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Existence of Hilfer Fractional Stochastic Differential Equations with Nonlocal Conditions and Delay via Almost Sectorial Operators

Sivajiganesan Sivasankar [1], Ramalingam Udhayakumar [1,*], Velmurugan Subramanian [2], Ghada AlNemer [3,*] and Ahmed M. Elshenhab [4,5]

1 Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632014, India
2 Department of Mathematics, School of Arts, Sciences, Humanities and Education, SASTRA Deemed to be University, Thanjavur 613401, India
3 Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
4 School of Mathematics, Harbin Institute of Technology, Harbin 150001, China
5 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
* Correspondence: udhayaram.v@gmail.com (R.U.); gnnemer@pnu.edu.sa (G.A.)

Abstract: In this article, we examine the existence of Hilfer fractional (HF) stochastic differential systems with nonlocal conditions and delay via almost sectorial operators. The major methods depend on the semigroup of operators method and the *Mönch* fixed-point technique via the measure of noncompactness, and the fundamental theory of fractional calculus. Finally, to clarify our key points, we provide an application.

Keywords: Hilfer fractional evolution system; measure of noncompactness; nonlocal condition; fixed-point theorem; almost sectorial operators

MSC: 34K30; 34K50; 47H08; 47H10

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1. Introduction

We analyse the nonlocal stochastic differential equations with HF derivative and almost sectorial operators

$$D_{0^+}^{\lambda,\mu} z(\wp) = \widetilde{A} z(\wp) + \mathcal{F}(\wp, z_\wp) + \mathcal{H}(\wp, z_\wp)\frac{dW(\wp)}{d\wp}, \qquad \wp \in \mathrm{J}' = (0,d], \tag{1}$$

$$I_{0^+}^{(1-\lambda)(1-\mu)} z(0) + N(z_\wp) = \xi \in L^2(\Lambda, B_H), \quad \wp \in (-\infty, 0], \tag{2}$$

where $\widetilde{A}$ denotes the almost sectorial operator that also generates an analytic semigroup $\{T(\wp), \wp \geq 0\}$ on $\mathcal{X}$. $D_{0^+}^{\lambda,\mu}$ stands for the HFD of order λ, $0 < \lambda < 1$ and type μ, $0 \leq \mu \leq 1$. Let $z(\cdot)$ be the state in a Hilbert space $\mathcal{X}$ with $\|\cdot\|$. The histories $z_\wp : (-\infty, 0] \to B_H$, $z_\wp(a) = z(\wp + a)$, $a \leq 0$ are associated with phase space B_H. Set $\mathrm{J} = [0,d]$, and let $\mathcal{F} : \mathrm{J} \times B_H \to \mathcal{X}$ and $\mathcal{H} : \mathrm{J} \times B_H \to L_2^0(\mathcal{U}, \mathcal{X})$ be the $\mathcal{X}$-valued functions and nonlocal term $N : B_H \to \mathcal{X}$.

In 1695, the concept of fractional calculus was presented as an important branch of mathematics. It took place at about the same time as the creation of classical calculus. Investigators have shown that various nonlocal events in the disciplines of architecture and biological sciences may be effectively expressed using fractional calculus. High-viscosity, nonlinear cycles in self-comparable and porous frameworks, fluid movement analogous to diffusion, heat flow, glasses, compressibility, and other areas are among the most popular applications of fractional calculus. Because analytic setups are often challenging to obtain, the useful application of fractional calculus in these fields has encouraged many investigators to consider their options. With an expanding variety of applications in economics, inorganic chemistry, neurobiology, compressibility, pharmacology, operations research, data analysis, etc., fractional calculus is receiving more attention from the scientific community. Additionally, it has been demonstrated that fractional differential equations may be helpful modelling tools in various scientific and engineering domains. Recent years have seen a significant advancement in the field of fractional differential equations. For more information, consult the bibliography by Kilbas et al. [1], Miller and Ross [2], Podlubny [3], Lakshmikantham et al. [4], Zhou [5], as well as the papers [6–16] and the references therein.

Furthermore, stochastic partial differential equations have drawn much interest since they were first used to mathematically simulate various events in the humanities and natural sciences [17]. Since noise or uncontrolled fluctuations are inherent and plentiful in natural and artificial systems, stochastic models should be studied rather than deterministic ones. Stochastic differential equations (SDEs) include unpredictability in the mathematical representation of a certain occurrence. The application of SDEs in finite and infinite dimensions to describe diverse phenomena in population dynamics, physics, electrical engineering, geography, psychology, biochemistry, and some other domains of physics and technology has lately attracted a lot of attention; refer to [18–24] for a broad introduction to stochastic differential equations and their applications.

Hilfer [25] initiated another type of fractional order derivative, which involved the R-L and Caputo fractional derivatives. Additionally, through conceptual predictions of laboratories in solid materials, chemical industries, sets of structures designed, architecture, and several other areas, the importance and implications of the HFD have been identified. Gu and Trujillo [26] recently used a fixed-point method and a noncompact measure approach to demonstrate that the Hilfer fractional derivative evolution issue had an integral solution. They created the most recent parameter $\mu \in [0,1]$ and a fractional parameter λ, so that $\mu = 0$ produced the R-L derivative and $\lambda = 1$ produced the Caputo derivative, to indicate the derivative's order. Hilfer fractional calculus has been the subject of several academic works, especially [20,27–29]. Researchers found a mild solution for HFD systems using almost sectorial operators and a fixed-point technique, according to [30–33].

Researchers are employing almost sectorial operators to advance fractional existence for fractional calculus. Researchers have developed a novel method for locating mild solutions for the system under investigation. Researchers have also established a theory that uses fractional calculus, semigroup operators, multivalued maps, the measure of noncompactness, the transfer function, the Wright function, and the fixed-point technique to infer different features of linked semigroups formed by almost sectorial operators. For further information, we can refer to [30,33–37]. In [30–32], researchers used Schauder's fixed-point theorem to arrive at their conclusions via almost sectorial operators. Researchers have recently used the nondense fields, cosine families, semigroup theory, numerous fixed-point approaches, and the measure of noncompactness to build fractional differential systems with nonlocal conditions with or without delay. The authors in [38,39] established their results via the *Mönch* fixed-point technique with the measure of noncompactness.

In 2017, Yang et al. [29] explored the existence of mild solutions for a class of HF evolution equations with nonlocal conditions in a Banach space, by employing the semigroup principle, fixed-point strategies, and the measure of noncompactness. Recent research has focused on the existence of mild solutions and controllability outcomes of Hilfer Fractional differential equations (HFDEs) with delay, using the measure of noncompactness [38,39]. By utilizing Krasnoselskii's fixed-point theorem, Dineshkumar et al. [20] developed a special collection of required criteria for the approximate controllability of an HF neutral stochastic delay integrodifferential system. In earlier research, Vijayakumar et al. [40] improved the idea of HFDEs to analyse infinite delays. The authors also discussed the appropriate presumptions necessary to prove the existence of mild solutions and the approximate controllability of HFDEs with delay in this paper. Nonetheless, most definitely, the study of the existence of HF stochastic differential systems with nonlocal conditions and infinite delay via almost sectorial operators using the measure of noncompactness outlined in this article has not been comprehended, and this encourages the present paper.

The remainder of the document is structured as follows: In Section 2, we cover the principles of fractional calculus, semigroups, phase spaces, almost sectorial operators, and measure of noncompactness. In Section 3, we present the existence of a mild solution to the considered system. Finally, to clarify our key points, we provide an application in Section 4.

2. Preliminaries

In this section, the essential preliminaries, fundamental definitions, notations, and lemmas of fractional calculus that are needed to establish the main results are presented.

The following important properties of $\widetilde{A}^{\eta}$ is discussed.

Theorem 1 (see [12]).

