

Article

Chen Inequalities for Spacelike Submanifolds in Statistical Manifolds of Type Para-Kähler Space Forms

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Abstract: In this paper, we prove some inequalities between intrinsic and extrinsic curvature invariants, namely involving the Chen first invariant and the mean curvature of totally real and holomorphic spacelike submanifolds in statistical manifolds of type para-Kähler space forms. Furthermore, we investigate the equality cases of these inequalities. As illustrations of the applications of the above inequalities, we consider a few examples.

Keywords: Chen inequality; statistical manifold of type para-Kähler space form; totally real submanifold



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1. Introduction

A fundamental challenge in submanifold theory is to obtain simple relationships between the main intrinsic and extrinsic invariants of submanifolds [1]. There is an increased interest to provide answers of this open problem establishing some types of geometric inequalities (see, e.g., [2–6]).

The study of Chen invariants started in 1993, when Chen investigated basic inequalities for submanifolds in real space forms, now called the *Chen inequalities* [7]. An insightful and comprehensive study on Chen inequalities can be discovered in [2]. Recently, illustrations of some selected research works on Chen invariants are revealed in [8–10]. Actual solutions of the above problem are focused on inequalities for submanifolds in a *statistical manifold*, concept introduced by Amari [11] in 1985 in the context of information geometry. The statistical manifolds also have applications in physics, machine learning, statistics, etc. Recently, Chen et al. established a Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature [12]. Moreover, Aytimur et al. studied Chen inequalities for statistical submanifolds of Kähler-like statistical manifolds [13].

The para-complex numbers (hyperbolic numbers) are introduced by Graves in 1845 [14] in the form $z = x + yj$, where x and y are real numbers, $j^2 = 1$ and $j \neq 1$. Later, the *para-Kähler geometry* is concerned with the study of para-Kähler structures. The notion of *para-Kähler manifold*, first called *stratified space* by Rashevskij [15], is clearly defined in 1949 by Ruse [16] and Rozenfeld [17]. This geometry has applications in areas of mathematics, mechanics, physics [18], and general theory of relativity [19]. Mihai et al. investigated skew-symmetric vector fields on CR-submanifolds of para-Kähler manifolds [20]. Recently, the concept of Codazzi-para-Kähler structure was introduced by Fei and Zhang in [21] in order to represent the interaction of Codazzi couplings with para-Kähler geometry. Very recently, Vilcu studied statistical manifolds endowed with almost product structures and para-Kähler-like statistical submersions [22]. Moreover, Chen et al. established Casorati inequalities for totally real spacelike submanifolds in statistical manifolds of type para-Kähler space forms [23].

The topic of *totally real* and *Lagrangian submanifolds* of Kähler manifolds has been studied extensively (see, e.g., [24–27]). However, just a few results are devoted to the context of para-Kähler manifolds. Chen established optimal Chen inequalities for Lagrangian submanifolds of the flat para-Kähler manifold (E^{2n}, g, P) [28]. Furthermore, the Lagrangian H-umbilical submanifolds of para-Kähler manifolds are explored in [29,30].

In this paper, we prove some inequalities between intrinsic and extrinsic curvature invariants, namely involving the Chen first invariant and the mean curvature of totally real and holomorphic spacelike submanifolds in statistical manifolds of type para-Kähler space forms. In this respect, we use a calculus of optimization theory. In addition, the case of equalities is demonstrated and some examples are revealed.

2. Preliminaries

A *statistical manifold* is a semi-Riemannian manifold $(\bar{M}, \bar{g})$, endowed with a pair of torsion-free affine connections $(\bar{\nabla}, \bar{\nabla}^*)$ which satisfy the formula:

$$X \bar{g}(Y, Z) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}_X^* Z),$$

for any $X, Y, Z \in \Gamma(T\bar{M})$, where $\Gamma(T\bar{M})$ is the set of smooth tangent vector fields on $\bar{M}$. Denote by $(\bar{M}, \bar{g}, \bar{\nabla})$ a statistical manifold [31]. The connections $\bar{\nabla}$ and $\bar{\nabla}^*$ are called *conjugate (dual) connections*. An obvious property of these dual connections is $\bar{\nabla} = (\bar{\nabla}^*)^*$. The pair $(\bar{\nabla}, \bar{g})$ is named statistical structure.

It follows that $(\bar{M}, \bar{g}, \bar{\nabla}^*)$ is also a statistical manifold. Denote by $\bar{\nabla}^0$ the Levi-Civita connection of $\bar{M}$ defined by $\bar{\nabla}^0 = \frac{\bar{\nabla} + \bar{\nabla}^*}{2}$ [32].

