mathematics

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Equiareal Parameterization of Triangular Bézier Surfaces

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Abstract: Parameterization is the key property of a parametric surface and significantly affects many kinds of applications. To improve the quality of parameterization, equiareal parameterization minimizes the equiareal energy, which is presented as a measure to describe the uniformity of iso-parametric curves. With the help of the binary Möbius transformation, the equiareal parameterization is extended to the triangular Bézier surface on the triangular domain for the first time. The solution of the corresponding nonlinear minimization problem can be equivalently converted into solving a system of bivariate polynomial equations with an order of three. All the exact solutions of the equations can be obtained, and one of them is chosen as the global optimal solution of the minimization problem. Particularly, the coefficients in the system of equations can be explicitly formulated from the control points. Equiareal parameterization keeps the degree, control points, and shape of the triangular Bézier surface unchanged. It improves the distribution of iso-parametric curves only. The iso-parametric curves from the new expression are more uniform than the original one, which is displayed by numerical examples.

Keywords: equiareal parameterization; equiareal energy; binary Möbius transformation; triangular Bézier surface; computer-aided geometric design

MSC: 65D17; 65D18; 68U05; 68U07

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Citation: Chen, J.; Kong, X.; Xu, H. Equiareal Parameterization of Triangular Bézier Surfaces. *Mathematics* **2022**, *10*, 4620. https://doi.org/10.3390/math10234620

Academic Editor: Jakub Nalepa

Received: 1 November 2022
Accepted: 4 December 2022
Published: 6 December 2022

1. Introduction

Parametric curves and surfaces play an increasingly important role in computer-aided geometric design [1]. The quality of their parameterization has a profound impact on their subsequent manipulations. Many applications of geometry processing strongly depend on the parameterization. Therefore, many scholars reparameterize the curves and surfaces without changing their shapes [2–10]. If the points or the iso-parametric curves are far from being uniform, these reparameterizations make them evenly distribute as far as possible with evenly distributed parameter values in the parameter domain.

For parametric curves, good parameterization can effectively improve the efficiency of interpolation [11], intersection [12], and real-time compensation of CNC [13]. Based on the intuitive geometric significance of the arc length parameter, [2,3] used the Möbius transformation to optimally parameterize the Bézier curve by minimizing the L2 norm between the parametric flow and the unit speed. The authors of [4] obtained the optimal parameterization with the piecewise polynomial/rational transformation. The explicit analytical solution for an optimal parameterized rational quadratic Bézier curve was obtained in [5]. In addition, the uniformity of the angular speed was also developed to optimize the quality of the parameterization [6].

Similar to the curves, the parameterization of surfaces is important in many surface algorithms such as rendering [14], tessellation [15], sampling [16], texture mapping [17], and so on. Based on the earlier work on curves, the reparameterization has been extensively applied to the tensor-product surface on the rectangular domain in the past decade. Using rational bilinear transformation, rational Bézier surfaces were reparameterized to improve the uniformity and orthogonality of iso-parametric curves with the cost of degree elevation [7]. Then, a new quality measure for the parameterization, called equiareal energy, was introduced to reparameterize the NURBS surface with the Möbius transformation [8] and composite Möbius transformations [9]. Recently, the equiareal energy was also used to generate a hierarchical NURBS surface for an area-preserving parameterization with the freeform transformation [10].

However, the existing reparameterization methods of surfaces are only for tensor-product surfaces, and there are no relevant results for triangular Bézier surfaces. The triangular Bézier surface is also an important tool in geometric modeling. It can effectively deal with the interpolation of scattered data points and can be widely used in any topological structure [1,18,19]. The parameterization of the triangular Bézier surface also greatly affects the results of many surface algorithms. A better parameterization is also widely desired. The qualities of the parameterization and the uniformity of iso-parametric curve distribution are compared in Figure 1. Therefore, the reparameterization method of triangular Bézier surfaces is as important as the existing work on tensor-product surfaces.

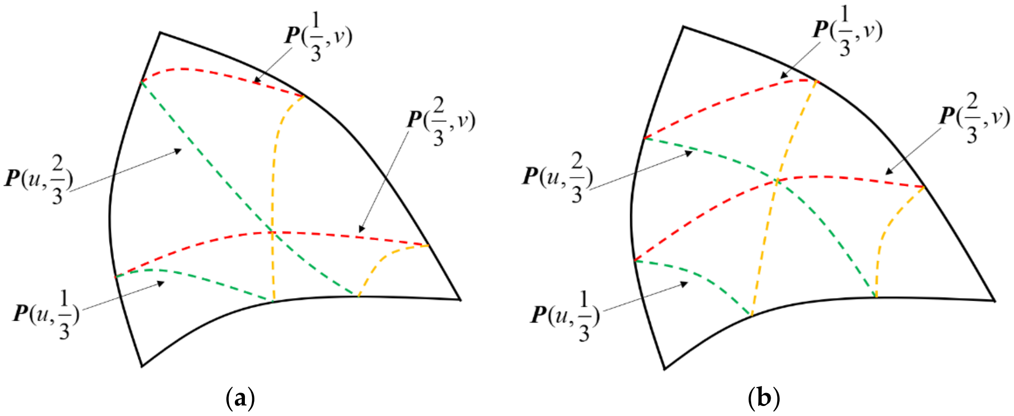


Figure 1. Different parameterizations have different iso-parametric curve distributions: (**a**) poor and (**b**) good parameterization.