1. *Suppose $0 < \eta \leq 1$, and the accompanying $\mathcal{X}_{\eta} = D(\widetilde{A}^{\eta})$ is a Banach space with $\|z\|_{\eta} = \|\widetilde{A}^{\eta} z\|$, $z \in \mathcal{X}_{\chi}$.*
2. *Assume $0 < \gamma < \eta \leq 1$, and the accompanying $D(\widetilde{A}^{\eta}) \to D(\widetilde{A}^{\gamma})$ and the technique are compact while $\widetilde{A}$ is compact.*
3. *For all $0 < \eta \leq 1$, there exists $C_{\eta} > 0$ such that*

$$\|\widetilde{A}^{\eta} \mathsf{S}(\wp)\| \leq \frac{C_{\eta}}{\wp^{\eta}}, \; 0 < \wp \leq d.$$

The family of all highly quantifiable, square-integrable, $\mathcal{X}$-valued random components, specified as $L_2(\Lambda, \mathcal{X})$, is a Banach space associated with $\|z(\cdot)\|_{L_2(\Lambda,\mathcal{X})} = \left(E\|z(.,W)\|^2\right)^{\frac{1}{2}}$, where E is identified as $E(z) = \int_{\Lambda} z(W) dP$. A necessary subspace of $L_2(\Lambda, \mathcal{X})$ is provided by

$$L_2^0(\mathfrak{s}, \mathcal{X}) = \{z \in L_2(\Lambda, \mathcal{X}),\; z \text{ is } \mathscr{F}_0 - \text{measurable}\}.$$

Let $\mathsf{C} : \mathrm{J} \to \mathcal{X}$ be the collection of all continuous functions, where $\mathrm{J} = [0, d]$ and $\mathrm{J}' = (0, d]$ with $d > 0$. Take $Y = \left\{z \in \mathsf{C} : \lim_{\wp \to 0} \wp^{1-\mu+\lambda\mu-\lambda\vartheta} z(\wp) \text{ exists and finite } \right\}$, which is the Banach space and its norm on $\|\cdot\|_Y$, defined as $\|z\|_Y = \sup_{\wp \in \mathrm{J}'} \left\{\wp^{1-\mu+\lambda\mu-\lambda\vartheta} \|z(\wp)\|\right\}$. Set $\mathcal{B}_P(\mathrm{J}) = \{u \in \mathsf{C} \text{ such that } \|u\| \leq P\}$. We note that, if $z(\wp) = \wp^{-1+\mu-\lambda\mu+\lambda\vartheta} y(\wp)$, $\wp \in (0, d]$, then $z \in Y$ iff $y \in \mathsf{C}$ and $\|z\|_Y = \|y\|$. We introduce $\mathcal{H}$ with $\|\mathcal{H}\|_{L^p(\mathrm{J},\mathbb{R}^+)}$ through $\mathcal{H} \in L^p(\mathrm{J}, \mathbb{R}^+)$ for all p through $1 \leq p \leq \infty$. The functions $\mathcal{H} : \mathrm{J} \times B_H \to \mathcal{X}$, which are the Bochner integrable functions with norm $\|\mathcal{H}\|_{L^p(\mathrm{J},\mathcal{X})}$, are also specified by $L^p(\mathrm{J}, \mathcal{X})$.

Definition 1 (see [5]). *The fractional integral of order λ for the function $\mathcal{H} : [d, \infty) \to \mathbb{R}$ having the lower bound d is introduced as*

$$I^{\lambda}_{d+} \mathcal{F}(\wp) = \frac{1}{\Gamma(\lambda)} \int_d^{\wp} \frac{\mathcal{F}(\nu)}{(\wp - \nu)^{1-\lambda}} d\nu, \quad \wp > 0,\; \lambda \in \mathbb{R}^+.$$

Definition 2 (see [5]). *The R-L derivative has order $\lambda > 0$, $k - 1 \leq \lambda < k$, $k \in \mathbb{N}$, and its function $\mathcal{H} : [d, +\infty) \to \mathbb{R}$ is described as*

$${}^{L}D^{\lambda}_{d+} \mathcal{F}(\wp) = \frac{1}{\Gamma(k - \lambda)} \frac{d^k}{d\wp^k} \int_d^{\wp} \frac{\mathcal{F}(\nu)}{(\wp - \nu)^{\lambda+1-k}} d\nu,\; \wp > d,\; \nu \in \mathbb{R}^+.$$

Definition 3 (see [5]). *The Caputo derivative has order $\lambda > 0$, $k - 1 \leq \lambda < k$, $k \in \mathbb{N}$, and its function $\mathcal{F} : [d, +\infty) \to \mathbb{R}$ is classified by*

$${}^{C}D^{\lambda}_{d+} \mathcal{F}(\wp) = \frac{1}{\Gamma(k - \lambda)} \int_d^{\wp} \frac{\mathcal{F}^k(\nu)}{(\wp - \nu)^{\lambda+1-k}} d\nu = I^{k-\lambda}_{d+} \mathcal{H}^k(\wp),\; \wp > d,\; \nu \in \mathbb{R}^+.$$

Definition 4 (see [25]). *The* HFD *of order* $0 < \lambda < 1$ *and type* $\mu \in [0,1]$ *for the function* $\mathcal{H} : [d, +\infty) \to \mathbb{R}$ *is*

$$D_{d^+}^{\lambda,\mu}\mathcal{F}(\wp) = [I_{d^+}^{(1-\lambda)\mu} D(I_{d^+}^{(1-\lambda)(1-\mu)}\mathcal{F})](\wp).$$

Remark 1.

1. *Suppose* $\mu = 0,\ 0 < \lambda < 1$, *and* $d = 0$, *therefore the* HFD *corresponds to the conventional R-L fractional derivative:*

$$D_{0^+}^{\lambda,0}\mathcal{F}(\wp) = \frac{d}{d\nu} I_{0+}^{1-\lambda}\mathcal{F}(\wp) =^{L} D_{0+}^{\lambda}\mathcal{F}(\wp).$$

2. *Suppose* $\mu = 1,\ 0 < \lambda < 1$, *and* $d = 0$, *therefore the* HFD *corresponds to the conventional Caputo fractional derivative:*

$$D_{0^+}^{\lambda,1}\mathcal{F}(\wp) = I_{0^+}^{1-\lambda}\frac{d}{d\nu}\mathcal{F}(\wp) =^{C} D_{0+}^{\lambda}\mathcal{F}(\wp).$$

Now, we describe the abstract phase space B_H. Let $w : (-\infty, 0] \to (0, +\infty)$ be continuous along $l = \int_{-\infty}^{0} w(\wp)d\wp < +\infty$. Now, for every $n > 0$, we have

$$B = \{\delta : [-n, 0] \to \mathcal{X} \text{ such that } \delta(\wp) \text{ is measurable and bounded}\},$$

and set the space B with

$$\|\delta\|_{[-n,0]} = \sup_{\tau \in [-n,0]} \|\delta(\tau)\|, \text{ for all } \delta \in \mathcal{B}.$$

Now, we define

$$B_H = \Big\{\delta : (-\infty, 0] \to \mathcal{X} \text{ such that for any } n > 0,\ \delta|_{[-n,0]} \in \mathcal{B} \text{ and } \int_{-\infty}^{0} w(\tau)\|\delta\|_{[\tau,0]}d\tau < +\infty\Big\}.$$

If B_H is endowed with

$$\|\delta\|_{B_H} = \int_{-\infty}^{0} w(\tau)\|\delta\|_{[\tau,0]}d\tau, \text{ for all } \delta \in B_H,$$

then $(B_H, \|\cdot\|)$ is a Banach space.

Presently, we define the space

$$B_H' = \{z : (-\infty, d] \to \mathcal{X} \text{ such that } z|_{\mathrm{J}} \in \complement,\ \xi \in B_H\}.$$

Consider the seminorm $\|\cdot\|_d$ in B_H' defined by

$$\|z\|_d = \|\xi\|_{B_H} + \sup\{\|z(\tau)\| : \tau \in [0, d]\},\ z \in B_H'.$$

Lemma 1. *Suppose* $z \in B_H'$, *then for all* $\wp \in \mathrm{J}$, $z_\wp \in B_H$. *Moreover,*

$$l|z(\wp)| \le \|z_\wp\|_{B_H} \le \|\xi\|_{B_H} + l \sup_{r \in [0,\wp]} |z(r)|,$$

where $l = \int_{-\infty}^{0} w(\wp)d\wp < \infty$.

Definition 5 (see [35]). *For $0 < \vartheta < 1$, $0 < \omega < \frac{\pi}{2}$, we determine the family of closed linear operators $\Theta_{\omega}^{\vartheta}$, the region $S_{\omega} = \{\theta \in \mathbb{C}\backslash\{0\}$ with $|arg\ \theta| \leq \omega\}$ and $\widetilde{A} : D(\widetilde{A}) \subset \mathcal{X} \to \mathcal{X}$ which satisfy:*

(i) $\sigma(\widetilde{A}) \subseteq S_{\omega}$;

(ii) $\|(\theta - \widetilde{A})^{-1}\| \leq \mathcal{K}_{\delta}|\nu|^{-\vartheta}$, *for all $\omega < \delta < \pi$ and there exists a constant $\mathcal{K}_{\delta}$.*

Then, $\widetilde{A} \in \Theta\omega^{-\vartheta}$ is identified as an almost sectorial operator on $\mathcal{X}$.

