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Article

Necessary and Sufficient Conditions for Normalized Wright Functions to Be in Certain Classes of Analytic Functions

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Abstract: In this paper, the function classes $\mathcal{SP}_p(\sigma,\nu)$ and $\mathcal{UCSP}(\sigma,\nu)$ are investigated for the normalized Wright functions. More precisely, several sufficient and necessary conditions are provided so that the aforementioned functions are in these classes. Furthermore, several corollaries will follow from our results.

Keywords: wright functions; analytic function; uniformly spiral-like; uniformly convex spiral-like

MSC: 30C45

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions that are defined on the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, which can be written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

and $\mathcal{T} \subset \mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad z \in \mathbb{U}. \tag{2}$$

A function $f \in \mathcal{A}$ is called a spiral-like function if

$$\Re\left(e^{-i\sigma}\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{U},$$

where $|\sigma| < \pi/2$. Further, a function $f \in \mathcal{A}$ is called a convex spiral-like function if $zf'(z)$ is a spiral-like function.

Definition 1. *A function $f \in \mathcal{A}$ belongs to the subclass $\mathcal{SP}_p(\sigma,\nu)$ of the class of spiral-like functions if the following condition is verified.*

$$\Re\left\{e^{-i\sigma}\left(\frac{zf'(z)}{f(z)}\right)\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right| + \nu \qquad (z \in \mathbb{U}; |\sigma| < \pi/2\,; 0 \le \nu < 1).$$

Definition 2. *A function $f \in \mathcal{A}$ is in the subclass $\mathcal{UCSP}(\sigma,\nu)$ of the class of convex spiral-like functions if $zf'(z) \in \mathcal{SP}_p(\sigma,\nu)$.*

The abovementioned subclasses were introduced by Selvaraj and Geetha, see [1]. For functions $f \in \mathcal{T}$, let us define the following subclasses.

$$\mathcal{SP}_p\mathcal{T}(\sigma,\nu) = \mathcal{SP}_p(\sigma,\nu) \cap \mathcal{T}$$

and

$$\mathcal{UCSPT}(\sigma,\nu) = \mathcal{UCSP}(\sigma,\nu) \cap \mathcal{T}.$$

In particular, we see that $\mathcal{SP}_p(\sigma,0) = \mathcal{SP}_p(\sigma)$ is the class of uniformly spiral-like functions and $\mathcal{UCSP}(\sigma,0) = \mathcal{UCSP}(\sigma)$ is the class of uniformly convex spiral-like functions. These subclasses were introduced by Ravichandran et al. [2]. Further, Rønning [3] introduced and investigated the subclasses $\mathcal{SP}_p(0,0) = \mathcal{SP}$ and $\mathcal{UCSP}(0,0) = \mathcal{UCV}$. For further fascinating developments of a few linked uniformly spiral-like and uniformly convex spiral-like subclasses, readers may be referred to [4–11].

Special functions have been used extensively in many practical applications in physics, mathematics, and engineering. Recently, special functions have found many connections with geometric function theory, see [12–26]. In this work, we consider the Wright function, which is a well-known special function defined as

$$\varphi(\gamma,\delta;z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\gamma n+\delta)} \frac{z^n}{n!}, \quad \gamma > -1, \delta, z \in \mathbb{C}. \tag{3}$$

Wright functions play a substantial role in many areas, including the asymptotic theory of partitions, Mikusinski operational calculus, Green functions, and partial differential equations of fractional order, see, for example [27–29].

Remark 1. *For $\gamma = 1$ and $\delta = p+1$, the Bessel functions $J_p(z)$ can be written as Wright functions $\varphi(1, p+1; -z^2/4)$, where*

$$J_p(z) = \left(\frac{z}{2}\right)^p \varphi(1, p+1; \frac{-z^2}{4}) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+p+1)} \frac{(z/2)^{2n+p}}{n!}.$$

Further, for $\gamma > 0$ and $p > -1$, the generalized Bessel function (or the Bessel–Wright function) $J_p^{\gamma}(z) \equiv \varphi(\gamma, p+1; -z)$.