The aim of this study was to present the equiareal parameterization of the triangular Bézier surface for the first time, such that the equiareal energy is minimized. The equiareal energy measures the deviation from the uniformity of the iso-parameter curves. The minimum value of the energy function results in the equiareal parameterization. Different from the tensor-product surfaces, it is difficult to extend the same reparameterization methods from curves to triangular Bézier surfaces. This is mainly because the expression of triangular Bézier surfaces is completely different from that of curves or tensor-product surfaces [1]. The equiareal energy of triangular Bézier surfaces is more complicated than that of tensor-product surfaces.

In this paper, we try to make the reparameterization, algorithm procedure, and expression of the resulting surface uncomplicated. The equiareal energy is simplified to a definite integral of a polynomial with the help of the binary Möbius transformation, which can modify the distribution of iso-parameter curves and maintain the shape of the triangular Bézier surface. Thus, the nonlinear optimization problem, which minimizes the equiareal energy, is transformed into solving a system of bivariate polynomial equations with an order of three. Finally, all the exact solutions of the equations can be obtained. One of the exact solutions is chosen as the global optimal solution of the nonlinear optimization problem, and the corresponding equiareal parameterization is obtained.

In summary, the process of the equiareal parameterization is uncomplicated. The main contributions of this paper are as follows:

- With the help of the binary Möbius transformation, a nonlinear optimization problem is explicitly formulated to improve the equiareality of the triangular Bézier surface.

- The global optimal solution of the optimization problem is obtained by solving the bivariate polynomial equations with an order of three.
- That global optimal solution directly leads to the equiareal parameterization.

The rest of this paper is organized as follows: In Section 2, we describe the triangular Bézier surfaces and the binary Möbius transformation. In Section 3, we introduce the equiareal parameterization by the minimization of the equiareal energy. In Sections 4 and 5, we show the explicit expressions of the equiareal energy and equations. In Section 6, using the exact solutions of the equations, we obtain the equiareal parameterization from the global optimal solution of the minimization problem. Several examples present more uniform iso-parametric curves across the triangular Bézier surfaces in Section 7. Section 8 concludes the paper.

2. Triangular Bézier Surfaces and the Binary Möbius Transformations

In this section, we introduce the triangular Bézier surfaces and the binary Möbius transformation. The binary Möbius transformation is a reparameterization technology specifically for triangular Bézier surfaces.

A triangular Bézier surface in $\boldsymbol{R}^3$ is formulated by

$$\boldsymbol{P}(u,v) = \sum_{i+j\le n} B^n_{i,j}(u,v)\boldsymbol{P}_{i,j},\ 0 \le u,v,u+v \le 1. \tag{1}$$

where $\boldsymbol{P}_{i,j}$ are the control points, and the basis functions $B^n_{i,j}(u,v)$ are the nth-degree binary Bernstein polynomials defined by

$$B^n_{i,j}(u,v) = \frac{n!}{i!j!(n-i-j)!}u^i v^j (1-u-v)^{n-i-j},\ 0 \le u,v,u+v \le 1. \tag{2}$$

The binary Möbius transformations [20] of the triangular Bézier surface are defined by

$$\begin{aligned} u = u(s,t) = \tfrac{\alpha s}{(1-s-t)+\alpha s+\beta t}, v = v(s,t) = \tfrac{\beta t}{(1-s-t)+\alpha s+\beta t}, \\ \alpha,\beta > 0, 0 \le s,t,s+t \le 1. \end{aligned} \tag{3}$$

The positive parameters α and β determine the transformations, which is a bijection. Based on the binary Möbius transformations, the reparameterization converts the triangular Bézier surface $\boldsymbol{P}(u, v)$ to the triangular rational Bézier surface $\boldsymbol{R}(s, t)$, which is derived in the following form

$$\boldsymbol{R}(s,t) = \boldsymbol{P}(u(s,t),v(s,t)) = \frac{\sum\limits_{i+j\le n} B^n_{i,j}(s,t)\alpha^i\beta^j\boldsymbol{P}_{i,j}}{\sum\limits_{i+j\le n} B^n_{i,j}(s,t)\alpha^i\beta^j},\ 0 \le s,t,s+t \le 1. \tag{4}$$

The new reparameterized surface $\boldsymbol{R}(s, t)$ is a rational triangular Bézier surface that has the same geometric shape as the triangular Bézier surface $\boldsymbol{P}(u, v)$. Furthermore, $\boldsymbol{R}(s, t)$ has the same control points $\boldsymbol{P}_{i,j}$ as $\boldsymbol{P}(u, v)$ in light of expression (1). Meanwhile, the weights $\omega_{i,j}$ of $\boldsymbol{R}(s, t)$ can also be obtained as $\left\{\alpha^i\beta^j\right\}_{i+j\le n}$ for the control points $\boldsymbol{P}_{i,j}$.

The binary Möbius transformations do not change the degree, control points, or geometric shape of the original surface $\boldsymbol{P}(u, v)$, but only change the distribution of the iso-parametric lines on the surface. Different values of α and β will have different effects on the distribution of iso-parametric lines and the surface parameterization.

Different from the Möbius transformations of tensor-product surfaces [8,9], the binary Möbius transformations in (3) are more complex and flexible. For example, the variation in α in transformations of tensor-product surfaces [8,9] changes the distribution of iso-parametric u-lines only. Meanwhile, the variation in α in the binary Möbius transformations can change not only the distribution of iso-parametric u-lines but also the distribution of iso-parametric v-lines. Figure 2 shows the different (u, v) curves in the definition domain when we change the value of α only.

