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On Estimating the Parameters of the Beta Inverted Exponential Distribution under Type-II Censored Samples

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Abstract: This article aims to consider estimating the unknown parameters, survival, and hazard functions of the beta inverted exponential distribution. Two methods of estimation were used based on type-II censored samples: maximum likelihood and Bayes estimators. The Bayes estimators were derived using an informative gamma prior distribution under three loss functions: squared error, linear exponential, and general entropy. Furthermore, a Monte Carlo simulation study was carried out to compare the performance of different methods. The potentiality of this distribution is illustrated using two real-life datasets from difference fields. Further, a comparison between this model and some other models was conducted via information criteria. Our model performs the best fit for the real data.

Keywords: Bayes estimators; beta inverted exponential distribution; generalized hypergeometric function; loss functions; maximum likelihood estimators; simulation study; type-II censored samples

MSC: 62F10; 62F15; 62G05; 62N01; 65C05

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1. Introduction

In many life-testing and reliability studies for engineering or medical sciences, the information on failure times for all experimental units may not be obtained ultimately by the experimenter. Due to this, there are many situations in which it is pre-planned to remove units before failure, and these obtained data are called censored data. The most common censoring schemes in life-testing experiments are type-I and type-II censoring schemes. The type-II censoring scheme is used often in toxicology experiments and life-testing applications, where it has proven to save time and money. Many authors have addressed Bayesian and non-Bayesian estimations based on type-II censored samples or different types of samples, including [1], who derived maximum likelihood estimation (MLE) and Bayes estimation under different types of loss functions for exponentiated Weibull distribution based on type-II censored samples. Prakash [2] discussed the properties of the Bayes estimator and the minimax estimator of the parameter of the inverted exponential distribution. The moments of the lower record value and the estimation of the parameter were presented, based on a series of observed record values by the maximum likelihood (ML). Furthermore, Dey and Dey [3] derived the MLE of the generalized inverted exponential distribution parameters in the case of the progressive type-II censoring scheme with binomial removals. Singh and Goel [4] studied a three-parameter beta inverted exponential distribution (BIED). They derived the non-central moments, inverse moments, moment-generating function, inverse-moment-generating function, and mode. Furthermore, they examined the distributional properties of order statistics. Moreover, a statistical inference about the distribution parameters based on a complete sample was investigated. Garg et al. [5] studied the MLE of the parameters and the expected Fisher information under a random censoring model of the generalized inverted exponential distribution. Bakoban and Abu-Zinadah [6] considered the four-parameter beta generalized inverted exponential distribution for complete samples. In their research, the MLE, the Fisher information matrix, and the confidence interval were found. Besides that, the Monte Carlo simulation was discussed to illustrate the theoretical results of the estimation. Finally, applications on real datasets were provided. Aldahlan [7] applied the ML method to estimate the inverse Weibull inverse exponential distribution parameters. One real dataset about time between failures for repairable items was applied. This article focuses on estimation methods based on the type-II censoring scheme. Two estimation methods were used to estimate the unknown parameters for the beta inverted exponential distribution (BIED): MLE and Bayes estimation. The proposed distribution has three parameters (scale parameter λ and shape parameters α and β). The cumulative distribution function (CDF) and the probability density function (PDF) of BIED, respectively, are:

$$F(x) = \frac{1}{B(\alpha,\beta)} \int_0^{e^{\frac{-\lambda}{x}}} \omega^{\alpha-1}(1-\omega)^{\beta-1} \quad d\omega, \qquad x>0, \alpha, \beta, \lambda > 0, \tag{1}$$

and:

$$f(x) = \frac{\lambda}{x^2 B(\alpha,\beta)} e^{\frac{-\alpha\lambda}{x}} [1-e^{\frac{-\lambda}{x}}]^{\beta-1}, \qquad x>0, \alpha, \beta, \lambda > 0, \tag{2}$$

where $B(\alpha,\beta) = \int_0^1 w^{\alpha-1} (1-w)^{\beta-1}\, dw$ is the beta function. Equation (1) could also be written as a regularized incomplete beta function:

$$F(x) = I_{e^{\frac{-\lambda}{x}}}(\alpha,\beta) = \frac{B(e^{\frac{-\lambda}{x}},\alpha,\beta)}{B(\alpha,\beta)}, \qquad x>0, \alpha, \beta, \lambda > 0, \tag{3}$$

where $B(y,\alpha,\beta)$ is the incomplete beta function, such that:

$$B(y,\alpha,\beta) = \int_0^y \omega^{\alpha-1} (1-\omega)^{\beta-1}\, d\omega, \qquad 0 \leqslant y \leqslant 1,\ \alpha\, \beta > 0. \tag{4}$$

The inverse of CDF is called the quantile function and is given by:

$$x = Q(u) = \frac{-\lambda}{\log\,[I_u^{-1}(\alpha,\beta)]}, \qquad 0 < u < 1. \tag{5}$$

The survival and hazard function of the BIED, respectively, are given by:

$$S(x) = 1 - I_{e^{\frac{-\lambda}{x}}}(\alpha,\beta), \qquad x>0, \alpha, \beta, \lambda > 0, \tag{6}$$

and:

$$h(x) = \frac{\lambda}{x^2 B(\alpha,\beta)} \frac{e^{\frac{-\alpha\lambda}{x}} [1-e^{\frac{-\lambda}{x}}]^{\beta-1}}{1 - I_{e^{\frac{-\lambda}{x}}}(\alpha,\beta)}, \qquad x>0, \alpha, \beta, \lambda > 0. \tag{7}$$

The layout of this article is as follows: In Section 2, the estimation of the unknown parameters for the BIED under type-II censored samples is introduced. A simulation study is discussed in Section 3. In Section 4, an application with real data is provided. Finally, the conclusion is given in Section 5.

2. Method of Estimation

In this section, we derive the ML and Bayesian estimators for the unknown parameters of the BIED based on type-II censored samples.

