



Article Fixed/Predefined-Time Synchronization of Complex-Valued Stochastic BAM Neural Networks with Stabilizing and Destabilizing Impulse

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Abstract: This article is mainly concerned with the fixed-time and predefined-time synchronization problem for a type of complex-valued BAM neural networks with stochastic perturbations and impulse effect. First, some previous fixed-time stability results on nonlinear impulsive systems in which stabilizing and destabilizing impulses were separately analyzed are extended to a general case in which the stabilizing and destabilizing impulses can be handled simultaneously. Additionally, using the same logic, a new predefined-time stability lemma for stochastic nonlinear systems with a general impulsive effect is obtained by using the inequality technique. Then, based on these novel results, two novel controllers are implemented to derive some simple fixed/predefined-time synchronization criteria for the considered complex-valued impulsive BAM neural networks with stochastic perturbations using the non-separation method. Finally, two numerical examples are given to demonstrate the feasibility of the obtained results.

Keywords: stochastic perturbation; impulse effect; complex-valued BAM neural network; fixed-time synchronization; predefined-time synchronization

MSC: 34A37; 34D06; 34D20

1. Introduction

The BAM neural network was initially introduced in 1987 [1], and consists of two layers of neurons, each of which is linked to all the neurons in the other layer, while the neurons in the same layer do not have any connection to each other. Due to the powerful associative memory and information association ability of BAM neural network, it has been widely studied by scholars in recent years [2–4], who have applied them to many key areas, such as pattern recognition, secure communication, and automatic control engineering [5–8], etc. For example, Ref. [3] investigated decentralized event-triggered stability analysis of neutral-type BAM neural networks with Markovian jump parameters and mixed time varying delays. Ref. [5] investigated the fixed-time (FXT) synchronization of complex-valued memristive BAM neural networks with leakage delays and its application to image encryption and decryption problem. Ref. [6] concerned the asymptotic anti-synchronization issue of memristive BAM neural networks, and then applied their results to a network security communication study. In Ref. [7], the authors investigated the finite-time (FNT) projection synchronization problem of memristive BAM neural networks together with application to image encryption.

The recent decades witnessed the wide investigation of various types of synchronization issue of neural networks with or without stochastic fluctuations including asymptotic synchronization, exponential synchronization, general decay synchronization, FNT synchronization, FXT synchronization, and predefined-time (PDT) synchronization. Among them, FNT synchronization has attracted a great concern because it can achieve the synchronization aim in a FNT and has good robustness and anti-interference properties [9–11].



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Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). However, the settling time (ST, which can be seen as the minimum upper bound of system state to reach zero) of FNT synchronization is extensively relevant to the initial values of system, and thus it is unable to exert its superiority when the initial values of a system are unknown or can not be obtained, which undoubtedly limits the specific application of the FNT control techniques. In order to cope with this difficult issue, Polyakov [12] introduced FXT stability concept, in which its ST does not depend on the initial values of the system but only relevant on the system parameters and the controller gains. On this basis, a great number of scholars have studied FXT synchronization of various types of nonlinear systems [13–17].

It is worth noting that, however, the FXT synchronization also has some limitations, such as its ST still being dependent on the system parameters, and it can therefore not be specified in advance. However, in some special engineering applications, such as DC Microgrid [18], secure communication [19], etc., it is required that the system states achieve the synchronization aim within a pre-defined time. Thus, another type of synchronization, PDT synchronization, has been introduced and well studied in the past few years [17]. The one advantage of PDT synchronization is that its ST can be an arbitrary number and can be scheduled in advance.

In addition, with the continuous development of neural network theory, it has been found that real-valued neural networks have some limitations when solving many practical application problems, for example XOR [20], reaction-convection-diffusion systems [21,22], etc. Thus, as a extension of real-valued neural network, the complex-valued neural network has been adopted and investigated in depth since they have a more complex structure and richer properties, which can solve the above problems more easily and effectively. Currently, two methods have been introduced and used to analyze the dynamical behaviors of complex-valued neural network: one is the separation method, in which the real part and imaginary part of network are divided into two real-valued systems and then the dynamics of the these systems are studied separately [23–25]. The separation method is feasible, but there are some disadvantages, such as, after separation, the dimensionality of the system becomes two-fold and there is a need to design controllers for the real and imaginary parts of the system separately, which greatly increases the computational effort and control cost. The other is the non-separated approach, i.e., discussing the dynamical behavior of the original complex-valued neural network directly on the complex domain [17,26]. Compared to the separation method, the non-separation method is much simpler, less computationally intensive, and easier to implement. Based on this, this paper uses a nonseparated approach to discuss the synchronization of the considered complex-valued BAM neural network.

In addition, it is worth noting that in neural network systems, synaptic transmission can be seen as a noisy process of random fluctuation from the release of neurotransmitters and other probabilistic causes [27]. That is, there might exist stochastic disturbances in neural network systems. Currently, there are many excellent studies on the synchronization of neural networks with stochastic perturbations [28–36]. Such as, in Refs. [35,36], the FXT/PDT synchronization for a class of fuzzy neural networks with stochastic perturbations and the PDT synchronization of time-delayed BAM neural networks with stochastic fluctuations are discussed, respectively. However, in some actual situations, such as in the opening and closing of the operation buttons, the change of frequency or sudden noise may cause an unexpected change in the system state, i.e., the impulse effect [33,37–39]. Therefore, investigating the synchronization of impulsive neural networks with stochastic fluctuations is of great theoretical significance. So far, however, there are few results on the FXT/PDT synchronization of complex-valued neural networks with both of these two effects. Especially, there are very seldom works on the PDT synchronization of these kinds of networks due to the lack of available PDT stability results for impulsive systems, not to mention the PDT stability of nonlinear systems with both impulsive effects and stochastic perturbations.

Inspired by the above analysis, in this paper, we focused on the FXT/PDT synchronization issue of a class of complex-valued BAM neural networks with both impulsive effects and stochastic perturbations. The main contributions of our paper are mainly reflected in the following four aspects. (1) Some new unified FXT/PDT stability results for nonlinear impulsive systems with stabilizing and destabilizing impulses are introduced via employing the inequality technique. (2) Some simple FXT/PDT synchronization criteria for complex-valued BAM neural networks with stochastic perturbations and general impulse effects are derived via designing novel controllers. (3) The ST estimation obtained in the current study is more accurate compared to some early published works [15,40,41]. (4) Unlike the traditional separation method [5,25], this paper uses a non-separation method to deal with the FXT/PDT synchronization of considered complex-valued BAM neural networks, which is more simple analytically and can effectively reduce the computational and control burden.

Notation 1. In this article, the set of natural numbers is denoted by N, the set of all real numbers is denoted by R, R^+ stands to the set of non-negative numbers, R^n is the set consisting of all n-dimensional real vectors. C is the set of complex numbers. C^n denotes the n-dimensional complex vector-valued space. For any $\lambda \in C$, $\overline{\lambda}$ is the conjugate of λ . $Re(\lambda)$ and $Im(\lambda)$ represent separately the real and imaginary parts of λ . $|\lambda| = \sqrt{\lambda\overline{\lambda}}$. (Ω, F, P) denotes a complete filtered probability space, in which the filtration $F = \{\mathscr{F}_t; t \in R^+\}$ satisfies the usual condition that \mathscr{F}_0 contains all P-null sets in \mathscr{F} . E(.) denotes the mathematical expectation corresponding to the probability measure P. The sign function of λ is $[\lambda] = sign(Re(\lambda)) + isign(Im(\lambda))$. $\|\cdot\|$ denotes the Euclidean norm, that is $\psi = (\psi_1, \psi_2, \dots, \psi_n)^T \in R^n$, $\|\psi\| = \sqrt{\sum_{i=1}^n |\psi_i|^2}$.

2. System Description and Preliminary Knowledge

Consider the following complex-valued BAM neural network with stochastic perturbations

$$\begin{cases} dx_{i}(t) = \left[-a_{i}x_{i}(t) + \sum_{j=1}^{m} c_{ij}\bar{h}_{j}(y_{j}(t)) + I_{i}(t)\right]dt + \varrho_{i}(t,x_{i}(t))d\omega(t), & t \neq t_{\tau}, \\ dy_{j}(t) = \left[-b_{j}y_{j}(t) + \sum_{i=1}^{n} d_{ji}\bar{g}_{i}(x_{i}(t)) + J_{j}(t)\right]dt + \tilde{\varrho}_{j}(t,y_{j}(t))d\omega(t), & t \neq t_{\tau}, \end{cases}$$
(1)
$$\Delta x_{i}(t_{\tau}^{+}) = x_{i}(t_{\tau}^{+}) - x_{i}(t_{\tau}^{-}) = (\xi_{i} - 1)x_{i}(t_{\tau}), & t = t_{\tau}, \\ \Delta y_{j}(t_{\tau}^{+}) = y_{j}(t_{\tau}^{+}) - y_{j}(t_{\tau}^{-}) = (\xi_{j}^{-} - 1)y_{j}(t_{\tau}), & t = t_{\tau}, \end{cases}$$

where $\tau \in N$, $i \in \tilde{I} \triangleq \{1, \dots, n\}$, $j \in \tilde{J} \triangleq \{1, \dots, m\}$, positive integers n and m denote the number of neurons from the neural domains X_u and Y_u , respectively. $x_i(t) \in C$ and $y_j(t) \in C$ stand for the state variables of the *i*th neuron of neural domain X_u and the *j*th neuron of neural domain Y_u , respectively. $a_i \in C$ and $b_j \in C$ represent the self-inhibition rate of the *i*th neuron in the neural domain X_u and the *j*th neuron in the neural domain Y_u , respectively. $c_{ij} \in C$ and $d_{ji} \in C$ are the synaptic connection weights. $\bar{h}_j(.) \in C$ and $\bar{g}_i(.) \in$ C denote the activation functions. $I_i(t) \in C$ and $J_j(t) \in C$ stand for the external inputs. $q_i(t) : C \times R^n \to C$ and $\tilde{q}_j(t) : C \times R^n \to C$ denote the noise intensity function. $\omega(t) \in C$ denotes a Brown motion given on the probability space (Ω, F, P) . ξ_i and $\tilde{\xi}_j$ are positive constants. For all $\tau \in N$, $x_i(t_{\tau}^+) = \lim_{t \to t_{\tau} \to 0} x_i(t)$, $x_i(t_{\tau}^-) = \lim_{t \to t_{\tau} \to 0} x_i(t) = x_i(t_{\tau})$, $y_j(t_{\tau}^+) = \lim_{t \to t_{\tau} \to 0} y_j(t)$, $y_j(t_{\tau}^-) = \lim_{t \to t_{\tau} \to 0} y_j(t) = y_j(t_{\tau})$ denote the impulse jumps at the impulse moments t_{τ} . The inpulse sequence $\{t_{\tau}\}_{\tau \in N}$ satisfies $t_1 < t_2 < \ldots < t_{\tau} < \ldots$ and $\lim_{\tau \to +\infty} t_{\tau} = +\infty$. The initial conditions of system (1) are $x_i(0) = x_i^0 \in C$ and $y_i(0) = y_i^0 \in C$.