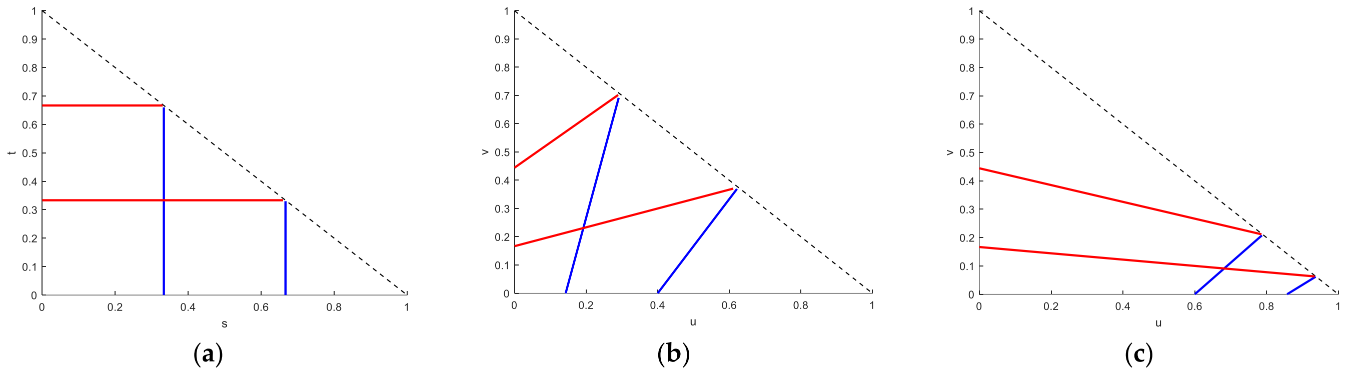


Figure 2. Both the (u, v) curves in the definition domain variety when we change the value of α only. (**a**) Uniform distribution of (s, t) curves. (**b**) (u, v) curves for $\alpha = 1/3$, $\beta = 0.4$. (**c**) (u, v) curves for $\alpha = 3$, $\beta = 0.4$.

3. Equiareal Parameterization

In this section, we first introduce the equiareal energy, which measures the deviation from the uniformity of the iso-parameter curves. The equiareal energy is connected to the binary Möbius transformations. The equiareal parameterization seeks the positive parameters α and β of the binary Möbius transformations, which minimizes the equiareal energy.

The equiareal energy e [8–10] of the triangular Bézier surface in (1) is defined as follows:

$$e = \iint\limits_{D} \left(EG - F^2\right) \mathrm{d}u\mathrm{d}v, E = \boldsymbol{P}_u \cdot \boldsymbol{P}_u, F = \boldsymbol{P}_u \cdot \boldsymbol{P}_v, G = \boldsymbol{P}_v \cdot \boldsymbol{P}_v. \tag{5}$$

where $\boldsymbol{P}_u = \frac{\partial}{\partial u}\boldsymbol{P}(u, v)$ and $\boldsymbol{P}_v = \frac{\partial}{\partial v}\boldsymbol{P}(u, v)$ are the partial derivative vectors of surface $\boldsymbol{P}(u, v)$, and $D = \{(u, v)|0 \le u, v, u + v \le 1\}$. The symbols E, F, and G are from the first fundamental form of the surface.

The equiareal energy e measures the deviation in the current surface parameterization from its uniform parameterization, which makes the iso-parametric lines uniformly distributed [8–10]. Different values of α and β in binary Möbius transformations result in a different equiareal energy e. Hence, we denote the equiareal energy $e(\alpha, \beta)$ of the re-parameterized surface $\boldsymbol{R}(s, t)$ as follows:

$$\begin{aligned} e(\alpha, \beta) &= \iint\limits_{D} \left(EG - F^2\right) \mathrm{d}s\mathrm{d}t, \\ &= \iint\limits_{D} \left(\|\boldsymbol{R}_s(s,t)\|^2 \|\boldsymbol{R}_t(s,t)\|^2 - \|\boldsymbol{R}_s(s,t)\boldsymbol{R}_t(s,t)\|^2\right) \mathrm{d}s\mathrm{d}t. \end{aligned} \tag{6}$$

where $\boldsymbol{R}_s = \frac{\partial}{\partial s}\boldsymbol{R}(s, t)$ and $\boldsymbol{R}_t = \frac{\partial}{\partial t}\boldsymbol{R}(s, t)$ are the partial derivative vectors of surface $\boldsymbol{R}(s, t)$.

The equiareal parameterization chooses the values of α and β in the binary Möbius transformations in (3), such that the equiareal energy $e(\alpha, \beta)$ becomes as small as possible. The goal of this study was to solve the following optimization problem with respect to α and β:

$$\begin{aligned} &\min_{\alpha,\beta} e(\alpha, \beta) = \iint\limits_{D} \left(\|\boldsymbol{R}_s(s,t)\|^2 \|\boldsymbol{R}_t(s,t)\|^2 - \|\boldsymbol{R}_s(s,t)\boldsymbol{R}_t(s,t)\|^2\right) \mathrm{d}s\mathrm{d}t, \\ &\text{s.t.} \alpha > 0, \beta > 0. \end{aligned} \tag{7}$$

The equiareal energy $e(\alpha, \beta)$ is a continuous function of α and β. By setting the partial derivatives of the equiareality energy to zero [21], the solution of α and β to obtain the minimum of the equiareal energy $e(\alpha, \beta)$ is identified by the roots of the two equations

$$\begin{cases} \frac{\partial}{\partial \alpha} e(\alpha, \beta) = 0, \\ \frac{\partial}{\partial \beta} e(\alpha, \beta) = 0. \end{cases} \tag{8}$$

Considering the expression of $\boldsymbol{R}(s, t)$ in (4) is a rational function, the integrand in (6) is also a fraction. This means that the equiareal energy $e(\alpha, \beta)$ is highly nonlinear, and the exact integration is too expensive. It is almost impossible to obtain the expressions of the equations in (8). In other words, there is no closed-form solution for the equations. Hence, the key of the equiareal parameterization is to simplify the double definite integral in (6) and obtain the explicit expressions of the equations in (8). They are presented in Sections 4 and 5.