Proposition 1 (see [35]). *Suppose $z \in \Theta_{\omega}^{-\vartheta}$, for $0 < \vartheta < 1$ and $0 < \omega < \frac{\pi}{2}$. Next, the following conditions are satisfied:*

- ⋇ $T(\wp)$ *is analytic and* $\frac{d^k}{d\wp^k}T(\wp) = (-\widetilde{A})^k T(\wp), \quad \wp \in S_{\frac{\pi}{2}-\omega}$;
- ⋇ $T(\wp + \nu) = T(\wp)T(\nu), \quad$ *for all* $\nu, \wp \in S_{\frac{\pi}{2}-\omega}$;
- ⋇ $\|T(\wp)\|_{L(\mathcal{X})} \leq \kappa_0 \wp^{\vartheta-1}$, $\wp > 0$; *where the constant* $\kappa_0 > 0$;
- ⋇ *The* $D(\widetilde{A}^{\theta}) \subset \Sigma_T$ *provided* $\theta > 1 - \vartheta$, *if* $\Sigma_T = \{z \in \mathcal{X} : \lim_{\wp \to 0^+} T(\wp)z = z\}$;
- ⋇ $(\nu - \widetilde{A})^{-1} = \int_0^{\infty} e^{-\upsilon\nu} T(\nu)d\nu$, $\nu \in \mathbb{C}$ *and* $Re(\nu) > 0$.

Definition 6 (see [41]). *Define the wright function $W_{\lambda}(\beta)$ by*

$$W_{\lambda}(\beta) = \sum_{k \in \mathbb{N}} \frac{(-\beta)^{k-1}}{\Gamma(1 - \lambda k)(k-1)!}, \ \beta \in \mathbb{C}. \tag{3}$$

with the following property

$$\int_0^{\infty} \theta^{\iota} W_{\lambda}(\theta) d\theta = \frac{\Gamma(1+\iota)}{\Gamma(1+\lambda\iota)}, \quad \text{for } \iota \geq 0.$$

Theorem 2 (see [5]). *If $\wp > 0$, for all $d > 0$, the continuity is uniform on $[d, \infty)$, then $\mathsf{S}_{\lambda}(\wp)$ and $\mathsf{Q}_{\lambda}(\wp)$ are continuous in the uniform operator topology.*

Lemma 2 (see [41]). *If $\{T_{\gamma}(\wp)\}_{\wp>0}$ is a compact operator, then $\{\mathsf{S}_{\gamma,\delta}(\wp)\}_{\wp>0}$ and $\{\mathsf{Q}_{\gamma}(\wp)\}_{\wp>0}$ are also compact linear operators.*

Lemma 3 (see [26]). *System (1)–(2) is identical to an integral equation presented by*

$$z(\wp) = \frac{\xi(0) - N(z_{\wp}))}{\Gamma(\mu(1-\lambda)} \wp^{-(1-\lambda)(\mu-1)} + \frac{1}{\Gamma(\lambda)} \int_0^{\wp} (\wp - \nu)^{\lambda-1} [\widetilde{A} z_{\nu} d\nu + \mathcal{F}(\nu, z_{\nu}) d\nu + \mathcal{H}(\nu, z_{\nu}) dW(\nu)].$$

Definition 7 (see [26]). *Let $z(\wp)$ be the solution of the integral equation provided by (3), then $z(\wp)$ satisfies*

$$z(\wp) = \mathsf{S}_{\lambda,\mu}(\wp)[\xi(0) - N(z_{\wp})] + \int_0^{\wp} \mathsf{K}_{\lambda}(\wp - \nu)\mathcal{F}(\nu, z_{\nu}) d\nu + \int_0^{\wp} \mathsf{K}_{\lambda}(\wp - \nu)\mathcal{H}(\nu, z_{\nu}) dW(\nu), \ \wp \in \mathrm{J}, \tag{4}$$

where $\mathsf{S}_{\lambda,\mu}(\wp) = I_0^{\mu(1-\lambda)} \mathsf{K}_{\lambda}(\wp)$, $\mathsf{K}_{\lambda}(\wp) = \wp^{\lambda-1}\mathsf{Q}_{\lambda}(\wp)$ *and* $\mathsf{Q}_{\lambda}(\wp) = \int_0^{\infty} \lambda\epsilon W_{\lambda}(\epsilon) T(\wp^{\lambda}\epsilon) d\epsilon$.

Definition 8 (see [13]). *A stochastic process $z : (-\infty, d] \to \mathcal{X}$ is said to be a mild solution of the proposed system (1)–(2), provided $I_{0^+}^{(1-\lambda)(1-\mu)} z(0) + N(z_\wp) = \xi \in L^2(\Lambda, B_H)$, $\wp \in (-\infty, 0]$ and the following integral equation*

$$z(\wp) = \mathsf{S}_{\lambda,\mu}(\wp)\left[\xi(0) - N(z_\wp)\right] + \int_0^\wp (\wp - \nu)^{\lambda-1} \mathsf{Q}_\lambda(\wp - \nu) \mathcal{F}(\nu, z_\nu) d\nu \\ + \int_0^\wp (\wp - \nu)^{\lambda-1} \mathsf{Q}_\lambda(\wp - \nu) \mathcal{H}(\nu, z_\nu) dW(\nu), \ \wp \in \mathrm{J}$$

is satisfied.

Lemma 4 (see [30]).

1. $\mathsf{K}_\lambda(\wp)$ *and* $\mathsf{S}_{\lambda,\mu}(\wp)$ *are strongly continuous, for* $\wp > 0$.
2. $\mathsf{K}_\lambda(\wp)$ *and* $\mathsf{S}_{\lambda,\mu}(\wp)$ *are bounded linear operators on* $\mathcal{X}$*, for any fixed* $\wp \in S_{\frac{\pi}{2}-\omega}$*, and we have*

$$\left\|\mathsf{K}_\lambda(\wp) z\right\| \le \kappa_p \wp^{-1+\lambda\vartheta} \|z\|, \qquad \left\|\mathsf{Q}_\lambda(\wp) z\right\| \le \kappa_p \wp^{-\lambda+\lambda\vartheta} \|z\|, \\ \left\|\mathsf{S}_{\lambda,\mu}(\wp) z\right\| \le \frac{\Gamma(\vartheta)}{\Gamma(\mu(1-\lambda)+\lambda\vartheta)} \kappa_p \wp^{-1+\mu-\lambda\mu+\lambda\vartheta} \|z\|.$$

We now review a few ideas related to the Hausdorff *MNC*.

Definition 9. *For a bounded set $\mathbb{X}$ in a Banach space $\mathcal{X}$, the Hausdorff MNC β is denoted as*

$$\beta(\mathbb{X}) = \inf\{\epsilon > 0 : \mathbb{X} \textit{ can be connected with a finite number of balls with radii } \epsilon\}. \tag{5}$$

Lemma 5 (see [42]). *Suppose $\mathcal{X}$ is a Banach space and $\mathbb{X}, \mathbb{Y} \subseteq \mathcal{X}$ are bounded. Consequently, the following characteristics are satisfied:*

(i) $\mathbb{X}$ *is precompact iff* $\beta(\mathbb{X}) = 0$;
(ii) $\beta(\mathbb{X}) = \beta(\overline{\mathbb{X}}) = \beta(conv(\mathbb{X}))$*, where* $\overline{\mathbb{X}}$ *and* $conv(\mathbb{X})$ *are the closure and convex hull of* $\mathbb{X}$*, respectively;*
(iii) *If* $\mathbb{X} \subseteq \mathbb{Y}$ *then* $\beta(\mathbb{X}) \le \beta(\mathbb{Y})$;
(iv) $\beta(\mathbb{X} + \mathbb{Y}) \le \beta(\mathbb{X}) + \beta(\mathbb{Y})$*, such that* $\mathbb{X} + \mathbb{Y} = \{a_1 + a_2 : a_1 \in \mathbb{X}, a_2 \in \mathbb{Y}\}$;
(v) $\beta(\mathbb{X} \cup \mathbb{Y}) \le \max\{\beta(\mathbb{X}), \beta(\mathbb{Y})\}$;
(vi) $\beta(\gamma\mathbb{X}) = |\gamma|\beta(\mathbb{X})$*, for all* $\gamma \in \mathbb{R}$*, when* $\mathcal{X}$ *is a real Banach space;*
(vii) *Suppose the operator* $\Psi : D(\Psi) \subseteq \mathcal{X} \to \mathcal{X}_1$ *is Lipschitz continuous with constant* κ_1*, then we know* $\wp(\Psi(\mathbb{X})) \le \kappa_1 \beta(\mathbb{X})$ $\forall$ *bounded subset* $\mathbb{X} \subset D(\Psi)$*, where* $\mathcal{X}_1$ *is the another Banach space and* $\wp$ *represents the Hausdorff MNC in* $\mathcal{X}_1$.