The response system of the above drive system is described as follows:

$$d\tilde{x}_{i}(t) = \left[-a_{i}\tilde{x}_{i}(t) + \sum_{j=1}^{m} c_{ij}\bar{h}_{j}(\tilde{y}_{j}(t)) + I_{i}(t) + u_{i}(t)\right]dt + \varrho_{i}(t,\tilde{x}_{i}(t))d\omega(t), \quad t \neq t_{\tau},$$

$$d\tilde{y}_{j}(t) = \left[-b_{j}\tilde{y}_{j}(t) + \sum_{i=1}^{n} d_{ji}\bar{g}_{i}(\tilde{x}_{i}(t)) + J_{j}(t) + v_{j}(t)\right]dt + \tilde{\varrho}_{j}(t,\tilde{y}_{j}(t))d\omega(t), \quad t \neq t_{\tau},$$

$$\Delta\tilde{x}_{i}(t_{\tau}) = \tilde{x}_{i}(t_{\tau}^{+}) - \tilde{x}_{i}(t_{\tau}^{-}) = (\xi_{i} - 1)\tilde{x}_{i}(t_{\tau}), \quad t = t_{\tau},$$

$$\Delta\tilde{y}_{j}(t_{\tau}) = \tilde{y}_{j}(t_{\tau}^{+}) - \tilde{y}_{j}(t_{\tau}^{-}) = (\xi_{j}^{-} - 1)\tilde{y}_{j}(t_{\tau}), \quad t = t_{\tau},$$
(2)

where $\tilde{x}_i(t) \in C$ and $\tilde{y}_i(t) \in C$ denote the state variables of the response system. $u_i(t)$ and $v_j(t)$ are the controllers that will be designed in the next section. The initial values of system (2) are $\tilde{x}_i(0) = \tilde{x}_i^0 \in C$ and $\tilde{y}_i(0) = \tilde{y}_i^0 \in C$.

Let $s_i(t) = \tilde{x}_i(t) - x_i(t)$ and $r_j(t) = \tilde{y}_j(t) - y_j(t)$, then the error systems of (1) and (2) can be described as

$$\begin{cases} ds_{i}(t) = [-a_{i}s_{i}(t) + \sum_{j=1}^{m} c_{ij}h_{j}(r_{j}(t)) + u_{i}(t)]dt + \sigma_{i}(t,s_{i}(t))d\omega(t), & t \neq t_{\tau}, \\ dr_{j}(t) = [-b_{j}r_{j}(t) + \sum_{i=1}^{n} d_{ji}g_{i}(s_{i}(t)) + v_{j}(t)]dt + \tilde{\sigma}_{j}(t,r_{j}(t))d\omega(t), & t \neq t_{\tau}, \\ \Delta s_{i}(t_{\tau}) = s_{i}(t_{\tau}^{+}) - s_{i}(t_{\tau}^{-}) = (\xi_{i} - 1)s_{i}(t_{\tau}), & t = t_{\tau}, \\ \Delta r_{j}(t_{\tau}) = r_{j}(t_{\tau}^{+}) - r_{j}(t_{\tau}^{-}) = (\xi_{j} - 1)r_{j}(t_{\tau}), & t = t_{\tau}, \end{cases}$$
(3)

where $\sigma_i(t, s_i(t)) = \varrho_i(t, \tilde{x}_i(t)) - \varrho_i(t, x_i(t)), \tilde{\sigma}_j(t, r_j(t)) = \tilde{\varrho}_j(t, \tilde{y}_j(t)) - \tilde{\varrho}_j(t, y_j(t)), h_j(r_j(t)) = \bar{h}_j(\tilde{y}_j(t)) - \bar{h}_j(y_j(t))$ and $g_i(s_i(t)) = \bar{g}_i(\tilde{x}_i(t)) - \bar{g}_i(x_i(t)).$

The following are our assumptions about complex-valued activation functions and stochastic noise intensity functions.

Assumption 1. The complex-valued activation functions $\bar{h}_j(t)$ and $\bar{g}_i(t)$ satisfy the Lipschitz condition, that is for $\forall z_1, z_2 \in C$ ($z_1 \neq z_2$), there are scalers $L_i^g > 0$ and $L_i^h > 0$ such that

$$|\bar{g}_i(z_1) - \bar{g}_i(z_2)| \le L_i^g |z_1 - z_2|, \quad |\bar{h}_j(z_1) - \bar{h}_j(z_2)| \le L_j^h |z_1 - z_2|.$$

where $i \in \tilde{I}$ and $j \in \tilde{J}$.

Assumption 2. For any complex-valued constant *z*, there exist positive constants η_i and $\tilde{\eta}_j > 0$ such that $\sigma_i(t)$ and $\tilde{\sigma}_j(t)$ satisfy

$$\frac{1}{2}\sigma_i(t,z)\overline{\sigma_i(t,z)} \leq \eta_i |z|^2, \quad \frac{1}{2}\tilde{\sigma_j}(t,z)\overline{\tilde{\sigma_j}(t,z(t))} \leq \tilde{\eta_j} |z|^2.$$

Definition 1 ([42]). If there exists a positive integer τ^0 and a constant $\nu_{\tau} > 0$, such that

$$\frac{t_2 - t_1}{\nu_{\tau}} - \tau^0 \le \Xi_{\kappa}(t_1, t_2) \le \frac{t_2 - t_1}{\nu_{\tau}} + \tau^0,$$

where $\Xi_{\kappa}(t_1, t_2)$ is the number of impulses of impulse sequence $\kappa = \{t_{\tau}\} \in N$ in time interval (t_1, t_2) , then the impulse sequence $\kappa = \{t_{\tau}\}_{\tau \in N}$ is said to have an average impulse interval ν_{τ} . Consider the following complex-valued stochastic nonlinear system

$$\begin{cases} d\psi(t) = f(t,\psi(t))dt + \sigma(t,\psi(t))d\omega(t), \quad \psi(0) = \psi_0, \quad t \neq t_\tau, \\ \Delta x|_{t=t_\tau} = J_k(t_\tau,\psi(t_\tau)), \quad t_\tau \in N, \end{cases}$$
(4)

where $\psi(t) = (\psi_1(t), \psi_2(t), \dots, \psi_n(t))^T \in \mathbb{C}^n$ is the state vector of the system. $f(\cdot) : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$ and $\sigma(\cdot) : \mathbb{C} \times \mathbb{R}^n \to \mathbb{C}^n$ are the continuous function given in advance and satisfy f(0) = 0, $\sigma(0) = 0$. $\omega(t) \in \mathbb{C}$ denotes the Brown motion. $J_k : \mathbb{R}^+ \times \mathbb{C}^n \to \mathbb{C}^n$ is a continuous function. For

all $\tau \in N$, $\Delta \psi(t_{\tau}) = \psi(t_{\tau}^+) - \psi(t_{\tau}^-)$ denotes the impulse jump at the impulse moment t_{τ} , and $\psi(t_{\tau}^+) = \lim_{t \to t_{\tau} + 0} \psi(t)$, $\psi(t_{\tau}^-) = \lim_{t \to t_{\tau} - 0} \psi(t) = \psi(t_{\tau})$. The impulsive sequence $\{t_{\tau}\}_{\tau \in N}$ satisfies $0 \le t_0 < t_1 < \ldots < t_{\tau} < \ldots$ and $\lim_{\tau \to +\infty} t_{\tau} = +\infty$.

Definition 2 ([43]). Let $C^{1,2}(R^+ \times C^n; R^+)$ be the set of all non-negative functions $V(t, \psi, \bar{\psi})$ on $R^+ \times C^n$ its partial derivative for t is continuous and second-order partial derivatives for ψ and $\bar{\psi}$ exist, then for each $V \in C^{1,2}(R^+ \times C^n; R^+)$, its operator $\pounds V(t, \psi, \bar{\psi})$ is defined as

$$\begin{split} \mathcal{E}V(t,\psi,\bar{\psi}) = & \frac{\partial V(t,\psi,\bar{\psi})}{\partial t} + \frac{\partial V(t,\psi,\bar{\psi})}{\partial \psi} f(t) + \frac{\partial V(t,\psi,\bar{\psi})}{\partial \bar{\psi}} \bar{f}(t) + \frac{1}{2} \sigma^{T}(t) \frac{\partial^{2} V(t,\psi,\bar{\psi})}{\partial \psi^{2}} \sigma(t) \\ & + \sigma^{T}(t) \frac{\partial^{2} V(t,\psi,\bar{\psi})}{\partial \psi \partial \bar{\psi}} \bar{\sigma}(t) + \frac{1}{2} \bar{\sigma}^{T}(t) \frac{\partial^{2} V(t,\psi,\bar{\psi})}{\partial \bar{\psi}^{2}} \bar{\sigma}(t). \end{split}$$

Definition 3 ([44]). The zero solution of system (4) is called to be FXT stable in probability, if solution $\psi(t, \psi_0)$ which corresponds to initial condition $\psi_0 \in C^n$ satisfies

- (1) For any initial condition $\psi_0(\neq 0) \in C^n$, there exists a ST function $T(\psi_0, \kappa) = \inf\{t | \psi(t, \psi_0) = 0\}$ such that $Pro\{T(\psi_0, \kappa) < \infty\} = 1$.
- (2) Stability in probability: for every pair of $\varepsilon \in (0, 1)$ and v > 0, there exists a $\delta = \delta(\varepsilon, v) > 0$ such that $Pro\{|\psi(t, \psi_0)| \le v \text{ for all } t \ge 0\} \ge 1 \varepsilon$, whenever $|\psi_0| < \delta$.
- (3) The mathematical expectation of $T(\psi_0, \kappa)$ is bounded by a constant $T_{\varepsilon} > 0$ such that

$$E(T(\psi_0, \kappa)) \leq T_{\varepsilon}, \quad \forall \psi_0 \in C^n.$$

Definition 4 ([35]). For any initial value $\psi_0 \in C^n$, the given positive constants T_c , if the zero solution of system (4) is FXT stable in probability, and satisfies $T(\psi_0, \kappa) \leq T_c$ for any $T_c > 0$, the zero solution of (4) is called to be PDT stable in probability.

