(u(t)) - \frac{\partial u}{\partial t}\right\|_{C^{\beta}(\overline{\Omega})}\right) \\ &\leq C\left(\|u(t)\|_{C^{\beta}(\overline{\Omega})} + \|g(u(t))\|_{C^{\beta}(\overline{\Omega})} + \left\|\frac{\partial u}{\partial t}\right\|_{C^{1+\beta}(\overline{\Omega})}\right) \\ &\leq C(C_1(\beta) + C_3(\beta) + C_2(\beta)) \quad \text{for all } t \geq 1, \end{aligned} \tag{58}$$

where the constant $C > 0$ is independent of $t$.

Therefore, the desired inequality (57) follows from inequality (58) with

$$C_4(\beta) := C(C_1(\beta) + C_2(\beta) + C_3(\beta)).$$

The proof of Claim 5 is complete. □

**Step 2**: On the other hand, we consider the original linear *parabolic problem* (see the semilinear parabolic problem (5)):

$$\begin{cases} \frac{\partial u}{\partial t} - Au(t) = g(u(t)) & \text{in } \Omega \times (0,1], \\ Bu(t) = 0 & \text{on } \Gamma \times (0,1], \\ u(0) = u_0 & \text{in } \overline{\Omega}. \end{cases} \tag{59}$$

By applying Theorems A1 (inequality (A3)) and A2 (inequality (A4)) to the linear parabolic problem (59), we obtain that the Hölder norm $\|u(t)\|_{C^{2+\beta}(\overline{\Omega})}$ is uniformly bounded for all $0 \leq t \leq 1$, since $u_0 \in C^{2+\beta}(\overline{\Omega})$ for $\beta < 2\alpha - 1 - \frac{n}{p} < \alpha$.

**Step 3**: Summing up, we have proved that the Hölder norm $\|u(t)\|_{C^{2+\beta}(\overline{\Omega})}$ is uniformly bounded for all $t \geq 0$.

The proof of Theorem 2 is now complete.

## 5. Applications to Diffusive Logistic Equations in Population Dynamics

In this section, as an application of Theorem 2, we study the dynamics of a population inhabiting a strongly heterogeneous environment that is modeled by a class of diffusive logistic equations with Dirichlet and Neumann boundary conditions of the form

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) = d\,\Delta w(x,t) + (m(x) - h(x)w(x,t))w(x,t) & \text{in } \Omega \times (0,\infty), \\ Bw(x',t) = a(x')\frac{\partial w}{\partial \mathbf{n}} + b(x')w = 0 & \text{on } \Gamma \times (0,\infty), \\ w(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{60}$$

Here:

(1) $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \ldots + \partial^2/\partial x_n^2$ is the Laplacian.
(2) $d$ is a positive parameter.
(3) $m(x)$ is a real-valued function on $\overline{\Omega}$.
(4) $h(x)$ is a non-negative function on $\overline{\Omega}$.
(5) Either $a(x') \equiv 1$ and $b(x') \equiv 0$ on $\Gamma$ (Neumann cases) or $a(x') \equiv 0$ and $b(x') \equiv 1$ on $\Gamma$ (Dirichlet case).
(6) $\mathbf{n}$ is the unit outward normal to the boundary $\Gamma = \partial\Omega$ (see Figure 1).

The purpose of this section is to discuss the changes that occur in the structure of positive solutions of the steady state as the parameter $\lambda = 1/d$ varies near the first eigenvalue $\lambda_1(m)$ under the condition that:

(M1) The function $m(x)$ belongs to the space $L^\infty(\Omega)$ and the set $\{x \in \Omega : m(x) > 0\}$ has *positive measure*.

This section is an expanded and revised version of the previous work Taira [14–16].

We begin with our motivation and some of the modeling process leading to the semilinear parabolic initial boundary value problem (60). The basic interpretation of the various terms in the semilinear parabolic problem (60) may be stated as follows (see Tables 1 and 2):

(i) The solution $w(x,t;u_0)$ represents the population density of a species inhabiting a region $\Omega$.
(ii) The members of the population are supposed to move about $\Omega$ via the type of random walks occurring in *Brownian motion* that is modeled by the diffusive term $d\Delta$; hence $d$ represents the rate of diffusive dispersal. For large values of $d$ the population spreads more rapidly than for small values of $d$.
(iii) The local rate of change in the population density is described by the density-dependent term $m(x) - h(x)u$.
(iv) The term $m(x)$ describes the rate at which the population would grow or decline at the location $x$ in the absence of crowding or limitations on the availability of resources. The sign of $m(x)$ will be positive on favorable habitats for population growth and negative on unfavorable ones. Specifically $m(x)$ may be considered as a food source or any resource which will be good in some areas and bad in others.
(v) The term $-h(x)u$ describes the effects of crowding on the growth rate of the population at the location $x$; these effects are supposed to be independent of those determining the growth rate. The size of the coefficient of intraspecific competition $h(x)$ describes the strength of the effects of crowding within the population.

(vi) In terms of biology, the homogeneous Dirichlet condition represents that $\Omega$ is surrounded by a completely hostile exterior such that any member of the population which reaches the boundary dies immediately; in other words, the exterior of the domain is deadly to the population ($a(x') \equiv 0$ and $b(x') \equiv 1$ on $\Gamma$).
(vii) If the boundary acts as a *barrier*, so that individuals reaching the boundary simply return to the interior, a Neumann boundary condition results ($a(x') \equiv 1$ and $b(x') \equiv 0$ on $\Gamma$).
(viii) If the exterior is hostile but not completely deadly, a mixed or Robin boundary condition results ($a(x') \equiv 1$ and $b(x') \geq 0$ on $\Gamma$), and the analysis is similar.
(ix) A biological interpretation of our main results (Theorems 6 and 9) is that when the environment has an impassable boundary and is on the average unfavorable, then high diffusion rates have the same effect as they always have when the boundary is deadly; but if the boundary is impassable and the environment is on the average neutral or favorable, then the population can persist, no matter what its rate of diffusion.

**Table 1.** A biological meaning of each term in the semilinear initial boundary value problem (60).

| Term | Biological Interpretation |
|---|---|
| $\Omega$ | Terrain |
| $w(x,t)$ | Population density of a species inhabiting the terrain |
| $\Delta$ | A member of the population moves about the terrain via the type of random walks occurring in Brownian motion |
| $d$ | Rate of diffusive dispersal |
| $m(x)$ | Intrinsic growth rate |
| $h(x)$ | Coefficient of intraspecific competition |

**Table 2.** A biological meaning of boundary conditions in the semilinear initial boundary value problem (60).

| Boundary Condition | Biological Interpretation |
|---|---|
| Dirichlet case | Completely hostile (deadly) exterior |
| Neumann case | Barrier |
| Robin or mixed-type case | Hostile but not completely deadly exterior |

In order to study the semilinear parabolic initial boundary value problem (60), we may view it as generating a *dynamical system*. The semilinear parabolic problem (60) admits a unique classical solution for sufficiently small times. However, comparison theorems based on the maximum principle guarantee the existence of global solutions in time, since the nonlinearity we are dealing with is sublinear. Our approach is to observe that whether our model (60) predicts persistence or extinction for the population is determined by the nature of the steady states. Our models are shown to possess a unique positive steady state, that is, a unique positive solution of the semilinear elliptic boundary value problem

$$\begin{cases} d\,\Delta u(x) + (m(x) - h(x)u(x))u(x) = 0 & \text{in } \Omega, \\ u(x') = 0 & \text{on } \Gamma. \end{cases} \tag{61}$$

A solution $u \in C^2(\overline{\Omega})$ of the semilinear elliptic problem (61) is said to be *non-trivial* if it does not identically equal zero on $\overline{\Omega}$. A non-trivial solution $u$ is called a *positive solution* if it is strictly positive everywhere in $\Omega$.

The object of the analysis is to determine how the spatial arrangement of favorable and unfavorable habitats affects the population being modeled. In fact, we show that the semilinear parabolic problem (60) admits a unique positive steady state which is a global attractor for non-negative solutions provided $d$ is sufficiently small (see part (ii) of Theorem 7), so that the population persists, and further we show that the zero solution is a global attractor for non-negative solutions if $d$ is sufficiently large (see part (i) of Theorem 7), so that the population tends to extinction.