Observe that the Wright function $\varphi(\gamma,\delta,z) \notin \mathcal{A}$. To overcome this shortcoming, in this work, we consider the following normalized Wright functions:

$$\Psi^{(1)}(\gamma,\delta;z) := \Gamma(\delta) z \varphi(\gamma,\delta;z) = \sum_{n=0}^{\infty} \frac{\Gamma(\delta)}{\Gamma(\gamma n+\delta)} \frac{z^{n+1}}{n!}, \quad \gamma > -1, \delta > 0, z \in \mathbb{U}$$

and

$$\begin{aligned}\Psi^{(2)}(\gamma,\delta;z) &:= \Gamma(\gamma+\delta)\left(\varphi(\gamma,\delta;z) - \frac{1}{\Gamma(\delta)}\right) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma n+\gamma+\delta)} \frac{z^{n+1}}{(n+1)!}, \quad \gamma > -1, \gamma+\delta > 0, z \in \mathbb{U}.\end{aligned}$$

It is easily verified that

$$\Psi^{(1)}(\gamma,\delta;z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\delta)}{\Gamma(\gamma(n-1)+\delta)} \frac{z^n}{(n-1)!}, \quad \gamma > -1, \delta > 0, z \in \mathbb{U}, \tag{4}$$

and

$$\Psi^{(2)}(\gamma,\delta;z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma(n-1)+\gamma+\delta)}\frac{z^n}{n!},\ \ \gamma>-1,\gamma+\delta>0,z\in\mathbb{U}. \tag{5}$$

Moreover, note that $\Psi^{(1)}(\gamma,\delta;z)$ and $\Psi^{(2)}(\gamma,\delta;z)$ satisfy the following equations.

$$\gamma z(\Psi^{(1)}(\gamma,\delta;z))'=(\delta-1)\Psi^{(1)}(\gamma,\delta-1;z)+(\gamma+\delta+1)\Psi^{(1)}(\gamma,\delta;z),$$

$$\Psi^{(2)}(\gamma,\delta;z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma(n-1)+\gamma+\delta)}\frac{z^n}{n!},\gamma>-1,\gamma+\delta>0,z\in\mathbb{U},$$

and

$$z(\Psi^{(2)}(\gamma,\delta;z))'=\Psi^{(1)}(\gamma,\gamma+\delta;z)\text{ and }V'_{\gamma,\delta}(z)=\frac{\Gamma(\delta)}{\Gamma(\gamma+\delta)}V_{\gamma,\gamma+\delta}(z), \tag{6}$$

where $V_{\gamma,\delta}(z)=\frac{\Psi^{(1)}(\gamma,\delta;z)}{z}$.

Furthermore, for the normalized Bessel function $\bar{J}_p(z)$, we have

$$-\Psi^{(1)}(1,p+1;-z)=\bar{J}_p(z):=\Gamma(p+1)z^{1-p/2}J_p(2\sqrt{z}). \tag{7}$$

In recent years, several researchers have used the normalized Wright functions (see [30–33]) to obtain some necessary and sufficient conditions so that they are in certain classes of analytic functions with negative coefficients. Motivated with the aforementioned works, several sufficient and necessary conditions are provided in the present work for the normalized Wright functions $\Psi^{(1)}(\gamma,\delta;z)$ and $\Psi^{(2)}(\gamma,\delta;z)$ so that they are in classes $\mathcal{SP}_p(\sigma,\nu)$ and $\mathcal{UCSP}(\sigma,\nu)$. Many findings in the literature have been improved by our main results, and new techniques have been added to the proofs, including geometric proof.

To achieve our targeted results, we demand the following lemma.

Lemma 1 ([1])**.** *A sufficient and necessary condition for a function f given by (1) to be in the function class $\mathcal{SP}_p(\sigma,\nu)$ and a function f given by (2) to be in the function class $\mathcal{SP}_p\mathcal{T}(\sigma,\nu)$ is*

$$\sum_{n=2}^{\infty}(2n-\cos\sigma-\nu)|a_n|\le\cos\sigma-\nu\qquad(|\sigma|<\pi/2\,;0\le\nu<1). \tag{8}$$

Further, a necessary and sufficient condition for a function f given by (1) to be in the function class $\mathcal{UCSP}(\sigma,\nu)$ and a function f given by (2) to be in the function class $\mathcal{UCSPT}(\sigma,\nu)$ is

$$\sum_{n=2}^{\infty}n(2n-\cos\sigma-\nu)|a_n|\le\cos\sigma-\nu\qquad(|\sigma|<\pi/2\,;0\le\nu<1). \tag{9}$$

2. Necessary and Sufficient Conditions for the Normalized Wright Functions to Be in $\mathcal{SP}_p(\sigma,\nu)$ and $\mathcal{UCSP}(\sigma,\nu)$

Theorem 1. *The function $\Psi^{(1)}(\gamma,\delta;z)\in\mathcal{SP}_p(\sigma,\nu)$ if the next condition is verified for $\gamma\ge1$.*

$$\delta(\cos\sigma-\nu)+(\cos\sigma+\nu-2)(\delta+1)\left(e^{\frac{1}{\delta+1}}-1\right)-2e^{\frac{1}{\delta+1}}\ge0.$$