Lemma 1 ([26]). *For any* $z_0 \in C$ *, one has*

 $z_0 + \overline{z_0} = 2Re(z_0) \le 2|z_0|, \quad [z_0]z_0 + [z_0]\overline{z_0} \ge 2|z_0|.$

Lemma 2 ([45]). *Assume* $\alpha_1, \alpha_2, ..., \alpha_e \ge 0, 0 < \delta \le 1, \zeta > 1$, *then*

$$\sum_{i=1}^e \alpha_i^{\delta} \ge (\sum_{i=1}^e \alpha_i)^{\delta}, \quad \sum_{i=1}^e \alpha_i^{\zeta} \ge n^{1-\zeta} (\sum_{i=1}^e \alpha_i)^{\zeta}.$$

Lemma 3 ([40,41]). Suppose there exists a function $V(t, \psi(t))$ satisfying V(t, 0) = 0, such that the following two statements hold

(*i*) $\alpha_1 \| \psi(t) \|^2 \le V(t, \psi(t)) \le \alpha_2 \| \psi(t) \|^2$, for any $t \in R^+$, (*ii*)

$$\begin{cases} \pounds V(t,\psi(t)) \le KV(t,\psi(t)) - \mu V^{\delta}(t,\psi(t)) - \lambda V^{\gamma}(t,\psi(t)), \quad t \neq t_{\tau}, \\ V(t_{\tau}^+,\psi(t_{\tau}^+)) \le \xi V(t_{\tau},\psi(t_{\tau})), \quad t = t_{\tau}, \end{cases}$$
(5)

where $K < \min\{\mu, \lambda, -\frac{\ln \xi}{\nu_{\tau}}\}$, α_1 , α_2 , μ , ξ and λ are positive scalers, $0 < \delta < 1$ and $\gamma > 1$ are constants, then the origin for system (4) is FXT stable in probability, and the ST T_1 is estimated by

$$T_1 = \frac{1}{\eta(1-\gamma)} \ln\left(1 - \frac{\eta}{\lambda \omega}\right) + \frac{1}{\eta(1-\delta)} \ln\left(\frac{\mu}{\mu - \eta \pi^2}\right),$$

where $\eta = K + \frac{\ln \xi}{\nu_{\tau}}, \omega = \xi^{-\tau^0(1-\gamma)sign(1-\xi)}, \pi = \xi^{-\tau^0(1-\delta)sign(1-\xi)}.$

Proof. By combining the similar analysis employed in Theorem 3.1 in Ref. [30] and Theorems 1 and 2 in Ref. [41], we can prove Lemma 3 by constructing an impulsive

comparison system and using some inequality techniques. The detailed proof of Lemma 3 is omitted here to save space. \Box

Remark 1. In fact, in a recent study [41], the authors investigated the FXT stable issue of deterministic nonlinear system with impulsive effects by using the well-known comparison principle of impulsive differential inequality and some analysis methods. However, due to the limitation of the applied analysis method in Ref. [41], it discusses the impulses separately as stabilizing and destabilizing impulsives, which was done in Theorems 1 and 2 in Ref. [41]. In Lemma 3, however, we have combined the results of these two Theorems by introducing a novel term sign $(1 - \xi)$, which reduces the mathematical expression of ST. In addition, Lemma 3 consider the stochastic perturbations besides impulsive effects. From this aspect, the FXT stable result concluded by Lemma 3 is more general and has a better applicability.

Lemma 4. If there exists Lyapunov function $V(t, \psi(t))$ such that: (i) $\alpha_1 \| \psi(t) \|^2 \le V(t, \psi(t)) \le \alpha_2 \| \psi(t) \|^2$, for any $t \in \mathbb{R}^+$, $z \in \mathbb{R}^n$. (ii)

$$\begin{cases} \pounds(V)(t,\psi(t)) \le \frac{T_0}{T_c} \Big(KV(t,\psi(t)) - \mu V^{\delta}(t,\psi(t)) - \lambda V^{\gamma}(t,\psi(t)) \Big), & t \neq t_{\tau}, \\ V(t_{\tau}^+,\psi(t_{\tau}^+)) \le \xi V(t_{\tau},\psi(t_{\tau})), & t = t_{\tau}, \end{cases}$$
(6)

where $T_0 = \frac{1}{\lambda(\gamma-1)\omega} + \frac{\pi^2}{\mu(1-\delta)}$, T_c is an arbitrary number given in advance. $\alpha_1 > 0$, $\alpha_2 > 0$, $K < \min\{\mu, \lambda, -\frac{T_c}{T_0}\left(\frac{\ln\xi}{\nu_{\tau}}\right)\}$, positive constants μ , ξ , λ , δ and γ are given in Lemma 3, then the zero solution of the system (4) is PDT stable in probability.

Proof. It follows from Lemma 3 that the zero solution of the system (4) under conditions (i) and (ii) are FXT stable in probability and its ST satisfies

$$T_2 = \frac{1}{\eta(1-\gamma)} \ln\left(1 - \frac{\eta}{\frac{T_0}{T_c}\lambda\omega}\right) + \frac{1}{\eta(1-\delta)} \ln\left(\frac{\frac{T_0}{T_c}\mu}{\frac{T_0}{T_c}\mu - \eta\pi^2}\right),$$

where $\eta = \frac{T_0}{T_c}K + \frac{\ln \xi}{v_{\tau}}$. According to the inequality $1 - \frac{1}{x} < \ln x < x - 1(x > 0)$, we have

$$T_{2} \leq \frac{1}{\eta(1-\gamma)} \left(-\frac{\eta}{\frac{T_{0}}{T_{c}}\lambda\omega} \right) + \frac{1}{\eta(1-\delta)} \left(1 - \frac{\frac{I_{0}}{T_{c}}\mu - \eta\pi^{2}}{\frac{T_{0}}{T_{c}}\mu} \right)$$
$$\leq \frac{\frac{T_{c}}{T_{0}}}{\lambda(\gamma-1)\omega} + \frac{\pi^{2}\frac{T_{c}}{T_{0}}}{\mu(1-\delta)}$$
$$= \frac{T_{c}}{T_{0}} \left(\frac{1}{\lambda(\gamma-1)\omega} + \frac{\pi^{2}}{\mu(1-\delta)} \right)$$
$$= T_{c}.$$

The proof is achieved. \Box

Remark 2. In some circumstance where the initial conditions of the system are not accessible, the PDT synchronization has better application prospects than FXT synchronization due to its ST being independent of the initial values and system parameters. Presently, some novel results have been introduced on PDT synchronization of nonlinear systems without the impulsive effect [19,46]. However, research results on PDT synchronization of nonlinear systems with impulses are still very few due to the inconvenience caused by impulsive gains, in particularly, there are almost no studies on PDT synchronization of nonlinear systems with both impulses and stochastic perturbations. Here, we derived the PDT stability criteria for these kinds of systems by applying the novel inequality technique.

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3. Main Results

3.1. Fixed Time Synchronization

In this subsection, the FXT stable given in Lemma 4 will be applied to discuss FXT synchronization of complex-valued impulsive BAM neural networks with stochastic perturbations. To make the systems (1) and (2) achieve FXT synchronization, we only need to prove that the error system (3) is stable in FXT. To do this, the following control protocol is developed

$$\begin{cases} u_{i}(t) = -\chi_{i}[s_{i}(t)]|s_{i}(t)|^{\theta} - \zeta_{i}[s_{i}(t)]|s_{i}(t)|^{\theta}, \\ v_{j}(t) = -\rho_{j}[r_{j}(t)]|r_{j}(t)|^{\theta} - \gamma_{j}[r_{j}(t)]|r_{j}(t)|^{\theta}, \end{cases}$$
(7)

where $i \in \tilde{I}$, $j \in \tilde{J}$, χ_i , ζ_i , ρ_j , γ_j are positive real numbers, $0 < \theta < 1$ and $\vartheta > 1$. Before giving the main theorem, for convenience, we also denote

$$\begin{split} p_{i} &= \eta_{i} - Re(a_{i}) + \frac{1}{2} \sum_{j=1}^{m} |c_{ij}| L_{j}^{h} + \frac{1}{2} \sum_{j=1}^{m} |d_{ji}| L_{i}^{g}, \\ q_{j} &= \eta_{j} - Re(b_{j}) + \frac{1}{2} \sum_{i=1}^{n} |d_{ji}| L_{i}^{g} + \frac{1}{2} \sum_{i=1}^{n} |c_{ij}| L_{j}^{h}, \\ \alpha &= 2^{\frac{\theta+1}{2}} \min\{\min_{i \in \tilde{I}} \{\chi_{i}\}, \min_{j \in \tilde{J}} \{\rho_{j}\}\}, \quad \beta = 2 \min\left\{n^{\frac{1-\theta}{2}} \min_{i \in \tilde{I}} \{\zeta_{i}\}, m^{\frac{1-\theta}{2}} \min_{j \in \tilde{J}} \{\gamma_{j}\}\right\}, \\ \xi &= \max\{\max_{i \in \tilde{I}} \{\xi_{i}^{2}\}, \max_{j \in \tilde{J}} \{\xi_{j}^{2}\}\}, \quad k = \max_{i \in \tilde{I}, j \in \tilde{J}} \{2p_{i}, 2q_{j}\}. \end{split}$$