### 5.1. Dirichlet Eigenvalue Problems with Indefinite Weights

It is known that many of the qualitative aspects of the analysis depend crucially on the size of the first positive eigenvalue $\lambda_1(m)$ for the *linearized Dirichlet eigenvalue problem* with an indefinite weight function $m(x)$ and a positive parameter $\lambda = 1/d$:

$$\begin{cases} -\Delta\,\phi(x) = \lambda\, m(x)\,\phi(x) & \text{in } \Omega, \\ \phi(x') = 0 & \text{on } \Gamma. \end{cases} \tag{62}$$

The next theorem asserts the existence of the first positive eigenvalue $\lambda_1(m)$ of the Dirichlet problem (62), implying persistence for the population (see Manes–Micheletti [17], de Figueiredo [18]):

**Theorem 5** (the Dirichlet case)**.** *If the* intrinsic growth rate $m(x)$ *satisfies condition (M1), then the first eigenvalue $\lambda_1(m)$ of the Dirichlet problem* (62) *is positive and simple, and its corresponding eigenfunction $\phi_1(x)$ may be chosen to be strictly positive everywhere in $\Omega$. Moreover, no other eigenvalues have positive eigenfunctions:*

$$\begin{cases} -\Delta\,\phi_1(x) = \lambda_1(m)\, m(x)\phi_1(x) & \textit{in } \Omega, \\ \phi_1(x) > 0 & \textit{in } \Omega, \\ \phi_1(x') = 0 & \textit{on } \Gamma. \end{cases} \tag{63}$$

Some important remarks are in order:

**Remark 1.**

1° *By the Rayleigh principle (see Manes–Micheletti [17], de Figueiredo [18]), we know that the first eigenvalue $\lambda_1(m)$ is given by the variational formula*

$$\lambda_1(m) = \inf\left\{ \frac{\int_\Omega |\nabla\phi(x)|^2\,dx}{\int_\Omega m(x)\phi(x)^2\,dx} : \phi \in W_0^{1,2}(\Omega),\ \int_\Omega m(x)\phi(x)^2\,dx > 0 \right\}. \tag{64}$$

*Here $W_0^{1,2}(\Omega) = H_0^1(\Omega)$ is the closure of smooth functions with compact support in $\Omega$ in the Sobolev space $W^{1,2}(\Omega) = H^1(\Omega)$.*

2° *By Formula* (64)*, we find that the first eigenvalue $\lambda_1(m)$ is strictly decreasing with respect to the weight $m(x)$ in the following sense (see [19] (Proposition 8.3)): If $m_1(x) \leq m_2(x)$ almost everywhere in $\Omega$, then the corresponding first eigenvalues $\lambda_1(m_1)$ and $\lambda_1(m_2)$ satisfy the relation*

$$\lambda_1(m_1) \geq \lambda_1(m_2).$$

*If the inequality is strict on a set of positive measure, it follows that $\lambda_1(m_1) > \lambda_1(m_2)$.*

A biological interpretation of Theorem 5 (the Dirichlet case) may be stated as follows:

(i) If there is a favorable region, then the models we consider predict persistence for a population since the existence of the first positive eigenvalue is equivalent to the existence of a positive density function describing the distribution of the population of $\Omega$.

(ii) The size of the first eigenvalue $\lambda_1(m)$ is of crucial importance; increasing $\lambda_1(m)$ imposes a more stringent condition on the diffusion rate $d$ if the population is to persist, since $0 < d < 1/\lambda_1(m)$ (see Theorem 6).

(iii) It is worthwhile to point out here that the first eigenvalue $\lambda_1(m)$ will tend to be smaller in situations where favorable and unfavorable habitats are closely intermingled (producing cancellation effects), and larger when the favorable region consists of a relatively small number of relatively large isolated components.

*5.2. Diffusive Logistic Dirichlet Problems*

In this subsection, by using Theorem 2 we study the following semilinear parabolic initial boundary value problem with homogeneous *Dirichlet* condition:

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) = d\,\Delta w(x,t) + (m(x) - h(x)w(x,t))w(x,t) & \text{in } \Omega \times (0,\infty), \\ w(x',t) = 0 & \text{on } \Gamma \times (0,\infty), \\ w(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{65}$$

To do so, we consider the *logistic Dirichlet problem* (61) with $d := 1/\lambda$:

$$\begin{cases} -\Delta\, u(x) = \lambda(m(x) - h(x)u(x))u(x) & \text{in } \Omega, \\ u(x') = 0 & \text{on } \Gamma. \end{cases} \tag{66}$$

We suppose that the coefficient of intraspecific competition $h(x)$ is a non-negative function in the space $C^1(\overline{\Omega})$, and let

$$\Omega^+(h) = \{x \in \Omega : h(x) > 0\},$$

and

$$\Omega_0(h) = \Omega \setminus \overline{\Omega^+(h)}.$$

In this paper, we study the case where $h(x) > 0$ on the boundary $\partial\Omega$. More precisely, our *structural condition* on the coefficient of intraspecific competition $h(x)$ is stated as follows (see Figure 4):

(Z1) The open set $\Omega_0(h)$ consists of a *finite* number of connected components with boundary of class $C^1$, say $\Omega_0^k(h)$, $1 \leq k \leq N$, which are bounded away from the boundary $\partial\Omega$.

$$\Omega_0(h) = \bigcup_{k=1}^{N} \Omega_0^k(h).$$

This structural condition is inspired by Ouyang [20], Theorem 2.

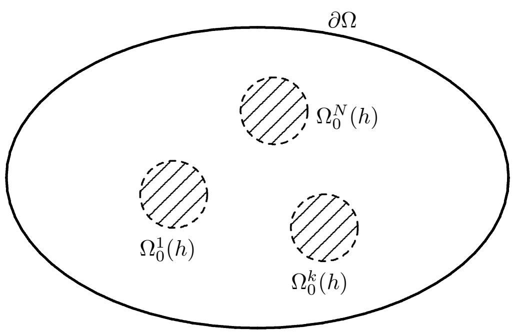


**Figure 4.** The structural condition (Z1) on the coefficient of intraspecific competition $h(x)$.

We consider the Dirichlet eigenvalue problem with indefinite weight function $m(x)$ in each connected component $\Omega_0^k(h)$

$$\begin{cases} -\Delta\, \psi(x) = \mu\, m(x)\, \psi(x) & \text{in } \Omega_0^k(h), \\ \psi(x') = 0 & \text{on } \partial\Omega_0^k(h), \end{cases} \tag{67}$$

where $\partial\Omega_0^k(h)$ denotes the boundary of $\Omega_0^k(h)$.

In this paper, we suppose that

(Z2) Each set $\{x \in \Omega_0^k(h) : m(x) > 0\}$ has *positive measure* for $1 \leq k \leq N$,

and let

$$\mu_1\left(\Omega_0^k(h)\right) = \text{the } \textit{first eigenvalue} \text{ of the Dirichlet eigenvalue problem (67).}$$

By applying Theorem 1 with

$$\Omega := \Omega_0^k(h) \quad \text{for } 1 \leq k \leq N,$$
$$(M1) := (Z2),$$

we obtain that the first eigenvalue $\lambda_1\left(\Omega_0^k(h)\right)$ is positive and algebraically simple:

$$\mu_1\left(\Omega_0^k(h)\right) > 0 \quad \text{for } 1 \leq k \leq N. \tag{68}$$

Moreover, by the Rayleigh principle ([18], Proposition 1.10; [21], Proposition 3.4) we know that the first eigenvalue $\mu_1\left(\Omega_0^k(h)\right)$ is given by the variational formula

$$\mu_1\left(\Omega_0^k(h)\right) = \inf\left\{\frac{\int_{\Omega_0^k(h)} |\nabla\psi(x)|^2\,dx}{\int_{\Omega_0^k(h)} m(x)\psi(x)^2\,dx} : \psi \in H_0^1\left(\Omega_0^k(h)\right),\ \int_{\Omega_0^k(h)} m(x)\psi(x)^2\,dx > 0\right\}.$$

By virtue of assertion (68), we can associate with the open set $\Omega_0(h)$ a positive number $\mu_1(\Omega_0(h))$ as follows:

$$\mu_1(\Omega_0(h)) = \min\left\{\mu_1\left(\Omega_0^1(h)\right), \mu_1\left(\Omega_0^2(h)\right), \ldots, \mu_1\left(\Omega_0^N(h)\right)\right\}. \tag{69}$$