Suppose M is an m -dimensional submanifold of a $2n$ -dimensional statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla})$ with g the induced metric on M , and ∇ the induced connection on M . Then (M, g, ∇) is likewise a statistical manifold.

The *formulas of Gauss* [31] are represented by the expressions:

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \bar{\nabla}_X^* Y &= \nabla_X^* Y + h^*(X, Y),\end{aligned}$$

for any $X, Y \in \Gamma(TM)$, where the bilinear and symmetric $(0, 2)$ -tensors h and h^* are called the *imbedding curvature tensor* of M in $\bar{M}$ with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$, respectively.

Denote by R and $\bar{R}$ the $(0, 4)$ -curvature tensors for the connections ∇ and $\bar{\nabla}$, respectively.

Then, for the vector fields X, Y, Z , and W tangent to M , the *equation of Gauss* on the connection $\bar{\nabla}$ is given by [32]:

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) + \bar{g}(h(X, Z), h^*(Y, W)) \\ &- \bar{g}(h^*(X, W), h(Y, Z)).\end{aligned}\quad (1)$$

Similarly, let R^* and $\bar{R}^*$ be the $(0, 4)$ -curvature tensors for the connections ∇^* and $\bar{\nabla}^*$, respectively.

Hence, the *equation of Gauss* on the connection $\bar{\nabla}^*$ becomes [32]:

$$\begin{aligned}\bar{g}(\bar{R}^*(X, Y)Z, W) &= g(R^*(X, Y)Z, W) + \bar{g}(h^*(X, Z), h(Y, W)) \\ &- \bar{g}(h(X, W), h^*(Y, Z)),\end{aligned}\quad (2)$$

for any vector fields X, Y, Z , and W tangent to M .

Denote by S the *statistical curvature tensor field* on the statistical manifold (M, g, ∇) defined by [31]:

$$S(X, Y)Z = \frac{1}{2}\{R(X, Y)Z + R^*(X, Y)Z\},\quad (3)$$

for any $X, Y, Z \in \Gamma(TM)$. Clearly, S is skew-symmetric. Thus S is a Riemann-curvature-like-tensor [33].

For a non-degenerate two-dimensional subspace π of the tangent space $T_x M$, at a point $x \in M$, the *sectional curvature* of (M, ∇, g) [31] is given by:

$$K(\pi) = K(X \wedge Y) = \frac{g(S(X, Y)Y, X)}{g(X, X)g(Y, Y) - g^2(X, Y)}, \quad (4)$$

where $\{X, Y\}$ is a basis of π .

Denote by K_0 the sectional curvature of the Levi-Civita connection ∇^0 on M .

The *scalar curvature* τ of (M, ∇, g) at a point $x \in M$ is defined by the expression:

$$\tau(x) = \sum_{1 \leq i < j \leq m} K(e_i \wedge e_j) = \sum_{1 \leq i < j \leq m} g(S(e_i, e_j)e_j, e_i), \quad (5)$$

where $\{e_1, \dots, e_m\}$ is an orthonormal frame at x .

Denote by τ_0 the scalar curvature of the Levi-Civita connection ∇^0 on M .

Let $\{e_1, \dots, e_m\}$ and $\{e_{m+1}, \dots, e_{2n}\}$ be orthonormal bases of the tangent space $T_x M$ and $T_x^\perp M$, respectively, at a point $x \in M$. Then, the *mean curvature* vector fields of M for $\bar{\nabla}$ and $\bar{\nabla}^*$ are defined by, respectively:

$$H = \frac{1}{m} \sum_{i=1}^m h_{ii}, \quad H^* = \frac{1}{m} \sum_{i=1}^m h_{ii}^*.$$

For the Levi-Civita connection $\bar{\nabla}^0$, we denote by $h^0 = \frac{h+h^*}{2}$ the *second fundamental form*, and by

$$H^0 = \frac{H + H^*}{2} \quad (6)$$

the *mean curvature* vector field of M .

Next, the *squared mean curvatures* of M for ∇ and ∇^* are given by:

$$\|H\|^2 = \frac{1}{m^2} \sum_{\alpha=m+1}^{2n} \left(\sum_{i=1}^m h_{ii}^\alpha \right)^2, \quad \|H^*\|^2 = \frac{1}{m^2} \sum_{\alpha=m+1}^{2n} \left(\sum_{i=1}^m h_{ii}^{*\alpha} \right)^2,$$

where $h_{ij}^\alpha = g(h(e_i, e_j), e_\alpha)$ and $h_{ij}^{*\alpha} = g(h^*(e_i, e_j), e_\alpha)$, for $i, j \in \{1, \dots, m\}$, $\alpha \in \{m+1, \dots, 2n\}$.