2.1. Maximum Likelihood Estimation

Assume that $X_1, X_2, \ldots, X_n$ are a random sample from the BIED and the order statistics of this sample are $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$. The rth observations were chosen in type-II censored sample ($r < n$). The likelihood function of the type-II censored sample is given by (see [8]):

$$L(\underset{\sim}{\Theta}|\underset{\sim}{x}) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_i)\, [1 - F(x_r)]^{n-r}, \tag{8}$$

By substituting Equations (2) and (3) in (8), the likelihood function for the vector $\underset{\sim}{\Theta} = (\alpha,\, \beta,\, \lambda)$ is given by:

$$L(\underset{\sim}{\Theta}|\underset{\sim}{x}) = \frac{n!}{(n-r)!} \frac{\lambda^r}{[B(\alpha,\beta)]^r} \prod_{i=1}^{r} \frac{1}{x_i^2} \left(e^{-\sum_{i=1}^{r} \frac{\alpha\lambda}{x_i}}\right) \prod_{i=1}^{r} [1 - e^{\frac{-\lambda}{x_i}}]^{\beta-1} \left[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]^{(n-r)}. \tag{9}$$

Then, the log-likelihood function can be expressed as follows:

$$\begin{aligned} \ell = \log\left[\frac{n!}{(n-r)!}\right] + r\log\lambda - r\log[B(\alpha,\beta)] - 2\sum_{i=1}^{r}\log x_i - \sum_{i=1}^{r}\frac{\alpha\lambda}{x_i} \\ + (\beta-1)\sum_{i=1}^{r}\log[1 - e^{\frac{-\lambda}{x_i}}] + (n-r)\log\left[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right], \end{aligned} \tag{10}$$

The partial derivatives of the log-likelihood function with respect to α, β, and λ are given as:

$$\frac{\partial\ell}{\partial\alpha} = \frac{-r}{B(\alpha,\beta)}\left[\frac{\partial}{\partial\alpha}B(\alpha,\beta)\right] - \sum_{i=1}^{r}\frac{\lambda}{x_i} + \frac{(n-r)}{\left[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\left(\frac{\partial\left[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}{\partial\alpha}\right), \tag{11}$$

where:

$$\begin{aligned} \frac{\partial}{\partial\alpha}B(\alpha,\beta) &= \frac{\Gamma'(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta) - \Gamma(\alpha)\Gamma(\beta)\frac{\partial}{\partial\alpha}\Gamma(\alpha+\beta)}{[\Gamma(\alpha+\beta)]^2} \\ &= B(\alpha,\beta)\,[\psi(\alpha) - \psi(\alpha+\beta)], \end{aligned}$$

According to Abramowitz and Stegun [9], $\psi(z) = \frac{d\,[\log\Gamma(z)]}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}$ is called the Psi (or digamma) function. Then:

$$\frac{\partial\ell}{\partial\alpha} = -r[\psi(\alpha) - \psi(\alpha+\beta)] - \sum_{i=1}^{r}\frac{\lambda}{x_i} + \frac{(n-r)}{\left[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\left(\frac{\partial\left[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}{\partial\alpha}\right),$$

By using the Leibniz integral rule to find $\left(\frac{\partial\left[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}{\partial\alpha}\right)$ (see [9]), then:

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha} =& -r[\psi(\alpha) - \psi(\alpha+\beta)] - \sum_{i=1}^{r}\frac{\lambda}{x_i} + \frac{(n-r)}{\left[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\left\{-\frac{1}{B(\alpha,\beta)}\right. \\ &\left.\times\left[\int_0^{e^{-\frac{\lambda}{x_r}}} w^{\alpha-1}(1-w)^{\beta-1}\log(w)\,dw - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)B(\alpha,\beta)[\psi(\alpha) - \psi(\alpha+\beta)]\right]\right\}. \end{aligned}$$

Let $U = (1 - W)$. Then:

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha} =& -r[\psi(\alpha) - \psi(\alpha+\beta)] - \sum_{i=1}^{r}\frac{\lambda}{x_i} + \frac{(n-r)}{\left[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\left\{-\frac{1}{B(\alpha,\beta)}\right. \\ &\left.\times\left[\int_{1-e^{-\frac{\lambda}{x_r}}}^{1}(1-u)^{\alpha-1}u^{\beta-1}\log(1-u)\,du - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)B(\alpha,\beta)[\psi(\alpha) - \psi(\alpha+\beta)]\right]\right\}. \end{aligned}$$

By using $\log(1-u) = -\sum_{k=1}^{\infty} \frac{u^k}{k}$ and taking $Y = 1 - U$, we obtain the partial derivatives of α:

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} =& -r[\psi(\alpha) - \psi(\alpha+\beta)] - \sum_{i=1}^{r} \frac{\lambda}{x_i} + \frac{(n-r)}{[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]} \Bigg\{ \frac{1}{B(\alpha,\beta)} \\ &\times \sum_{k=1}^{\infty} \frac{1}{k} B(e^{\frac{-\lambda}{x_r}}; \alpha, \beta+k) + I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)[\psi(\alpha) - \psi(\alpha+\beta)] \Bigg\}.\end{aligned} \tag{12}$$

Equation (12) can be rewritten as:

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} =& -r[\psi(\alpha) - \psi(\alpha+\beta)] - \sum_{i=1}^{r} \frac{\lambda}{x_i} + \frac{(n-r)}{[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]} \Bigg(\Gamma(\alpha)\, \beta_{(\alpha)}\, (e^{\frac{-\lambda}{x_r}})^{\alpha} \\ &\times {}_3\widetilde{F}_2(\alpha, \alpha, 1-\beta; \alpha+1, \alpha+1; e^{\frac{-\lambda}{x_r}}) - [\log(e^{\frac{-\lambda}{x_r}}) - \psi(\alpha) \\ &+ \psi(\alpha+\beta)]\, I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta) \Bigg),\end{aligned} \tag{13}$$

where ${}_p\widetilde{F}_q(a_1,\ldots,a_p; b_1,\ldots,b_q; z) = \frac{{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)}{\Gamma(b_1)\ldots\Gamma(b_q)}$ is called the regularized hypergeometric function and ${}_pF_q(a_1,\ldots,a_p; b_1,\ldots,b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k,\ldots,(a_p)_k}{(b_1)_k,\ldots,(b_q)_k} \frac{z^k}{k!}$ is the generalized hypergeometric function, and $(a)_n = a(a+1),\ldots,(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ denotes the ascending factorial (Pochhammer symbol) (see [10]).

Next,

$$\frac{\partial \ell}{\partial \beta} = \frac{-r}{B(\alpha,\beta)} [\frac{\partial}{\partial \beta} B(\alpha,\beta)] + \sum_{i=1}^{r} \log[1 - e^{\frac{-\lambda}{x_i}}] + \frac{(n-r)}{[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]} \left(\frac{\partial [1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]}{\partial \beta} \right), \tag{14}$$

where:

$$\begin{aligned}\frac{\partial}{\partial \beta} B(\alpha,\beta) &= \frac{\Gamma(\alpha)\Gamma'(\beta)\Gamma(\alpha+\beta) - \Gamma(\alpha)\Gamma(\beta)\frac{\partial}{\partial \beta}\Gamma(\alpha+\beta)}{[\Gamma(\alpha+\beta)]^2} \\ &= B(\alpha,\beta)\,[\psi(\beta) - \psi(\alpha+\beta)],\end{aligned}$$

then:

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} &= -r[\psi(\beta) - \psi(\alpha+\beta)] + \sum_{i=1}^{r} \log[1 - e^{\frac{-\lambda}{x_i}}] + \frac{(n-r)}{[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]} \left(\frac{\partial [1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]}{\partial \beta} \right), \\ &= -r[\psi(\beta) - \psi(\alpha+\beta)] + \sum_{i=1}^{r} \log[1 - e^{\frac{-\lambda}{x_i}}] + \frac{(n-r)}{[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]} \Bigg\{ - \frac{1}{B(\alpha,\beta)} \\ &\quad \times \Bigg[\int_0^{e^{-\frac{\lambda}{x_r}}} w^{\alpha-1}(1-w)^{\beta-1} \log(1-w)\, dw - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\, B(\alpha,\beta) \\ &\quad \times [\psi(\beta) - \psi(\alpha+\beta)] \Bigg] \Bigg\}.\end{aligned}$$

By using $\log(1-u) = -\sum_{k=1}^{\infty} \frac{u^k}{k}$, we obtain:

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} &= -r[\psi(\beta) - \psi(\alpha+\beta)] + \sum_{i=1}^{r} \log[1 - e^{\frac{-\lambda}{x_i}}] + \frac{(n-r)}{[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]} \\ &\times \left\{ \frac{1}{B(\alpha,\beta)} \sum_{k=1}^{\infty} \frac{1}{k} B(e^{-\frac{\lambda}{x_r}}; \alpha+k, \beta) + I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\, [\psi(\beta) - \psi(\alpha+\beta)] \right\}. \end{aligned} \tag{15}$$

Equation (15) can be rewritten as:

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} =& -r[\psi(\beta) - \psi(\alpha+\beta)] + \sum_{i=1}^{r} \log[1 - e^{\frac{-\lambda}{x_i}}] + \frac{(n-r)}{[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]} \left\{ -(1 - e^{\frac{-\lambda}{x_r}})^{\beta} \right. \\ &\times \Gamma(\beta)\, \alpha_{(\beta)}\, {}_3\widetilde{F}_2(\beta,\beta,1-\alpha; 1+\beta, 1+\beta; 1 - e^{\frac{-\lambda}{x_r}}) - I_{(1-e^{\frac{-\lambda}{x_r}})}(\beta,\alpha) \\ &\left. \times [-\log(1 - e^{\frac{-\lambda}{x_r}}) + \psi(\beta) - \psi(\alpha+\beta)] \right\}. \end{aligned} \tag{16}$$

and:

$$\frac{\partial \ell}{\partial \lambda} = \frac{r}{\lambda} - \sum_{i=1}^{r} \frac{\alpha}{x_i} + (\beta-1) \sum_{i=1}^{r} \frac{e^{\frac{-\lambda}{x_i}}}{[1 - e^{\frac{-\lambda}{x_i}}] x_i} + \frac{(n-r)}{[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]} \left(\frac{\partial [1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]}{\partial \lambda} \right), \tag{17}$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{r}{\lambda} - \sum_{i=1}^{r} \frac{\alpha}{x_i} + (\beta-1) \sum_{i=1}^{r} {x_i}^{-1} (e^{\frac{\lambda}{x_i}} - 1)^{-1} + \frac{(n-r)}{[1 - I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)]} \left(\frac{e^{\frac{-\alpha\lambda}{x_r}} [1 - e^{\frac{-\lambda}{x_r}}]^{\beta-1}}{x_r\, B(\alpha,\beta)} \right). \tag{18}$$

After equating Equations (13), (16) and (18) with zero and solving them, simultaneously, the MLE of α, β, and λ could be found using the ***Newton–Raphson*** method via ***Mathematica 11***. Furthermore, the invariance property of the ML is used to estimate $S(x_0)$ and $h(x_0)$.

2.2. Bayes Estimation

Bayes estimators for the BIED are obtained based on type-II censored samples in this subsection. Singh and Goel [4] derived the Bayes estimators for the BIED based on complete samples under the SE loss function. They considered the gamma prior distribution for the unknown BIED parameters. The prior distribution is denoted by $\pi(\theta)$, which tells us what is known about θ without observing the data. Bayes theorem is based on the posterior distribution, which is defined as $\pi^*(\theta|\underset{\sim}{x})$ and given by (see [11]):

$$\pi^*(\theta|\underset{\sim}{x}) = \frac{L(\underset{\sim}{x}|\theta)\pi(\theta)}{\int L(\underset{\sim}{x}|\theta)\pi(\theta)\, d\theta}, \tag{19}$$

where θ is continuous and $L(\underset{\sim}{x}|\theta)$ is the likelihood function. Furthermore, Equation (19) could be written as:

$$\pi^*(\theta|\underset{\sim}{x}) = k\, L(\underset{\sim}{x}|\theta)\pi(\theta), \tag{20}$$

where k is called the normalizing constant, necessary to ensure that the posterior distribution $\pi^*(\theta|\underset{\sim}{x})$ integrates or sums to one.

Here, we derive the Bayes estimates for α, β, and λ under three types of loss functions: squared error (SE), linear exponential (LINEX), and general entropy (GE). Moreover, four cases are considered first when α is unknown, while β and λ are known. A second case is when β is unknown, while α, λ are known. A third case is when the scale parameter λ is unknown. Finally, a fourth case is when both β and λ are unknown. Two techniques are used to compute the estimates: the standard Bayes and importance sampling techniques for the first three cases. The last case is computed via the importance sampling technique.

2.2.1. Case 1: Bayes Estimators When α Is Unknown

Assume α is unknown and has the following prior distribution $\alpha \sim Gamma(a, b)$; thus, the prior for α is given by:

$$\pi(\alpha) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha}, \qquad \alpha > 0, a, b > 0. \tag{21}$$

By combining (9) and (21), the posterior distribution of the unknown parameter α is given by:

$$\pi^*(\alpha|\underline{x}) = k_1\,\alpha^{a-1}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\right]^r e^{-\left[\sum_{i=1}^r \frac{\lambda}{x_i}+b\right]\alpha} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}, \tag{22}$$

where k_1 is the normalizing constant, defined as:

$$k_1^{-1} = \int_0^\infty \alpha^{a-1}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\right]^r e^{-\left[\sum_{i=1}^r \frac{\lambda}{x_i}+b\right]\alpha} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\,d\alpha. \tag{23}$$

Therefore, the Bayes estimator of α, denoted by $\varphi(\alpha)$, is obtained under three types of loss functions and two techniques as follows.

i. SE Loss Function

The symmetric loss function SE is defined as:

$$\hat{\varphi}_{SE}(\theta) = E(\varphi(\theta)|\underline{x}) = \int \varphi(\theta)\pi^*(\theta|\underline{x})d\theta \tag{24}$$