Theorem 3 (see [14]). *If $\{v_k\}_{k=1}^\infty$ is a series of Bochner integrable functions from J to $\mathcal{X}$ by the measurement $\|v_k(\wp)\| \le \beta(\wp)$, for each $\wp \in \mathcal{V}$ and for all $k \ge 1$, where $\beta \in L^1(\mathrm{J}, \mathbb{R})$, then the function $\omega(\wp) = \beta(\{v(\wp) : k \ge 1\})$ is in $L^1(\mathrm{J}, \mathbb{R})$ and satisfies*

$$\beta\left(\left\{ \int_0^\wp v_k(\nu) d\nu : \ k \ge 1 \right\}\right) \le 2 \int_0^\wp \omega(\nu) d\nu.$$

Lemma 6. *Let $\mathbb{X} \subset \mathcal{X}$ be a bounded set, then there exists a countable set $\mathbb{X}_0 \subset \mathbb{X} \ni \beta(\mathbb{X}) \le 2\beta(\mathbb{X}_0)$.*

Definition 10 (see [42]). *Suppose E^+ is the positive cone of an ordered Banach space $(E, \le)$. Let Ω be the function denoted on the collection of all bounded subset of the Banach space $\mathcal{X}$ by values in E^+; it is known as the MNC on $\mathcal{X}$ iff $\Omega(conv(\nu)) = \Omega(\nu)$ for each bounded subset $\nu \subset \mathcal{X}$, where $conv(\nu)$ denotes the closed convex hull of ν.*

Lemma 7 (see [43]). *Let G be a closed convex subset of a Banach space $\mathcal{X}$ and $0 \in G$. Suppose $F : G \to \mathcal{X}$ is a continuous map that satisfies Mönch's condition, i.e., suppose $G_1 \subset G$ is countable and $G_1 \subset \overline{co}(\{0\} \cup F(G_1)) \implies \overline{G_1}$ is compact. Then, F has a fixed point in G.*

3. Existence of a Mild Solution

This section deals with the existence of a mild solution for the proposed system (1)–(2), using *Mönch's* fixed-point Theorem 7. The following are the essential hypotheses to prove the main theorems.

(H_1) Let $\widetilde{A}$ be the almost sectorial operator of the analytic semigroup $T(\wp)$, $\wp > 0$ in $\mathcal{X}$ such that $\|T(\wp)\| \leq \mathsf{K}_1$ where $\mathsf{K}_1 \geq 0$ is the constant.

(H_2) The function $\mathcal{F} : \mathrm{J} \times B_H \to \mathcal{X}$ satisfies:

(a) Carathéodory condition: $\mathcal{F}(\cdot, z)$ is strongly measurable for all $z \in B_H$, $\mathcal{F}(\wp, \cdot)$ is continuous for a.e. $\wp \in \mathrm{J}$, and $\mathcal{F}(\cdot, \cdot) : [0, S] \to \mathcal{X}$ is strongly measurable;

(b) There exist a constant $0 < \lambda_1 < \lambda$, $m_1 \in L^{\frac{1}{\lambda_1}}(\mathrm{J}, \mathbb{R}^+)$, and nondecreasing continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\mathcal{F}(\wp, z)\| \leq m_1(\wp) f(\wp^{1-\mu+\lambda\mu-\lambda\vartheta}\|z\|)$, $z \in \mathcal{X}$, $\wp \in \mathrm{J}$, where f satisfies $\liminf_{k\to\infty} \frac{\psi(k)}{k} = 0$;

(c) There exist a constant $0 < \lambda_2 < \lambda$ and $e_1 \in L^{\frac{1}{\lambda_2}}(\mathrm{J}, \mathbb{R}^+)$ such that, for all bounded subsets $M \subset \mathcal{X}$, $\beta(\mathcal{F}(\wp, M)) \leq e_1(\wp)\beta(M)$ for a.e. $\wp \in \mathrm{J}$.

(H_3) The function $\mathcal{H} : \mathrm{J} \times B_H \to L_2^0(\mathcal{U}, \mathcal{X})$ satisfies:

(a) Carathéodory condition: $\mathcal{H}(\cdot, z)$ is strongly measurable for all $z \in B_H$, $\mathcal{H}(\wp, \cdot)$ is continuous for a.e. $\wp \in \mathrm{J}$, and $\mathcal{H}(\cdot, \cdot) : [0, S] \to L_2^0(\mathcal{U}, \mathcal{X})$ is strongly measurable;

(b) There exist a constant $0 < \lambda_3 < \lambda$, $m_2 \in L^{\frac{1}{\lambda_3}}(\mathrm{J}, \mathbb{R}^+)$, and nondecreasing continuous function $\hbar : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\|\mathcal{H}(\wp, z)\| \leq m_2(\wp)\hbar(\wp^{1-\mu+\lambda\mu-\lambda\vartheta}\|z\|)$, $z \in \mathcal{X}$, $\wp \in \mathrm{J}$, where $\hbar$ satisfies $\liminf_{k\to\infty} \frac{\sigma(k)}{\sigma} = 0$;

(c) There exist a constant $0 < \lambda_4 < \lambda$ and $e_2 \in L^{\frac{1}{\lambda_4}}(\mathrm{J}, \mathbb{R}^+)$ such that, for all bounded subsets $M \subset \mathcal{X}$, $\beta(\mathcal{H}(\wp, M)) \leq e_2(\wp)\beta(M)$ for a.e. $\wp \in \mathrm{J}$.

(H_4) The function $N : C(\mathrm{J}, \mathcal{X}) \to \mathcal{X}$ is a continuous, compact operator and there exists a value $L_1 > 0$ such that $\|N(z_1) - N(z_2)\| \leq L_1\|z_1 - z_2\|$.

Theorem 4. *If $(H_1) - (H_4)$ holds, then the HF stochastic system (1)–(2) has a unique solution on J provided $\xi(0) \in D(\widetilde{A}^\theta)$ with $\theta > 1 + \vartheta$.*

Proof. Let us assume that the operator $\Psi : B'_H \to B'_H$, defined as

$$\Psi(z(\wp)) = \begin{cases} \Psi_1(\wp),\ (-\infty, 0], \\ \mathsf{S}_{\lambda,\mu}(\wp)\left[\xi(0) - N(z_\wp)\right] + \int_0^\wp (\wp - \nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp - \nu)\mathcal{F}(\nu, z_\nu)d\nu \\ + \int_0^\wp (\wp - \nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp - \nu)\mathcal{H}(\nu, z_\nu)dW(\nu),\ \wp \in \mathrm{J}. \end{cases} \tag{6}$$

For $\Psi_1 \in B_H$, we define $\widehat{\Psi}$ by

$$\widehat{\Psi}(\wp) = \begin{cases} \Psi_1(\wp), & \wp \in (-\infty, 0], \\ \mathsf{S}_{\lambda,\mu}(\wp)\xi(0), & \wp \in \mathrm{J}, \end{cases}$$

then $\widehat{\Psi} \in B'_H$. Let $z(\wp) = \wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v(\wp) + \widehat{\Psi}(\wp)]$, $\infty < \wp \leq d$. It is straightforward to demonstrate that z satisfies (8) iff v satisfies v_0 and

$$\begin{aligned} v(\wp) = &-\mathsf{S}_{\lambda,\mu}(\wp)N\big(\wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\wp + \widehat{\Psi}_\wp]\big) \\ &+ \int_0^\wp (\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)\mathcal{F}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)d\nu \\ &+ \int_0^\wp (\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)\mathcal{H}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)dW(\nu). \end{aligned}$$

Let $B''_H = \{v \in B'_H : v_0 \in B_H\}$. For all $v \in B'_H$,

$$\begin{aligned} \|v\|_b =& \|v_0\|_{B_H} + \sup\{\|v(\nu)\| : 0 \leq \nu \leq d\} \\ =& \sup\{\|v(\nu)\| : 0 \leq \nu \leq d\}. \end{aligned}$$

Therefore, $(B''_H, \|\cdot\|)$ is a Banach space.