**Remark 2.** *It should be noticed (see Chavel [22] (p. 18, Corollary 1); López-Gómez [19] (Section 8.1)) that the value $\mu_1(\Omega_0(h))$ tends to be smaller in situations where favorable and unfavorable habits are closely intermingled, and larger when the favorable region consists of a relatively small number of relatively large isolated components.*

Now we can state our main result that is a generalization of Cantrell–Cosner [23] (Theorems 2.1 and 2.3), Hess–Kato [24] (Theorem 2) and Hess [25] (Theorem 27.1) to the case where the *coefficient of intraspecific competition $h(x)$* may vanish in $\Omega$ (see Figure 4 as above):

**Theorem 6** (the logistic Dirichlet case). *Let $m(x) \in C^\theta(\overline{\Omega})$ for $0 < \theta < 1$. Suppose that condition (M1) and the structural conditions (Z1) and (Z2) are satisfied. Then the logistic Dirichlet problem* (66) *has a unique positive solution $u(\lambda) \in C^{2+\theta}(\overline{\Omega})$ for every $\lambda \in (\lambda_1(m), \mu_1(\Omega_0(h)))$. For any $\lambda \geq \mu_1(\Omega_0(h))$, there exists no positive solution of the logistic Dirichlet problem* (66). *Moreover, we have the assertions (see Figure 5 below)*

$$\lim_{\lambda\uparrow\mu_1(\Omega_0(h))} \|u(\lambda)\|_{L^2(\Omega)} = +\infty,$$
$$\lim_{\lambda\downarrow\lambda_1(m)} \|u(\lambda)\|_{C^{2+\theta}(\overline{\Omega})} = 0.$$

A biological interpretation of Theorem 6 (the logistic Dirichlet case) may be stated as follows (see Figure 5):

(i) If the environment has a completely hostile boundary, then the models we consider predict persistence for a population if its diffusion rate $d$ is below the critical value $1/\lambda_1(m)$ depending on the *intrinsic growth rate* $m(x)$ and if it is above the critical value $1/\mu_1(\Omega_0(h))$ depending on the coefficient $h(x)$ describing the strength of the crowding effects.

(ii) In a certain sense, the most favorable situations will occur if there is a relatively large favorable region (with good resources and without crowding effects) located some distance away from the boundary of $\Omega$.

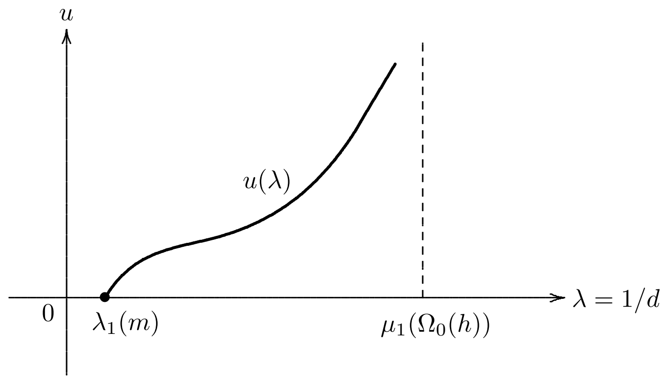


**Figure 5.** A biological interpretation of Theorem 6 (the logistic Dirichlet case).

Some important remarks are in order:

**Remark 3.**

1° *Theorem 6 may be proved by using the* super-sub-solution method *just as in the proof of Fraile et al. [26] (Theorems 3.5 and 4.6), with a weaker assertion*

$$\lim_{\lambda \uparrow \mu_1(\Omega_0(h))} \|u(\lambda)\|_{C(\overline{\Omega})} = +\infty.$$

2° *Theorem 6 asserts that the assertions hold true if the dimension $n$ is greater than* 2 *($n \geq 3$). It should be emphasized that an estimate of the growth rate of the total size $\|u(\lambda)\|_{L^1(\Omega)} = \int_\Omega u(\lambda)\,dx$ of the positive steady states $u(\lambda)$ as $\lambda \uparrow \mu_1(\Omega_0(h))$ is of crucial importance from the viewpoint of population dynamics.*

3° *López-Gómez–Sabina de Lis [27] analyze the pointwise growth to infinity of positive solutions of the logistic Dirichlet problem in the case where $m(x) \equiv 1$ in $\Omega$ (see [27], Theorems 4.2 and 4.3). Furthermore, García-Melián et al. [28] study the pointwise behavior and the uniqueness of positive solutions of nonlinear elliptic boundary value problems of general sublinear type, and give the exact limiting profile of the positive solutions (see [28], Theorem 3.1, Corollary 3.3 and Theorem 6.4). Their numerical computations confirm and illuminate the above bifurcation diagram (Figure 5).*

**Remark 4.** *Suppose that*

$$h(x) > 0 \quad \text{on } \overline{\Omega}, \tag{70}$$

*and that the* intrinsic growth rate $m(x)$ *satisfies condition (M1). Then, by arguing as in the proof of Cantrell–Cosner [23] (Theorem 4.1) we can give an estimate of the decay rate of the total size of the positive steady states $u(\lambda)$:*

$$\int_\Omega u(\lambda)\,dx \leq \left(1 - \frac{\lambda_1(m)}{\lambda}\right) |\Omega|^{2/3} \frac{\left(\int_\Omega m^+(x)^3\,dx\right)^{1/3}}{\min_{x\in\overline{\Omega}} h(x)} \quad \text{for all } \lambda > \lambda_1(m).$$

*Here $|\Omega|$ is the volume of $\Omega$ and*

$$m^+(x) = \max\{m(x),\ 0\} \quad \textit{for } x \in \Omega.$$

*Therefore, we find that the quantity*

$$|\Omega|^{2/3} \frac{\left(\int_\Omega m^+(x)^3\, dx\right)^{1/3}}{\min_{x\in\overline{\Omega}} h(x)}$$

*is the carrying capacity of the population.*

*5.3. Stability for Positive Solutions of Diffusive Logistic Dirichlet Problems*

Secondly, by using Theorem 2 we can study the asymptotic stability properties for positive solutions of the logistic Dirichlet problem (66) (see [4] (Section 6)):

$$\begin{cases} -\Delta\, u(x) = \lambda(m(x) - h(x)u(x))u(x) & \text{in } \Omega, \\ u(x') = 0 & \text{on } \Gamma. \end{cases}$$

In this case, the dynamics of a population inhabiting a strongly heterogeneous environment is modeled by the semilinear parabolic initial boundary value problem Equation (65) with homogeneous Dirichlet condition

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) = d\, \Delta w(x,t) + (m(x) - h(x)w(x,t))w(x,t) & \text{in } \Omega \times (0,\infty), \\ w(x',t) = 0 & \text{on } \Gamma \times (0,\infty), \\ w(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

In order to study the semilinear parabolic problem (65), we may view it as generating a dynamical system. To do so, we consider the semilinear parabolic problem (65) with $d := 1/\lambda$:

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) = \frac{1}{\lambda}\, \Delta w(x,t) + (m(x) - h(x)w(x,t))w(x,t) & \text{in } \Omega \times (0,\infty), \\ w(x',t) = 0 & \text{on } \Gamma \times (0,\infty), \\ w(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{71}$$

It is known (see [5] (p. 320, Theorems 5.2 and 5.3) and [29] (Proposition 3.4, Lemma 4.2, Theorem 4.5)) that the semilinear parabolic problem (71) admits a unique classical global solution $w(x,t;u_0)$ for each initial value $u_0 \in C^{2+\theta}(\overline{\Omega})$ satisfying the *compatibility conditions*

$$\begin{cases} u_0(x) \geq 0 & \text{in } \Omega, \\ u_0(x') = 0 & \text{on } \Gamma. \end{cases} \tag{72}$$

A positive solution $w_0(x)$ of the logistic Dirichlet problem (66) is said to be *globally asymptotically stable* if we have the assertion

$$\max_{x\in\overline{\Omega}} |w(x,t;u_0) - w_0(x)| \longrightarrow 0 \quad \text{as } t \to \infty$$

for each non-trivial initial value $u_0 \in C^{2+\theta}(\overline{\Omega})$ satisfying the compatibility conditions (72).