A tensor field $P \neq \pm I$ of type $(1, 1)$, satisfying $P^2 = I$, where I is the identity tensor field on $\bar{M}$, is named an *almost product structure* on $\bar{M}$.

An *almost para-Hermitian manifold* denoted by $(\bar{M}, P, \bar{g})$ [2] is a manifold $\bar{M}$ equipped with an almost product structure P and a semi-Riemannian metric $\bar{g}$ performing:

$$\bar{g}(PX, PY) = -\bar{g}(X, Y), \quad (7)$$

for all vector fields X, Y on $\bar{M}$. Notice that the dimension of $(\bar{M}, P, \bar{g})$ is even.

If $(\bar{M}, P, \bar{g})$ satisfies the formula $\bar{\nabla}P = 0$, then it is called a *para-Kähler manifold* [2], where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$.

An *almost para-Hermitian-like manifold* $(\bar{M}, P, \bar{g})$ [22] is a semi-Riemannian manifold $(\bar{M}, \bar{g})$ equipped with an almost product structure P satisfying:

$$\bar{g}(PX, Y) + \bar{g}(X, P^*Y) = 0, \quad (8)$$

for all vector fields X, Y on $\bar{M}$, where P^* is $(1, 1)$ -tensor field on $\bar{M}$.

A *para-Kähler-like statistical manifold* [22] is defined as an almost para-Hermitian-like manifold $(\bar{M}, P, \bar{g})$ endowed with a statistical structure $(\bar{\nabla}, \bar{g})$ such that $\bar{\nabla}P = 0$. It follows that the para-Kähler-like statistical manifolds are the generalization case of the para-Kähler manifolds.

A statistical manifold of type para-Kähler space form [22] is defined as a para-Kähler-like statistical manifold $(\bar{M}, \bar{\nabla}, P, \bar{g})$ where the curvature tensor $\bar{R}$ of the connection $\bar{\nabla}$ satisfies:

$$\begin{aligned} \bar{R}(X, Y)Z = & \frac{c}{4} \{ \bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(PY, Z)PX \\ & - \bar{g}(PX, Z)PY + \bar{g}(X, PY)PZ - \bar{g}(PX, Y)PZ \}, \end{aligned} \quad (9)$$

for any vector fields X, Y , and Z and a real constant c .

A submanifold M in an almost para-Hermitian (like) manifold $(\bar{M}, P, \bar{g})$ is called *totally real* if P maps each tangent space $T_x M$ into its corresponding normal space $T_x^\perp M$.

A submanifold M in an almost para-Hermitian (like) manifold $(\bar{M}, P, \bar{g})$ is called *holomorphic (or invariant)* submanifold if $P(T_x M) = T_x M, x \in M$.

We consider the following constrained extremum problem

$$\min_{x \in M} f(x), \quad (10)$$

where M is a submanifold of a (semi)-Riemannian manifold $(\bar{M}, \bar{g})$, and $f : \bar{M} \rightarrow \mathbb{R}$ is a function of differentiability class C^2 .

Theorem 1 ([34]). *If M is complete and connected, $(\text{grad} f)(x_0) \in T_{x_0}^\perp M$ for a point $x_0 \in M$, and the bilinear form $\mathcal{F} : T_{x_0} M \times T_{x_0} M \rightarrow \mathbb{R}$ defined by:*

$$\mathcal{F}(X, Y) = \text{Hess}(f)(X, Y) + \bar{g}(h(X, Y), \text{grad} f), \quad (11)$$

is positive definite in x_0 , then x_0 is the optimal solution of the problem (10), where h is the second fundamental form of M .

Remark 1 ([34]). *If the bilinear form $\mathcal{F}$ defined by (11) is positive semi-definite on the submanifold M , then the critical points of $f|_M$ are global optimal solutions of the problem (10).*

3. Main Inequalities

Theorem 2. (i) *The Chen first invariant of a holomorphic spacelike submanifold M of dimension m in a statistical manifold of type para-Kähler space form $(\bar{M}, \bar{\nabla}, P, \bar{g})$ of dimension $2n$ satisfies the inequality:*

$$\begin{aligned} \tau - K(\pi) \geq & 2(\tau_0 - K_0(\pi)) + \frac{c}{8} \sum_{1 \leq i < j \leq m} \{ g^2(e_i, Pe_j) + g^2(Pe_i, e_j) + 2g(e_i, Pe_i)g(e_j, Pe_j) \\ & - 4g(e_i, Pe_j)g(Pe_i, e_j) \} - \frac{c}{8} \{ g^2(e_1, Pe_2) + g^2(Pe_1, e_2) + 2g(e_1, Pe_1)g(e_2, Pe_2) \\ & - 4g(e_1, Pe_2)g(e_2, Pe_1) \} + \frac{(m-2)(m+1)c}{8} - \frac{m^2(m-2)}{4(m-1)} (\|H\|^2 + \|H^*\|^2). \end{aligned} \quad (12)$$