For $P > 0$, choose $\mathcal{B}_P = \{v \in B''_H : \|v\|_d \leq P\}$, then $\mathcal{B}_P \subset B''_H$ is uniformly bounded, and for $v \in \mathcal{B}_P$, according to Lemma 1,

$$\begin{aligned} \|v_\wp + \widehat{\Psi}_\wp\|_{B_H} &\leq \|v_\wp\|_{B_H} + \|\widehat{\Psi}\|_{B_H} \\ &\leq l\left(P + \frac{\Gamma(\vartheta)}{\Gamma(\mu(1-\lambda)+\lambda\vartheta)}\kappa_P \wp^{-1+\mu-\lambda\mu+\lambda\vartheta}\right) + \|\Psi_1\|_{B_H} \\ &= P'. \end{aligned}$$

Introducing an operator $\Omega : B''_H \to B''_H$, defined by

$$\Omega v(\wp) = \begin{cases} 0, \ \wp \in (-\infty, 0], \\ -\mathsf{S}_{\lambda,\mu}(\wp)N\big(\wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\wp + \widehat{\Psi}_\wp]\big) \\ + \int_0^\wp (\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)\mathcal{F}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)d\nu \\ + \int_0^\wp (\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)\mathcal{H}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)dW(\nu), \quad \wp \in \mathrm{J}. \end{cases}$$

Next, we show that Ω has a fixed point.

Step 1: We have to prove that there exists a positive value P such that $\Omega(\mathcal{B}_P(\mathrm{J})) \subseteq \mathcal{B}_P(\mathrm{J})$. Assume the statement is false, i.e., for all $P > 0$, there exists $v^P \in \mathcal{B}_P(\mathrm{J})$, but $\Omega(v^P)$ is not in $\mathcal{B}_P(\mathrm{J})$, that is,

$$
\begin{aligned}
E\left\|v^P\right\|^2 \leq P < E&\left\| \sup_{\wp\in[0,d]} \wp^{1-\mu+\lambda\mu-\lambda\vartheta}\big(\Omega v^P(\wp)\big)\right\|^2 \\
&\leq \sup_{\wp\in[0,d]} E\left\|\wp^{1-\mu+\lambda\mu-\lambda\vartheta}\Big[-\mathsf{S}_{\lambda,\mu}(\wp)N\big(\wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v^P_\wp+\widehat{\Psi}_\wp]\big)\right. \\
&\quad+\int_0^\wp(\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu \\
&\quad\left.+\int_0^\wp(\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v^P_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\Big]\right\|^2 \\
&\leq 3d^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\Big[E\left\|\mathsf{S}_{\lambda,\mu}(\wp)N\big(\wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v^P_\wp+\widehat{\Psi}_\wp]\big)\right\|^2 \\
&\quad+E\left\|\int_0^\wp(\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v^P_\nu+\widehat{\Psi}_\nu]\big)d\nu\right\|^2 \\
&\quad+E\left\|\int_0^\wp(\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v^P_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\right\|^2\Big] \\
&\leq 3d^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\Big[\|\mathsf{S}_{\lambda,\mu}(\wp)\|^2\big[L_1^2\|v^P_\wp+\widehat{\Psi}_\wp\|^2+\|N(0)\|^2\big] \\
&\quad+\int_0^\wp(\wp-\nu)^{2(\lambda-1)}\|\mathsf{Q}_\lambda(\wp-\nu)\|^2E\|\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v^P_\nu+\widehat{\Psi}_\nu]\big)d\nu\|^2 \\
&\quad+\int_0^\wp(\wp-\nu)^{2(\lambda-1)}\|\mathsf{Q}_\lambda(\wp-\nu)\|^2E\|\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v^P_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\|^2\Big] \\
&\leq 3d^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\Big[\|\mathsf{S}_{\lambda,\mu}(\wp)\|^2\big[L_1^2\|v^P_\wp+\widehat{\Psi}_\wp\|^2+\|N(0)\|^2\big] \\
&\quad+\kappa_p^2\int_0^\wp(\wp-\nu)^{2(\lambda-1)}(\wp-\nu)^{2(-\lambda+\lambda\vartheta)}m_1^2(d)f^2(P')d\nu \\
&\quad+Tr(Q)\kappa_p^2\int_0^\wp(\wp-\nu)^{2(\lambda-1)}(\wp-\nu)^{2(-\lambda+\lambda\vartheta)}m_2^2(d)\hbar^2(P')d\nu\Big] \\
&\leq 3d^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\Big[\left(\frac{\Gamma(\vartheta)}{\Gamma(\mu(1-\lambda)+\lambda\vartheta)}\right)^2\kappa_p^2d^{2(-1+\mu-\lambda\mu+\lambda\vartheta)}\big[L_1^2P'^2+\|N(0)\|^2\big] \\
&\quad+\left(\frac{d^{\lambda\vartheta}}{\lambda\vartheta}\right)^2\kappa_p^2m_1^2(d)f^2(P')+Tr(Q)\left(\frac{d^{\lambda\vartheta}}{\lambda\vartheta}\right)^2\kappa_p^2m_2^2(d)\hbar^2(P')\Big] \\
&\leq 3\kappa_p^2d^{2(1-\mu+\lambda\mu-\lambda\vartheta)}M^*,
\end{aligned}
$$

where

$$
\begin{aligned}
M^*=&\left(\frac{\Gamma(\vartheta)}{\Gamma(\mu(1-\lambda)+\lambda\vartheta)}\right)^2d^{2(-1+\mu-\lambda\mu+\lambda\vartheta)}\big[L_1^2P'^2+\|N(0)\|^2\big] \\
&+\left(\frac{d^{\lambda\vartheta}}{\lambda\vartheta}\right)^2\kappa_p^2m_1^2(d)f^2(P')+Tr(Q)\left(\frac{d^{\lambda\vartheta}}{\lambda\vartheta}\right)^2m_2^2(d)\hbar^2(P').
\end{aligned}
$$

The above inequality is divided by P and applying the limit as $P\to\infty$, we obtain $1\leq 0$, which is the contradiction. Therefore, $\Omega(\mathcal{B}_P(\mathrm{J}))\subseteq\mathcal{B}_P(\mathrm{J})$.

Step 2: The operator Ω is continuous on $\mathcal{B}_P(\mathrm{J})$ since Ω maps $\mathcal{B}_P(\mathrm{J})$ into $\mathcal{B}_P(\mathrm{J})$. For any $v^k,\ v\in\mathcal{B}_P(\mathrm{J}),\ k=0,1,2,\ldots$ such that $\lim_{k\to\infty}v^k=v$, we have $\lim_{k\to\infty}v^k(\wp)=v(\wp)$ and $\lim_{k\to\infty}\wp^{1-\mu+\lambda\mu-\lambda\vartheta}v^k(\wp)=\wp^{1-\mu+\lambda\mu-\lambda\vartheta}v(\wp)$.

By (H_2),

$$\begin{aligned}\mathcal{F}(\wp, z_k(\wp)) &= \mathcal{F}\big(\wp, \wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v^k(\wp) + \widehat{\Upsilon}(\wp)]\big) \to \mathcal{F}\big(\wp, \wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v(\wp) + \widehat{\Upsilon}(\wp)]\big) \\ &= \mathcal{F}(\wp, z(\wp)) \text{ as } k \to \infty.\end{aligned}$$

Take

$$F_k(\nu) = \mathcal{F}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu^k + \widehat{\Upsilon}_\nu]\big) \text{ and } F(\nu) = \mathcal{F}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Upsilon}_\nu]\big).$$

Then, we may derive the following using hypotheses (H_2) and Lebesgue's dominated convergence principle.

$$\int_0^\wp (\wp-\nu)^{2(\lambda-1)} \|\mathrm{Q}_\lambda(\wp-\nu)\|^2 E\|F_k(\nu) - F(\nu)\|^2 d\nu \to 0 \text{ as } k \to \infty,\ \wp \in \mathrm{J}. \tag{7}$$