Theorem 1. Based on Assumptions 1 and 2, if the inequality $k < \min\left\{\alpha, \beta, -\frac{\ln \xi}{\nu_{\tau}}\right\}$ is satisfied, then the systems (1) and (2) with control scheme (7) are FXT synchronized in probability, and the ST is estimated by

$$T_{3} = \frac{2}{\eta(1-\vartheta)} \ln\left(1-\frac{\eta}{\beta\breve{\omega}}\right) + \frac{2}{\eta(1-\vartheta)} \ln\left(\frac{\alpha}{\alpha-\eta\breve{\pi}^{2}}\right),$$

where $\eta = k + \frac{\ln\xi}{\nu_{\tau}}, \,\breve{\omega} = \xi^{-\tau^{0}(1-\frac{\vartheta+1}{2})sign(1-\xi)}$ and $\breve{\pi} = \xi^{-\tau^{0}(1-\frac{\vartheta+1}{2})sign(1-\xi)}.$

Proof. Constructing the Lyapunov function as

$$V(t) = V_1(t) + V_2(t),$$

where

$$V_1(t) = \frac{1}{2} \sum_{i=1}^n \overline{s_i(t)} s_i(t), \quad V_2(t) = \frac{1}{2} \sum_{j=1}^m \overline{r_j(t)} r_j(t).$$
(8)

Calculating the $\pounds V_1(t)$ along the trajectory of the system (3), yields

$$\begin{split} \mathcal{E}V_{1}(t) &= \frac{1}{2} \sum_{i=1}^{n} \overline{s_{i}(t)} \{-a_{i}s_{i}(t) + \sum_{j=1}^{m} c_{ij}h_{j}(r_{j}(t)) - \chi_{i}[s_{i}(t)]|s_{i}(t)|^{\theta} - \zeta_{i}[s_{i}(t)]|s_{i}(t)|^{\theta} \} \\ &+ \frac{1}{2} \sum_{i=1}^{n} s_{i}(t) \{\overline{-a_{i}s_{i}(t)} + \sum_{j=1}^{m} \overline{c_{ij}h_{j}(r_{j}(t))} - \overline{\chi_{i}[s_{i}(t)]|s_{i}(t)|^{\theta}} - \overline{\zeta_{i}[s_{i}(t)]|s_{i}(t)|^{\theta}} \} \\ &+ \frac{1}{2} \sum_{i=1}^{n} \sigma_{i}(t,s_{i}(t))\overline{\sigma_{i}(t,s_{i}(t))} \end{split}$$

$$=\sum_{i=1}^{n}(-Re(a_{i}))s_{i}(t)\overline{s_{i}(t)} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\overline{s_{i}(t)}c_{ij}h_{j}(r_{j}(t)) + s_{i}(t)\overline{c_{ij}h_{j}(r_{j}(t))}\right)$$
$$-\frac{1}{2}\sum_{i=1}^{n}\chi_{i}\left(\overline{s_{i}(t)}[s_{i}(t)] + s_{i}(t)\overline{[s_{i}(t)]}\right)|s_{i}(t)|^{\theta}$$
$$-\frac{1}{2}\sum_{i=1}^{n}\zeta_{i}\left(\overline{s_{i}(t)}[s_{i}(t)] + s_{i}(t)\overline{[s_{i}(t)]}\right)|s_{i}(t)|^{\theta} + \frac{1}{2}\sum_{i=1}^{n}\sigma_{i}(t,s_{i}(t))\overline{\sigma_{i}(t,s_{i}(t))}.$$

From Lemma 1 and Assumption 1, we can get

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} \left(\overline{s_{i}(t)}c_{ij}h_{j}(r_{j}(t)) + s_{i}(t)\overline{c_{ij}h_{j}(r_{j}(t))}\right) \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |\overline{s_{i}(t)}c_{ij}h_{j}(r_{j}(t))| \\
\leq \sum_{i=1}^{n}\sum_{j=1}^{m} (|c_{ij}|L_{j}^{h}|r_{j}(t)||s_{i}(t)|) \\
\leq \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} |c_{ij}|L_{j}^{h}(|r_{j}(t)|^{2} + |s_{i}(t)|^{2}).$$
(9)

In view of Assumption 2, we have

$$\frac{1}{2}\sum_{i=1}^{n}\sigma_{i}(t,s_{i}(t))\overline{\sigma_{i}(t,s_{i}(t))} \leq \sum_{i=1}^{n}\eta_{i}|s_{i}(t)|^{2}.$$
(10)

According to Lemma 2, one has

$$-\frac{1}{2}\sum_{i=1}^{n}\chi_{i}\left(\overline{s_{i}(t)}[s_{i}(t)]+s_{i}(t)\overline{[s_{i}(t)]}\right)|s_{i}(t)|^{\theta} \leq -\sum_{i=1}^{n}\chi_{i}|s_{i}(t)|^{\theta+1},$$
(11)

$$-\frac{1}{2}\sum_{i=1}^{n}\zeta_{i}\left(\overline{s_{i}(t)}[s_{i}(t)]+s_{i}(t)\overline{[s_{i}(t)]}\right)|s_{i}(t)|^{\vartheta} \leq -\sum_{i=1}^{n}\zeta_{i}|s_{i}(t)|^{\vartheta+1}.$$
(12)

By combining above inequalities, we obtain

$$\mathcal{E}V_{1}(t) \leq \sum_{i=1}^{n} (-Re(a_{i}))|s_{i}(t)|^{2} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} |c_{ij}|L_{j}^{h}(|r_{j}(t)|^{2} + |s_{i}(t)|^{2}) - \sum_{i=1}^{n} \chi_{i}|s_{i}(t)|^{\theta+1} - \sum_{i=1}^{n} \zeta_{i}|s_{i}(t)|^{\theta+1} + \sum_{i=1}^{n} \eta_{i}|s_{i}(t)|^{2} = \sum_{i=1}^{n} \left(-Re(a_{i}) + \frac{1}{2} \sum_{j=1}^{m} |c_{ij}|L_{j}^{h} + \eta_{i}\right)|s_{i}(t)|^{2} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} |c_{ij}|L_{j}^{h}|r_{j}(t)|^{2} - \sum_{i=1}^{n} \chi_{i}|s_{i}(t)|^{\theta+1} - \sum_{i=1}^{n} \zeta_{i}|s_{i}(t)|^{\theta+1}.$$

$$(13)$$

Similarly, for $V_2(t)$ it is not difficult to get

$$\mathcal{L}V_{2}(t) \leq \sum_{j=1}^{m} \left(-Re(b_{j}) + \frac{1}{2} \sum_{i=1}^{n} |d_{ji}| L_{i}^{g} + \tilde{\eta}_{j} \right) |r_{j}(t)|^{2} + \frac{1}{2} \sum_{j=1}^{m} \sum_{i=1}^{n} |d_{ji}| L_{i}^{g} |s_{i}(t)|^{2} - \sum_{j=1}^{m} \rho_{j} |r_{j}(t)|^{\theta+1} - \sum_{j=1}^{m} \gamma_{j} |r_{j}(t)|^{\theta+1}.$$

$$(14)$$

Therefore, finally we have

$$\begin{split} \mathcal{E}V(t) &= \mathcal{E}V_{1}(t) + \mathcal{E}V_{2}(t) \\ &\leq \sum_{i=1}^{n} \left(\eta_{i} - Re(a_{i}) + \frac{1}{2} \sum_{j=1}^{m} |c_{ij}|L_{j}^{h} + \frac{1}{2} \sum_{j=1}^{m} |c_{ij}|L_{j}^{S}| \right) |s_{i}(t)|^{2} \\ &+ \sum_{j=1}^{m} \left(\bar{\eta}_{j} - Re(b_{j}) + \frac{1}{2} \sum_{i=1}^{n} |d_{ij}|L_{i}^{S} + \frac{1}{2} \sum_{i=1}^{n} |c_{ij}|L_{j}^{h}| \right) |r_{j}(t)|^{2} \\ &- \sum_{i=1}^{n} \chi_{i} |s_{i}(t)|^{\theta+1} - \sum_{j=1}^{m} \rho_{j} |r_{j}(t)|^{\theta+1} - \sum_{i=1}^{n} \zeta_{i} |s_{i}(t)|^{\theta+1} - \sum_{j=1}^{m} \gamma_{j} |r_{j}(t)|^{\theta+1} \right) \\ &\leq \sum_{i=1}^{n} p_{i} |s_{i}(t)|^{2} + \sum_{j=1}^{m} q_{j} |r_{j}(t)|^{2} \\ &- \min_{i \in I} \{\chi_{i}\} \left(\sum_{i=1}^{n} |s_{i}(t)|^{\theta+1} \right) - \min_{j \in I} \{\rho_{j}\} \left(\sum_{j=1}^{m} |r_{j}(t)|^{\theta+1} \right) \\ &- \min_{i \in I} \{\zeta_{i}\} \left(\sum_{i=1}^{n} |s_{i}(t)|^{2} + \max_{j \in J} \{2q_{i}\} \frac{1}{2} \sum_{j=1}^{m} |s_{i}(t)|^{2} + \max_{j \in J} \{2q_{j}\} \frac{1}{2} \sum_{j=1}^{m} |s_{i}(t)|^{2} \right)^{\frac{\theta+1}{2}} \\ &- \min_{i \in I} \{\chi_{i}\} \left(\sum_{i=1}^{n} |s_{i}(t)|^{2} \right)^{\frac{\theta+1}{2}} - \min_{1 \leq j \leq m} \{\gamma_{j}\} m^{\frac{1-\theta}{2}} \left(\sum_{j=1}^{m} |r_{j}(t)|^{2} \right)^{\frac{\theta+1}{2}} \\ &- \min_{i \in I} \{\zeta_{i}\} n^{\frac{1-\theta}{2}} \left(\sum_{i=1}^{n} |s_{i}(t)|^{2} \right)^{\frac{\theta+1}{2}} - \min_{1 \leq j \leq m} \{\gamma_{i}\} m^{\frac{1-\theta}{2}} \left(\sum_{j=1}^{m} |r_{j}(t)|^{2} \right)^{\frac{\theta+1}{2}} \\ &\leq k(V_{1}(t) + V_{2}(t)) - \alpha \left((V_{1}(t))^{\frac{\theta+1}{2}} + (V_{2}(t))^{\frac{\theta+1}{2}} \right) - 2^{\frac{\theta-1}{2}} \beta \left((V_{1}(t))^{\frac{\theta+1}{2}} + (V_{2}(t))^{\frac{\theta+1}{2}} \right) \\ &\leq kV(t) - \alpha (V_{1}(t) + V_{2}(t))^{\frac{\theta+1}{2}} - \beta (V_{1}(t) + V_{2}(t))^{\frac{\theta+1}{2}} \\ &= kV(t) - \alpha V(t)^{\frac{\theta+1}{2}} - \beta V(t)^{\frac{\theta+1}{2}} \\ &= kV(t) - \alpha V(t)^{\frac{\theta+1}{2}} - \beta V(t)^{\frac{\theta+1}{2}} \\ &= kV(t)^{2} + \frac{1}{2} \sum_{i=1}^{n} \overline{s_{i}(t^{\frac{1}{2}})} s_{i}(t^{\frac{1}{2}}) + V_{2}(t^{\frac{1}{2}}) \\ &= \frac{1}{2} \sum_{i=1}^{n} \overline{s_{i}(t^{\frac{1}{2}})} s_{i}(t^{\frac{1}{2}}) + \frac{1}{2} \sum_{i=1}^{m} \overline{r_{i}(t^{\frac{1}{2}})} r_{i}(t^{\frac{1}{2}}) \\ &= \frac{1}{2} \sum_{i=1}^{n} \overline{s_{i}(t^{\frac{1}{2}})} s_{i}(t^{\frac{1}{2}}) + \frac{1}{2} \sum_{i=1}^{m} \overline{r_{i}(t^{\frac{1}{2}})} r_{i}(t^{\frac{1}{2}}) \\ &= \frac{1}{2} \sum_{i=1}^{n} \overline{s_{i}(t^{\frac{1}{2}})} s_{i}(t^{\frac{1}{2}}) + \frac{1}{2} \sum_{i=1}^{m} \overline{s_{i}(t^{\frac{1}{2}})} r_{i}(t^{\frac{1}{2}}) \\ &= \frac{1}{2} \sum_{i=1}^{n} \overline{s_{i}(t^{\frac{1}{2})}} s_{i}$$