For nonlinear equations in (8), some iterative methods can be used to obtain a numerical solution by specifying an initial value [7–10]. This numerical solution is regarded as a local optimal solution of the nonlinear minimization problem (7). The equations in (8) are polynomial; all the exact solutions of the equations can be obtained. The global optimal solution is determined from them. More details are shown in Section 6.

4. Expression of Equiareal Energy $e(\alpha, \beta)$

In this section, we achieve the explicit expressions of the equiareal energy $e(\alpha, \beta)$ in order to solve the nonlinear programming problems in (7). The equiareal energy $e(\alpha, \beta)$ is eventually reduced to a double integral whose integrand is a polynomial of the integral variable.

We can see that $\boldsymbol{R}_s(s,t)$ and $\boldsymbol{R}_t(s,t)$ in (6) are the partial derivative vectors of rational surface $\boldsymbol{R}(s, t)$ in (4). The expression of the integrand in (6) is very complex, making the expression of the definite integral $e(\alpha, \beta)$ almost impossible to be displayed explicitly.

The goal of this section is to obtain the explicit expression of equiareal energy $e(\alpha, \beta)$ out of (6). Considering that $\boldsymbol{P}(u, v)$ is polynomial and $\boldsymbol{R}(s, t)$ is fractional, we use a back substitution in the integral. The integrand function in (6) is converted to the expression of $\boldsymbol{P}(u, v)$. Correspondingly, the definite integral about (s, t) in (6) is converted to the definite integral about (u, v).

With the help of chain rule, we have

$$\begin{aligned} \boldsymbol{R}_s(s,t) &= \tfrac{\partial}{\partial s}\boldsymbol{P}(u(s,t),v(s,t)) = \boldsymbol{P}_u(u,v)\tfrac{\partial u}{\partial s} + \boldsymbol{P}_v(u,v)\tfrac{\partial v}{\partial s}, \\ \boldsymbol{R}_t(s,t) &= \tfrac{\partial}{\partial t}\boldsymbol{P}(u(s,t),v(s,t)) = \boldsymbol{P}_u(u,v)\tfrac{\partial u}{\partial t} + \boldsymbol{P}_v(u,v)\tfrac{\partial v}{\partial t}. \end{aligned} \tag{9}$$

By substituting (9) into the integrand in (6), we have

$$\begin{aligned} \|\boldsymbol{R}_s(s,t)\|^2\|\boldsymbol{R}_t(s,t)\|^2 &= \left(\boldsymbol{P}_u(u,v)\tfrac{\partial u}{\partial s} + \boldsymbol{P}_v(u,v)\tfrac{\partial v}{\partial s}\right)^2\left(\boldsymbol{P}_u(u,v)\tfrac{\partial u}{\partial t} + \boldsymbol{P}_v(u,v)\tfrac{\partial v}{\partial t}\right)^2, \\ \|\boldsymbol{R}_s(s,t)\boldsymbol{R}_t(s,t)\|^2 &= \left(\left(\boldsymbol{P}_u(u,v)\tfrac{\partial u}{\partial s} + \boldsymbol{P}_v(u,v)\tfrac{\partial v}{\partial s}\right)\cdot\left(\boldsymbol{P}_u(u,v)\tfrac{\partial u}{\partial t} + \boldsymbol{P}_v(u,v)\tfrac{\partial v}{\partial t}\right)\right)^2. \end{aligned} \tag{10}$$

Now, the integrand in (6) can be presented as follows:

$$\begin{aligned} &\|\boldsymbol{R}_s(s,t)\|^2\|\boldsymbol{R}_t(s,t)\|^2 - \|\boldsymbol{R}_s(s,t)\boldsymbol{R}_t(s,t)\|^2 \\ &= \left(\|\boldsymbol{P}_u(u,v)\|^2\|\boldsymbol{P}_v(u,v)\|^2 - \|\boldsymbol{P}_u(u,v)\boldsymbol{P}_v(u,v)\|^2\right)\left(\left(\tfrac{\partial u}{\partial s}\tfrac{\partial v}{\partial t} - \tfrac{\partial u}{\partial t}\tfrac{\partial v}{\partial s}\right)^2\right) \\ &= \left(\|\boldsymbol{P}_u(u,v)\|^2\|\boldsymbol{P}_v(u,v)\|^2 - \|\boldsymbol{P}_u(u,v)\boldsymbol{P}_v(u,v)\|^2\right)\left(\tfrac{\partial(u,v)}{\partial(s,t)}\right)^2 \end{aligned} \tag{11}$$

where $\frac{\partial(u,v)}{\partial(s,t)}$ is the Jacobian determinant [21].