Then, the Bayes estimator of $\varphi(\alpha)$ under the SE loss function, denoted by $\hat{\varphi}_{SSE_C}(\alpha)$, and can be found using Equations (24) and (22).

$$\hat{\varphi}_{SSE_C}(\alpha) = k_1\int_0^\infty \varphi(\alpha)\,\alpha^{a-1}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\right]^r e^{-\left[\sum_{i=1}^r \frac{\lambda}{x_i}+b\right]\alpha} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\,d\alpha. \tag{25}$$

where k_1^{-1} is defined in Equation (23).

ii. LINEX Loss Function

Varian [12] proposed the LINEX loss function as follows:

$$\hat{\varphi}_{LE}(\theta) = -\frac{1}{\tau}\log\left[E_\theta(e^{-\tau\varphi(\theta)}|\underline{x})\right] = -\frac{1}{\tau}\log\left[\int e^{-\tau\varphi(\theta)}\pi^*(\theta|\underline{x})\,d\theta\right], \tag{26}$$

The Bayes estimator of $\varphi(\alpha)$ under the LINEX loss function, denoted by $\hat{\varphi}_{SLE_C}(\alpha)$, can be found by using Equations (26) and (22).

$$\begin{aligned}\hat{\varphi}_{SLE_C}(\alpha) = -\frac{1}{\tau}\log\Bigg[k_1\int_0^\infty e^{-\tau\varphi(\alpha)}\alpha^{a-1}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\right]^r e^{-\left[\sum_{i=1}^r \frac{\lambda}{x_i}+b\right]\alpha} \\ \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\,d\alpha\Bigg],\end{aligned} \tag{27}$$

where k_1^{-1} is defined in (23).

iii. GE Loss Function

According to Calabria and Pulcini [13], the GE loss function of $\varphi(\theta)$ can be defined as:

$$\hat{\varphi}_{GE}(\theta) = \left[E_\theta(\varphi(\theta)^{-q}|\underline{x})\right]^{\frac{-1}{q}} = \left[\int \varphi(\theta)^{-q}\pi(\theta|\underline{x})d\theta\right]^{\frac{-1}{q}}, \tag{28}$$

The Bayes estimator of $\varphi(\alpha)$ under the GE loss function, denoted by $\hat{\varphi}_{SGE_C}(\alpha)$, can be found by using Equations (22) and (28).

$$\begin{aligned}\hat{\varphi}_{SGE_C}(\alpha) = \Bigg[k_1 \int_0^\infty [\varphi(\alpha)]^{-q}\alpha^{a-1}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\right]^r e^{-\left[\sum_{i=1}^r \frac{\lambda}{x_i}+b\right]\alpha} \\ \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]} d\alpha\Bigg]^{\frac{-1}{q}},\end{aligned} \tag{29}$$

where k_1^{-1} is defined in (23).

The Bayes estimator of α using the importance sampling technique can be derived by rewriting the posterior density function in Equation (22); thus:

$$\pi^*(\alpha|\underline{x}) \propto \frac{\left(\sum_{i=1}^r \frac{\lambda}{x_i}+b\right)^a}{\Gamma(a)}\alpha^{a-1}e^{-\left[\sum_{i=1}^r \frac{\lambda}{x_i}+b\right]\alpha}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\right]^r e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}.$$

The posterior density function of α can be considered as:

$$\pi^*(\alpha|\underline{x}) \propto Gamma(a, \sum_{i=1}^r \frac{\lambda}{x_i}+b)\, g_1(\alpha|\underline{x}),$$

where:

$$g_1(\alpha|\underline{x}) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\right]^r e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}. \tag{30}$$

The Bayes estimators of $\varphi(\alpha)$ under the SE, LINEX, and GE loss functions based on the importance sampling technique, denoted by $\hat{\varphi}_{ISE_C}(\alpha)$, $\hat{\varphi}_{ILE_C}(\alpha)$, and $\hat{\varphi}_{IGE_C}(\alpha)$, respectively, could be found using the following Algorithm 1.

Algorithm 1 Importance sampling technique when α is unknown based on type-II censored samples.

1. Generate $\alpha_i \sim$ Gamma (a, $\sum_{i=1}^r \frac{\lambda}{x_i}+b$).
2. Repeat Step 1 to obtain $\alpha_1, \alpha_2, \ldots, \alpha_N$.
3. Calculate the values.

$$\hat{\varphi}_{ISE_C}(\alpha) = \frac{\sum_{j=1}^N \varphi(\alpha_j) g_1(\alpha_j|\underline{x})}{\sum_{j=1}^N g_1(\alpha_j|\underline{x})} \tag{31}$$

$$\hat{\varphi}_{ILE_C}(\alpha) = -\frac{1}{\tau}\log\left[\frac{\sum_{j=1}^N e^{-\tau\varphi(\alpha_j)} g_1(\alpha_j|\underline{x})}{\sum_{j=1}^N g_1(\alpha_j|\underline{x})}\right] \tag{32}$$

$$\hat{\varphi}_{IGE_C}(\alpha) = \left[\frac{\sum_{j=1}^N [\varphi(\alpha_j)]^{-q} g_1(\alpha_j|\underline{x})}{\sum_{j=1}^N g_1(\alpha_j|\underline{x})}\right]^{-\frac{1}{q}} \tag{33}$$

where

$$g_1(\alpha_j|\underline{x}) = \left[\frac{\Gamma(\alpha_j+\beta)}{\Gamma(\alpha_j)}\right]^r e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha_j,\beta)\right]} \tag{34}$$

The Bayes estimators of α are found numerically under these three loss functions by the ***NIntegrate*** function via ***Mathematica 11***.

2.2.2. Case 2: Bayes Estimators When β Is Unknown

Suppose β is unknown and has the following prior distribution $\beta \sim Gamma(c,d)$, given by:

$$\pi(\beta) = \frac{d^c}{\Gamma(c)}\beta^{c-1}e^{-d\beta}, \qquad \beta > 0,\ c,\ d > 0. \tag{35}$$

By combining (9) and (35), the posterior distribution of the unknown parameter β is given by:

$$\pi^*(\beta|\underline{x}) = k_2\,\beta^{c-1}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}, \tag{36}$$

where k_2 is the normalizing constant and can be written as:

$$k_2^{-1} = \int_0^\infty \beta^{c-1}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\,d\beta. \tag{37}$$

Therefore, the Bayes estimator of β, denoted by $\varphi(\beta)$, is obtained under three types of loss function and two techniques as follows.

i. SE Loss Function

The Bayes estimator of $\varphi(\beta)$ under the SE loss function, denoted by $\hat{\varphi}_{SSE_C}(\beta)$, can be found by using Equations (24) and (36).