$$\leq \sum_{i=1}^{2} \max_{i \in \tilde{I}} \{\tilde{\xi}_{i}^{2}\} \sum_{i=1}^{n} \overline{s_{i}(t_{\tau})} s_{i}(t_{\tau}) + \frac{1}{2} \max_{1 \le j \le m} \{\tilde{\xi}_{j}^{2}\} \sum_{j=1}^{m} \overline{r_{j}(t_{\tau})} r_{j}(t_{\tau})$$

$$\leq \max\{\max_{i \in \tilde{I}} \{\tilde{\xi}_{i}^{2}\}, \max_{1 \le j \le m} \{\tilde{\xi}_{j}^{2}\}\} \left(\frac{1}{2} \sum_{i=1}^{n} \overline{s_{i}(t)} s_{i}(t) + \frac{1}{2} \sum_{j=1}^{m} \overline{r_{j}(t)} r_{j}(t)\right)$$

$$= \xi V(t_{\tau}).$$
(16)

Thus, in view of Equations (15) and (16), we obtain from Lemma 3 that the error system (3) can achieve FXT stable in probability with ST T_3 , so that systems (1) and (2) achieve FXT synchronization in probability, and its ST obtained as

$$T_3 = \frac{2}{\eta(1-\vartheta)}\ln(1-\frac{\eta}{\beta\breve{\omega}}) + \frac{2}{\eta(1-\theta)}\ln(\frac{\alpha}{\alpha-\eta\breve{\pi}^2}).$$

The proof is finished. \Box

In Theorem 1 we consider the FXT synchronization of complex-valued BAM neural networks with both impulsive and stochastic effects. Since there is no stochastic perturbations in original drive-response systems, they become

$$\begin{aligned} dx_{i}(t) &= \left[-a_{i}x_{i}(t) + \sum_{j=1}^{m} c_{ij}h_{j}(y_{j}(t)) + I_{i}(t)\right]dt, \quad t \neq t_{\tau}, \\ dy_{j}(t) &= \left[-b_{j}y_{j}(t) + \sum_{i=1}^{n} d_{ji}g_{i}(x_{i}(t)) + J_{j}(t)\right]dt, \quad t \neq t_{\tau}, \\ \Delta x_{i}(t_{\tau}^{+}) &= x_{i}(t_{\tau}^{+}) - x_{i}(t_{\tau}^{-}) = (\xi_{i} - 1)x_{i}(t_{\tau}), \quad t = t_{\tau}, \\ \Delta y_{j}(t_{\tau}^{+}) &= y_{j}(t_{\tau}^{+}) - y_{j}(t_{\tau}^{-}) = (\tilde{\xi}_{j} - 1)y_{j}(t_{\tau}), \quad t = t_{\tau}, \end{aligned}$$
(17)

$$\begin{cases} d\tilde{x}_{i}(t) = \left[-a_{i}\tilde{x}_{i}(t) + \sum_{j=1}^{m} c_{ij}h_{j}(\tilde{y}_{j}(t)) + I_{i}(t) + u_{i}(t)\right]dt, & t \neq t_{\tau} \\ d\tilde{y}_{j}(t) = \left[-b_{j}\tilde{y}_{j}(t) + \sum_{i=1}^{n} d_{ji}g_{i}(\tilde{x}_{i}(t)) + J_{j}(t) + v_{j}(t)\right]dt, & t \neq t_{\tau} \\ \Delta\tilde{x}_{i}(t_{\tau}^{+}) = \tilde{x}_{i}(t_{\tau}^{+}) - \tilde{x}_{i}(t_{\tau}^{-}) = (\xi_{i} - 1)\tilde{x}_{i}(t_{\tau}), & t = t_{\tau}, \\ \Delta\tilde{y}_{j}(t_{\tau}^{+}) = \tilde{y}_{j}(t_{\tau}^{+}) - \tilde{y}_{j}(t_{\tau}^{-}) = (\xi_{j}^{-} - 1)\tilde{y}_{j}(t_{\tau}), & t = t_{\tau}. \end{cases}$$
(18)

At this time, by denoting

$$\begin{split} \tilde{p}_i &= -Re(a_i) + \frac{1}{2}\sum_{j=1}^m |c_{ij}|L_j^h + \frac{1}{2}\sum_{j=1}^m |d_{ji}|L_i^g, \ i \in \tilde{I}, \\ \tilde{q}_j &= -Re(b_j) + \frac{1}{2}\sum_{i=1}^n |d_{ji}|L_i^g + \frac{1}{2}\sum_{i=1}^n |c_{ij}|L_j^h, \ j \in \tilde{J}, \end{split}$$

then we have a following result from the Theorem 1.

Corollary 1. Suppose that Assumption 1 and the inequality $\tilde{k} \stackrel{\circ}{=} \max_{i \in \tilde{I}, j \in \tilde{J}} \{2\tilde{p}_i, 2\tilde{q}_j\} < \{\alpha, \beta, -\frac{\ln \xi}{\nu_\tau}\}$ is satisfied, then, under the controller (7), the system (17) and (18) will achieve FXT stabilization with a ST

$$T_4 = rac{2}{ ilde\eta(1-artheta)}\ln(1-rac{ ilde\eta}{etaec\omega}) + rac{2}{ ilde\eta(1- heta)}\ln(rac{lpha}{lpha- ilde\etaec\pi^2}),$$

where $\tilde{\eta} = \tilde{k} + \frac{\ln \xi}{\nu_{\tau}}$, $\check{\omega}$ and $\check{\pi}$ are defined above. Further, if there is no impulsive effect in systems (1) and (2), then they will degenerate to the following form

$$dx_{i}(t) = [-a_{i}x_{i}(t) + \sum_{j=1}^{m} c_{ij}h_{j}(y_{j}(t)) + I_{i}(t)]dt + \varrho_{i}(t, x_{i}(t))d\omega(t), \quad t \neq t_{\tau},$$

$$dy_{j}(t) = [-b_{j}y_{j}(t) + \sum_{i=1}^{n} d_{ji}g_{i}(x_{i}(t)) + J_{j}(t)]dt + \tilde{\varrho}_{j}(t, y_{j}(t))d\omega(t), \quad t \neq t_{\tau},$$
(19)

$$d\tilde{x}_{i}(t) = [-a_{i}\tilde{x}_{i}(t) + \sum_{j=1}^{m} c_{ij}h_{j}(\tilde{y}_{j}(t)) + I_{i}(t) + u_{i}(t)]dt + \sigma_{i}(t,\tilde{x}_{i}(t))d\omega(t), \quad t \neq t_{\tau},$$

$$d\tilde{y}_{j}(t) = [-b_{j}\tilde{y}_{j}(t) + \sum_{i=1}^{n} d_{ji}g_{i}(\tilde{x}_{i}(t)) + J_{j}(t) + v_{j}(t)]dt + \tilde{\sigma}_{j}(t,\tilde{y}_{j}(t))d\omega(t), \quad t \neq t_{\tau}.$$
(20)

In this case, for FXT synchronization of (19) and (20), we have a following result from the Theorem 1 and Lemma 2 of Ref. [9].

Corollary 2. Under the Assumptions 1 and 2, if the parameter k in Theorem 1 is satisfied $k < \min\{\alpha, \beta\}$, then the systems (19) and (20) will FXT synchronized in probability with ST $E[T(s_0, r_0)] < T_5$, where $E[T(s_0, r_0)] < T_5$ given as

$$T_{5} \triangleq \begin{cases} T_{5}^{1} = \frac{2}{k\epsilon(1-\nu)} \ln\left(1-\frac{k}{\alpha}\left(\frac{\alpha}{\beta}\right)^{\epsilon}\right), & k < 0, \\ T_{5}^{2} = \frac{2\pi}{(\nu-\theta)\alpha} \left(\frac{\alpha}{\beta}\right)^{\epsilon} csc(\epsilon\pi), & k = 0, \\ T_{5}^{3} = \frac{2\pi csc(\epsilon\pi)}{\beta(\nu-\theta)} \left(\frac{\beta}{\alpha-k}\right)^{1-\epsilon} I\left(\frac{\beta}{\zeta},\epsilon,1-\epsilon\right) \\ + \frac{\pi csc(\epsilon\pi)}{\alpha(\nu-\theta)} \left(\frac{\alpha}{\beta-k}\right)^{\epsilon} I\left(\frac{\alpha}{\zeta},1-\epsilon,\epsilon\right), & 0 < k < \min\{\beta,\alpha\} \end{cases}$$

where $\epsilon = \frac{(1-\theta)}{(\nu-\theta)}$, $\varsigma = \beta + \alpha - k$. Particularly, if $\nu + \theta = 2$, ST is estimated more precisely as $E[T(s_0, r_0)] < T_6$, where

$$T_{6} \triangleq \begin{cases} T_{6}^{1} = \frac{1}{\nu - 1} \frac{2}{\sqrt{\Lambda}} \left(\frac{\pi}{2} + \arctan\left(\frac{k}{\sqrt{\Lambda}}\right) \right), & -2\sqrt{\alpha\beta} < k < 2\sqrt{\alpha\beta} \\ T_{6}^{2} = \frac{2}{k(\nu - 1)}, & k = -2\sqrt{\alpha\beta}, \\ T_{6}^{3} = \frac{1}{(\nu - 1)\sqrt{-\Lambda}} \ln \frac{k + \sqrt{-\Lambda}}{k - \sqrt{-\Lambda}}, & k < -2\sqrt{\alpha\beta}, \end{cases}$$

where $\Lambda = 4\alpha\beta - k^2$.