Proof. Since

$$\Psi^{(1)}(\gamma,\delta;z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\delta)}{\Gamma(\gamma(n-1)+\delta)}\frac{z^n}{(n-1)!},$$

then by (8), it suffices to show that

$$\sum_{n=2}^{\infty}\frac{(2n-\cos\sigma-\nu)\Gamma(\delta)}{\Gamma(\gamma(n-1)+\delta)}\frac{1}{(n-1)!}\le\cos\sigma-\nu.$$

Let

$$k_1(\gamma,\delta;\sigma,\nu)=\sum_{n=2}^{\infty}\frac{(2n-\cos\sigma-\nu)\Gamma(\delta)}{\Gamma(\gamma(n-1)+\delta)}\frac{1}{(n-1)!}.$$

Setting $n=(n-1)+1$ and by direct computations, we obtain

$$k_1(\gamma,\delta;\sigma,\nu)=\sum_{n=2}^{\infty}\frac{2\Gamma(\delta)}{(n-2)!\Gamma(\gamma(n-1)+\delta)}+\sum_{n=2}^{\infty}\frac{(2-\cos\sigma-\nu)\Gamma(\delta)}{(n-1)!\Gamma(\gamma(n-1)+\delta)}.$$

By the assumption, $\gamma\geq 1$, and hence $\Gamma(n-1+\delta)\leq\Gamma(\gamma(n-1)+\delta)$, for $n\in\mathbb{C}$, remain true and is equipollent to

$$\frac{\Gamma(\delta)}{\Gamma(\gamma(n-1)+\delta)}\leq\frac{1}{(\delta)_{n-1}},\ \ n\in\mathbb{C}. \tag{10}$$

Here, $(\delta)_0=1$ and $(\delta)_n=\frac{\Gamma(n+\delta)}{\Gamma(\delta)}=\delta(\delta+1)(\delta+2)\cdots(\delta+n-1)$ is the well known Pochhammer symbol.

By using Equation (10), we obtain

$$k_1(\gamma,\delta;\sigma,\nu)\leq\sum_{n=2}^{\infty}\frac{2}{(\delta)_{n-1}(n-2)!}+\sum_{n=2}^{\infty}\frac{(2-\cos\sigma-\nu)}{(\delta)_{n-1}(n-1)!}.$$

Further, the inequality

$$(\delta)_{n-1}=\delta(\delta+1)(\delta+2)\cdots(\delta+n-1)\geq\delta(\delta+1)^{n-2},\ \ n\in\mathbb{C} \tag{11}$$

remain true and this is equipollent to $\frac{1}{(\delta)_{n-1}}\leq\frac{1}{\delta(\delta+1)^{n-2}}$, $n\in\mathbb{C}$.

Using Equation (11), we obtain

$$\begin{aligned}k_1(\gamma,\delta;\sigma,\nu)&\leq\sum_{n=2}^{\infty}\frac{2}{\delta(\delta+1)^{n-2}(n-2)!}+\sum_{n=2}^{\infty}\frac{(2-\cos\sigma-\nu)}{\delta(\delta+1)^{n-2}(n-1)!}\\&=\frac{2}{\delta}e^{\frac{1}{\delta+1}}+\frac{\delta+1}{\delta}(2-\cos\sigma-\nu)\left(e^{\frac{1}{\delta+1}}-1\right)\leq\cos\sigma-\nu.\end{aligned}$$

Hence,

$$\delta(\cos\sigma-\nu)+(\cos\sigma+\nu-2)(\delta+1)\left(e^{\frac{1}{\delta+1}}-1\right)-2e^{\frac{1}{\delta+1}}\geq 0.$$

This accomplishes the proof of Theorem 1. □

Theorem 2. *The normalized Wright function $\Psi^{(1)}(\gamma,\delta;z)\in\mathcal{UCSP}(\sigma,\nu)$ if the following condition is verified for $\gamma\geq 1$.*

$$\delta(\delta+1)(\cos\sigma-\nu)-2e^{\frac{1}{\delta+1}}+(\delta+1)(\cos\sigma+\nu-6)e^{\frac{1}{\delta+1}}+(\cos\sigma+\nu-2)(\delta+1)^2\left(e^{\frac{1}{\delta+1}}-1\right)\geq 0.$$