To simplify the expression of the equiareal energy $e(\alpha, \beta)$, we transform the integration variable in (6) back to the original parameters (u, v). Considering $\frac{\partial(u,v)}{\partial(s,t)}$ is the Jacobian determinant, we have

$$\frac{\partial(u,v)}{\partial(s,t)} = \frac{1}{\frac{\partial(s,t)}{\partial(u,v)}}. \tag{12}$$

Meanwhile, according to the relationship between area elements before and after variable replacement in the double integral [21], we have

$$\mathrm{d}s\mathrm{d}t = \frac{\partial(s,t)}{\partial(u,v)}\mathrm{d}u\mathrm{d}v. \tag{13}$$

By substituting (11)–(13) into the double integral in (6), we have

$$\begin{aligned} e(\alpha,\beta) &= \iint\limits_D \left(\|\boldsymbol{R}_s(s,t)\|^2\|\boldsymbol{R}_t(s,t)\|^2 - \|\boldsymbol{R}_s(s,t)\boldsymbol{R}_t(s,t)\|^2\right)\mathrm{d}s\mathrm{d}t \\ &= \iint\limits_D \left(\|\boldsymbol{P}_u(u,v)\|^2\|\boldsymbol{P}_v(u,v)\|^2 - \|\boldsymbol{P}_u(u,v)\boldsymbol{P}_v(u,v)\|^2\right)\frac{1}{\left(\frac{\partial(s,t)}{\partial(u,v)}\right)^2}\mathrm{d}s\mathrm{d}t \\ &= \iint\limits_D \left(\|\boldsymbol{P}_u(u,v)\|^2\|\boldsymbol{P}_v(u,v)\|^2 - \|\boldsymbol{P}_u(u,v)\boldsymbol{P}_v(u,v)\|^2\right)\frac{1}{\left(\frac{\partial(s,t)}{\partial(u,v)}\right)^2}\frac{\partial(s,t)}{\partial(u,v)}\mathrm{d}u\mathrm{d}v \\ &= \iint\limits_D \left(\|\boldsymbol{P}_u(u,v)\|^2\|\boldsymbol{P}_v(u,v)\|^2 - \|\boldsymbol{P}_u(u,v)\boldsymbol{P}_v(u,v)\|^2\right)\frac{1}{\frac{\partial(s,t)}{\partial(u,v)}}\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{14}$$

Now, we obtain the equiareal energy $e(\alpha, \beta)$

$$e(\alpha,\beta) = \iint\limits_D \left(\|\boldsymbol{P}_u(u,v)\|^2\|\boldsymbol{P}_v(u,v)\|^2 - \|\boldsymbol{P}_u(u,v)\boldsymbol{P}_v(u,v)\|^2\right)\frac{1}{\left(\frac{\partial s}{\partial u}\frac{\partial t}{\partial v} - \frac{\partial s}{\partial v}\frac{\partial t}{\partial u}\right)}\mathrm{d}u\mathrm{d}v. \tag{15}$$

The partial derivatives $\frac{\partial s}{\partial u}$, $\frac{\partial t}{\partial v}$, $\frac{\partial s}{\partial v}$, $\frac{\partial t}{\partial u}$ can be expressed with the help of the inverse transformations of (3), which are shown as follows:

$$\begin{gathered} s(u,v) = \frac{\beta u}{\alpha\beta(1-u-v)+\alpha v+\beta u},\ t(u,v) = \frac{\alpha v}{\alpha\beta(1-u-v)+\alpha v+\beta u}, \\ 0 \le u, v, u+v \le 1. \end{gathered} \tag{16}$$

The partial derivatives $\frac{\partial s}{\partial u}$, $\frac{\partial t}{\partial v}$, $\frac{\partial s}{\partial v}$, $\frac{\partial t}{\partial u}$ are shown as follows:

$$\begin{aligned} \frac{\partial s}{\partial u} &= \frac{\alpha\beta(\beta+v-\beta v)}{(\alpha\beta(1-u-v)+\alpha v+\beta u)^2},\ \frac{\partial s}{\partial v} = \frac{\alpha\beta u(\beta-1)}{(\alpha\beta(1-u-v)+\alpha v+\beta u)^2}, \\ \frac{\partial t}{\partial u} &= \frac{\alpha\beta v(\alpha-1)}{(\alpha\beta(1-u-v)+\alpha v+\beta u)^2},\ \frac{\partial t}{\partial v} = \frac{\alpha\beta(\alpha+u-\alpha u)}{(\alpha\beta(1-u-v)+\alpha v+\beta u)^2}. \end{aligned} \tag{17}$$

Finally, the equiareal energy $e(\alpha, \beta)$ can be described as follows:

$$e(\alpha,\beta) = \iint\limits_D \left(\|\boldsymbol{P}_u\|^2\|\boldsymbol{P}_v\|^2 - \|\boldsymbol{P}_u\boldsymbol{P}_v\|^2\right)\frac{(\alpha\beta(1-u-v)+\alpha v+\beta u)^3}{\alpha^2\beta^2}\mathrm{d}u\mathrm{d}v. \tag{18}$$

Considering the expression of $\boldsymbol{P}(u, v)$ in (1) is polynomial, $e(\alpha, \beta)$ in (18) is a double integral of polynomial, which is much simpler than the one in (6).

5. Expressions of the Equations

The solution of the optimization problem in (7) can be transformed to solve the equations in (8). When the expression of the equiareal energy $e(\alpha, \beta)$ in (6) is reduced to (18), the expressions of the equations in (8) can also be accordingly simplified. The goal in this section is to obtain the explicit expressions of the equations in (8).