Remark 3. During the synchronization analysis of complex-valued NNs, there are some works that divided the original complex-valued networks into real and imaginary networks and then investigated their synchronization performance separately [19,21,24,47]. Undoubtedly, this will double the dimension of the system and increase the control burden. In this article, we investigated the FXT synchronization of the considered stochastic complex-valued stochastic NNs with general impulses using the non-separation method. Compared to the separation method used in Refs. [19,21,24,47], it makes the theocratical analysis much easier and helps to simplify the control process.

3.2. Predefined Time Synchronization

In this subsection, we will investigate the PDT synchronization of drive-response networks (1) and (2) by designing a new controller given as follows

$$\begin{cases} u_{i}(t) = -\frac{T_{0}}{T_{c}} (\chi_{i}[s_{i}(t)]|s_{i}(t)|^{\theta} + \zeta_{i}[s_{i}(t)]|s_{i}(t)|^{\theta}), \\ v_{j}(t) = -\frac{T_{0}}{T_{c}} (\rho_{j}[r_{j}(t)]|r_{j}(t)|^{\theta} + \gamma_{j}[r_{j}(t)]|r_{j}(t)|^{\theta}), \end{cases}$$

$$(21)$$

where $j \in \tilde{J}$, $i \in \tilde{I}$, T_c is PDT given in advance, $T_0 = \frac{2}{\beta(\vartheta-1)\omega_0} + \frac{2\pi_0^2}{\alpha(1-\vartheta)}$, $\omega_0 = \xi^{-0.5\tau^0(1-\vartheta)sign(1-\xi)}$, $\pi_0 = \xi^{-0.5\tau^0(1-\vartheta)sign(1-\xi)}$. χ_i , ζ_i , ρ_j , γ_j are positive scalars, $0 < \theta < 1$ and $\vartheta > 1$.

Theorem 2. Under the Assumptions 1 and 2, if the inequality $k < \min\left\{\alpha, \beta, -\frac{T_c}{T_0} \frac{\ln \xi}{\nu_{\tau}}\right\}$ is satisfied, then systems (1) and (2) are PDT synchronized in probability via controller (21).

Proof. Define the Lyapunov function as

$$V(t) = V_1(t) + V_2(t),$$
(22)

where

$$V_1(t) = \frac{1}{2} \sum_{i=1}^n \overline{s_i(t)} s_i(t), \quad V_2(t) = \frac{1}{2} \sum_{j=1}^m \overline{r_j(t)} r_j(t).$$
(23)

Calculating $\pounds V_1(t)$ for $V_1(t)$ along the trajectory of the system (3), we get

$$\begin{split} \mathcal{L}V_{1}(t) &= \frac{1}{2} \sum_{i=1}^{n} \overline{s_{i}(t)} \Biggl\{ -a_{i}s_{i}(t) + \sum_{j=1}^{m} c_{ij}h_{j}(r_{j}(t)) - \frac{T_{0}}{T_{c}}\chi_{i}[s_{i}(t)]|s_{i}(t)|^{\theta} - \frac{T_{0}}{T_{c}}\zeta_{i}[s_{i}(t)]|s_{i}(t)|^{\theta} \Biggr\} \\ &+ \frac{1}{2} \sum_{i=1}^{n} s_{i}(t) \Biggl\{ \overline{-a_{i}s_{i}(t)} + \sum_{j=1}^{m} \overline{c_{ij}h_{j}(r_{j}(t))} - \frac{T_{0}}{T_{c}}\overline{\chi_{i}[s_{i}(t)]|s_{i}(t)|^{\theta}} - \frac{T_{0}}{T_{c}}\overline{\zeta_{i}[s_{i}(t)]|s_{i}(t)|^{\theta}} \Biggr\} \\ &+ \frac{1}{2} \sum_{i=1}^{n} \sigma_{i}(t,s_{i}(t))\overline{\sigma_{i}(t,s_{i}(t))} \\ &= \sum_{i=1}^{n} (-Re(a_{i}))s_{i}(t)\overline{s_{i}(t)} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\overline{s_{i}(t)}c_{ij}h_{j}(r_{j}(t)) + s_{i}(t)\overline{c_{ij}h_{j}(r_{j}(t))}\right) \Biggr) \\ &- \frac{1}{2} \frac{T_{0}}{T_{c}} \sum_{i=1}^{n} \chi_{i} \left(\overline{s_{i}(t)}[s_{i}(t)] + s_{i}(t)\overline{[s_{i}(t)]}\right)|s_{i}(t)|^{\theta} \\ &- \frac{1}{2} \frac{T_{0}}{T_{c}} \sum_{i=1}^{n} \zeta_{i} \left(\overline{s_{i}(t)}[s_{i}(t)] + s_{i}(t)\overline{[s_{i}(t)]}\right)|s_{i}(t)|^{\theta} \\ &+ \frac{1}{2} \sum_{i=1}^{n} \sigma_{i}(t,s_{i}(t))\overline{\sigma_{i}(t,s_{i}(t))}. \end{split}$$

In view of Lemma 1 and Assumption 1, we obtain

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} \left(\overline{s_{i}(t)}c_{ij}h_{j}(r_{j}(t)) + s_{i}(t)\overline{c_{ij}h_{j}(r_{j}(t))}\right) \leq \sum_{i=1}^{n}\sum_{j=1}^{m}|\overline{s_{i}(t)}c_{ij}h_{j}(r_{j}(t))| \\
\leq \sum_{i=1}^{n}\sum_{j=1}^{m}(|c_{ij}|L_{j}^{h}|r_{j}(t)||s_{i}(t)|) \\
\leq \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}|c_{ij}|L_{j}^{h}(|r_{j}(t)|^{2} + |s_{i}(t)|^{2}).$$
(25)

In addition by Assumption 2, we have

$$\frac{1}{2}\sum_{i=1}^{n}\sigma_{i}^{T}(t,s_{i}(t))\overline{\sigma_{i}(t,s_{i}(t))} \leq \sum_{i=1}^{n}\eta_{i}|s_{i}(t)|^{2}.$$
(26)

According to Lemma 2, one has

$$-\frac{1}{2}\frac{T_0}{T_c}\sum_{i=1}^n \chi_i \Big(\overline{s_i(t)}[s_i(t)] + s_i(t)\overline{[s_i(t)]}\Big)|s_i(t)|^{\theta} \le -\frac{T_0}{T_c}\sum_{i=1}^n \chi_i|s_i(t)|^{\theta+1},$$
(27)

$$-\frac{1}{2}\frac{T_0}{T_c}\sum_{i=1}^n \zeta_i \Big(\overline{s_i(t)}[s_i(t)] + s_i(t)\overline{[s_i(t)]}\Big)|s_i(t)|^{\vartheta} \le -\frac{T_0}{T_c}\sum_{i=1}^n \zeta_i|s_i(t)|^{\vartheta+1}.$$
 (28)

By introducing Equations (25)–(28) into (24), we get

$$\mathcal{L}V_{1}(t) \leq \sum_{i=1}^{n} (-Re(a_{i}))|s_{i}(t)|^{2} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} |c_{ij}|L_{j}^{h} (|r_{j}(t)|^{2} + |s_{i}(t)|^{2}) - \sum_{i=1}^{n} \frac{T_{0}}{T_{c}} \chi_{i}|s_{i}(t)|^{\theta+1} - \sum_{i=1}^{n} \frac{T_{0}}{T_{c}} \zeta_{i}|s_{i}(t)|^{\theta+1} + \sum_{i=1}^{n} \eta_{i}|s_{i}(t)|^{2}$$

$$=\sum_{i=1}^{n} \left(-Re(a_{i}) + \frac{1}{2} \sum_{j=1}^{m} |c_{ij}| L_{j}^{h} + \eta_{i}\right) |s_{i}(t)|^{2} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} |c_{ij}| L_{j}^{h} |r_{j}(t)|^{2} - \sum_{i=1}^{n} \frac{T_{0}}{T_{c}} \chi_{i} |s_{i}(t)|^{\theta+1} - \sum_{i=1}^{n} \frac{T_{0}}{T_{c}} \zeta_{i} |s_{i}(t)|^{\theta+1}.$$
(29)