Proof. Since

$$\Psi^{(1)}(\gamma,\delta;z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\delta)}{\Gamma(\gamma(n-1)+\delta)}\frac{z^n}{(n-1)!},$$

then by (9), it suffices to show that

$$\sum_{n=2}^{\infty}\frac{(2n^2-n(\cos\sigma+\nu))\Gamma(\delta)}{\Gamma(\gamma(n-1)+\delta)}\frac{1}{(n-1)!}\leq\cos\sigma-\nu.$$

Let

$$k_2(\gamma,\delta;\sigma,\nu)=\sum_{n=2}^{\infty}\frac{(2n^2-n(\cos\sigma+\nu))\Gamma(\delta)}{\Gamma(\gamma(n-1)+\delta)}\frac{1}{(n-1)!}.$$

Setting $n = (n-1)+1$, $n^2 = (n-1)(n-2)+3(n-1)+1$ and by straightforward computation, we have

$$k_2(\gamma,\delta;\sigma,\nu) = \sum_{n=3}^{\infty} \frac{2\Gamma(\delta)}{(n-3)!\Gamma(\gamma(n-1)+\delta)} + \sum_{n=2}^{\infty} \frac{(6-\cos\sigma-\nu)\Gamma(\delta)}{(n-2)!\Gamma(\gamma(n-1)+\delta)}$$
$$+ \sum_{n=2}^{\infty} \frac{(2-\cos\sigma-\nu)\Gamma(\delta)}{(n-1)!\Gamma(\gamma(n-1)+\delta)}.$$

Using Equations (10) and (11), we obtain

$$k_2(\gamma,\delta;\sigma,\nu) \le \sum_{n=3}^{\infty} \frac{2}{\delta(\delta+1)^{n-2}(n-3)!} + \sum_{n=2}^{\infty} \frac{(6-\cos\sigma-\nu)}{(n-2)!\delta(\delta+1)^{n-2}} + \sum_{n=2}^{\infty} \frac{(2-\cos\sigma-\nu)}{(n-1)!\delta(\delta+1)^{n-2}}$$
$$= \frac{2}{\delta(\delta+1)}e^{\frac{1}{\delta+1}} + \frac{6-\cos\sigma-\nu}{\delta}e^{\frac{1}{\delta+1}} + \frac{(2-\cos\sigma-\nu)(\delta+1)}{\delta}\left(e^{\frac{1}{\delta+1}}-1\right) \le \cos\sigma-\nu.$$

Hence,

$$\delta(\delta+1)(\cos\sigma-\nu) - 2e^{\frac{1}{\delta+1}} + (\delta+1)(\cos\sigma+\nu-6)e^{\frac{1}{\delta+1}} + (\cos\sigma+\nu-2)(\delta+1)^2\left(e^{\frac{1}{\delta+1}}-1\right) \ge 0.$$

This accomplish the proof of Theorem 2. □

By taking $\nu = 0$ in Theorems 1 and 2, we immediately reach the next consequences.

Corollary 1. *The function $\Psi^{(1)}(\gamma,\delta;z) \in \mathcal{SP}_p(\sigma)$ if the following condition is verified for $\gamma \ge 1$.*

$$\delta\cos\sigma + (\cos\sigma-2)(\delta+1)\left(e^{\frac{1}{\delta+1}}-1\right) - 2e^{\frac{1}{\delta+1}} \ge 0.$$

Corollary 2. *The function $\Psi^{(1)}(\gamma,\delta;z) \in \mathcal{UCSP}(\sigma)$ if the following condition is verified for $\gamma \ge 1$.*

$$\delta(\delta+1)\cos\sigma - 2e^{\frac{1}{\delta+1}} + (\delta+1)(\cos\sigma-6)e^{\frac{1}{\delta+1}} + (\cos\sigma-2)(\delta+1)^2\left(e^{\frac{1}{\delta+1}}-1\right) \ge 0.$$

By picking $\sigma = 0$ in Corollary 1 and in Corollary 2, we immediately arrive at the next consequences.

Corollary 3. *Let $\gamma \ge 1$ and $\delta > x_0 \cong 3.60234$ where x_0 is the numerical root of*

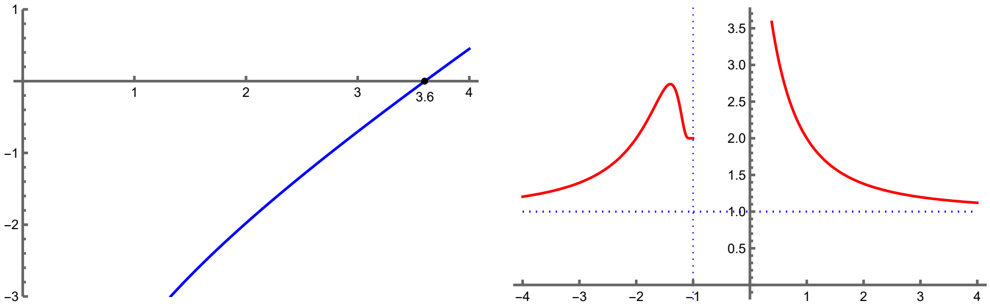
$$\left(2-e^{\frac{1}{x+1}}\right)x - 3e^{\frac{1}{x+1}} + 1 = 0,$$

then $\Psi^{(1)}(\gamma,\delta;z) \in \mathcal{SP}$.