The next theorem, due to [16] (Theorem 1.3), describes the asymptotic stability properties for positive solutions of the logistic Dirichlet problem (66) (see Cantrell–Cosner [23] (Theorems 2.1 and 4.9), Fraile et al. [26] (Theorem 3.7)):

**Theorem 7** (the logistic Dirichlet case)**.** *Let $m(x) \in C^\theta(\overline{\Omega})$ for $0 < \theta < 1$. Suppose that condition (M1) and the structural conditions (Z1) and (Z2) are satisfied. Then we have the following three assertions (i)–(iii) (see Figure 6 below):*

(i) *The zero solution of the logistic Dirichlet problem* (66) *is* globally asymptotically stable *if* $\lambda$ *is so small that*

$$0 < \lambda < \lambda_1(m).$$

*In this case, we can give an estimate of the decay rate of the total size of the population*

$$\int_\Omega w(x,t;u_0)\,dx \tag{73}$$
$$\leq \exp\left[-\left(\frac{1}{\lambda} - \frac{1}{\lambda_1(m)}\right)\lambda_1(1)\,t\right]|\Omega|^{1/2}\left(\int_\Omega u_0(x)^2\,dx\right)^{1/2} \quad \textit{for all } t > 0.$$

(ii) *A positive solution* $u(\lambda)$ *of the logistic Dirichlet problem* (66) *is* globally asymptotically stable *for each* $\lambda$ *satisfying the condition*

$$\lambda_1(m) < \lambda < \mu_1(\Omega_0(h)).$$

(iii) *If* $\lambda$ *is so large that*

$$\lambda > \mu_1(\Omega_0(h)),$$

*then we have the assertion*

$$\max_{x\in\overline{\Omega}} |w(x,t;u_0)| \longrightarrow \infty \quad \textit{as } t \to \infty \tag{74}$$

*for each non-trivial initial value* $u_0 \in C^{2+\theta}(\overline{\Omega})$ *satisfying the compatibility conditions* (72).

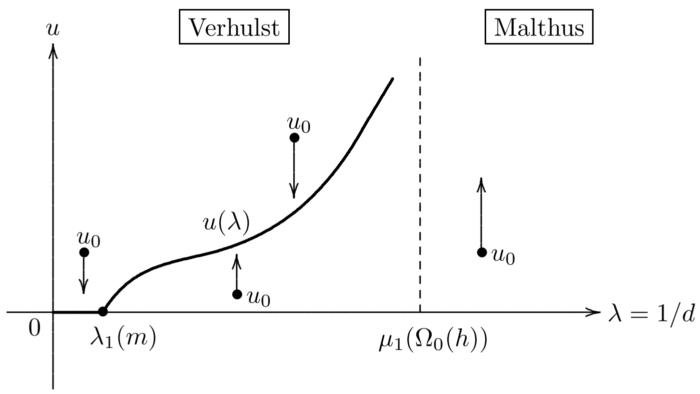


**Figure 6.** A biological interpretation of parts (i)–(iii) of Theorem 7 (the logistic Dirichlet case): Malthus versus Verhulst.

A biological interpretation of Theorem 7 (the logistic Dirichlet case) may be stated as follows (see Figure 6):

(i) A population will grow exponentially until limited by lack of available resources if the diffusion rate $d = 1/\lambda$ is below the critical value $1/\mu_1(\Omega_0(h))$ (assertion (74) in part (iii)); this idea is generally credited to the English economist Thomas Robert Malthus (1776–1834).
(ii) If the diffusion rate $d = 1/\lambda$ is above the critical value $1/\mu_1(\Omega_0(h))$, then the model obeys the logistic equation introduced by the Belgian mathematical biologist Pierre François Verhulst (1804–1849) around 1840 (the decay estimate (73) in part (i)).

### *5.4. Heuristic Approach to Diffusive Logistic Dirichlet Problems via the Semenov Approximation*

This subsection is adapted from Taira [30]. For simplicity, we suppose that the coefficient of intraspecific competition $h(x)$ satisfies the condition (70)

$$h(x) > 0 \quad \text{on } \overline{\Omega},$$

and further that the *intrinsic growth rate* $m(x)$ satisfies condition (M1). First, we rewrite the logistic Dirichlet problem (66) in the form

$$\begin{cases} -\Delta\, u(x) = \lambda(m(x) - h(x)\, u(x))u(x) & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ u(x') = 0 & \text{on } \Gamma. \end{cases}$$

Namely, we consider the logistic Dirichlet problem (66) as the Dirichlet eigenvalue problem with the *weight* $m(x) - h(x)\, u$.

However, Theorem 5 asserts that the first eigenvalue $\lambda_1(m)$ is the *unique* eigenvalue of the Dirichlet eigenvalue problem (62) corresponding to a positive eigenfunction $\phi_1(x)$. Now we suppose that the solution $u$ is of the form

$$u = C(\lambda)\, \phi_1 \quad \text{for } \lambda > \lambda_1(m),$$

where $C(\lambda)$ is a non-zero constant. Then we have the formulas

$$-\Delta u = -C(\lambda)\, \Delta\phi_1 = C(\lambda)\, \lambda_1(m)\, m(x)\, \phi_1 \quad \text{in } \Omega$$

and

$$\lambda(m(x) - h(x)\, u)u = \lambda(m(x) - h(x)\, u)C(\lambda)\, \phi_1 \quad \text{in } \Omega.$$

This implies that

$$\lambda\, m(x) - \lambda\, h(x)\, u = \lambda_1(m)\, m(x) \quad \text{in } \Omega,$$

so that

$$u = u(\lambda) = \frac{m(x)}{h(x)}\left(1 - \frac{\lambda_1(m)}{\lambda}\right) \quad \text{in } \Omega.$$

Therefore, we obtain that the bifurcation solution curve $(\lambda, u)$ of the logistic Dirichlet problem (66) is "formally" given by Formula (75), called the *Semenov approximation* in Chemistry ([31]),

$$u(\lambda) = \frac{m(x)}{h(x)}\left(1 - \frac{\lambda_1(m)}{\lambda}\right) \quad \text{for } \lambda > \lambda_1(m). \tag{75}$$

In view of Formula (75) and Figure 7 below, we find that the quantity

$$\ell := \frac{\max_{\overline{\Omega}} m}{\min_{\overline{\Omega}} h} = \frac{\max_{\overline{\Omega}} m^+}{\min_{\overline{\Omega}} h}$$

is the *carrying capacity* of the environment under condition (70).

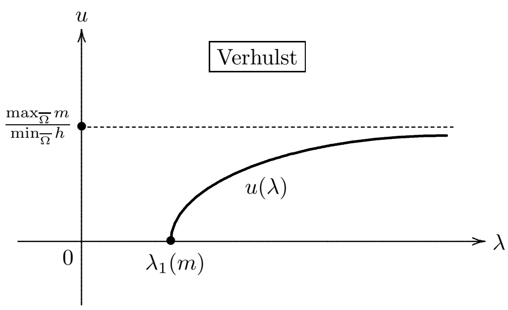


**Figure 7.** The *formal* positive solution curve $(\lambda, u(\lambda))$ for $\lambda > \lambda_1(m)$ under condition (70) (the logistic Dirichlet case).

*5.5. Diffusive Logistic Neumann Problems*

In this subsection, by using Theorem 2 we study the following semilinear parabolic initial boundary value problem with homogeneous *Neumann* condition of the form

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) = d\,\Delta w(x,t) + (m(x) - h(x)w(x,t))w(x,t) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial w}{\partial \mathbf{n}}(x',t) = 0 & \text{on } \Gamma \times (0,\infty), \\ w(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{76}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

In order to study the semilinear initial boundary value problem (76), we may view it as generating a *dynamical system*. To do so, we consider the semilinear parabolic problem (76) with $d := 1/\lambda$:

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) = \frac{1}{\lambda}\,\Delta w(x,t) + (m(x) - h(x)w(x,t))w(x,t) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial w}{\partial \mathbf{n}}(x',t) = 0 & \text{on } \Gamma \times (0,\infty), \\ w(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{77}$$

It is known (see [5] (p. 320, Theorems 5.2 and 5.3) and [29] (Proposition 3.4, Lemma 4.2, Theorem 4.5)) that the semilinear parabolic problem (77) admits a unique classical global solution $w(x,t;v_0)$ for each initial value $v_0 \in C^{2+\theta}(\overline{\Omega})$ satisfying the *compatibility conditions*

$$\begin{cases} v_0(x) \geq 0 & \text{in } \Omega, \\ \frac{\partial v_0}{\partial \mathbf{n}}(x') = 0 & \text{on } \Gamma. \end{cases} \tag{78}$$

The analysis of the semilinear parabolic problem (76) with homogeneous Neumann condition may be somewhat different since the operator $-\Delta$ with homogeneous Neumann condition has zero as an eigenvalue. However, the same general approach to the semilinear parabolic initial boundary value problem (65) with homogeneous Dirichlet condition can still be used (see Hess [25]).