$$\begin{aligned}\hat{\varphi}_{SSE_C}(\beta) = k_2\int_0^\infty \varphi(\beta)\beta^{c-1}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} \\ \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\,d\beta,\end{aligned} \tag{38}$$

where k_2^{-1} is defined in Equation (37).

ii. LINEX Loss Function

The Bayes estimator of $\varphi(\beta)$ under the LINEX loss function, denoted by $\hat{\varphi}_{SLE_C}(\beta)$, can be found by using Equations (26) and (36).

$$\begin{aligned}\hat{\varphi}_{SLE}(\beta) = -\frac{1}{\tau}\log\Bigg[k_2\int_0^\infty e^{-\tau\varphi(\beta)}\beta^{c-1}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r \\ \times e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\,d\beta\Bigg],\end{aligned} \tag{39}$$

where k_2^{-1} is defined in (37).

iii. GE Loss Function

The Bayes estimator of $\varphi(\beta)$ under the GE loss function, denoted by $\hat{\varphi}_{SGE_C}(\beta)$, can be found by using Equations (28) and (36).

$$\begin{aligned}\hat{\varphi}_{SGE_C}(\beta) = \Bigg[k_2\int_0^\infty [\varphi(\beta)]^{-q}\,\beta^{c-1}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r \\ \times e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\,d\beta\Bigg]^{\frac{-1}{q}},\end{aligned} \tag{40}$$

where k_2^{-1} is defined in (37).

The Bayes estimator of β using the importance sampling technique can be derived by rewriting the posterior density function in Equation (36); thus:

$$\pi^*(\beta|\underline{x}) \propto \frac{(d-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}])^c}{\Gamma(c)}\beta^{c-1}\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r \times e^{-\left[d-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta}\, e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}.$$

The posterior density function of β can be considered as:

$$\pi^*(\beta|\underline{x}) \propto Gamma(c, d-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}])g_2(\beta|\underline{x}),$$

where:

$$g_2(\beta|\underline{x}) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}. \tag{41}$$

The Bayes estimators of $\varphi(\beta)$ under the SE, LINEX, and GE loss functions based on the importance sampling technique, denoted by $\hat{\varphi}_{ISE_C}(\beta)$, $\hat{\varphi}_{ILE_C}(\beta)$, and $\hat{\varphi}_{IGE_C}(\beta)$, respectively, could be found using the following Algorithm 2.

Algorithm 2 Importance sampling technique when β is unknown based on type-II censored samples.

1. Generate $\beta_i \sim$ Gamma(c, d-$\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]$).
2. Repeat Step 1 to obtain $\beta_1, \beta_2, \ldots, \beta_N$.
3. Calculate the values.

$$\hat{\varphi}_{ISE_C}(\beta) = \frac{\sum_{j=1}^{N}\varphi(\beta_j)g_2(\beta_j|\underline{x})}{\sum_{j=1}^{N}g_2(\beta_j|\underline{x})} \tag{42}$$

$$\hat{\varphi}_{ILE_C}(\beta) = -\frac{1}{\tau}\log\left[\frac{\sum_{j=1}^{N}e^{-\tau\varphi(\beta_j)}g_2(\beta_j|\underline{x})}{\sum_{j=1}^{N}g_2(\beta_j|\underline{x})}\right] \tag{43}$$

$$\hat{\varphi}_{IGE_C}(\beta) = \left[\frac{\sum_{j=1}^{N}[\varphi(\beta_j)]^{-q}g_2(\beta_j|\underline{x})}{\sum_{j=1}^{N}g_2(\beta_j|\underline{x})}\right]^{-\frac{1}{q}} \tag{44}$$

where

$$g_2(\beta_j|\underline{x}) = \left[\frac{\Gamma(\alpha+\beta_j)}{\Gamma(\beta_j)}\right]^r e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta_j)\right]} \tag{45}$$

The Bayes estimators of β under the three loss functions cannot be computed analytically through the two techniques used. They can be found numerically using the ***NIntegrate*** function via ***Mathematica 11***.

2.2.3. Case 3: Bayes Estimators When λ Is Unknown

Suppose that the scale parameter λ is unknown and has the following prior distribution $\lambda \sim Gamma(f, \nu)$, given by:

$$\pi(\lambda) = \frac{\nu^f}{\Gamma(f)}\lambda^{f-1}e^{-\nu\lambda}, \qquad \lambda > 0,\ f,\ \nu > 0. \tag{46}$$

By combining (9) and (46), the posterior distribution of the unknown parameter λ is given by:

$$\pi^*(\lambda|\underline{x}) = k_3\, \lambda^{f+r-1} e^{-\left[\sum_{i=1}^{r} \frac{\alpha}{x_i}+\nu\right]\lambda} e^{(\beta-1)\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}, \tag{47}$$

where k_3 is the normalizing constant and can be written as:

$$k_3^{-1} = \int_0^\infty \lambda^{f+r-1} e^{-\left[\sum_{i=1}^{r} \frac{\alpha}{x_i}+\nu\right]\lambda} e^{(\beta-1)\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\, d\lambda. \tag{48}$$

The Bayes estimator of λ is indicated by $\varphi(\lambda)$, and this estimator is obtained under three types of loss function and two techniques as follows.

i. SE Loss Function

The Bayes estimator of $\varphi(\lambda)$ under the SE loss function, denoted by $\hat{\varphi}_{SSE_C}(\lambda)$, can be found by using Equations (24) and (47).

$$\begin{aligned}\hat{\varphi}_{SSE_C}(\lambda) = k_3 \int_0^\infty \varphi(\lambda)\, \lambda^{f+r-1} e^{-\left[\sum_{i=1}^{r} \frac{\alpha}{x_i}+\nu\right]\lambda} e^{(\beta-1)\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]} \\ \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\, d\lambda,\end{aligned} \tag{49}$$

where k_3^{-1} is defined in Equation (48).

ii. LINEX Loss Function

The Bayes estimator of $\varphi(\lambda)$ under the LINEX loss function, denoted by $\hat{\varphi}_{SLE_C}(\lambda)$, can be found by using Equations (26) and (47).

$$\begin{aligned}\hat{\varphi}_{SLE_C}(\lambda) = -\frac{1}{\tau}\log\Bigg[k_3 \int_0^\infty e^{-\tau\varphi(\lambda)} \lambda^{f+r-1} e^{-\left[\sum_{i=1}^{r} \frac{\alpha}{x_i}+\nu\right]\lambda} \\ \times e^{(\beta-1)\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\, d\lambda\Bigg],\end{aligned} \tag{50}$$

where k_3^{-1} is defined in (48).

iii. GE Loss Function

The Bayes estimator of $\varphi(\lambda)$ under the GE loss function, denoted by $\hat{\varphi}_{SGE_C}(\lambda)$, can be found by using Equations (28) and (47).

$$\begin{aligned}\hat{\varphi}_{SGE_C}(\lambda) = \Bigg[k_3 \int_0^\infty [\varphi(\lambda)]^{-q} \lambda^{f+r-1} e^{-\left[\sum_{i=1}^{r} \frac{\alpha}{x_i}+\nu\right]\lambda} \\ \times e^{(\beta-1)\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}\, d\lambda\Bigg]^{\frac{-1}{q}}.\end{aligned} \tag{51}$$

where k_3^{-1} is defined in (48).