Proof. Let $y = \left(2-e^{\frac{1}{x+1}}\right)x - 3e^{\frac{1}{x+1}} + 1$, $x > 0$. By straightforward computation, we have

$$y\prime = 2 - \left(1 - \frac{x+3}{(x+1)^2}\right)e^{\frac{1}{x+1}}.$$

From the graph of the function $g(x) = y\prime$, we immediately can observe that $g(x) = y\prime(x) > 0$ for each $x > 0$ (see Figure 1b).



(**a**) (**b**)

Figure 1. The graph of the function: (**a**) $f(x) = y = \left(2 - e^{\frac{1}{x+1}}\right)x - 3e^{\frac{1}{x+1}} + 1$ and (**b**) $g(x) = y\prime = 2 - \left(1 - \frac{x+3}{(x+1)^2}\right)e^{\frac{1}{x+1}}$.

Thus, $y(x)$ is an increasing function for $x > 0$.

Moreover, the graph of the function $y(x)$ shows that the equation

$$\left(2 - e^{\frac{1}{x+1}}\right)x - 3e^{\frac{1}{x+1}} + 1 = 0,$$

has a numerical root that equals $x_0 = 3.60234$ (see Figure 1a).

Therefore, $\left(2 - e^{\frac{1}{\delta+1}}\right)\delta - 3e^{\frac{1}{\delta+1}} + 1 \geq 0$ for every $\delta \geq x_0$.

Thus, the proof is finished. $\square$

Corollary 4. *Let $\gamma \geq 1$ and $\delta > x_1 \cong 7.01251$ where x_1 is the numerical root of*

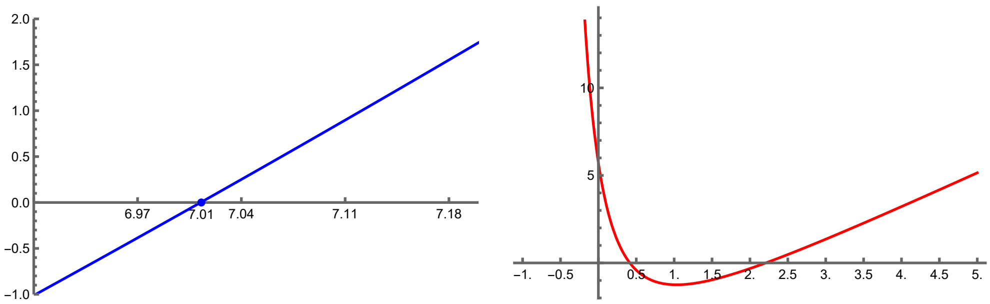
$$\left(2 - e^{\frac{1}{x+1}}\right)x^2 + \left(3 - 7e^{\frac{1}{x+1}}\right)x - 8e^{\frac{1}{x+1}} + 1 = 0,$$

then $\Psi^{(1)}(\gamma, \delta; z) \in \mathcal{UCV}$.

Proof. Let $y = \left(2 - e^{\frac{1}{x+1}}\right)x^2 + \left(3 - 7e^{\frac{1}{x+1}}\right)x - 8e^{\frac{1}{x+1}} + 1$, $x > 0$. By straightforward computation, we have

$$y\prime = (4x+3) - \left(2x + 7 - \frac{x^2 + 7x + 8}{(x+1)^2}\right)e^{\frac{1}{x+1}}.$$

From the graph of function $g(x) = y\prime$, we can immediately observe that $g(x) = y\prime(x) > 0$ for each $x > 2.25$ (see Figure 2b).



(**a**) (**b**)

Figure 2. The graph of function: (**a**) $f(x) = y = \left(2 - e^{\frac{1}{x+1}}\right)x^2 + \left(3 - 7e^{\frac{1}{x+1}}\right)x - 8e^{\frac{1}{x+1}} + 1$ and (**b**) $g(x) = y\prime = (4x+3) - \left(2x + 7 - \frac{x^2+7x+8}{(x+1)^2}\right)e^{\frac{1}{x+1}}$.