(ii) *The Chen first invariant of a totally real spacelike submanifold M of dimension m in a statistical manifold of type para-Kähler space form $(\bar{M}, \bar{\nabla}, P, \bar{g})$ of dimension $2n$ satisfies the inequality:*

$$\tau - K(\pi) \geq 2(\tau_0 - K_0(\pi)) + \frac{(m-2)(m+1)c}{8} - \frac{m^2(m-2)}{4(m-1)} (\|H\|^2 + \|H^*\|^2). \quad (13)$$

Moreover, the equality cases of (12) and (13) hold identically at all points $x \in M$ if and only if we have:

$$\begin{aligned} h_{1j}^\alpha &= h_{2j}^\alpha = h_{ij}^\alpha = 0, \\ h_{1j}^{*\alpha} &= h_{2j}^{*\alpha} = h_{ij}^{*\alpha} = 0, \\ h_{11}^\alpha + h_{22}^\alpha &= h_{33}^\alpha = \dots = h_{mm}^\alpha, \\ h_{11}^{*\alpha} + h_{22}^{*\alpha} &= h_{33}^{*\alpha} = \dots = h_{mm}^{*\alpha}, \end{aligned}$$

for any $\alpha \in \{m+1, \dots, 2n\}$ and any $i, j \in \{3, \dots, m\}$, $i < j$.

Proof. For $x \in M$, consider $\{e_1, \dots, e_m\}$ and $\{e_{m+1}, \dots, e_{2n}\}$ orthonormal bases of $T_x M$ and $T_x^\perp M$, respectively. Let $\{e_1, e_2\}$ be an orthonormal basis in a plane section π at x , where the sectional curvature of $T_x M$ is minimum. The Chen first invariant is defined by the expression:

$$\delta_M(x) = \tau(x) - K(\pi).$$

The sectional curvature $K(\pi)$ of the plane section π is given by:

$$K(\pi) = \frac{1}{2} [g(R(e_1, e_2)e_2, e_1) + g(R^*(e_1, e_2)e_2, e_1)]. \quad (14)$$

From the Formulas (1) and (9), we achieve:

$$\begin{aligned} g(R(e_1, e_2)e_2, e_1) &= \frac{c}{4} \{1 + g^2(e_1, Pe_2) + g(e_2, Pe_2)g(e_1, Pe_1) \\ &\quad - 2g(Pe_1, e_2)g(e_1, Pe_2)\} + \sum_{\alpha=m+1}^{2n} (h_{11}^{*\alpha}h_{22}^\alpha - h_{12}^{*\alpha}h_{12}^\alpha). \end{aligned} \quad (15)$$

From the Formulas (2) and (9), we have:

$$\begin{aligned} g(R^*(e_1, e_2)e_2, e_1) &= -g(R(e_1, e_2)e_1, e_2) = -\frac{c}{4} \{-1 - g^2(Pe_1, e_2) - g(e_2, Pe_2)g(e_1, Pe_1) \\ &\quad + 2g(Pe_1, e_2)g(e_1, Pe_2)\} + \sum_{\alpha=m+1}^{2n} (h_{11}^\alpha h_{22}^{*\alpha} - h_{12}^{*\alpha}h_{12}^\alpha). \end{aligned} \quad (16)$$

Replacing (15) and (16) in (14), we find:

$$\begin{aligned} K(\pi) &= \frac{c}{4} \{1 + \frac{g^2(e_1, Pe_2)}{2} + \frac{g^2(Pe_1, e_2)}{2} + g(e_2, Pe_2)g(e_1, Pe_1) \\ &\quad - 2g(Pe_1, e_2)g(e_1, Pe_2)\} + \frac{1}{2} \sum_{\alpha=m+1}^{2n} (h_{11}^\alpha h_{22}^{*\alpha} + h_{11}^{*\alpha}h_{22}^\alpha - 2h_{12}^\alpha h_{12}^{*\alpha}). \end{aligned} \quad (17)$$

From $2h^0 = h + h^*$, it follows that (17) becomes:

$$\begin{aligned} K(\pi) &= \frac{c}{4} \{1 + \frac{g^2(e_1, Pe_2)}{2} + \frac{g^2(Pe_1, e_2)}{2} + g(e_2, Pe_2)g(e_1, Pe_1) \\ &\quad - 2g(Pe_1, e_2)g(e_1, Pe_2)\} + 2 \sum_{\alpha=m+1}^{2n} [h_{11}^{0\alpha}h_{22}^{0\alpha} - (h_{12}^{0\alpha})^2] \\ &\quad - \frac{1}{2} \sum_{\alpha=m+1}^{2n} \{[h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2] + [h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2]\}. \end{aligned} \quad (18)$$