By (H_3),

$$\begin{aligned}\mathcal{H}(\wp, z_k(\wp)) &= \mathcal{H}\big(\wp, \wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v^k(\wp) + \widehat{\Upsilon}(\wp)]\big) \to \mathcal{H}\big(\wp, \wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v(\wp) + \widehat{\Upsilon}(\wp)]\big) \\ &= \mathcal{H}(\wp, z(\wp)) \text{ as } k \to \infty.\end{aligned}$$

Take

$$H_k(\nu) = \mathcal{H}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu^k + \widehat{\Upsilon}_\nu]\big) \text{ and } H(\nu) = \mathcal{H}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Upsilon}_\nu]\big).$$

Then, from hypotheses (H_3) and Lebesgue's dominated convergence theorem, we arrive at

$$\int_0^\wp (\wp-\nu)^{2(\lambda-1)} \|\mathrm{Q}_\lambda(\wp-\nu)\|^2 E\|H_k(\nu) - H(\nu)\|^2 dW(\nu) \to 0 \text{ as } k \to \infty,\ \wp \in \mathrm{J}. \tag{8}$$

Take $\mathcal{N}_k(\wp) = N\big(\wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\wp^k + \widehat{\Upsilon}_\wp]\big)$ and $\mathcal{N}(\wp) = N\big(\wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\wp + \widehat{\Upsilon}_\wp]\big)$, from (H_4), we have

$$E\|\mathcal{N}_k(\wp) - \mathcal{N}(\wp)\|^2 \to 0 \text{ as } k \to \infty. \tag{9}$$

Now,

$$\begin{aligned}E\big\|\Omega v^k - \Omega v\big\|_d^2 \leq{}& 3\left(\frac{\Gamma(\vartheta)}{\Gamma(\mu(1-\lambda)+\lambda\vartheta)}\right)^2 \kappa_p^2 d^{2(-1+\mu-\lambda\mu+\lambda\vartheta)} E\big\|\mathcal{N}_k(\wp) - \mathcal{N}(\wp)\big\|^2 \\ &+ 3\kappa_p^2\left(\frac{d^{\lambda\vartheta}}{\lambda\vartheta}\right)^2 E\big\|F_k(\nu) - F(\nu)\big\|^2 d\nu \\ &+ 3Tr(Q)\kappa_p^2\left(\frac{d^{\lambda\vartheta}}{\lambda\vartheta}\right)^2 E\big\|H_k(\nu) - H(\nu)\big\|^2 d\nu.\end{aligned}$$

Using (7)–(9), we obtain

$$E\big\|\Omega v^k - \Omega v\big\|_d^2 \to 0 \text{ as } k \to \infty.$$

Therefore, Ω is continuous on $\mathcal{B}_P$.

Step 3: After that, we have to demonstrate that Ω is equicontinuous.

For $z \in \mathcal{B}_P(\mathrm{J})$, and $0 \leq \wp_1 < \wp_2 \leq d$, we have

$$
\begin{aligned}
&E\big\|\Omega z(\wp_2)-\Omega z(\wp_1)\big\|^2\\
&\quad=E\bigg\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\bigg(-\mathsf{S}_{\lambda,\mu}(\wp_2)N\big(\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_2}+\widehat{\Psi}_{\wp_2}]\big)\\
&\qquad+\int_0^{\wp_2}(\wp_2-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\\
&\qquad+\int_0^{\wp_2}(\wp_2-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\bigg)\\
&\qquad-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\bigg(-\mathsf{S}_{\lambda,\mu}(\wp_1)N\big(\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_1}+\widehat{\Psi}_{\wp_1}]\big)\\
&\qquad+\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_1-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\\
&\qquad+\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_1-\nu)\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\bigg)\bigg\|^2\\
&\quad\le 6E\bigg\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\mathsf{S}_{\lambda,\mu}(\wp_2)N\big(\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_2}+\widehat{\Psi}_{\wp_2}]\big)\\
&\qquad-\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\mathsf{S}_{\lambda,\mu}(\wp_2)N\big(\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_1}+\widehat{\Psi}_{\wp_1}]\big)\bigg\|^2\\
&\qquad+6E\bigg\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\mathsf{S}_{\lambda,\mu}(\wp_2)N\big(\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_1}+\widehat{\Psi}_{\wp_1}]\big)\\
&\qquad-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\mathsf{S}_{\lambda,\mu}(\wp_1)N\big(\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_1}+\widehat{\Psi}_{\wp_1}]\big)\bigg\|^2\\
&\qquad+9E\bigg\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_2-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\\
&\qquad-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\bigg\|^2\\
&\qquad+9E\bigg\|\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\\
&\qquad-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_1-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\bigg\|^2\\
&\qquad+9E\bigg\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int_{\wp_1}^{\wp_2}(\wp_2-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\bigg\|^2\\
&\qquad+9E\bigg\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_2-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\\
&\qquad-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\bigg\|^2\\
&\qquad+9E\bigg\|\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\\
&\qquad-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_1-\nu)\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\bigg\|^2\\
&\qquad+9E\bigg\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int_{\wp_1}^{\wp_2}(\wp_2-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\bigg\|^2\\
&\quad\le\sum_{i=1}^{8}I_i.
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= 6E\Big\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\mathsf{S}_{\lambda,\mu}(\wp_2)N\big(\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_2}+\widehat{\Psi}_{\wp_2}]\big)\\
&\quad-\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\mathsf{S}_{\lambda,\mu}(\wp_2)N\big(\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_1}+\widehat{\Psi}_{\wp_1}]\big)\Big\|^2\\
&\le 6\wp_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\Big\|\mathsf{S}_{\lambda,\mu}(\wp_2)\Big\|^2E\Big\|N\big(\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_2}+\widehat{\Psi}_{\wp_2}]\big)-N\big(\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_1}+\widehat{\Psi}_{\wp_1}]\big)\Big\|^2.
\end{aligned}
$$

From hypotheses (H_3) and (8), we obtain that I_1 tends to 0 as $\wp_2 \to \wp_1$.

$$
\begin{aligned}
I_2 &= 6E\Big\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\mathsf{S}_{\lambda,\mu}(\wp_2)N\big(\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_1}+\widehat{\Psi}_{\wp_1}]\big)\\
&\quad-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\mathsf{S}_{\lambda,\mu}(\wp_1)N\big(\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_1}+\widehat{\Psi}_{\wp_1}]\big)\Big\|^2\\
&\le 6\Big\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\mathsf{S}_{\lambda,\mu}(\wp_2)-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\mathsf{S}_{\lambda,\mu}(\wp_1)\Big\|^2E\Big\|N\big(\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\wp_1}+\widehat{\Psi}_{\wp_1}]\big)\Big\|^2.
\end{aligned}
$$

By the strong continuity of $\mathsf{S}_{\lambda,\mu}(\wp)$ and (H_4), we get $I_2 \to 0$ as $\wp_2 \to \wp_1$.

$$
\begin{aligned}
I_3 &= 9E\Big\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_2-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\\
&\quad-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\Big\|^2\\
&\le 9E\Big\|\Big(\int_0^{\wp_1}\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_2-\nu)^{\lambda-1}-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_1-\nu)^{\lambda-1}\Big)\\
&\qquad\qquad\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\Big\|^2\\
&\le 9\Big\|\int_0^{\wp_1}\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_2-\nu)^{\lambda-1}-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_1-\nu)^{\lambda-1}\Big\|^2\\
&\qquad\qquad\Big\|\mathsf{Q}_\lambda(\wp_2-\nu)\Big\|^2E\Big\|\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)\Big\|^2d\nu\\
&\le 9\kappa_p^2\Big\|\int_0^{\wp_1}\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_2-\nu)^{\lambda-1}-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_1-\nu)^{\lambda-1}\Big\|^2\\
&\qquad\qquad(\wp_2-\nu)^{2(-\lambda+\lambda\vartheta)}m_1^2(d)f^2(P')d\nu.
\end{aligned}
$$

This implies $I_3 \to 0$ as $\wp_2 \to \wp_1$.

$$
\begin{aligned}
I_4 &= 9E\Big\|\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\\
&\quad-\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_1-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu\Big\|^2\\
&\le 9E\Big\|\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\wp_1}(\wp_1-\nu)^{\lambda-1}\big[\mathsf{Q}_\lambda(\wp_2-\nu)-\mathsf{Q}_\lambda(\wp_1-\nu)\big]
\end{aligned}
$$

$$\mathcal{F}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)d\nu\Big\|^2$$
$$\leq 9\wp_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)} \int_0^{\wp_1} (\wp_1-\nu)^{2(\lambda-1)} \Big\| \big[\mathsf{Q}_\lambda(\wp_2-\nu) - \mathsf{Q}_\lambda(\wp_1-\nu)\big] \Big\|^2 E\Big\|\mathcal{F}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)\Big\|^2 d\nu$$
$$\leq 9\wp_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)} \int_0^{\wp_1} (\wp_1-\nu)^{2(\lambda-1)} \Big\| \big[\mathsf{Q}_\lambda(\wp_2-\nu) - \mathsf{Q}_\lambda(\wp_1-\nu)\big] \Big\|^2 m_1^2(d) f^2(P')d\nu.$$