Similarly, for $V_2(t)$, it is not difficult to get

Therefore, we have

$$\begin{split} \mathcal{E}V(t) =& \mathcal{E}V_{1}(t) + \mathcal{E}V_{2}(t) \\ &\leq \sum_{i=1}^{n} \left(\eta_{i} - Re(a_{i}) + \frac{1}{2} \sum_{j=1}^{m} |c_{ij}|L_{i}^{\hbar} + \frac{1}{2} \sum_{j=1}^{m} |d_{ji}|L_{i}^{g} \right) |s_{i}(t)|^{2} \\ &+ \sum_{j=1}^{m} \left(\bar{\eta}_{j} - Re(b_{j}) + \frac{1}{2} \sum_{i=1}^{n} |d_{ji}|L_{i}^{g} + \frac{1}{2} \sum_{i=1}^{n} |c_{ij}|L_{j}^{h} \right) |r_{j}(t)|^{2} \\ &- \sum_{j=1}^{m} \rho_{j} \frac{T_{0}}{T_{c}} |r_{j}(t)|^{\theta+1} - \sum_{i=1}^{n} \chi_{i} \frac{T_{0}}{T_{c}} |s_{i}(t)|^{\theta+1} - \sum_{j=1}^{m} \gamma_{j} \frac{T_{0}}{T_{c}} |r_{j}(t)|^{\theta+1} \\ &- \sum_{i=1}^{n} \zeta_{i} \frac{T_{0}}{T_{c}} |s_{i}(t)|^{\theta+1} \\ &\leq \sum_{i=1}^{n} p_{i} |s_{i}(t)|^{2} + \sum_{j=1}^{m} q_{j} |r_{j}(t)|^{2} \\ &- \min\{\frac{T_{0}}{T_{c}} \chi_{i}\} \left(\sum_{i=1}^{n} |s_{i}(t)|^{\theta+1} \right) - \min\{\frac{T_{0}}{T_{c}} \zeta_{i}\} \left(\sum_{j=1}^{n} |s_{i}(t)|^{\theta+1} \right) \\ &- \min\{\frac{T_{0}}{T_{c}} \rho_{j}\} \left(\sum_{j=1}^{m} |s_{i}(t)|^{2} + \max_{j \in \overline{j}} \{2q_{j}\} \frac{1}{2} \sum_{j=1}^{m} |r_{j}(t)|^{2} \\ &- \min\{\frac{T_{0}}{T_{c}} \chi_{i}\} \left(\sum_{i=1}^{n} |s_{i}(t)|^{2} + \max_{j \in \overline{j}} \{2q_{j}\} \frac{1}{2} \sum_{j=1}^{m} |r_{j}(t)|^{2} \\ &- \min\{\frac{T_{0}}{T_{c}} \chi_{i}\} \left(\sum_{i=1}^{n} |s_{i}(t)|^{2} + \max_{j \in \overline{j}} \{2q_{j}\} \frac{1}{2} \sum_{j=1}^{m} |r_{j}(t)|^{2} \\ &- \min\{\frac{T_{0}}{T_{c}} \rho_{j}\} \left(\sum_{i=1}^{m} |r_{j}(t)|^{2} \right) \frac{\theta^{\pm 1}}{2} \\ &- \min\{\frac{T_{0}}{T_{c}} \rho_{j}\} \left(\sum_{j=1}^{m} |r_{j}(t)|^{2} \right) \frac{\theta^{\pm 1}}{2} \\ &- \min\{\frac{T_{0}}{T_{c}} \rho_{j}\} \left(\sum_{j=1}^{m} |r_{j}(t)|^{2} \right) \frac{\theta^{\pm 1}}{2} \\ &- \min\{\frac{T_{0}}{T_{c}} \rho_{j}\} \left(\sum_{j=1}^{m} |r_{j}(t)|^{2} \right) \frac{\theta^{\pm 1}}{2} \\ &- \min\{\frac{T_{0}}{T_{c}} \rho_{j}\} \left(\sum_{j=1}^{m} |r_{j}(t)|^{2} \right) \frac{\theta^{\pm 1}}{2} \\ &- \min\{\frac{T_{0}}{T_{c}} (V_{1}(t) + V_{2}(t)) - \alpha \frac{T_{0}}{T_{c}} (V_{1}(t) \right) \frac{\theta^{\pm 1}}{2} \right) \\ &\leq kV(t) - \alpha \frac{T_{0}}{T_{c}} (V_{1}(t) + V_{2}(t)) \frac{\theta^{\pm 1}}{2} \\ &- \beta \frac{T_{0}}{T_{c}} \left(kV(t) - \alpha V(t) \frac{\theta^{\pm 1}}{2} - \beta V(t) \frac{\theta^{\pm 1}}{2} \right). \end{split}$$

When $t = t_{\tau}$

$$\begin{split} V(t_{\tau}^{+}) = &V_{1}(t_{\tau}^{+}) + V_{2}(t_{\tau}^{+}) \\ = &\frac{1}{2} \sum_{i=1}^{n} \overline{s_{i}(t_{\tau}^{+})} s_{i}(t_{\tau}^{+}) + \frac{1}{2} \sum_{j=1}^{m} \overline{r_{j}(t_{\tau}^{+})} r_{j}(t_{\tau}^{+}) \\ = &\frac{1}{2} \max_{i \in \tilde{I}} \{\xi_{i}^{2}\} \sum_{i=1}^{n} \overline{s_{i}(t_{\tau})} s_{i}(t_{\tau}) + \frac{1}{2} \max_{1 \le j \le m} \{\tilde{\xi}_{j}^{-2}\} \sum_{j=1}^{m} \overline{r_{j}(t_{\tau})} r_{j}(t_{\tau}) \\ \leq &\max\{\max_{i \in \tilde{I}} \{\xi_{i}^{2}\}, \max_{1 \le j \le m} \{\tilde{\xi}_{j}^{-2}\}\} \left(\frac{1}{2} \sum_{i=1}^{n} \overline{S}_{i} s_{i} + \frac{1}{2} \sum_{j=1}^{m} \overline{r_{j}} r_{j}\right) \\ = &\xi V(t_{\tau}). \end{split}$$

Therefore, from Lemma 4, we conclude that the error system (3) will be PDT stable in probability, so the systems (1) and (2) can be PDT synchronized in probability.

Similar to Corollaries 1 and 2, when there is no impulsive effects or stochastic perturbations in systems (1) and (2), we have following results from Theorem 2 and Lemma 3 of Ref. [9]. \Box

Corollary 3. Under the Assumption 1, if the inequality $\tilde{k} < \{\alpha, \beta, -\frac{T_c}{T_0} \frac{\ln \xi}{\nu_{\tau}}\}$ is satisfied, then the systems (17) and (18) can be PDT synchronized in probability via controller (21), where parameter \tilde{k} is given in Corollary 1.

Corollary 4. Suppose that Assumptions 1 and 2 hold true. If the parameter of Theorem 2 satisfied the inequality $k < \min\left\{\alpha, \beta, -\frac{T_c}{T_0} \frac{\ln \xi}{v_\tau}\right\}$, where T_0 was chosen as $T_0 = T_5$ when $\theta + \vartheta = 2$ and $T_0 = T_6$ when $\theta + \vartheta \neq 2$, then the system (19) and (20) can be PDT synchronized in probability via controller (21).

Remark 4. Currently there are many works on the PDT synchronization of various types of chaotic systems with or without stochastic perturbations such as [19,26,35]. However, due to the lack of available PDT stability results for nonlinear impulsive systems, there are few works on the PDT synchronization of chaotic nonlinear systems with impulsive effects, not to mention those with both impulsive effects and stochastic perturbations. In this paper, for the first time, we have considered the PDT synchronization of complex-valued stochastic BAM neural networks with a general type of impulsive effects, where the impulsive gains can be stabilizing or destabilizing. Since, compared to FXT synchronization, the ST of in PDT synchronization can be given in advance on the basis of application requirement and it has nothing to do with the initial vales and intrinsic system parameters, the PDT synchronization studied in Section 3.2 has a broader application background.

4. Numerical Examples and Simulations

Now we verify the FXT and PDT synchronization criteria obtained in the above section by giving two relevant numerical examples.

Example 1. For n = m = 2, consider the following complex-valued impulse BAM neural network with stochastic perturbations:

$$dx_{i}(t) = [-a_{i}x_{i}(t) + \sum_{j=1}^{2} c_{ij}\bar{h}_{j}(y_{j}(t))]dt + \varrho_{i}(t,x_{i}(t))d\omega(t), t \neq t_{\tau},$$

$$dy_{j}(t) = [-b_{j}y_{j}(t) + \sum_{i=1}^{2} d_{ji}\bar{g}_{i}(x_{i}(t))]dt + \tilde{\varrho}_{j}(t,y_{j}(t))d\omega(t), t \neq t_{\tau},$$

$$\Delta x_{i}(t_{\tau}^{+}) = (\xi_{i} - 1)x_{i}(t_{\tau}), t = t_{\tau},$$

$$\Delta y_{j}(t_{\tau}^{+}) = (\xi_{j} - 1)y_{j}(t_{\tau}), t = t_{\tau},$$
(31)

its related parameters are chosen as $a_1 = 0.3744 - 1.0464i$, $a_2 = 0.3072 - 1.4592i$, $b_1 = 1.8 * [0.45 - 1.07i, 0; 0, 0.22 - 1.43i], b_2 = 0.3960 - 2.5740i, c_{11} = 0.624 + 1.812i,$ $c_{12} = -1.656 + 0.120i, c_{21} = -1.740 + 2.040i, c_{22} = 0.624 - 1.452i, d_{11} = 0.624 + 1.812i,$ $d_{12} = -1.656 + 0.120i, \ d_{21} = -1.740 + 2.040i, \ d_{22} = 0.624 - 1.452i, \ \bar{h}_1(u) = \bar{h}_2(u) = 0.624 - 1.452i, \ \bar{h}_2(u) = 0.624 - 1.452i, \$ $\bar{g}_1(u) = \bar{g}_2(u) = 1.2 \tanh(Re(u)) + 1.2 \tanh(Im(u))i, \ \varrho_1(t, \cdot) = 0.080 - 0.680i,$ $\varrho_2(t,\cdot) = 0.282 + 0.826i, \ \tilde{\varrho}_1 = 0.258 + 0.444i, \ \tilde{\varrho}_2 = -0.432 - 1.016i.$

By setting the impulsive moments as $t_{\tau} = 2\tau$ for $\tau \in N$ and impulsive gains $\xi_i = -0.42$ and $\tilde{\xi}_i = 0.45$, then the MATLAB simulations of system (31) with initial conditions $x_1^0 = -0.63 + 0.82i$, $x_2^0 = 0.98 - 0.71i$, $y_1^0 = 0.46 - 0.38i$ and $y_2^0 = -1.27 + 0.82i$ are shown in Figure 1, which indicates that system (31) has a chaotic attractor.



Figure 1. The chaotic attractor of the real and imaginary parts of system (31).