First, we consider the *linearized Neumann eigenvalue problem* with an indefinite weight function $m(x)$ and a real parameter $\nu = 1/d$:

$$\begin{cases} -\Delta\,\psi(x) = \nu\, m(x)\,\psi(x) & \text{in } \Omega, \\ \frac{\partial \psi}{\partial \mathbf{n}}(x') = 0 & \text{on } \Gamma. \end{cases} \tag{79}$$

We discuss the structure of positive solutions of the eigenvalue problem (79) under the condition that:

(M2) The *intrinsic growth rate* $m(x)$ belongs to the Hölder space $C^\theta(\overline{\Omega})$ for $0 < \theta < 1$ and it attains *both* positive and negative values in $\Omega$.

If condition (M2) is satisfied, then the Neumann eigenvalue problem (79) admits a unique *non-zero*, eigenvalue $\nu_1(m)$ having a positive eigenfunction. More precisely, we have the following theorem (see Brown–Lin [32] (Theorem 3.13) and Senn–Hess [33] (Theorems 2 and 3)):

**Theorem 8** (the Neumann case)**.** *If the intrinsic growth rate $m(x)$ satisfies condition (M2), then the Neumann eigenvalue problem (79) admits a unique* non-zero, *eigenvalue $\nu_1(m)$ having a positive eigenfunction. More precisely, we have the following three assertions (i)–(iii):*

*(i)* *If $\int_\Omega m(x)\,dx < 0$, then the smallest, non-zero eigenvalue $\nu_1(m)$ of the Neumann problem (79) is positive and simple, and its corresponding eigenfunction $\psi_1(x) \in C^{2+\theta}(\overline{\Omega})$ may be chosen to be strictly positive everywhere in $\Omega$. Moreover, no other positive eigenvalues have positive eigenfunctions. The eigenvalue 0 is simple and has the positive eigenfunction $\psi_1(x) \equiv 1$ in $\Omega$.*

(ii) *If $\int_\Omega m(x)\,dx > 0$, then the largest, non-zero eigenvalue $\nu_1(m)$ of the Neumann problem (79) is negative and simple, and its corresponding eigenfunction $\psi_1(x) \in C^{2+\theta}(\overline{\Omega})$ may be chosen to be strictly positive everywhere in $\Omega$. Moreover, no other negative eigenvalues have positive eigenfunctions. The eigenvalue 0 is simple and has the positive eigenfunction $\psi_1(x) \equiv 1$ in $\Omega$.*

(iii) *If $\int_\Omega m(x)\,dx = 0$, then the eigenvalue 0 of the Neumann problem (79) is the only eigenvalue having the positive eigenfunction $\psi_1(x) \equiv 1$ in $\Omega$.*

Next we study the following *steady state problem* with a parameter $d = 1/\nu$:

$$\begin{cases} -\Delta v(x) = \nu(m(x) - h(x)v(x))v(x) & \text{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}}(x') = 0 & \text{on } \Gamma. \end{cases} \tag{80}$$

This problem is the *logistic Neumann problem*.

Then we have the following generalization of Hess [25] (Example 28.6) to the case where the coefficient of intraspecific competition $h(x)$ may vanish in $\Omega$ under the structural conditions (Z1) and (Z2) (see Fraile et al. [26] (Theorem 3.7), Senn [34] (Theorem 3.2)):

**Theorem 9** (the logistic Neumann case)**.** *Suppose that condition (M2) and the structural conditions (Z1) and (Z2) are satisfied. Then we have the following two assertions (i) and (ii):*

(i) *If $\int_\Omega m(x)\,dx < 0$, the logistic Neumann problem (80) has a unique positive solution $v(\lambda) \in C^{2+\theta}(\overline{\Omega})$ for every $\lambda \in (\nu_1(m), \mu_1(\Omega_0(h)))$. For any $\lambda \geq \mu_1(\Omega_0(h))$, there exists no positive solution of the semilinear problem (79). Moreover, we have the assertions*

$$\lim_{\lambda \uparrow \mu_1(\Omega_0(h))} \|v(\lambda)\|_{L^2(\Omega)} = +\infty,$$
$$\lim_{\lambda \downarrow \nu_1(m)} \|v(\lambda)\|_{C^{2+\theta}(\overline{\Omega})} = 0.$$

*In a neighborhood of the point $(0,0)$ the solution set of the logistic Neumann problem (80) just consists of the two lines of trivial solutions (see Figure 8 below).*

(ii) *$\int_\Omega m(x)\,dx \geq 0$, the logistic Neumann problem (80) has a unique positive solution $v(\lambda) \in C^{2+\theta}(\overline{\Omega})$ for every $\lambda \in (0, \mu_1(\Omega_0(h)))$. For each $\lambda \geq \mu_1(\Omega_0(h))$, there exists no positive solution of the logistic Neumann problem (80). Moreover, we have the assertions*

$$\lim_{\lambda \uparrow \mu_1(\Omega_0(h))} \|v(\lambda)\|_{L^2(\Omega)} = +\infty,$$
$$\lim_{\lambda \downarrow 0} \|v(\lambda) - c\|_{C^{2+\theta}(\overline{\Omega})} = 0,$$

*where*

$$c = \begin{cases} \frac{\int_\Omega m(x)\,dx}{\int_\Omega h(x)\,dx} & \text{if } \int_\Omega m(x)\,dx > 0, \\ 0 & \text{if } \int_\Omega m(x)\,dx = 0. \end{cases}$$

*Namely, if $\int_\Omega m(x)\,dx > 0$, there occurs a* secondary bifurcation *from the line $\{0\} \times \mathbf{R}$ of trivial solutions at the point $(0,c)$ (see Figure 9 below). If $\int_\Omega m(x)\,dx = 0$, there are two curves bifurcating at the point $(0,0)$; the line $\{0\} \times \mathbf{R}$ of trivial solutions and the curve $\{(\lambda, u(\lambda)) : \lambda > 0\}$ (see Figure 10 below).*

A biological interpretation of Theorem 9 (the logistic Neumann case) may be stated as follows (see Figures 8–10):

(i) When the environment has an impassable boundary and is on the average unfavorable, then high diffusion rates have the same effect as they always have when the boundary is deadly (cf. Figure 5).

(ii) The behavior of solutions of the logistic Neumann problem (76) is similar to that of the problem (65) with homogeneous Dirichlet condition if $\int_\Omega m(x)\,dx < 0$. In fact, there is a positive eigenvalue with positive eigenfunction to act as a bifurcation point for positive steady states.

(iii) If the boundary is impassable and the environment is on the average neutral or favorable, then the population can persist, no matter what its rate of diffusion.

(iv) The behavior of solutions of the logistic Neumann problem (76) is similar to that of the problem (65) with homogeneous Dirichlet condition if $\int_\Omega m(x)\,dx < 0$. In fact, there is a positive eigenvalue with a positive eigenfunction to act as a bifurcation point for positive steady states (cf. Figure 6).

(v) If $\int_\Omega m(x)\,dx > 0$, there occurs a *secondary bifurcation* from the line $\{0\} \times \mathbf{R}$ of trivial solutions.

(vi) If $\int_\Omega m(x)\,dx = 0$, there are two curves bifurcating at the point $(0,0)$; the line $\{0\} \times \mathbf{R}$ of trivial solutions and the curve $\{(\lambda, v(\lambda)) : \lambda > 0\}$.

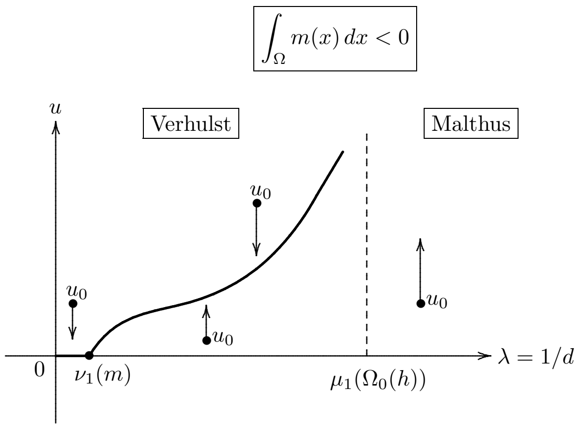


**Figure 8.** A biological interpretation of part (i) of Theorem 9 in the case where $\int_\Omega m(x)\,dx < 0$: Malthus versus Verhulst.