By substituting (18) into the equations in (8), we have

$$\begin{cases} \frac{\partial}{\partial \alpha} e(\alpha,\beta) = \frac{1}{\alpha^3\beta^2} \iint\limits_D \left(\|\boldsymbol{P}_u\|^2 \|\boldsymbol{P}_v\|^2 - \|\boldsymbol{P}_u \boldsymbol{P}_v\|^2 \right) (\alpha\beta(1-u-v) + \alpha v + \beta u)^2 (\alpha\beta(1-u-v) + \alpha v - 2\beta u) \mathrm{d}u\mathrm{d}v = 0, \\ \frac{\partial}{\partial \beta} e(\alpha,\beta) = \frac{1}{\alpha^2\beta^3} \iint\limits_D \left(\|\boldsymbol{P}_u\|^2 \|\boldsymbol{P}_v\|^2 - \|\boldsymbol{P}_u \boldsymbol{P}_v\|^2 \right) (\alpha\beta(1-u-v) + \alpha v + \beta u)^2 (\alpha\beta(1-u-v) - 2\alpha v + \beta u) \mathrm{d}u\mathrm{d}v = 0. \end{cases} \tag{19}$$

They can be written as a system of bivariate polynomial equations with order 3 as follows:

$$\begin{cases} d_{33} + 3d_{32} + 3d_{31} + d_{30} - 3d_{13} - 3d_{12} - 2d_{03} = 0, \\ d_{33} - 3d_{31} - 2d_{30} + 3d_{23} - 3d_{21} + 3d_{13} + d_{03} = 0, \end{cases} \tag{20}$$

where

$$d_{k,l} = \alpha^k \beta^l c_{k,l}, 0 \le k, l \le 3, k + l \ge 3.$$
$$c_{k,l} = \iint\limits_D B^3_{3-k,3-l}(u,v) \left(\|\boldsymbol{P}_u(u,v)\|^2 \|\boldsymbol{P}_v(u,v)\|^2 - \|\boldsymbol{P}_u(u,v)\boldsymbol{P}_v(u,v)\|^2 \right) \mathrm{d}u\mathrm{d}v. \tag{21}$$

There are 10 different coefficients $c_{k,l}$ in Equation (20). All of them are the double integrals in (21) and can be calculated by a numerical quadrature method. However, because the integrands in (21) are polynomials, the expressions of $c_{k,l}$ can be explicitly given. The partial derivative vectors $\boldsymbol{P}_u(u,v)$ and $\boldsymbol{P}_v(u,v)$ of the surface in (21) are obtained as

$$\boldsymbol{P}_u(u,v) = n \sum_{i+j \le n-1} B^{n-1}_{i,j}(u,v) \Delta_1 \boldsymbol{P}_{i,j}, \boldsymbol{P}_v(u,v) = n \sum_{i+j \le n-1} B^{n-1}_{i,j}(u,v) \Delta_2 \boldsymbol{P}_{i,j},$$
$$\Delta_1 \boldsymbol{P}_{i,j} = \boldsymbol{P}_{i+1,j} - \boldsymbol{P}_{i,j}, \Delta_2 \boldsymbol{P}_{i,j} = \boldsymbol{P}_{i,j+1} - \boldsymbol{P}_{i,j}. \tag{22}$$

In this way, the expressions of the coefficients $c_{k,l}$ can be presented as a Bernstein form.

$$c_{k,l} = n^4 \sum_{\substack{i_1+j_1 \\ \le n-1}} \sum_{\substack{i_2+j_2 \\ \le n-1}} \sum_{\substack{i_3+j_3 \\ \le n-1}} \sum_{\substack{i_4+j_4 \\ \le n-1}} \iint\limits_D B^{n-1}_{i_1 j_1}(u,v) B^{n-1}_{i_2 j_2}(u,v) B^{n-1}_{i_3 j_3}(u,v) B^{n-1}_{i_4 j_4}(u,v) B^3_{3-l,3-k}(u,v) f_{i_1 i_2 i_3 i_4, j_1 j_2 j_3 j_4} \mathrm{d}u\mathrm{d}v,$$
$$f_{i_1 i_2 i_3 i_4, j_1 j_2 j_3 j_4} = \begin{vmatrix} \Delta_1 \boldsymbol{P}_{i_1,j_1} \Delta_1 \boldsymbol{P}_{i_2,j_2} & \Delta_1 \boldsymbol{P}_{i_1,j_1} \Delta_2 \boldsymbol{P}_{i_3,j_3} \\ \Delta_1 \boldsymbol{P}_{i_2,j_2} \Delta_2 \boldsymbol{P}_{i_4,j_4} & \Delta_2 \boldsymbol{P}_{i_3,j_3} \Delta_2 \boldsymbol{P}_{i_4,j_4} \end{vmatrix}. \tag{23}$$

In order to have an easier computation of the double integrals in (20), we introduce the multiplication and integral properties of the binary Bernstein polynomials [1] as follows:

$$B^{n_1}_{i_1,j_1}(u,v) B^{n_2}_{i_2,j_2}(u,v) = \frac{\binom{n_1}{i_1, j_1}\binom{n_2}{i_2, j_2}}{\binom{n_1+n_2}{i_1+i_2, j_1+j_2}} B^{n_1+n_2}_{i_1+i_2, j_1+j_2}(u,v), \binom{n}{i, j} = \frac{n!}{i!j!(n-i-j)!}. \tag{24}$$

$$\iint\limits_D B^n_{i,j}(u,v) \mathrm{d}u\mathrm{d}v = \frac{1}{(n+1)(n+2)}. \tag{25}$$

By substituting (24) and (25) into the double integral in (23), we have the explicit expressions of the coefficients $c_{k,l}$ finally as follows:

$$c_{k,l} = \frac{n^3 \binom{3}{3-l, 3-k}}{4(4n+1)} \sum_{i_1+j_1 \le n-1} \sum_{i_2+j_2 \le n-1} \sum_{i_3+j_3 \le n-1} \sum_{i_4+j_4 \le n-1} \frac{\binom{n-1}{i_1, j_1}\binom{n-1}{i_2, j_2}\binom{n-1}{i_3, j_3}\binom{n-1}{i_4, j_4} f_{i_1 i_2 i_3 i_4, j_1 j_2 j_3 j_4}}{\binom{4n-1}{i_1+i_2+i_3+i_4+3-l, j_1+j_2+j_3+j_4+3-k}}. \tag{26}$$

To sum up, the values of all 10 coefficients in the bivariate polynomial equations in (26) can be directly obtained without integral calculation. This greatly reduces the complexity of the coefficient calculation. The bivariate polynomial equations are solved in the next section.