Now, we introduce the response system of system (31) as follows

$$d\tilde{x}_{i}(t) = [-a_{i}\tilde{x}_{i}(t) + \sum_{j=1}^{2} c_{ij}\bar{h}_{j}(\tilde{y}_{j}(t)) + u_{i}(t)]dt + \varrho_{i}(t,\tilde{x}_{i}(t))d\omega(t), t \neq t_{\tau},$$

$$d\tilde{y}_{j}(t) = [-b_{j}\tilde{y}_{j}(t) + \sum_{i=1}^{2} d_{ji}\bar{g}_{i}(\tilde{x}_{i}(t)) + v_{j}(t)]dt + \tilde{\varrho}_{j}(t,\tilde{y}_{j}(t))d\omega(t), t \neq t_{\tau},$$

$$\Delta\tilde{x}_{i}(t_{\tau}) = (\xi_{i} - 1)\tilde{x}_{i}(t_{\tau}), t = t_{\tau},$$

$$\Delta\tilde{y}_{j}(t_{\tau}) = (\xi_{j}^{2} - 1)\tilde{y}_{j}(t_{\tau}), t = t_{\tau},$$
(32)

where the corresponding parameter is the same as in system (31). By simple computation we can obtain that $L_i^g = L_j^{h} = 1$, $\eta_i = \tilde{\eta}_j = 0.5$, i, j = 1, 2. Thus, the Assumptions 1 and 2 are satisfied. To simplify the demonstration, we first reselect the impulsive moments as $t_{\tau} = 0.12\tau$ for $\tau \in N$, and then choose the control parameters as $\chi_1 = \chi_2 = 8.6, \zeta_1 = \zeta_2 = 5.6, \rho_1 = \rho_2 = 7.8, \gamma_1 = \gamma_2 = 6.8, \theta = 0.25, \vartheta = 1.12$, we can calculate that $\alpha = 12.0292$, $\beta = 10.7438$, $\xi = 0.2025$ and $\varsigma = k + \frac{\ln \xi}{\nu_{\tau}} = -3.4103 < 0$. This means that all the criteria of Theorem 1 hold true. Therefore, from Theorem 1, driveresponse impulsive neural networks (31) and (32) will achieve FXT time synchronization with ST $T_3 = 1.5424$. The time evolution of FXT synchronization errors between systems (31) and (32) are given in Figure 2, where the initial values of system (32) are arbitrarily taken in [-5, 5].



Figure 2. Evaluation of the real and imaginary parts of FXT synchronization errors s_i and r_i .

Next we consider the PDT synchronization of systems (31) and (32) under controller (21). By choosing $T_c = 1.32$, the condition $k < \min\{\alpha, \beta, -\frac{T_c}{T_0} \frac{\ln \xi}{\nu_\tau}\}$ in Theorem 2 is satisfied, where $T_0 = 1.7730$. Therefore, according to Theorem 2, the drive-response systems (31) and (32) are PDT synchronized in a PDT $T_c = 1.32$ that is less than $T_0 = 1.7730$ and $T_3 = 1.5424$. The time evolution of the PDT synchronization error between the systems (31) and (32) under the PDT controller is shown in Figure 3.



Figure 3. Evaluation of the real and imaginary parts of PDT synchronization errors s_i and r_i .



$$\begin{cases} dx_{i}(t) = \left[-a_{i}x_{i}(t) + \sum_{j=1}^{3} c_{ij}\bar{h}_{j}(y_{j}(t))\right]dt + \varrho_{i}(t,x_{i}(t))d\omega(t), \ t \neq t_{\tau}, \\ dy_{j}(t) = \left[-b_{j}y_{j}(t) + \sum_{i=1}^{3} d_{ji}\bar{g}_{i}(x_{i}(t))\right]dt + \tilde{\varrho}_{j}(t,y_{j}(t))d\omega(t), \ t \neq t_{\tau}, \\ \Delta x_{i}(t_{\tau}^{+}) = (\xi_{i}-1)x_{i}(t_{\tau}), \ t = t_{\tau}, \\ \Delta y_{j}(t_{\tau}^{+}) = (\xi_{j}-1)y_{j}(t_{\tau}), \ t = t_{\tau}, \end{cases}$$
(33)

where $\bar{h}_1(u) = \bar{h}_2(u) = \bar{h}_3(u) = \bar{g}_1(u) = \bar{g}_2(u) = \bar{g}_3(u) = 1.2 * (cos(Re(u)) + isin(Im(u))),$ $q_1 = q_2 = q_3 = 1.8 + 0.56i$ and $\omega(t)$ is a real-valued Brown motion. The other parameters of (33) are chosen as $c_{11} = 0.672 + 1.812i$, $c_{12} = -1.656 + 0.12i$, $c_{21} = -1.74 + 2.04i$, $c_{23} = 0.624 - 1.452i$, $c_{32} = -1.74 + 2.04i$, $c_{33} = 0.624 - 1.452i$, $c_{13} = c_{22} = c_{31} = 0$, $\begin{array}{l} d_{11} = 0.915 - 0.615i, \ d_{12} = -2.1 + 2.565i, \ d_{13} = 2.265 - 2.58i, \ d_{21} = -1.86 + 7.725i, \\ d_{22} = 0.435 - 2.685i, \ d_{23} = 1.725 - 2.04i, \ d_{31} = -1.8 + 2.256i, \ d_{32} = 2.46 + 4.725i, \\ d_{33} = 1.725 - 2.176i, \ a_1 = 0.48 - 1.308i, \ a_2 = 0.384 - 1.824i, \ a_3 = 0.624 - 1.584i, \\ b_1 = 0.54 - 1.284i, \ b_2 = 0.264 - 1.716i, \ b_3 = 0.504 - 1.356i. \ Figure 4 \ shows \ the \ MATLAB \ simulations \ of \ system \ (33) \ with \ initial \ values \ x_1(\theta) = -0.4 + 0.8i, \ x_2(\theta) = 0.9 - 0.7i, \\ x_3(\theta) = 0.29 - 0.47i, \ y_1(\theta) = 0.6 - 0.3i, \ y_2(\theta) = -1.2 + 0.8i, \ y_3(\theta) = -0.4 + 0.8i, \ and \\ \xi_i = 0.58, \ \bar{\xi}_j = 0.9, \ i, \ j = 1, 2, 3 \ which \ indicates \ that \ system \ (33) \ has \ a \ chaotic \ attractors. \end{array}$



Figure 4. The chaotic attractor of the real and imaginary parts of system (33).

The corresponding slave system is described as

$$d\tilde{x}_{i}(t) = [-a_{i}\tilde{x}_{i}(t) + \sum_{j=1}^{3} c_{ij}\bar{h}_{j}(\tilde{y}_{j}(t)) + u_{i}(t)]dt + \varrho_{i}(t,\tilde{x}_{i}(t))d\omega(t), t \neq t_{\tau},$$

$$d\tilde{y}_{j}(t) = [-b_{j}\tilde{y}_{j}(t) + \sum_{i=1}^{3} d_{ji}\bar{g}_{i}(\tilde{x}_{i}(t)) + v_{j}(t)]dt + \tilde{\varrho}_{j}(t,\tilde{y}_{j}(t))d\omega(t), t \neq t_{\tau},$$

$$\Delta\tilde{x}_{i}(t_{\tau}) = (\xi_{i} - 1)\tilde{x}_{i}(t_{\tau}), t = t_{\tau},$$

$$\Delta\tilde{y}_{j}(t_{\tau}) = (\xi_{j} - 1)\tilde{y}_{j}(t_{\tau}), t = t_{\tau},$$
(34)

where $a_i, b_i, c_{ij}, d_{ji}, \bar{h}_j, \bar{g}_i, \varrho_i, \tilde{\varrho}_j, i, j = 1, 2, 3$ are defined in system (33). It is not difficult to check that Assumptions 1 and 2 are satisfied with $L_i^g = L_j^h = 1$, $\eta_i = \tilde{\eta}_j = 2.2$ for $i, \ell = 1, 2, 3$. In addition, by simple computation, we get $k_1 = \max\{k_i\} = 9.6279$, $k_2 = \max\{\kappa_\ell\} = 9.7479, k = \max\{2k_1, 2k_2\} = 19.4957$. Choosing $\chi_1 = \chi_2 = \chi_3 = 4.2$, $\zeta_1 = \zeta_2 = \zeta_3 = 4.6$, $\rho_1 = \rho_2 = \rho_3 = 4.5$, $\gamma_1 = \gamma_2 = \gamma_3 = 3.7$, $\theta = 0.7$ and $\theta = 1.7$, then the conditions in Theorem 1 are satisfied. Therefore, according to Theorem 1, the master-slave systems (33) and (34) are FXT synchronized within T = 0.6743. Figure 5 shows the time changes of the solutions of errors dynamics between (33) and (34) under the controller (7), where the initial values of system (34) are randomly selected in [-8, 8].



Figure 5. Evaluation of the real and imaginary parts of FXT synchronization errors s_i and r_i .

Next we consider the PDT synchronization of systems (33) and (34) under controller (21). By choosing $T_c = 0.521$, the condition $k < \min\{\alpha, \beta, -\frac{T_p}{T_0} \frac{\ln \xi}{\nu_\tau}\}$ in Theorem 2 is satisfied, where $T_0 = 1.773$. Therefore, according to Theorem 2, the drive-response systems (31) and (32) are PDT synchronized in a PDT $T_p = 0.521$ that is less than $T_0 = 1.7730$ and $T_3 = 0.6743$. The time evolution of the PDT synchronization error between the systems (33) and (34) under the PDT controller is shown in Figure 6.



Figure 6. Evaluation of the real and imaginary parts of PDT synchronization errors s_i and r_i .

5. Conclusions

In this article, we considered the FXT and PDT synchronization issue of a class of complex-valued neural networks with stochastic perturbations and impulsive effects. First, some FXT and PDT stability results are introduced for general complex-valued stochastic nonlinear systems with impulsive effects. Then, based on these developed results, we have derived some novel FXT and PDT synchronization criteria for the considered complex-variable neural networks by designing novel controllers and employing some analysis methods. The introduced controllers are simple and they do not include a linear feedback term $k_i e_i(t)$, which is used by most of the recent published works on FXT and PDT synchronization studies, Thereby, the control burden is relaxed to some extent. In addition, the feasibility of the theocratical results are demonstrated by giving one numerical example and its numerical simulations. It is worth to mention that the devolved theoretical results of the paper will provide some insights to investigate the more complex types of neural networks

with impulsive effects such as Clifford-valued neural networks and quaternion-valued neural networks, etc.

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