Since $\mathsf{Q}_\lambda(\wp)$ is uniformly continuous in operator norm topology, we obtain $I_4 \to 0$ as $\wp_2 \to \wp_1$.

$$I_5 = 9E\Big\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta} \int_{\wp_1}^{\wp_2} (\wp_2-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{F}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)d\nu\Big\|^2$$
$$\leq 9\wp_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)} \int_{\wp_1}^{\wp_2} (\wp_2-\nu)^{2(\lambda-1)} \|\mathsf{Q}_\lambda(\wp_2-\nu)\|^2 E\|\mathcal{F}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)\|^2 d\nu$$
$$\leq 9\kappa_p^2\wp_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)} \int_{\wp_1}^{\wp_2} (\wp_2-\nu)^{2(\lambda\vartheta-1)} m_1^2(d) f^2(P')d\nu.$$

Integrating and $\wp_2 \to \wp_1 \Longrightarrow I_5 = 0$.

$$I_6 = 9E\Big\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta} \int_0^{\wp_1} (\wp_2-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{H}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)dW(\nu)$$
$$- \wp_1^{1-\mu+\lambda\mu-\lambda\vartheta} \int_0^{\wp_1} (\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{H}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)dW(\nu)\Big\|^2$$
$$\leq 9E\Big\|\Big(\int_0^{\wp_1} \wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_2-\nu)^{\lambda-1} - \wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_1-\nu)^{\lambda-1}\Big) \mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{H}\big(\nu, \nu^{(1+\lambda\vartheta)(1-\mu)}[v_\nu + \widehat{\Psi}_\nu]\big)dW(\nu)\Big\|^2$$
$$\leq 9\Big\|\int_0^{\wp_1} \wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_2-\nu)^{\lambda-1} - \wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_1-\nu)^{\lambda-1}\Big\|^2 \Big\|\mathsf{Q}_\lambda(\wp_2-\nu)\Big\|^2 E\Big\|\mathcal{H}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)\Big\|^2 dW(\nu)$$
$$\leq 9\kappa_p^2 Tr(Q)\Big\|\int_0^{\wp_1} \wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_2-\nu)^{\lambda-1} - \wp_1^{1-\mu+\lambda\mu-\lambda\vartheta}(\wp_1-\nu)^{\lambda-1}\Big\|^2 (\wp_2-\nu)^{2(-\lambda+\lambda\vartheta)} m_2^2(d)\hbar^2(P')d\nu.$$

This implies $I_6 \to 0$ as $\wp_2 \to \wp_1$.

$$I_7 = 9E\Big\|\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta} \int_0^{\wp_1} (\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{H}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)dW(\nu)$$
$$- \wp_1^{1-\mu+\lambda\mu-\lambda\vartheta} \int_0^{\wp_1} (\wp_1-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_1-\nu)\mathcal{H}\big(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu + \widehat{\Psi}_\nu]\big)dW(\nu)\Big\|^2$$
$$\leq 9E\Big\|\wp_1^{1-\mu+\lambda\mu-\lambda\vartheta} \int_0^{\wp_1} (\wp_1-\nu)^{\lambda-1}\big[\mathsf{Q}_\lambda(\wp_2-\nu) - \mathsf{Q}_\lambda(\wp_1-\nu)\big]$$

$$\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\Big\|^2$$

$$\leq 9Tr(Q)\wp_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int_0^{\wp_1}(\wp_1-\nu)^{2(\lambda-1)}\Big\|\big[\mathsf{Q}_\lambda(\wp_2-\nu)-\mathsf{Q}_\lambda(\wp_1-\nu)\big]\Big\|^2$$

$$E\Big\|\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)\Big\|^2 d\nu$$

$$\leq 9Tr(Q)\wp_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int_0^{\wp_1}(\wp_1-\nu)^{2(\lambda-1)}\Big\|\big[\mathsf{Q}_\lambda(\wp_2-\nu)-\mathsf{Q}_\lambda(\wp_1-\nu)\big]\Big\|^2 m_2^2(d)\hbar^2(P')d\nu.$$

Since $\mathsf{Q}_\lambda(\wp)$ is uniformly continuous in operator norm topology, we obtain $I_7 \to 0$ as $\wp_2 \to \wp_1$.

$$I_8 = 9E\Big\|\wp_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int_{\wp_1}^{\wp_2}(\wp_2-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp_2-\nu)\mathcal{H}\big(\nu,\nu^{(1+\lambda\vartheta)(1-\mu)}[v_\nu+\widehat{\Psi}_\nu]\big)dW(\nu)\Big\|^2$$

$$\leq 9Tr(Q)\wp_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int_{\wp_1}^{\wp_2}(\wp_2-\nu)^{2(\lambda-1)}\|\mathsf{Q}_\lambda(\wp_2-\nu)\|^2$$

$$E\|\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)\|^2 d\nu$$

$$\leq 9Tr(Q)\kappa_p^2\wp_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int_{\wp_1}^{\wp_2}(\wp_2-\nu)^{2(\lambda\vartheta-1)}m_2^2(d)\hbar^2(P')d\nu.$$

Integrating, we get $\wp_2 \to \wp_1 \Longrightarrow I_8 = 0$.
Therefore, Ω is equicontinuous on J.
Step 4: The *Mönch* conditions are true.
Consider $\Omega = \Omega_1 + \Omega_2 + \Omega_3$, where

$$\Omega_1 v(\wp) = -\mathsf{S}_{\lambda,\mu}(\wp)N\big(\wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\wp+\widehat{\Psi}_\wp]\big),$$
$$\Omega_2 v(\wp) = \int_0^{\wp}(\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)d\nu,$$
$$\Omega_3 v(\wp) = \int_0^{\wp}(\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)\mathcal{H}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu+\widehat{\Psi}_\nu]\big)dW(\nu).$$

Assume that $G_1 \subseteq \mathfrak{B}_P$ is countable and $G_1 \subset \overline{co}\big(\{0\}\cup F(G_1)\big)$. We show that β, the Hausdorff MNC, has the property $\beta(G_1) = 0$. Without loss of generality, we may suppose $G_1 = \{v^k\}_{k=1}^{\infty}$. Since $\Omega(G_1)$ is equicontinuous on J as well.
Applying Lemma 5, and the assumptions $(H_2)(c)$, $(H_3)(c)$, and (H_4), we get

$$\beta\big(\{\Omega_1 v^k(\wp)\}_{k=1}^{\infty}\big) \leq \beta\big\{-\mathsf{S}_{\lambda,\mu}(\wp)N\big(\wp^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\wp^k+\widehat{\Psi}_\wp]\big)\big\}_{k=1}^{\infty} = 0.$$

Since N is compact, then $\mathsf{S}_{\lambda,\mu}(\wp)$ is relatively compact.

$$\beta\big(\{\Omega_2 v^k(\wp)\}_{k=1}^{\infty}\big) \leq \beta\Big\{\int_0^{\wp}(\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)\mathcal{F}\big(\nu,\nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_\nu^k+\widehat{\Psi}_\nu]\big)d\nu\Big\}_{k=1}^{\infty}$$
$$\leq 2\int_0^{\wp}(\wp-\nu)^{\lambda-1}\mathsf{Q}_\lambda(\wp-\nu)e_1(\nu)\sup_{-\infty<\theta\leq 0}\beta\big(\{v_\wp^k(\theta)\}_{k=1}^{\infty}\big)d\nu$$
$$\leq 2\left(\frac{d^{\lambda\vartheta}}{\lambda\vartheta}\right)\|e_1\|_{L^{\frac{1}{\lambda_2}}(\mathrm{J},\mathbb{R}^+)}\sup_{-\infty<\theta\leq 0}\beta\big(\{v_\wp^k(\theta)\}_{k=1}^{\infty}\big),$$

$$\beta\big(\{\Omega_3 v^k(\wp)\}_{k=1}^{\infty}\big) \leq \beta\bigg\{ \int_0^{\wp} (\wp-\nu)^{\lambda-1} \mathrm{Q}_{\lambda}(\wp-\nu)\mathcal{H}(\nu, \nu^{1-\mu+\lambda\mu-\lambda\vartheta}[v_{\nu}^k + \widehat{\Psi}_{\nu}])dW(\nu)\bigg\}_{k=1}^{\infty}$$
$$\leq 2Tr(Q)\int_0^{\wp} (\wp-\nu)^{\lambda-1}\mathrm{Q}_{\lambda}(\wp-\nu)e_2(\nu) \sup_{-\infty<\theta\leq 0} \beta\big(\{v_{\wp}^k(\theta)\}_{k=1}^{\infty}\big)d\nu$$
$$\leq 2Tr(Q)\bigg(\frac{d^{\lambda\vartheta}}{\lambda\vartheta}\bigg)\|e_2\|_{L^{\frac{1}{\lambda_4}}(\mathrm{J},\mathbb{R}^+)} \sup_{-\infty<\theta\leq 0} \beta\big(\{v_{\wp}^k(\theta)\}_{k=1}^{\infty}\big).$$