6. Solution of Equiareal Parameterization

The key to the equiareal parameterization is how to seek the values of α and β from the solution of the bivariate polynomial equations in (20). Before this, we first discuss the existence of the solution to the equations when α and β are positive.

We assume that the surface in (1) is regular, which has no sharp points, edges, or self-intersections. We have [22]

$$\|\boldsymbol{P}_u\|^2\|\boldsymbol{P}_v\|^2 - \|\boldsymbol{P}_u\boldsymbol{P}_v\|^2 = \|\boldsymbol{P}_u \times \boldsymbol{P}_v\|^2 > 0, (u,v) \in D. \tag{27}$$

Thus, there exist two constants, C_1 and C_2, with

$$0 < C_1 < \|\boldsymbol{P}_u\|^2\|\boldsymbol{P}_v\|^2 - \|\boldsymbol{P}_u\boldsymbol{P}_v\|^2 < C_2, (u,v) \in D. \tag{28}$$

resulting from

$$\iint\limits_D \frac{(\alpha\beta(1-u-v)+\alpha v+\beta u)^3}{\alpha^2\beta^2}\mathrm{d}u\mathrm{d}v = \frac{1}{20}\left(1+(\alpha+\beta+\alpha\beta)\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+1\right)\right). \tag{29}$$

by setting

$$h(\alpha,\beta) = 1+(\alpha+\beta+\alpha\beta)\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+1\right). \tag{30}$$

From (28)–(30), we obtain the inequalities

$$\frac{1}{20}h(\alpha,\beta)C_1 < e(\alpha,\beta) < \frac{1}{20}h(\alpha,\beta)C_2. \tag{31}$$

Note that

$$\lim_{\substack{\alpha\to 0^+\\ \beta>0}} h(\alpha,\beta) = \lim_{\substack{\alpha\to +\infty\\ \beta>0}} h(\alpha,\beta) = \lim_{\substack{\alpha>0\\ \beta\to 0^+}} h(\alpha,\beta) = \lim_{\substack{\alpha>0\\ \beta\to +\infty}} h(\alpha,\beta) = +\infty. \tag{32}$$

We have

$$\lim_{\substack{\alpha\to 0^+\\ \beta>0}} e(\alpha,\beta) = \lim_{\substack{\alpha\to +\infty\\ \beta>0}} e(\alpha,\beta) = \lim_{\substack{\alpha>0\\ \beta\to 0^+}} e(\alpha,\beta) = \lim_{\substack{\alpha>0\\ \beta\to +\infty}} e(\alpha,\beta) = +\infty. \tag{33}$$

Because the equiareal energy $e(\alpha, \beta)$ is infinite when the α and β approach the "boundaries" of the domain of definition, then the optimization problem (7) must have a global minimum for $\alpha > 0$ and $\beta > 0$. Considering the continuity of the equiareal energy $e(\alpha, \beta)$ with respect to α and β, Equation (20) has at least one solution.

However, there is no guarantee that the solution of bivariate polynomial equations in (20) is unique. Because the equations are polynomial, we can obtained all the exact solutions (denoted by $\{(\alpha_i,\beta_i)\}_{i=1}^k$) rapidly using *wsolve* (a Maple package for solving system of polynomial equations) [23], which implements zero decomposition algorithms for systems of polynomial equations [24]. If the positive solution is not unique, all the solutions should be substituted into the expression of equiareal energy $e(\alpha, \beta)$ in (18). We choose the best pair of α and β (denoted by $(\alpha_{i_0},\beta_{i_0})$) to minimize the equiareal energy. In practice, we observe that the equations always have a unique positive solution only. We conclude that the equiareal parameterization is received.

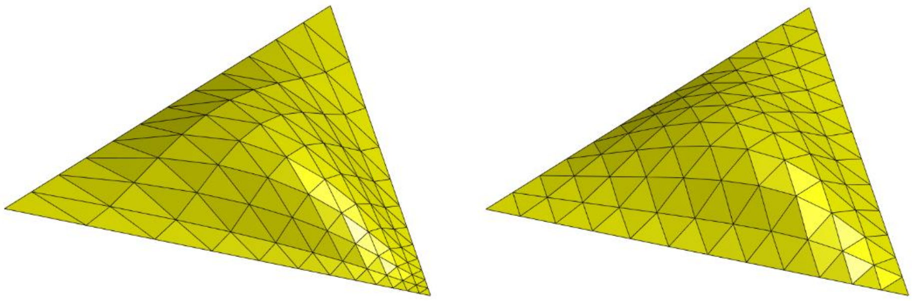
In summary, given a triangular Bézier surface $\boldsymbol{P}(u, v)$, the algorithm flow for the equiareal parameterization of triangular Bézier surfaces (denoted by Algorithm **EPT**) is presented as follows:

Algorithm EPT: The equiareal parameterization of triangular Bézier surfaces.