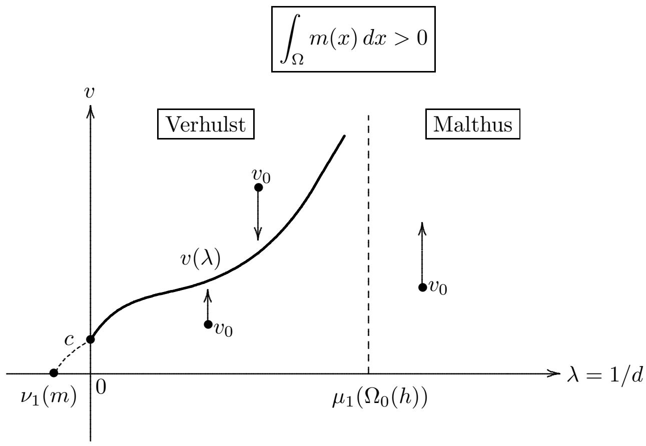


**Figure 9.** A biological interpretation of part (ii) of Theorem 9 in the case where $\int_\Omega m(x)\,dx > 0$: Malthus versus Verhulst.

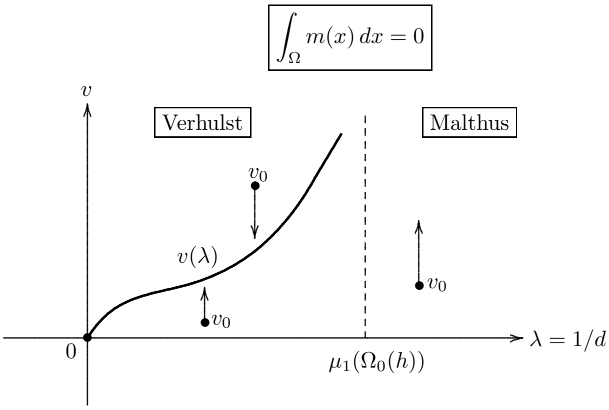


**Figure 10.** A biological interpretation of part (ii) of Theorem 9 in the case where $\int_\Omega m(x)\,dx = 0$: Malthus versus Verhulst.

More precisely, if the weight function $m(x)$ satisfies condition (M2), by using Cantrell–Cosner [23] (Theorem 4.1) and also Brown–Lin [32] (Theorem 3.13) we can prove the following *stability theorem* ([4], Theorem 1.5):

**Theorem 10** (the logistic Neumann case)**.** *Suppose that condition (M2) and the structural conditions (Z1) and (Z2) are satisfied. Then we have the following two assertions (i) and (ii):*

*(i)* *If $\int_\Omega m(x)\,dx < 0$, then we have the following three assertions (a)–(c) (see Figure 11 below):*

- *(a)* *The zero solution of the logistic Neumann problem* (80) *is* globally asymptotically stable *if $\lambda$ is so small that $0 < \lambda < \nu_1(m)$. In this case, we can obtain an estimate of the decay rate of the total size of the population*

$$\begin{aligned}&\int_\Omega w(x,t;v_0)\,dx \\ &\leq \exp\left[-\left(\frac{1}{\lambda} - \frac{1}{\nu_1(m)}\right)\nu_1(1)\,t\right]|\Omega|^{1/2}\left(\int_\Omega v_0(x)^2\,dx\right)^{1/2} \quad \textit{for all } t > 0.\end{aligned} \tag{81}$$

- *(b)* *A positive solution $v(\lambda)$ of the logistic Neumann problem* (80) *is* globally asymptotically stable *for each $\nu_1(m) < \lambda < \mu_1(\Omega_0(h))$.*
- *(c)* *If $\lambda$ is so large that $\lambda > \mu_1(\Omega_0(h))$, then we have the assertion*

$$\max_{x \in \overline{\Omega}} |w(x,t;v_0)| \longrightarrow \infty \quad \textit{as } t \to \infty \tag{82}$$

*for each non-trivial initial value $v_0 \in C^{2+\theta}(\overline{\Omega})$ satisfying the compatibility conditions* (78).

*(ii)* *If $\int_\Omega m(x)\,dx \geq 0$, then we have the following two assertions (d) and (e) (see Figures 12 and 13 below):*

- *(d)* *A positive solution $v(\lambda)$ of the logistic Neumann problem* (80) *is* globally asymptotically stable *for each $0 < \lambda < \mu_1(\Omega_0(h))$.*
- *(e)* *If $\lambda$ is so large that $\lambda > \mu_1(\Omega_0(h))$, then we have assertion* (82) *for each non-trivial initial value $v_0 \in C^{2+\theta}(\overline{\Omega})$ satisfying the compatibility conditions* (78).

Finally, we consider the case where the coefficient of intraspecific competition $h(x)$ satisfies the condition (70)

$$h(x) > 0 \quad \text{on } \overline{\Omega}.$$

If the weight function $m(x)$ satisfies condition (M2), then, by combining Theorem 2 and Brown–Lin [32] (Theorem 3.13) we can characterize the carrying capacity of the environment (see [23] (Theorem 4.1); [4] (Theorem 1.6)):

**Theorem 11** (the Neumann case)**.** *If conditions (M2) and* (70) *are satisfied, then we obtain that the quantity*

$$\ell = \frac{\max_{\overline{\Omega}} m}{\min_{\overline{\Omega}} h} = \frac{\max_{\overline{\Omega}} m^+}{\min_{\overline{\Omega}} h}$$

*is the* carrying capacity *of the environment (see Figures 11–13).*

**Remark 5.** *Suppose that condition* (70) *is satisfied in the case $\int_\Omega m(x)\, dx < 0$. Then, by using the variational formula of Brown–Lin [32] (Theorem 3.13) we can prove the following decay estimate of the total size of the positive steady states $v(\lambda)$ (see Figure 11):*

$$\int_\Omega v(\lambda)\, dx \leq \left(1 - \frac{\nu_1(m)}{\lambda}\right) |\Omega|^{2/3} \frac{\left(\int_\Omega m^+(x)^3\, dx\right)^{1/3}}{\min_{x \in \overline{\Omega}} h(x)}$$
$$\text{for all } \lambda > \nu_1(m).$$

*This proves that the quantity*

$$|\Omega|^{2/3} \frac{\left(\int_\Omega m^+(x)^3\, dx\right)^{1/3}}{\min_{x \in \overline{\Omega}} h(x)}$$

*is the* carrying capacity *of the population, just as in the Dirichlet case (see Remark 4).*

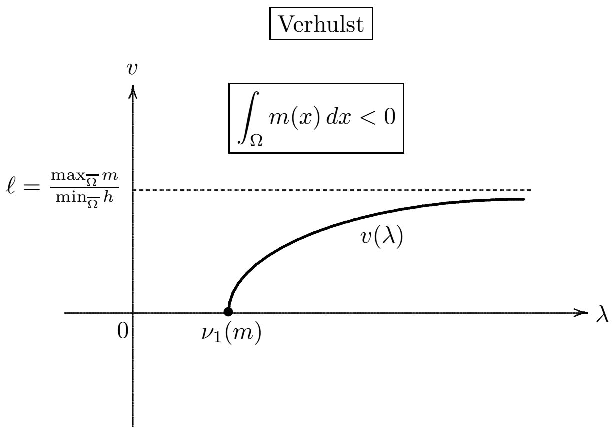


**Figure 11.** A biological interpretation of part (i) of Theorem 10 in the case where $\int_\Omega m(x)\, dx < 0$ under condition (70) (Verhulst).