Thus, we have

$$\beta\big(\{\Omega v^k(\wp)\}_{k=1}^{\infty}\big) \leq \beta\big(\{\Omega_1 v^k(\wp)\}_{k=1}^{\infty}\big) + \beta\big(\{\Omega_2 v^k(\wp)\}_{k=1}^{\infty}\big) + \beta\big(\{\Omega_3 v^k(\wp)\}_{k=1}^{\infty}\big)$$
$$\leq 2\bigg(\frac{d^{\lambda\vartheta}}{\lambda\vartheta}\bigg)\|e_1\|_{L^{\frac{1}{\lambda_2}}(\mathrm{J},\mathbb{R}^+)} \sup_{-\infty<\theta\leq 0} \beta\big(\{v_{\wp}^k(\theta)\}_{k=1}^{\infty}\big)$$
$$+ 2Tr(Q)\bigg(\frac{d^{\lambda\vartheta}}{\lambda\vartheta}\bigg)\|e_2\|_{L^{\frac{1}{\lambda_4}}(\mathrm{J},\mathbb{R}^+)} \sup_{-\infty<\theta\leq 0} \beta\big(\{v_{\wp}^k(\theta)\}_{k=1}^{\infty}\big)$$
$$\leq 2\big[\|e_1\|_{L^{\frac{1}{\lambda_2}}(\mathrm{J},\mathbb{R}^+)} + Tr(Q)\|e_2\|_{L^{\frac{1}{\lambda_4}}(\mathrm{J},\mathbb{R}^+)}\big]\beta\big(\{v^k(\wp)\}_{k=1}^{\infty}\big)$$
$$\leq M^*\beta\big(\{v^k(\wp)\}_{k=1}^{\infty}\big),$$

where $M^* = 2\big[\|e_1\|_{L^{\frac{1}{\lambda_2}}(\mathrm{J},\mathbb{R}^+)} + Tr(Q)\|e_2\|_{L^{\frac{1}{\lambda_4}}(\mathrm{J},\mathbb{R}^+)}\big]$.

Since G_1 and $\Omega(G_1)$ are equicontinuous for every J, it appears according to Lemma 5 that the inequality states that $\beta(\Omega G_1) \leq M^*\beta(G_1)$.

As a result, given the condition of *Mönch*'s technique, we obtain

$$\beta(G_1) \leq \beta\big(\overline{co}\{0\} \cup \Omega(G_1)\big) = \beta(\Omega G_1) \leq M^*\beta G_1.$$

Given that $M^* < 1$, we obtain $\beta(G_1) = 0$. Thus, G_1 is relatively compact. We know that Ω has a fixed point v in G_1 according to Lemma 7. The proof is completed. □

4. Example

Examine the HF stochastic differential system containing the nonlocal condition of the form

$$\begin{cases} D_{0^+}^{\frac{2}{3},\mu} z(\wp,\tau) = z_{\tau\tau}(\wp,\tau) + \gamma\Big(\wp, \int_{\infty}^{\wp} \chi_1(\nu-\wp)z(\wp,\tau)d\nu\Big) \\ \qquad\qquad + \chi\Big(\wp, \int_{\infty}^{\wp} \chi_2(\nu-\wp)z(\wp,\tau)dW(\nu)\Big), \\ z(\wp,0) = z(\wp,\pi) = 0, \ \wp \in \mathrm{J}, \\ I_{0^+}^{(1-\frac{2}{3})(1-\mu)} z(0,\tau) + \int_0^{\pi} \mathcal{N}(\alpha,\tau)z(\wp,\tau)d\alpha = z(0,\tau), \ \tau \in [0,\pi], \ \wp \in (-\infty,0), \end{cases} \tag{10}$$

where $D_{0^+}^{\frac{2}{3},\mu}$ denotes the HFD of order $\lambda = 2/3$, type μ and χ, χ_1, ρ and $\mathcal{N}$ are the required functions. Assume $W(\wp)$ is a one-dimensional normalized Brownian movement in $\mathcal{X}$ denoted by the smoothed probability area $(\Lambda, \mathscr{F}, P)$ and with $\|\cdot\|_{\mathcal{X}}$ to compose the system (10) in the abstract form of (1)–(2). To change this system into an abstract structure, let $\mathcal{X} = L^2[0,\pi]$ and $\widetilde{A} : D(\widetilde{A}) \subset \mathcal{X} \to \mathcal{X}$ is defined as $\widetilde{A}x = x'$ with

$$D(\widetilde{A}) = \big\{x \in \mathcal{X} : x, x' \text{ are absolutely continuous}, \ x'' \in \mathcal{X}, \ x(0) = x(\pi) = 0\big\}$$

and

$$\widetilde{A}x = \sum_{k=1}^{\infty} k^2 \langle x, \varrho_k\rangle \varrho_k, \ \varrho \in D(\widetilde{A}),$$

where $\varrho_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx)$, $k \in \mathbb{N}$ is the orthogonal set of eigenvectors of $\widetilde{A}$.

We know that $\widetilde{A}$ is the almost sectorial operator of the analytic semigroup $\{T(\wp),\ \wp \geq 0\}$ in $\mathcal{X}$, $T(\wp)$ is a noncompact semigroup on $\mathcal{X}$ with $\mu(T(\wp)B) \leq \mu(B)$, where μ denotes the Hausdorff MNC and there exists a constant $\mathcal{K}_1 \geq 1$ satisfying $\sup_{\wp\in \mathrm{J}} \|T(\wp)\| \leq \mathcal{K}_1$.

Define, $\mathcal{F} : \mathrm{J} \times B_H \to \mathcal{X}$, $\mathcal{H} : \mathrm{J} \times B_H \to L_2^0(\mathcal{U}, \mathcal{X})$, and $N : B_H \to \mathcal{X}$ are the appropriate functions, which satisfy the hypotheses $(H_1) - (H_4)$,

$$\mathcal{F}(\wp, z_\wp)(\tau), = \gamma\left(\wp, \int_\infty^\wp \chi_1(\nu - \wp)z(\wp, \tau)d\nu\right),$$
$$\mathcal{H}(\wp, z_\wp)(\tau), = \chi\left(\wp, \int_\infty^\wp \chi_2(\nu - \wp)z(\wp, \tau)dW(\nu)\right),$$
$$N(z_\wp)(\tau) = \int_0^\pi \mathcal{N}(\alpha, \tau)z(\wp, \tau)d\alpha.$$

We established some acceptance criteria for the aforementioned functions to demonstrate all of Theorem 4's assumptions, and we confirmed that the HF stochastic system (1)–(2) had a unique mild solution.

5. Conclusions

In this study, we concentrated on the existence of a mild solution of HF stochastic differential equations using nonlocal conditions and delay via an almost sectorial operator. The essential results were demonstrated by employing the findings and concepts belonging to almost sectorial operators, fractional calculus, the measure of noncompactness, and the fixed-point method. Finally, to explain the principle, we offered an example. In the years ahead, we will study the exact controllability of HF stochastic differential systems with infinite delay through almost sectorial operators by using the fixed-point approach.

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Abbreviations

The following abbreviations are used in this manuscript:

HF	Hilfer fractional
HFD	Hilfer fractional derivative
HFDEs	Hilfer fractional differential equations
MNC	Measure of noncompactness
SDEs	Stochastic differential equations
R-L	Riemann-Liouville

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