The Bayes estimator of λ using the importance sampling technique can be obtained by rewriting the posterior density function in Equation (47); thus:

$$\begin{aligned}\pi^*(\lambda|\underline{x}) \propto \frac{\left(\sum_{i=1}^{r} \frac{\alpha}{x_i}+\nu\right)^{f+r}}{\Gamma(f+r)} \lambda^{f+r-1} e^{-\left[\sum_{i=1}^{r} \frac{\alpha}{x_i}+\nu\right]\lambda} \\ \times e^{(\beta-1)\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}.\end{aligned}$$

The posterior density function of λ can be considered as:

$$\pi^*(\lambda|\underline{x}) \propto Gamma(f+r, \sum_{i=1}^{r} \frac{\alpha}{x_i} + \nu) g_3(\lambda|\underline{x}),$$

where:

$$g_3(\lambda|\underline{x}) = e^{(\beta-1)\sum_{i=1}^{r} \log[1-e^{\frac{-\lambda}{x_i}}]} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}. \tag{52}$$

The Bayes estimators of $\varphi(\lambda)$ under the SE, LINEX, and GE loss functions based on the importance sampling technique, denoted by $\hat{\varphi}_{ISE_C}(\lambda)$, $\hat{\varphi}_{ILE_C}(\lambda)$, and $\hat{\varphi}_{IGE_C}(\lambda)$, respectively, could be found using the following Algorithm 3.

Algorithm 3 Importance sampling technique when λ is unknown based on type-II censored samples.

1. Generate $\lambda_i \sim$ Gamma(f + r, $\sum_{i=1}^{r} \frac{\alpha}{x_i} + \nu$).
2. Repeat Step 1 to obtain $\lambda_1, \lambda_2, \ldots, \lambda_N$.
3. Calculate the values.

$$\hat{\varphi}_{ISE_C}(\lambda) = \frac{\sum_{j=1}^{N} \varphi(\lambda_j) g_3(\lambda_j|\underline{x})}{\sum_{j=1}^{N} g_3(\lambda_j|\underline{x})} \tag{53}$$

$$\hat{\varphi}_{ILE_C}(\lambda) = -\frac{1}{\tau} \log\left[\frac{\sum_{j=1}^{N} e^{-\tau\varphi(\lambda_j)} g_3(\lambda_j|\underline{x})}{\sum_{j=1}^{N} g_3(\lambda_j|\underline{x})}\right] \tag{54}$$

$$\hat{\varphi}_{IGE_C}(\lambda) = \left[\frac{\sum_{j=1}^{N} [\varphi(\lambda_j)]^{-q} g_3(\lambda_j|\underline{x})}{\sum_{j=1}^{N} g_3(\lambda_j|\underline{x})}\right]^{-\frac{1}{q}} \tag{55}$$

where

$$g_3(\lambda_j|\underline{x}) = e^{(\beta-1)\sum_{i=1}^{r} \log[1-e^{\frac{-\lambda_j}{x_i}}]} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda_j}{x_r}}}(\alpha,\beta)\right]}. \tag{56}$$

The Bayes estimators of λ under the three loss functions cannot be computed analytically through the two techniques used. It can be found numerically using the ***NIntegrate*** function via ***Mathematica 11***.

2.2.4. Case 4: Bayes Estimators When λ and β Are Unknown

Consider that both parameters λ and β are unknown. Suppose $\beta \sim Gamma(c, d)$ and $\lambda \sim Gamma(f, \nu)$. Therefore, the joint prior distribution is given by:

$$\pi(\lambda, \beta) = \frac{\nu^f}{\Gamma(f)} \lambda^{f-1} e^{-\nu\lambda} \frac{d^c}{\Gamma(c)} \beta^{c-1} e^{-d\beta}, \qquad \lambda,\ \beta > 0,\ f,\ \nu,\ c,\ d > 0. \tag{57}$$

By combining (9) and (57), the joint posterior distribution of the unknown parameters λ and β is given by:

$$\begin{aligned}\pi^*(\lambda, \beta|\underline{x}) = k_4\, \lambda^{f+r-1} e^{-\left[\sum_{i=1}^{r} \frac{\alpha}{x_i} + \nu\right]\lambda} \beta^{c-1} \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r e^{-\left[d - \sum_{i=1}^{r} \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} \\ \times\ e^{-\sum_{i=1}^{r} \log[1-e^{\frac{-\lambda}{x_i}}]} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]},\end{aligned} \tag{58}$$

where k_4 is the normalizing constant and can be written as:

$$k_4^{-1} = \int_0^\infty \int_0^\infty \lambda^{f+r-1} e^{-\left[\sum_{i=1}^r \frac{\alpha}{x_i}+\nu\right]\lambda} \beta^{c-1} \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} \times e^{-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]} d\lambda\, d\beta. \tag{59}$$

The Bayes estimators of λ and β are derived under three types of loss function as follows.

i. SE Loss Function

The Bayes estimators of $\varphi(\lambda,\beta)$ under the SE loss function, denoted by $\hat{\varphi}_{SSE_C}(\lambda,\beta)$, can be found by using Equations (24) and (58).

$$\hat{\varphi}_{SSE_C}(\lambda,\beta) = k_4 \int_0^\infty \int_0^\infty \varphi(\lambda,\beta) \lambda^{f+r-1} e^{-\left[\sum_{i=1}^r \frac{\alpha}{x_i}+\nu\right]\lambda} \beta^{c-1} \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r \times e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]} \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]} d\lambda\, d\beta, \tag{60}$$

where k_4^{-1} is defined in Equation (59).