Thus, $y(x)$ is an increasing function for $x > 2.25$.

Moreover, the graph of function $y(x)$ shows that the equation

$$\left(2-e^{\frac{1}{x+1}}\right)x^2+\left(3-7e^{\frac{1}{x+1}}\right)x-8e^{\frac{1}{x+1}}+1=0,$$

has a numerical root equals $x_1=7.01251$ (see Figure 1a).

Therefore, $\left(2-e^{\frac{1}{\delta+1}}\right)\delta^2+\left(3-7e^{\frac{1}{\delta+1}}\right)\delta-8e^{\frac{1}{\delta+1}}+1\geq 0$ for every $\delta\geq x_1$.

Thus, the proof is finished. □

Theorem 3. *The function $\Psi^{(2)}(\gamma,\delta;z)\in\mathcal{SP}_p(\sigma,\nu)$ if the next condition is verified for $\gamma\geq 1$.*

$$(\cos\sigma-\nu)(\gamma+\delta)+(\cos\sigma+\nu)(\gamma+\delta+1)^2\left(e^{\frac{1}{\gamma+\delta+1}}-1\right)+(\gamma+\delta+1)\left(2-\cos\sigma-\nu-2e^{\frac{1}{\gamma+\delta+1}}\right)\geq 0.$$

Proof. Since

$$\Psi^{(2)}(\gamma,\delta;z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma(n-1)+\gamma+\delta)}\frac{z^n}{n!},$$

then by (8), it suffices to show that

$$\sum_{n=2}^{\infty}\frac{(2n-\cos\sigma-\nu)\Gamma(\gamma+\delta)}{\Gamma(\gamma(n-1)+\gamma+\delta)}\frac{1}{n!}\leq\cos\sigma-\nu.$$

Let

$$k_3(\gamma,\delta;\sigma,\nu)=\sum_{n=2}^{\infty}\frac{(2n-\cos\sigma-\nu)\Gamma(\gamma+\delta)}{\Gamma(\gamma(n-1)+\gamma+\delta)}\frac{1}{n!}.$$

By straightforward computation, we have

$$k_3(\gamma,\delta;\sigma,\nu)=\sum_{n=2}^{\infty}\frac{2\Gamma(\gamma+\delta)}{(n-1)!\Gamma(\gamma(n-1)+\gamma+\delta)}-\sum_{n=2}^{\infty}\frac{(\cos\sigma+\nu)\Gamma(\gamma+\delta)}{n!\Gamma(\gamma(n-1)+\gamma+\delta)}.$$

Using Equations(10) and (11), we obtain

$$\begin{aligned}k_3(\gamma,\delta;\sigma,\nu)&\leq\sum_{n=2}^{\infty}\frac{2}{(\gamma+\delta)(\gamma+\delta+1)^{n-2}(n-1)!}-\sum_{n=2}^{\infty}\frac{\cos\sigma+\nu}{(\gamma+\delta)(\gamma+\delta+1)^{n-2}n!}\\&=\frac{2(\gamma+\delta+1)}{\gamma+\delta}\left(e^{\frac{1}{\gamma+\delta+1}}-1\right)-\frac{(\cos\sigma+\nu)(\gamma+\delta+1)^2}{\gamma+\delta}\left(e^{\frac{1}{\gamma+\delta+1}}-\frac{1}{\gamma+\delta+1}-1\right)\\&\leq\cos\sigma-\nu.\end{aligned}$$

Hence,

$$(\cos\sigma-\nu)(\gamma+\delta)+(\cos\sigma+\nu)(\gamma+\delta+1)^2\left(e^{\frac{1}{\gamma+\delta+1}}-1\right)+(\gamma+\delta+1)\left(2-\cos\sigma-\nu-2e^{\frac{1}{\gamma+\delta+1}}\right)\geq 0.$$

This accomplishes the proof of Theorem 3. □

If we set $\nu=0$ in Theorem 3, we immediately arrive at the next consequence.

Corollary 5. *The normalized Wright function $\Psi^{(2)}(\gamma,\delta;z)\in\mathcal{SP}_p(\sigma)$ if the following condition is verified for $\gamma\geq 1$.*

$$(\gamma+\delta)\cos\sigma+\cos\sigma(\gamma+\delta+1)^2\left(e^{\frac{1}{\gamma+\delta+1}}-1\right)+(\gamma+\delta+1)\left(2-\cos\sigma-2e^{\frac{1}{\gamma+\delta+1}}\right)\geq 0.$$

By picking $\sigma=0$ in Corollary 5, we immediately arrive at the next consequence.