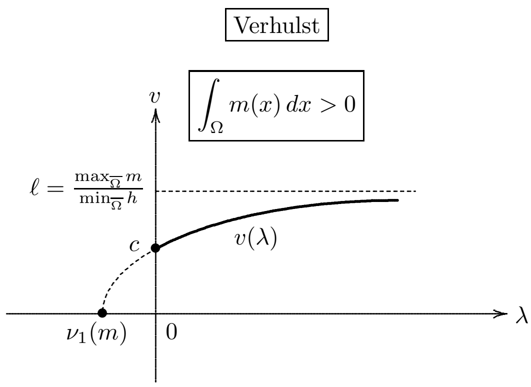


**Figure 12.** A biological interpretation of Theorem 10 in the case where $\int_\Omega m(x)\, dx > 0$ under condition (70) (Verhulst).

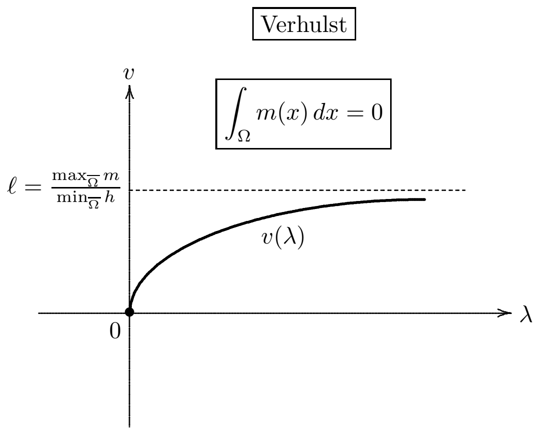


**Figure 13.** A biological interpretation of Theorem 10 in the case where $\int_\Omega m(x)\,dx = 0$ under condition (70) (Verhulst).

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## Appendix A. Classical Results for Linear Initial Boundary Value Problems of Parabolic Type

In this appendix we study linear initial boundary value problems of parabolic type in the framework of *Hölder spaces*. The material here is adapted from Ladyzhenskaya et al. [5] and Friedman [6].

Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ with boundary $\Gamma = \partial\Omega$ (see Figure A1) and let $Q_T = \Omega \times (0, T)$ be a cylinder in $\mathbf{R}^{n+1}$ (see Figure A2). In this section we consider the following two linear initial boundary value problems for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f & \text{in } Q_T, \\ u = \varphi & \text{on } \Gamma \times (0, T) \quad \text{(the Diriclet condition)}, \\ u|_{t=0} = u_0 & \text{on } \overline{\Omega}, \end{cases} \tag{A1}$$

and

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f & \text{in } Q_T, \\ \frac{\partial u}{\partial \beta} + b_0(x')u = \varphi & \text{on } \Gamma \times (0, T) \quad \text{(the oblique derivative condition)}, \\ u|_{t=0} = u_0 & \text{on } \overline{\Omega}. \end{cases} \tag{A2}$$

Here:

(1) $\beta$ is an outward pointing, *nowhere tangent* vector field of class $C^{1+\sigma}$ for $0 < \sigma < 1$ on the boundary $\Gamma$.

(2) $b_0 \in C^{1+\sigma}(\Gamma)$ and $b_0(x') \geq 0$ on $\Gamma$.

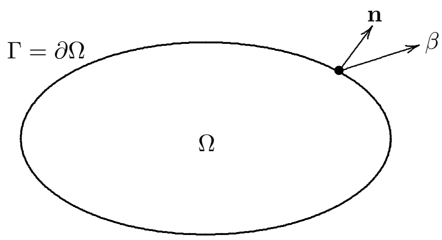


**Figure A1.** The vector field $\beta$ is outward and *nowhere tangent* to the boundary $\Gamma$.

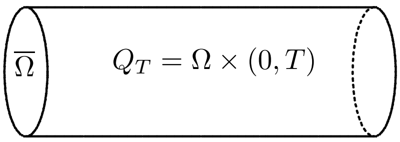


**Figure A2.** The cylindrical domain $Q_T = \Omega \times (0, T)$ and the lateral surface $\Gamma_T = \Gamma \times (0, T)$.

*Appendix A.1. Function Spaces for Equations of Parabolic Type*

In this subsection we introduce function spaces associated with the linear initial boundary value problems (A1) and (A2).

We consider $\sigma$-Hölder continuous functions on $\overline{\Omega} \times [0, \infty)$ where we use the metric

$$d\big((x,t),(x',t')\big) = \Big(|x-x'|^2 + |t-t'|\Big)^{1/2}$$

for the computation of the Hölder constant (see [6] (Chapter 3, Section 2)).

(I) The space $C^{\sigma,\sigma/2}(\overline{Q_T})$ for $0 < \sigma < 1$: First, we let

$$\begin{aligned} C^{\sigma,\sigma/2}(\overline{Q_T}) := &\text{ the space of continuous functions } u(x,t) \in C(\overline{Q_T}) \\ &\text{ that are } \sigma\text{-Hölder continuous with respect to } x \\ &\text{ and } \sigma/2\text{-Hölder continuous with respect to } t. \end{aligned}$$

We introduce the following two seminorms:

- $\langle u \rangle^{(\sigma)}_{x,Q_T} = \displaystyle\sup_{\substack{(x,t)\in Q_T,\\ (x',t)\in Q_T}} \frac{|u(x,t) - u(x',t)|}{|x-x'|^{\sigma}},$
- $\langle u \rangle^{(\sigma/2)}_{t,Q_T} = \displaystyle\sup_{\substack{(x,t)\in Q_T,\\ (x,t')\in Q_T}} \frac{|u(x,t) - u(x,t')|}{|t-t'|^{\sigma/2}},$

and the norm

$$\|u\|^{(\sigma)}_{Q_T} = \max_{(x,t)\in\overline{Q_T}} |u(x,t)| + \langle u \rangle^{(\sigma)}_{x,Q_T} + \langle u \rangle^{(\sigma/2)}_{t,Q_T}.$$

We remark that:

(1) $u \in C\big([0,T], C^{\sigma}(\overline{\Omega})\big) \Longrightarrow \langle u \rangle^{(\sigma)}_{x,Q_T} < \infty.$

(2) $u \in C^{\sigma/2}\big([0,T], C(\overline{\Omega})\big) \Longrightarrow \langle u \rangle^{(\sigma/2)}_{t,Q_T} < \infty.$

(II) The space $C^{1+\sigma,(1+\sigma)/2}(\overline{Q_T})$ for $0 < \sigma < 1$: Secondly, we let

$$\begin{aligned} C^{1+\sigma,(1+\sigma)/2}(\overline{Q_T}) := &\text{ the space of continuously differentiable functions} \\ &\ u(x,t) \in C^{1,0}(\overline{Q_T}) \text{ with respect to } x \text{ such that} \\ &\ \partial_x u(x,t) \text{ are } \sigma\text{-Hölder continuous with respect to } x \text{ and} \\ &\ \sigma/2\text{-Hölder continuous with respect to } t \text{ and further that} \\ &\ u(x,t) \text{ is } (1+\sigma)/2\text{-Hölder continuous with respect to } t. \end{aligned}$$

We introduce the following three seminorms:

- $\langle \partial_x u \rangle^{(\sigma)}_{x,Q_T} = \sup\limits_{\substack{(x,t)\in Q_T,\\ (x',t)\in Q_T}} \dfrac{|\partial_x u(x,t) - \partial_x u(x',t)|}{|x-x'|^{\sigma}}$,
- $\langle u \rangle^{((\sigma+1)/2)}_{t,Q_T} = \sup\limits_{\substack{(x,t)\in Q_T,\\ (x,t')\in Q_T}} \dfrac{|u(x,t) - u(x,t')|}{|t-t'|^{(1+\sigma)/2}}$,
- $\langle \partial_x u \rangle^{(\sigma/2)}_{t,Q_T} = \sup\limits_{\substack{(x,t)\in Q_T,\\ (x,t')\in Q_T}} \dfrac{|\partial_x u(x,t) - \partial_x u(x,t')|}{|t-t'|^{\sigma/2}}$,

and the norm

$$\begin{aligned}\|u\|^{(1+\sigma)}_{Q_T} &= \max_{(x,t)\in\overline{Q_T}} |u(x,t)| + \max_{(x,t)\in\overline{Q_T}} |\partial_x u(x,t)| \\ &\quad + \langle \partial_x u \rangle^{(\sigma)}_{x,Q_T} + \langle u \rangle^{((\sigma+1)/2)}_{t,Q_T} + \langle \partial_x u \rangle^{(\sigma/2)}_{t,Q_T}.\end{aligned}$$

We remark that:

(1) $u \in C\big([0,T], C^{1+\sigma}(\overline{\Omega})\big) \Longrightarrow \langle \partial_x u \rangle^{(\sigma)}_{x,Q_T} < \infty$.