Different forms of the Bayes estimator for $\varphi(\lambda,\beta)$ are obtained from Equation (60):

- When $\varphi(\lambda,\beta) = \lambda$, we obtain the Bayes estimator for λ, denoted by $\hat{\lambda}_{SSE_C}$, as:

$$\hat{\lambda}_{SSE_C} = k_4 \int_0^\infty \int_0^\infty \lambda^{f+r} e^{-\left[\sum_{i=1}^r \frac{\alpha}{x_i}+\nu\right]\lambda} \beta^{c-1} \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r \times e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]} \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]} d\lambda\, d\beta, \tag{61}$$

  where k_4^{-1} is defined in Equation (59);

- When $\varphi(\lambda,\beta) = \beta$, we obtain the Bayes estimator for β, denoted by $\hat{\beta}_{SSE_C}$, as:

$$\hat{\beta}_{SSE_C} = k_4 \int_0^\infty \int_0^\infty \lambda^{f+r-1} e^{-\left[\sum_{i=1}^r \frac{\alpha}{x_i}+\nu\right]\lambda} \beta^{c} \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]} \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]} d\lambda\, d\beta, \tag{62}$$

  where k_4^{-1} is defined in Equation (59).

ii. LINEX Loss Function

The Bayes estimator of $\varphi(\lambda,\beta)$ under the LINEX loss function, denoted by $\hat{\varphi}_{SLE_C}(\lambda,\beta)$, can be found by using Equations (26) and (58).

$$\hat{\varphi}_{SLE_C}(\lambda,\beta) = -\frac{1}{\tau}\log\left[k_4 \int_0^\infty \int_0^\infty e^{-\tau\varphi(\lambda,\beta)} \lambda^{f+r-1} e^{-\left[\sum_{i=1}^r \frac{\alpha}{x_i}+\nu\right]\lambda} \beta^{c-1} \times \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]} \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]} d\lambda\, d\beta\right], \tag{63}$$

where k_4^{-1} is defined in (59).

Different forms of the Bayes estimator for $\varphi(\lambda,\beta)$ are obtained from Equation (63):

- When $\varphi(\lambda,\beta)=\lambda$, we obtain the Bayes estimator for λ, denoted by $\hat{\lambda}_{SLE_C}$, as:

$$\begin{aligned}\hat{\lambda}_{SLE_C} = -\frac{1}{\tau}\log\Bigg[k_4 \int_0^\infty \int_0^\infty & \lambda^{f+r-1} e^{-\left[\sum_{i=1}^{r}\frac{\alpha}{x_i}+\nu+\tau\right]\lambda} \beta^{c-1} \\ & \times \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r e^{-\left[d-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]} \\ & \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]} d\lambda\, d\beta\Bigg],\end{aligned} \tag{64}$$

  where k_4^{-1} is defined in Equation (59);
- When $\varphi(\lambda,\beta)=\beta$, we obtain the Bayes estimator for β, denoted by $\hat{\beta}_{SLE_C}$, as:

$$\begin{aligned}\hat{\beta}_{SLE_C} = -\frac{1}{\tau}\log\Bigg[k_4 \int_0^\infty \int_0^\infty & \lambda^{f+r-1} e^{-\left[\sum_{i=1}^{r}\frac{\alpha}{x_i}+\nu\right]\lambda} \beta^{c-1} \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r \\ & \times e^{-\left[d-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]+\tau\right]\beta} e^{-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]} \\ & \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]} d\lambda\, d\beta\Bigg],\end{aligned} \tag{65}$$

  where k_4^{-1} is defined in Equation (59).

iii. GE Loss Function

The Bayes estimator of $\varphi(\lambda,\beta)$ under the GE loss function, denoted by $\hat{\varphi}_{SGE_C}(\lambda,\beta)$, can be found by using Equations (28) and (58).

$$\begin{aligned}\hat{\varphi}_{SGE_C}(\lambda,\beta) = \Bigg[k_4 \int_0^\infty \int_0^\infty & [\varphi(\lambda,\beta)]^{-q} \lambda^{f+r-1} e^{-\left[\sum_{i=1}^{r}\frac{\alpha}{x_i}+\nu\right]\lambda} \beta^{c-1} \\ & \times \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r e^{-\left[d-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]} \\ & \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]} d\lambda\, d\beta\Bigg]^{\frac{-1}{q}},\end{aligned} \tag{66}$$

where k_4^{-1} is defined in (59).

Different forms of the Bayes estimator for $\varphi(\lambda,\beta)$ are obtained from Equation (66):

- When $\varphi(\lambda,\beta)=\lambda$, we obtain the Bayes estimator for λ, denoted by $\hat{\lambda}_{SGE_C}$, as:

$$\begin{aligned}\hat{\lambda}_{SGE_C} = \Bigg[k_4 \int_0^\infty \int_0^\infty & \lambda^{f+r-q-1} e^{-\left[\sum_{i=1}^{r}\frac{\alpha}{x_i}+\nu\right]\lambda} \beta^{c-1} \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r \\ & \times e^{-\left[d-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda}{x_i}}]} \\ & \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]} d\lambda\, d\beta\Bigg]^{\frac{-1}{q}},\end{aligned} \tag{67}$$

  where k_4^{-1} is defined in Equation (59);

- When $\varphi(\lambda,\beta)=\beta$, we obtain the Bayes estimator for β, denoted by $\hat{\beta}_{SGE_C}$, as:

$$\begin{aligned}\hat{\beta}_{SGE_C} = \Bigg[& k_4 \int_0^\infty \int_0^\infty \lambda^{f+r-1} e^{-\left[\sum_{i=1}^r \frac{\alpha}{x_i}+\nu\right]\lambda} \beta^{c-q-1} \\ & \times \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} e^{-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]} \\ & \times e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]} d\lambda\, d\beta \Bigg]^{\frac{-1}{q}},\end{aligned} \tag{68}$$

where k_4^{-1} is defined in Equation (59).

The Bayes estimators of β and λ can be obtained using the importance sampling technique by rewriting the joint posterior density function of λ and β in Equation (58) as follows:

$$\begin{aligned}\pi^*(\lambda,\beta|\underline{x}) \propto & \frac{\left(\sum_{i=1}^r \frac{\alpha}{x_i}+\nu\right)^{f+r}}{\Gamma(f+r)} \lambda^{f+r-1} e^{-\left[\sum_{i=1}^r \frac{\alpha}{x_i}+\nu\right]\lambda} \\ & \times \frac{(d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}])^c}{\Gamma(c)} \beta^{c-1} e^{-\left[d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]\right]\beta} \\ & \times \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r \frac{e^{-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}}{(d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}])^c}.\end{aligned}$$

The joint posterior density function of β and λ can be considered as:

$$\begin{aligned}\pi^*(\lambda,\beta|\underline{x}) \propto & Gamma(f+r,\sum_{i=1}^r \frac{\alpha}{x_i}+\nu) \times Gamma(c, d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]) \\ & \times g_4(\lambda,\beta|\underline{x}),\end{aligned}$$

where:

$$g_4(\lambda,\beta|\underline{x}) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\right]^r \frac{e^{-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]} e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda}{x_r}}}(\alpha,\beta)\right]}}{(d-\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}])^c} \tag{69}$$

The Bayes estimators of $\varphi(\lambda,\beta)$ under the SE, LINEX, and GE loss functions based on the importance sampling technique, denoted by $\hat{\varphi}_{ISE_C}(\lambda,\beta)$, $\hat{\varphi}_{ILE_C}(\lambda,\beta)$, and $\hat{\varphi}_{IGE_C}(\lambda,\beta)$, respectively, could be found using the following Algorithm 4.