On the other hand, the scalar curvature of M , related to the sectional curvature K is given by:

$$\tau = \frac{1}{2} \sum_{1 \leq i < j \leq m} [g(R(e_i, e_j)e_j, e_i) + g(R^*(e_i, e_j)e_j, e_i)]. \quad (19)$$

From the Formulas (2) and (9), we obtain:

$$\begin{aligned} \sum_{1 \leq i < j \leq m} g(R(e_i, e_j)e_j, e_i) &= \frac{m(m-1)c}{8} + \frac{c}{4} \sum_{1 \leq i < j \leq m} [g^2(e_i, Pe_j) + g(e_i, Pe_i)g(e_j, Pe_j) \\ &\quad - 2g(Pe_i, e_j)g(e_i, Pe_j)] \\ &\quad + \sum_{1 \leq i < j \leq m} [g(h^*(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h^*(e_i, e_j))]. \end{aligned} \quad (20)$$

Similarly, we have:

$$\begin{aligned} \sum_{1 \leq i < j \leq m} g(R^*(e_i, e_j)e_j, e_i) &= - \sum_{1 \leq i < j \leq m} g(R(e_i, e_j)e_i, e_j) = \frac{m(m-1)c}{8} \\ &\quad + \frac{c}{4} \sum_{1 \leq i < j \leq m} [g(e_i, Pe_i)g(e_j, Pe_j) + g^2(e_j, Pe_i) \\ &\quad - 2g(e_i, Pe_j)g(e_j, Pe_i)] \\ &\quad - \sum_{1 \leq i < j \leq m} [g(h(e_i, e_i), h^*(e_j, e_j)) - g(h(e_i, e_j), h^*(e_i, e_j))]. \end{aligned} \quad (21)$$

Replacing (20) and (21) in (19), we find:

$$\begin{aligned} \tau &= \frac{m(m-1)c}{8} + \frac{c}{4} \sum_{1 \leq i < j \leq m} \left\{ \frac{g^2(e_i, Pe_j)}{2} + g(e_i, Pe_i)g(e_j, Pe_j) \right. \\ &\quad \left. - 2g(Pe_i, e_j)g(e_i, Pe_j) + \frac{g^2(e_j, Pe_i)}{2} \right\} \\ &\quad + \frac{1}{2} \sum_{\alpha=m+1}^{2n} \sum_{1 \leq i < j \leq m} (h_{ii}^\alpha h_{jj}^{*\alpha} + h_{jj}^\alpha h_{ii}^{*\alpha} - 2h_{ij}^\alpha h_{ij}^{*\alpha}) \\ &= \frac{m(m-1)c}{8} + \frac{c}{4} \sum_{1 \leq i < j \leq m} \left\{ \frac{g^2(e_i, Pe_j)}{2} + g(e_i, Pe_i)g(e_j, Pe_j) \right. \\ &\quad \left. - 2g(Pe_i, e_j)g(e_i, Pe_j) + \frac{g^2(e_j, Pe_i)}{2} \right\} \\ &\quad + \frac{1}{2} \sum_{\alpha=m+1}^{2n} \sum_{1 \leq i < j \leq m} [(h_{ii}^\alpha + h_{ii}^{*\alpha})(h_{jj}^\alpha + h_{jj}^{*\alpha}) - h_{ii}^\alpha h_{jj}^\alpha - h_{ii}^{*\alpha} h_{jj}^{*\alpha} \\ &\quad - (h_{ij}^\alpha + h_{ij}^{*\alpha})^2 + (h_{ij}^\alpha)^2 + (h_{ij}^{*\alpha})^2]. \end{aligned} \quad (22)$$

From $2h^0 = h + h^*$, the latter equation becomes:

$$\begin{aligned} \tau = & \frac{m(m-1)c}{8} + \frac{c}{4} \sum_{1 \leq i < j \leq m} \left\{ \frac{g^2(e_i, Pe_j)}{2} + g(e_i, Pe_i)g(e_j, Pe_j) \right. \\ & \left. - 2g(Pe_i, e_j)g(e_i, Pe_j) + \frac{g^2(e_j, Pe_i)}{2} \right\} \\ & + 2 \sum_{\alpha=m+1}^{2n} \sum_{1 \leq i < j \leq m} [h_{ii}^{0\alpha} h_{jj}^{0\alpha} - (h_{ij}^{0\alpha})^2] \\ & - \frac{1}{2} \sum_{\alpha=m+1}^{2n} \sum_{1 \leq i < j \leq m} \{ [h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2] + [h_{ii}^{*\alpha} h_{jj}^{*\alpha} - (h_{ij}^{*\alpha})^2] \}. \end{aligned} \quad (23)$$