Input: The control points of triangular Bézier surfaces.
Output: The parameters α and β for the binary Möbius transformations.
Step 1. Compute the coefficients $c_{k,l}$ in (23) and (26);
Step 2. Obtain all the solutions (denoted by $\{(\alpha_i, \beta_i)\}_{i=1}^{k}$) of the system of equations with the Maple package *wsolve*;
Step 3. By substituting $\{(\alpha_i, \beta_i)\}_{i=1}^{k}$ into the equiareal energy $e(\alpha, \beta)$ in (18), choose a positive solution (denoted by $(\alpha_{i_0}, \beta_{i_0})$), which makes the equiareal energy $e(\alpha_{i_0}, \beta_{i_0})$ to be the minimum in $\{e(\alpha_i, \beta_i)\}_{i=1}^{k}$.
By substituting $(\alpha_{i_0}, \beta_{i_0})$ into the binary Möbius transformations in (3), we obtain the equiareal parameterization.
Remark: Algorithm **EPT** is highly efficient. In practice, step 3 can be omitted, because the equations in (20) always have a unique positive solution only. The proof of the uniqueness of the solution will be our future work.

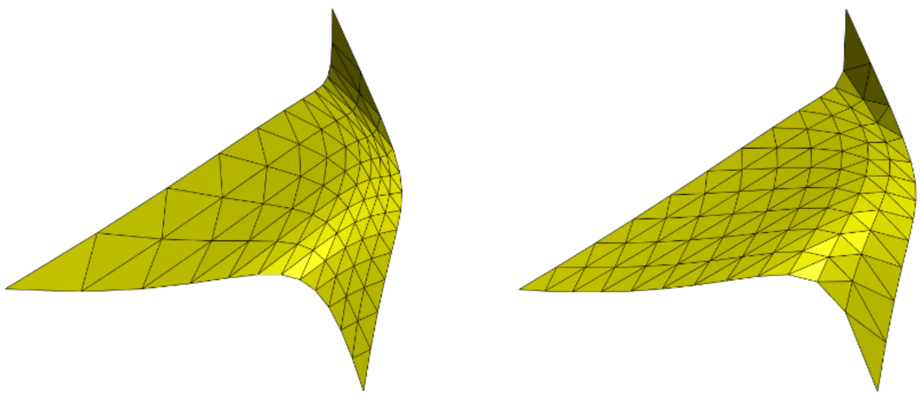
7. Experimental Results

Applying Algorithm **EPT**, Figures 3 and 4 illustrate the equiareal parameterization of a triangular Bézier surface. The original triangular Bézier surfaces in Figures 3a and 4a are far from being equiareal. With the help of equiareal parameterization, the uniformities of the iso-parametric curves in Figures 3b and 4b are considerably improved. The iso-parameter curve networks are shown to display the superiority of the reparameterization.



(**a**) (**b**)

Figure 3. Equiareal parameterization with $\alpha = 0.38$ and $\beta = 1.00$. (**a**) Original surface with $e = 11.25$. (**b**) Resultant surface with $e(\alpha, \beta) = 7.94$.



(**a**) (**b**)

Figure 4. Equiareal parameterization with $\alpha = 0.30$ and $\beta = 0.36$. (**a**) Original surface with $e = 22.15$. (**b**) Resultant surface with $e(\alpha, \beta) = 10.90$.

As mentioned above, the equations in (20) always have a unique positive solution, which is the global optimal solution of minimization problem (7), such that step 3 in Algorithm **EPT** is omitted. That shows our algorithm is simple and efficient.

8. Conclusions and Future Work

In order to improve the uniformity of iso-parametric curves for triangular Bézier surfaces, equiareal parameterization was presented in this paper. Based on the binary Möbius transformations, the equiareal energy can be minimized by solving a system of bivariate polynomial equations with order three. The global optimal solution of the corresponding nonlinear minimization problem can be obtained. Particularly, equiareal parameterization generates a ration triangular Bézier surface, which is also a common tool for geometry processing in CAD systems. The reparameterized surface has a clear and concise expression and maintains the degree, control points, and shape of the original surface. It modifies the distribution of iso-parameter curves only. Experimental examples were given to show the effectiveness of the method.

We will focus on two directions in the future. One is to improve the quality of the parameterization. In practice, the equiareal parameterization cannot uniformly distribute iso-parameter curves all over the surface. Some local parts may be not good enough. That is mainly because the binary Möbius transformations have only two degrees of freedom. In order to improve the parameterization, more degrees of freedom should be introduced into the transformations. The rational bilinear transformations [7], composite Möbius transformations [9], and the optimal freeform transformations [10] were presented on the rectangular domain with more degrees of freedom. The extension of these transformations to the triangular domain is left as our future work. Furthermore, constructing new energy functions to match these new parameterizations for some simple optimization problems is a more challenging task.

The other direction of our future work is the application of equiareal parameterization. In order to obtain a better result, we can apply equiareal parameterization to triangular Bézier surface algorithms such as rendering, tessellation, sampling, and texture mapping. If the parameterization is far from the equiareal parameterization, there are large distortions of the mapping from the definition domain to the surface. These distortions usually introduce many difficulties. Additionally, the equiareal parameterization uniformly triangulates the definition domain and leads to more robust and stable computations for those surface algorithms. How to combine equiareal parameterization and surface algorithms effectively is another goal of our future work.

Author Contributions: Conceptualization, J.C.; Methodology, J.C. and X.K.; Scripting, J.C. and X.K.; Experimental Validation, J.C.; Writing, J.C. and H.X.; Data Curation, X.K.; Formal Analysis, H.X.; Analysis of Results, H.X. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Natural Science Foundation of Zhejiang Province (nos. LY18F020026 and LY16F020020) and the Ningbo Natural Science Foundation (no. 2019A610033).

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the anonymous reviewers for their valuable suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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