Corollary 6. *Let $\gamma\geq 1$ and $\delta>x_2\cong 1.83392$ where x_2 is the numerical root of*

$$\left(e^{\frac{1}{x+1}}-1\right)x^2-e^{\frac{1}{x+1}}=0,$$

then $\Psi^{(2)}(\gamma,\delta;z) \in \mathcal{SP}$.

Proof. Let $y = \left(e^{\frac{1}{x+1}} - 1\right)x^2 - e^{\frac{1}{x+1}}$, $x > 0$. By straightforward computation, we have

$$y\prime = \left(2x - \frac{x^2-1}{(x+1)^2}\right)e^{\frac{1}{x+1}} - 2x.$$

From the graph of function $g(x) = y\prime$, we immediately can observe that $g(x) = y\prime(x) > 0$ for each $x > 0$ (see Figure 3b).

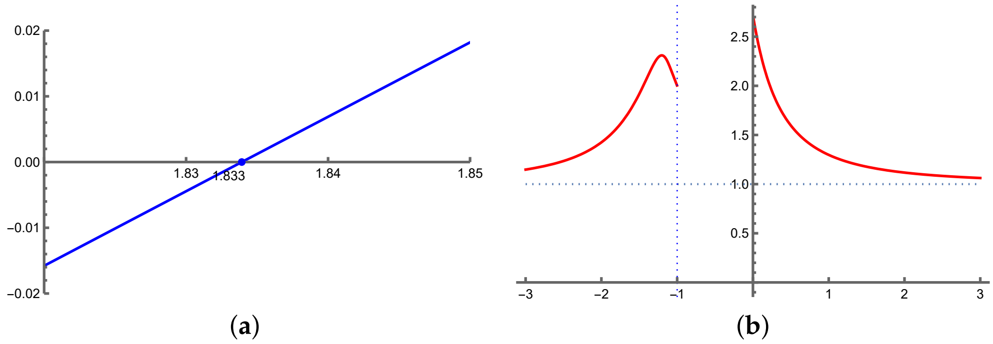


Figure 3. The graph of function: (**a**) $f(x) = y = \left(e^{\frac{1}{x+1}} - 1\right)x^2 - e^{\frac{1}{x+1}}$ and (**b**) $g(x) = y\prime = \left(2x - \frac{x^2-1}{(x+1)^2}\right)e^{\frac{1}{x+1}} - 2x$.

Thus, $y(x)$ is an increasing function for $x > 0$.

Moreover, the graph of function $y(x)$ shows that the equation

$$\left(e^{\frac{1}{x+1}} - 1\right)x^2 - e^{\frac{1}{x+1}} = 0,$$

has a numerical root equal to $x_2 = 1.83392$ (see Figure 3a).

Therefore, $\left(e^{\frac{1}{\delta+1}} - 1\right)\delta^2 - e^{\frac{1}{\delta+1}} \geq 0$ for every $\delta \geq x_2$.

Thus, the proof is finished. □

Theorem 4. *The function* $\Psi^{(2)}(\gamma,\delta;z) \in \mathcal{UCSP}(\sigma,\nu)$ *if the following condition is verified for* $\gamma \geq 1$.

$$(\gamma+\delta)(\cos\sigma - \nu) + (\cos\sigma + \nu - 2)(\gamma+\delta+1)\left(e^{\frac{1}{\gamma+\delta+1}} - 1\right) - 2e^{\frac{1}{\gamma+\delta+1}} \geq 0.$$

Proof. The function $\Psi^{(2)}(\gamma,\delta;z) \in \mathcal{UCSP}(\sigma,\nu)$ if $z(\Psi^{(2)}(\gamma,\delta;z))' \in \mathcal{SP}_p(\sigma,\nu)$, but from Equation (1) $z(\Psi^{(2)}(\gamma,\delta;z))' = \Psi^{(1)}(\gamma,\gamma+\delta;z)$. Then, making use of Theorem 1, the proof of the current theorem is finished. □

If we set $\nu = 0$ in Theorem 4, we immediately arrive at the next consequence.

Corollary 7. *The normalized Wright function* $\Psi^{(2)}(\gamma,\delta;z) \in \mathcal{UCSP}(\sigma)$ *if the following condition is verified for* $\gamma \geq 1$.

$$(\gamma+\delta)\cos\sigma + (\cos\sigma - 2)(\gamma+\delta+1)\left(e^{\frac{1}{\gamma+\delta+1}} - 1\right) - 2e^{\frac{1}{\gamma+\delta+1}} \geq 0.$$

By picking $\sigma = 0$ in Corollary 7, we immediately arrive at the next consequence.