Subtracting (18) from (22), the invariant δ_M can be written:

$$\begin{aligned} \delta_M = \tau - K(\pi) = & \frac{(m+1)(m-2)c}{8} + \frac{c}{4} \sum_{1 \leq i < j \leq m} \left\{ \frac{g^2(e_i, Pe_j)}{2} + g(e_i, Pe_i)g(e_j, Pe_j) \right. \\ & \left. - 2g(Pe_i, e_j)g(e_i, Pe_j) + \frac{g^2(e_j, Pe_i)}{2} \right\} + 2 \sum_{\alpha=m+1}^{2n} \sum_{1 \leq i < j \leq m} [h_{ii}^{0\alpha} h_{jj}^{0\alpha} - (h_{ij}^{0\alpha})^2] \\ & - \frac{1}{2} \sum_{\alpha=m+1}^{2n} \sum_{1 \leq i < j \leq m} \{ [h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2] + [h_{ii}^{*\alpha} h_{jj}^{*\alpha} - (h_{ij}^{*\alpha})^2] \} \\ & - \frac{c}{4} \left\{ \frac{g^2(e_1, Pe_2)}{2} + \frac{g^2(Pe_1, e_2)}{2} + g(e_2, Pe_2)g(e_1, Pe_1) \right. \\ & \left. - 2g(Pe_1, e_2)g(e_1, Pe_2) \right\} - 2 \sum_{\alpha=m+1}^{2n} [h_{11}^{0\alpha} h_{22}^{0\alpha} - (h_{12}^{0\alpha})^2] \\ & + \frac{1}{2} \sum_{\alpha=m+1}^{2n} \{ [h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2] + [h_{11}^{*\alpha} h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2] \}. \end{aligned} \quad (24)$$

Denote by $\mathcal{A}$ the expression:

$$\begin{aligned} \mathcal{A} = & \frac{(m+1)(m-2)c}{8} + \frac{c}{4} \sum_{1 \leq i < j \leq m} \left\{ \frac{g^2(e_i, Pe_j)}{2} + g(e_i, Pe_i)g(e_j, Pe_j) \right. \\ & \left. - 2g(Pe_i, e_j)g(e_i, Pe_j) + \frac{g^2(e_j, Pe_i)}{2} \right\} \\ & - \frac{c}{4} \left\{ \frac{g^2(e_1, Pe_2)}{2} + \frac{g^2(Pe_1, e_2)}{2} + g(e_2, Pe_2)g(e_1, Pe_1) - 2g(Pe_1, e_2)g(e_1, Pe_2) \right\} \\ & + 2(\tau_0 - K_0(\pi)). \end{aligned}$$

Moreover, δ_M becomes:

$$\begin{aligned} \delta_M = \mathcal{A} - & \frac{1}{2} \sum_{\alpha=m+1}^{2n} \left\{ \sum_{1 \leq i < j \leq m} [h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2] - h_{11}^\alpha h_{22}^\alpha + (h_{12}^\alpha)^2 \right\} \\ & - \frac{1}{2} \sum_{\alpha=m+1}^{2n} \left\{ \sum_{1 \leq i < j \leq m} [h_{ii}^{*\alpha} h_{jj}^{*\alpha} - (h_{ij}^{*\alpha})^2] - h_{11}^{*\alpha} h_{22}^{*\alpha} + (h_{12}^{*\alpha})^2 \right\}. \end{aligned} \quad (25)$$

It follows that:

$$\begin{aligned}
\delta_M &= \mathcal{A} - \frac{1}{2} \sum_{\alpha=m+1}^{2n} \left\{ \sum_{3 \leq i < j \leq m} [h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2] + \sum_{3 \leq j \leq m} [h_{11}^\alpha h_{jj}^\alpha - (h_{1j}^\alpha)^2] + \sum_{3 \leq j \leq m} [h_{22}^\alpha h_{jj}^\alpha - (h_{2j}^\alpha)^2] \right\} \\
&\quad - \frac{1}{2} \sum_{\alpha=m+1}^{2n} \left\{ \sum_{3 \leq i < j \leq m} [h_{ii}^{*\alpha} h_{jj}^{*\alpha} - (h_{ij}^{*\alpha})^2] + \sum_{3 \leq j \leq m} [h_{11}^{*\alpha} h_{jj}^{*\alpha} - (h_{1j}^{*\alpha})^2] + \sum_{3 \leq j \leq m} [h_{22}^{*\alpha} h_{jj}^{*\alpha} - (h_{2j}^{*\alpha})^2] \right\} \\
&\geq \mathcal{A} - \frac{1}{2} \sum_{\alpha=m+1}^{2n} \left\{ \sum_{3 \leq i < j \leq m} h_{ii}^\alpha h_{jj}^\alpha + \sum_{3 \leq j \leq m} h_{11}^\alpha h_{jj}^\alpha + \sum_{3 \leq j \leq m} h_{22}^\alpha h_{jj}^\alpha \right\} \\
&\quad - \frac{1}{2} \sum_{\alpha=m+1}^{2n} \left\{ \sum_{3 \leq i < j \leq m} h_{ii}^{*\alpha} h_{jj}^{*\alpha} + \sum_{3 \leq j \leq m} h_{11}^{*\alpha} h_{jj}^{*\alpha} + \sum_{3 \leq j \leq m} h_{22}^{*\alpha} h_{jj}^{*\alpha} \right\}.
\end{aligned} \tag{26}$$