(2) $u \in C^{\sigma/2}\big([0,T], C^{1}(\overline{\Omega})\big) \Longrightarrow \langle \partial_x u \rangle^{(\sigma/2)}_{t,Q_T} < \infty$.

(III) The space $C^{2+\sigma,1+\sigma/2}\big(\overline{Q_T}\big)$ for $0 < \sigma < 1$: Thirdly, we let

$C^{2+\sigma,1+\sigma/2}\big(\overline{Q_T}\big) :=$ the space of continuously differentiable functions $u(x,t) \in C^{2,1}\big(\overline{Q_T}\big)$ twice with respect to $x$ and once with respect to $t$ such that $\partial_x u(x,t)$ are $(1+\sigma)/2$-Hölder continuous with respect to $t$, that $\partial_t u(x,t)$ is $\sigma$-Hölder continuous with respect to $x$ and $\sigma/2$-Hölder continuous with respect to $t$ and further that $\partial_x^2 u(x,t)$ are $\sigma$-Hölder continuous with respect to $x$ and $\sigma/2$-Hölder continuous with respect to $t$.

We introduce the following five seminorms:

- $\langle \partial_t u \rangle^{(\sigma)}_{x,Q_T} = \sup\limits_{\substack{(x,t)\in Q_T,\\ (x',t)\in Q_T}} \dfrac{|\partial_t u(x,t) - \partial_t u(x',t)|}{|x-x'|^{\sigma}}$,
- $\left\langle \partial_x^2 u \right\rangle^{(\sigma)}_{x,Q_T} = \sup\limits_{\substack{(x,t)\in Q_T,\\ (x',t)\in Q_T}} \dfrac{\left|\partial_x^2 u(x,t) - \partial_x^2 u(x',t)\right|}{|x-x'|^{\sigma}}$,
- $\langle \partial_t u \rangle^{(\sigma/2)}_{t,Q_T} = \sup\limits_{\substack{(x,t)\in Q_T,\\ (x,t')\in Q_T}} \dfrac{|\partial_t u(x,t) - \partial_t u(x,t')|}{|t-t'|^{\sigma/2}}$,
- $\langle \partial_x u \rangle^{((1+\sigma)/2)}_{t,Q_T} = \sup\limits_{\substack{(x,t)\in Q_T,\\ (x,t')\in Q_T}} \dfrac{|\partial_x u(x,t) - \partial_x u(x,t')|}{|t-t'|^{(1+\sigma)/2}}$,
- $\left\langle \partial_x^2 u \right\rangle^{(\sigma/2)}_{t,Q_T} = \sup\limits_{\substack{(x,t)\in Q_T,\\ (x,t')\in Q_T}} \dfrac{\left|\partial_x^2 u(x,t) - \partial_x^2 u(x,t')\right|}{|t-t'|^{\sigma/2}}$,

and the norm

$$\begin{aligned}
&\|u\|_{Q_T}^{(2+\sigma)} \\
&= \max_{(x,t)\in\overline{Q_T}} |u(x,t)| + \max_{(x,t)\in\overline{Q_T}} |\partial_x u(x,t)| + \max_{(x,t)\in\overline{Q_T}} |\partial_t u(x,t)| + \max_{(x,t)\in\overline{Q_T}} \left|\partial_x^2 u(x,t)\right| \\
&\quad + \langle \partial_t u\rangle_{x,Q_T}^{(\sigma)} + \left\langle \partial_x^2 u\right\rangle_{x,Q_T}^{(\sigma)} + \langle \partial_t u\rangle_{t,Q_T}^{(\sigma/2)} + \langle \partial_x u\rangle_{t,Q_T}^{((1+\sigma)/2)} + \left\langle \partial_x^2 u\right\rangle_{t,Q_T}^{(\sigma/2)}.
\end{aligned}$$

We remark that:

(1) $u \in C\big([0,T], C^{2+\sigma}(\overline{\Omega})\big) \Longrightarrow \langle \partial_x^2 u\rangle_{x,Q_T}^{(\sigma)} < \infty.$

(2) $u \in C^{\sigma/2}\big([0,T], C^{2}(\overline{\Omega})\big) \Longrightarrow \langle \partial_x^2 u\rangle_{t,Q_T}^{(\sigma/2)} < \infty.$

(IV) The space $C^{2+\sigma,1+\sigma/2}\big(\overline{\Gamma_T}\big)$ for $0 < \sigma < 1$: Finally, we let

$$\begin{aligned}
C^{2+\sigma,1+\sigma/2}\big(\overline{\Gamma_T}\big) := &\text{ the space of functions } \psi \text{ defined on } \overline{\Gamma_T} \text{ such} \\
&\text{ that there exist functions } \Psi \in C^{2+\sigma,1+\sigma/2}\big(\overline{Q_T}\big) \\
&\text{ which coincide with } \psi \text{ on } \overline{\Gamma_T}.
\end{aligned}$$

We equip the Hölder space $C^{2+\sigma,1+\sigma/2}\big(\overline{\Gamma_T}\big)$ with the norm

$$\|\psi\|_{\Gamma_T}^{(2+\sigma)} = \inf \|\Psi\|_{Q_T}^{(2+\sigma)},$$

where the infimum is taken over all such $\Psi$.

*Appendix A.2. Unique Solvability Theorems for Linear Initial Boundary Value Problems of Parabolic Type*

In this subsection we formulate unique solvability theorems for problems (A1) and (A2) in the framework of Hölder spaces.

(I) The Dirichlet case: Let $0 < \mu < 1$ and $T > 0$. We suppose that

$$\begin{cases} f \in C^{\mu,\mu/2}\big(\overline{Q_T}\big), \\ \varphi \in C^{\mu+2,\mu/2+1}\big(\overline{\Gamma_T}\big), \\ u_0 \in C^{\mu+2}(\overline{\Omega}). \end{cases}$$

Then we have the following theorem ([5] (Chapter IV, Theorem 5.2)):

**Theorem A1** (the Dirichlet case)**.** *Suppose that the following* compatibility condition *is satisfied:*

$$\begin{aligned}
\varphi(x',0) &= u_0(x') \quad \textit{for all } x' \in \Gamma, \\
\Delta u_0(x') + f(x',0) &= \frac{\partial \varphi}{\partial t}(x',0) \quad \textit{for all } x' \in \Gamma.
\end{aligned}$$

*Then the linear initial boundary value problem* (A1) *has a unique solution*

$$u(x,t) \in C^{\mu+2,\mu/2+1}\big(\overline{\Omega}\times[0,T]\big).$$

*Moreover, we have the* a priori *estimate*

$$\|u\|_{Q_T}^{(\mu+2)} \le C_1\Big(\|f\|_{Q_T}^{(\mu)} + |\varphi|_{\Gamma_T}^{(\mu+2)} + \|u_0\|_{\Omega}^{(\mu+2)}\Big), \tag{A3}$$

*with a constant* $C_1 = C_1(T) > 0$.

(II) The regular oblique derivative case: Let $0 < \mu < 1$ and $T > 0$. We suppose that

$$\begin{cases} f \in C^{\mu,\mu/2}(\overline{Q_T}), \\ \varphi \in C^{\mu+1,(\mu+1)/2}(\overline{\Gamma_T}), \\ u_0 \in C^{\mu+2}(\overline{\Omega}). \end{cases}$$

Then we have the following theorem ([5] (Chapter IV, Theorem 5.3)):

**Theorem A2** (the regular oblique derivative case)**.** *Suppose that the following* compatibility condition *is satisfied:*

$$\frac{\partial u}{\partial \beta}(x') + b_0(x')u_0(x') = \varphi(x',0) \quad \textit{for all } x' \in \Gamma.$$

*Then the linear initial boundary value problem* (A2) *has a unique solution*

$$u(x,t) \in C^{\mu+2,\mu/2+1}(\overline{\Omega} \times [0,T]).$$

*Moreover, we have the* a priori *estimate*

$$\|u\|^{(\mu+2)}_{Q_T} \leq C_2 \left( \|f\|^{(\mu)}_{Q_T} + |\varphi|^{(\mu+1)}_{\Gamma_T} + \|u_0\|^{(\mu+2)}_{\Omega} \right), \tag{A4}$$

*with a constant* $C_2 = C_2(T) > 0$.

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