Corollary 8. *Let* $\gamma \geq 1$ *and* $\delta > x_0 \cong 3.60234$ *where* x_0 *is the numerical root of*

$$\left(2 - e^{\frac{1}{x+1}}\right)x - 3e^{\frac{1}{x+1}} + 1 = 0,$$

then $\Psi^{(2)}(\gamma,\delta;z) \in \mathcal{UCV}$.

3. Necessary and Sufficient Conditions for the Normalized Bessel Functions to Be in $\mathcal{SP}_p(\sigma,\nu)$ and $\mathcal{UCSP}(\sigma,\nu)$

If we set $\gamma = 1$, $\delta = p+1$ and $z = -z$ in Theorem 1, from Equation (7) we directly obtain the next results.

Theorem 5. *The function $\overline{J}_p(z) \in \mathcal{SP}_p(\sigma,\nu)$ if the next condition is verified.*

$$(p+1)(\cos\sigma - \nu) + (\cos\sigma + \nu - 2)(p+2)\left(e^{\frac{1}{p+2}} - 1\right) - 2e^{\frac{1}{p+2}} \geq 0.$$

If we set $\nu = 0$ in Theorem 5, we immediately arrive at the next consequence.

Corollary 9. *The normalized Bessel function $\overline{J}_p(z) \in \mathcal{SP}_p(\sigma)$ if the following condition is verified.*

$$(p+1)\cos\sigma + (\cos\sigma - 2)(p+2)\left(e^{\frac{1}{p+2}} - 1\right) - 2e^{\frac{1}{p+2}} \geq 0.$$

By picking $\sigma = 0$ in Corollary 9, we immediately arrive at the next consequence.

Corollary 10. *Let $p > x_0 - 1$, where $x_0 \cong 2.4898$ is the numerical root of*

$$\left(2 - e^{\frac{1}{x+1}}\right)x - 3e^{\frac{1}{x+1}} + 1 = 0,$$

then $\overline{J}_p(z) \in \mathcal{SP}$.

If we set $\gamma = 1$, $\delta = p+1$ and $z = -z$ in Theorem 2, from Equation (7) we directly have the next results.

Theorem 6. *The function $\overline{J}_p(z) \in \mathcal{UCSP}(\sigma,\nu)$ if the following condition is verified.*

$$\begin{aligned}(p+1)(p+2)(\cos\sigma - \nu) - 2e^{\frac{1}{p+2}} \quad &+ \quad (p+2)(\cos\sigma + \nu - 6)e^{\frac{1}{p+2}} \\ &+ \quad (\cos\sigma + \nu - 2)(p+2)^2\left(e^{\frac{1}{p+2}} - 1\right) \geq 0.\end{aligned}$$

If we set $\nu = 0$ in Theorem 6, we arrive at the next corollary.

Corollary 11. *The normalized Bessel function $\overline{J}_p(z) \in \mathcal{UCSP}(\sigma)$ if the following condition is verified.*

$$(p+1)(p+2)\cos\sigma - 2e^{\frac{1}{p+2}} + (p+2)(\cos\sigma - 6)e^{\frac{1}{p+2}} + (\cos\sigma - 2)(p+2)^2\left(e^{\frac{1}{p+2}} - 1\right) \geq 0.$$

By picking $\sigma = 0$ in Corollary 11, we arrive at the following corollary.

Corollary 12. *Let $p > x_1 - 1$, where $x_1 \cong 4.8523$ is the numerical root of*

$$\left(2 - e^{\frac{1}{x+1}}\right)x^2 + \left(3 - 7e^{\frac{1}{x+1}}\right)x - 6e^{\frac{1}{x+1}} + 1 = 0,$$

then $\overline{J}_p(z) \in \mathcal{UCV}$.

4. Conclusions

In the present work, we establish some sufficient and necessary conditions for the normalized Wright functions $\Psi^{(1)}(\gamma,\delta;z)$ and $\Psi^{(2)}(\gamma,\delta;z)$ so that they are in the subclasses of normalized analytic functions $\mathcal{SP}_p(\sigma, \nu)$ and $\mathcal{UCSP}(\sigma, \nu)$. Some interesting corollaries and applications of the results are also discussed. Using the normalized Wright functions $\Psi^{(1)}(\gamma, \delta; z)$ and $\Psi^{(2)}(\gamma, \delta; z)$ could inspire researchers to find new necessary and sufficient conditions for these functions so that they are in different subclasses of normalized analytic functions with negative coefficients defined in the open unit disk $\mathbb{U}$.

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