Let q_α be a quadratic form defined by $q_\alpha : \mathbb{R}^m \rightarrow \mathbb{R}$ for any $\alpha \in \{m+1, \dots, 2n\}$,

$$q_\alpha(h_{11}^\alpha, h_{22}^\alpha, \dots, h_{mm}^\alpha) = \sum_{3 \leq i < j \leq m} h_{ii}^\alpha h_{jj}^\alpha + \sum_{3 \leq j \leq m} (h_{11}^\alpha + h_{22}^\alpha) h_{jj}^\alpha$$

We investigate the constrained extremum problem

$$\max q_\alpha$$

with the condition

$$G : h_{11}^\alpha + h_{22}^\alpha + \dots + h_{mm}^\alpha = k^\alpha,$$

where k^α is a real constant.

We find the solution of the following system of first order partial derivatives:

$$\begin{cases}
\frac{\partial q_\alpha}{\partial h_{11}^\alpha} = \sum_{j=3}^m h_{jj}^\alpha = 0 \\
\frac{\partial q_\alpha}{\partial h_{22}^\alpha} = \sum_{j=3}^m h_{jj}^\alpha = 0 \\
\frac{\partial q_\alpha}{\partial h_{33}^\alpha} = h_{11}^\alpha + h_{22}^\alpha + \sum_{j=4}^m h_{jj}^\alpha = 0 \\
\dots\dots\dots \\
\frac{\partial q_\alpha}{\partial h_{mm}^\alpha} = h_{11}^\alpha + h_{22}^\alpha + \sum_{j=3}^{m-1} h_{jj}^\alpha = 0
\end{cases}$$

for any $\alpha \in \{m+1, \dots, 2n\}$.

The solutions of the above system are:

$$h_{11}^\alpha + h_{22}^\alpha = h_{33}^\alpha = \dots = h_{mm}^\alpha = \frac{k^\alpha}{m-1}.$$

For $p \in V$, consider $\mathcal{F}$ a 2-form, $\mathcal{F} : T_p V \times T_p V \rightarrow \mathbb{R}$ defined by:

$$\mathcal{F}(X, Y) = \text{Hess}(q_\alpha)(X, Y) + \langle h'(X, Y), (\text{grad} q_\alpha)(p) \rangle,$$

where h' is the second fundamental form of V in $\mathbb{R}^m$ and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^m$.

The Hessian matrix of q_α is given by:

$$\text{Hess}(q_\alpha) = \begin{pmatrix} 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}.$$

As $\sum_{i=1}^m U_i = 0$, for a vector field $X \in T_p V$, then the hyperplane V is totally geodesic in $\mathbb{R}^m$. Moreover, we see:

$$\begin{aligned} \mathcal{F}(X, Y) &= 2 \sum_{1 \leq i < j \leq m} U_i U_j - 2U_1 U_2 = \left(\sum_{i=1}^m U_i \right)^2 - \sum_{i=1}^m (U_i)^2 - 2U_1 U_2 \\ &= -(U_1 + U_2)^2 - \sum_{i=3}^m (U_i)^2 \leq 0. \end{aligned} \quad (27)$$

Using the Remark 1, adapted to our case, the critical point $(h_{11}^\alpha, \dots, h_{mm}^\alpha)$ of q_α is the global maximum point of the problem. Then we achieve:

$$q_\alpha \leq \frac{(m-2)}{2(m-1)} \left(\sum_{i=1}^m h_{ii}^\alpha \right)^2. \quad (28)$$

Similarly, we consider q_α^* be a quadratic form defined by $q_\alpha^* : \mathbb{R}^m \rightarrow \mathbb{R}$ for any $\alpha \in \{m+1, \dots, 2n\}$,

$$q_\alpha^*(h_{11}^{*\alpha}, h_{22}^{*\alpha}, \dots, h_{mm}^{*\alpha}) = \sum_{3 \leq i < j \leq m} h_{ii}^{*\alpha} h_{jj}^{*\alpha} + \sum_{3 \leq j \leq m} (h_{11}^{*\alpha} + h_{22}^{*\alpha}) h_{jj}^{*\alpha}.$$

We also study the constrained extremum problem

$$\max q_\alpha^*$$

with the condition

$$G^* : h_{11}^{*\alpha} + h_{22}^{*\alpha} + \dots + h_{mm}^{*\alpha} = k^{*\alpha},$$

where $k^{*\alpha}$ is a real constant. Therefore, we find:

$$q_\alpha^* \leq \frac{(m-2)}{2(m-1)} \left(\sum_{i=1}^m h_{ii}^{*\alpha} \right)^2. \quad (29)$$

Finally, δ_M from (26) has the expression:

$$\begin{aligned} \delta_M &\geq \mathcal{A} - \frac{1}{2} \sum_{\alpha=m+1}^{2n} \frac{(m-2)}{2(m-1)} \left(\sum_{i=1}^m h_{ii}^\alpha \right)^2 \\ &\quad - \frac{1}{2} \sum_{\alpha=m+1}^{2n} \frac{(m-2)}{2(m-1)} \left(\sum_{i=1}^m h_{ii}^{*\alpha} \right)^2. \end{aligned}$$

Moreover, δ_M satisfies the inequality:

$$\delta_M \geq \mathcal{A} - \frac{m^2(m-2)}{4(m-1)} (\|H\|^2 + \|H^*\|^2). \quad (30)$$

Consequently, the inequalities (12) and (13) are obtained.

The equality cases of the inequalities (12) and (13) hold if and only if we have equality sign in (26), (28), and (29). In this respect, we find:

$$\begin{aligned} h_{1j}^\alpha &= h_{2j}^\alpha = h_{ij}^\alpha = 0, \\ h_{1j}^{*\alpha} &= h_{2j}^{*\alpha} = h_{ij}^{*\alpha} = 0, \\ h_{11}^\alpha + h_{22}^\alpha &= h_{33}^\alpha = \dots = h_{mm}^\alpha, \\ h_{11}^{*\alpha} + h_{22}^{*\alpha} &= h_{33}^{*\alpha} = \dots = h_{mm}^{*\alpha}, \end{aligned}$$

for any $\alpha \in \{m+1, \dots, 2n\}$ and any $i, j \in \{3, \dots, m\}$, $i < j$. $\square$

4. Examples

Example 1. We point out that any para-Kähler manifold is a para-Kähler-like statistical manifold. Moreover, examples of statistical manifolds of type para-Kähler space forms can be illustrated among para-Kähler space forms. The flat para-Kähler space forms are represented by $\mathbb{E}_n^{2n}$ [35]. Delightful examples of spacelike Lagrangian submanifolds in $\mathbb{E}_n^{2n}$ can be discovered in [28,29]. The para-Kähler space forms of nonzero para-sectional curvature are investigated in [36].

Example 2. Let $\mathbb{R}^{2n}$ be a semi-Euclidean space of dimension $2n$, with the coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$, the flat affine connection ∇ and the pseudo-Riemannian metric g defined by

$$g = \sum_{i=1}^n (dx_i^2 - \alpha dy_i^2),$$

where $\alpha \neq 0$ is a real constant. The almost product structure P on $\mathbb{R}^{2n}$ is defined by

$$P(\partial_{x_i}) = \partial_{y_i}, \quad P(\partial_{y_i}) = \partial_{x_i}, \quad i = 1, \dots, n.$$

Then $(\mathbb{R}^{2n}, \nabla, P, g)$ is a statistical manifold of type para-Kähler space form. Moreover, P^* is expressed by

$$P^*(\partial_{x_i}) = \frac{1}{\alpha} \partial_{y_i}, \quad P^*(\partial_{y_i}) = \alpha \partial_{x_i}.$$

For X an open set of $\mathbb{R}^n$, define an isometric immersion $u : X \rightarrow \mathbb{R}^{2n}$ by

$$u(y_1, \dots, y_n) = (0, \dots, 0, y_1, \dots, y_n).$$

Then u can be represented as a spacelike Lagrangian submanifold of $(\mathbb{R}^{2n}, \nabla, P, g)$, where the above inequalities are satisfied.

5. Conclusions

In this article, we established new Chen inequalities for totally real and holomorphic spacelike submanifolds in statistical manifolds of type para-Kähler space form. Moreover, we examined the equality cases and we indicated a few examples.

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