Algorithm 4 Importance sampling technique when λ and β are unknown based on type-II censored samples.

1. Generate $\lambda_i \sim$ Gamma(f + r, $\sum_{i=1}^r \frac{\alpha}{x_i}+\nu$) and $\beta_i \sim$ Gamma(c, d-$\sum_{i=1}^r \log[1-e^{\frac{-\lambda}{x_i}}]$).
2. Repeat Step 1 to obtain $(\lambda_1,\beta_1),(\lambda_2,\beta_2),\ldots,(\lambda_N,\beta_N)$.
3. Calculate the values.

$$\hat{\varphi}_{ISE_C}(\lambda,\beta) = \frac{\sum_{j=1}^N \varphi(\lambda_j,\beta_j) g_4(\lambda_j,\beta_j|\underline{x})}{\sum_{j=1}^N g_4(\lambda_j,\beta_j|\underline{x})} \tag{70}$$

$$\hat{\varphi}_{ILE_C}(\lambda,\beta) = -\frac{1}{\tau}\log\left[\frac{\sum_{j=1}^N e^{-\tau\varphi(\lambda_j,\beta_j)} g_4(\lambda_j,\beta_j|\underline{x})}{\sum_{j=1}^N g_4(\lambda_j,\beta_j|\underline{x})}\right] \tag{71}$$

$$\hat{\varphi}_{IGE_C}(\lambda,\beta) = \left[\frac{\sum_{j=1}^N [\varphi(\lambda_j,\beta_j)]^{-q} g_4(\lambda_j,\beta_j|\underline{x})}{\sum_{j=1}^N g_4(\lambda_j,\beta_j|\underline{x})}\right]^{-\frac{1}{q}} \tag{72}$$

where

$$g_4(\lambda_j, \beta_j|\underline{x}) = \left[\frac{\Gamma(\alpha+\beta_j)}{\Gamma(\beta_j)}\right]^r \frac{e^{-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda_j}{x_i}}]}\, e^{(n-r)\log\left[1-I_{e^{\frac{-\lambda_j}{x_r}}}(\alpha,\beta_j)\right]}}{(d-\sum_{i=1}^{r}\log[1-e^{\frac{-\lambda_j}{x_i}}])^c}. \tag{73}$$

The Bayes estimators of λ and β under the three loss functions can be found numerically using the ***NIntegrate*** function via ***Mathematica 11***.

3. Simulation Study

Simulation studies were conducted using ***Mathematica 11*** to clarify the performance of the proposed estimators. Simulation results are given for the ML and Bayesian methods based on type-II censored samples. Furthermore, biases and mean-squared errors (MSE) were considered to illustrate the performance of the different estimators, defined as:

$$Bias(\hat{\theta}) = E[\hat{\theta}] - \theta \qquad MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2$$

The ML estimates of the parameters α, $\beta, \lambda, S(x_0)$ and $h(x_0)$ could be found using the following Algorithm 5.

Algorithm 5 ML method of the parameters α, $\beta, \lambda, S(x_0)$ and $h(x_0)$ based on type-II censored samples.

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1. For given true values selected as (α, β, λ), generate a random sample of size n from Equation (5).
2. Arrange Step 1 in ascending order to obtain $X_{1:n} < X_{2:n} < X_{3:n} < \cdots < X_{n:n}$.
3. Obtain the censored sample according to the censoring percentage.
4. Estimate the ML of the parameters α, β, and λ using the ***Newton–Raphson*** method to solve the equations given in (13), (16) and (18), simultaneously.
5. Compute the estimators of $S(x_0)$ and $h(x_0)$ from (6) and (7), respectively, using the estimates in Step 4.
6. Repeat Steps 1–5 1000 times.
7. Calculate the mean, bias, and MSE for each estimate.

Three different sets of true parameter values were selected to perform the simulation. Furthermore, different sample sizes $n = 30$, 50, and 100 and two censoring percentage of 80% and 90% were selected. Table 1 shows the ML estimates for α, β, λ, $S(x_0)$, $h(x_0)$, the bias, and the MSE for the true values ($\alpha = 0.8$, $\beta = 4$, $\lambda = 3$) and $S(x_0) = 0.72904$, $h(x_0) = 0.81786$ at $x_0 = 1$. The ML estimates for α, β, λ, $S(x_0)$, $h(x_0)$, the bias, and the MSE for the true values ($\alpha = 3$, $\beta = 0.8$, $\lambda = 3$) and $S(x_0) = 0.87604$, $h(x_0) = 0.05352$ at $x_0 = 5$ are summarized in Table 2. Moreover, for the true values ($\alpha = 3$, $\beta = 8$, $\lambda = 2$) and $S(x_0) = 0.22471$, $h(x_0) = 1.60828$ at $x_0 = 2$, the ML estimates for α, β, λ, $S(x_0)$, $h(x_0)$, the bias, and the MSE are listed in Table 3.

It is clear from Tables 1–3 that the MSEs of the estimates decrease as the sample size increases. Based on biases, the parameter λ was underestimated for all sets of true values, except in Table 2, the parameter λ was overestimated. Likewise, the parameter α was overestimated. Besides, we noticed that when the percentage of censoring was 90%, the MSE of the estimates in most cases was better.

Table 1. The ML estimates, bias, and MSE of the parameters α, $\beta, \lambda, S(x_0)$, and $h(x_0)$ for true parameter values ($\alpha = 0.8$, $\beta = 4$, $\lambda = 3$).

n	r		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{S}(x_0)$	$\hat{h}(x_0)$
30	**24**	MLE	1.27791	4.05604	2.45134	0.73519	0.83076
		Bias	0.47791	0.05604	−0.54866	0.00615	0.01291
		MSE	0.65026	1.07637	0.82283	0.00402	0.02662
	27	MLE	0.89924	4.01143	2.85928	0.72555	0.82323
		Bias	0.09924	0.01143	−0.14072	−0.00349	0.00537
		MSE	0.07609	0.92676	0.32039	0.00396	0.02497
50	**40**	MLE	0.88102	3.97067	2.90404	0.73132	0.81202
		Bias	0.08102	−0.02934	−0.09596	0.00227	−0.00584
		MSE	0.05673	0.28639	0.20724	0.00250	0.01299
	45	MLE	0.87403	3.95937	2.90266	0.73291	0.80861
		Bias	0.07403	−0.04063	−0.09734	0.00386	−0.00925
		MSE	0.03969	0.27514	0.19996	0.00258	0.01454
100	**80**	MLE	0.86352	3.99116	2.92567	0.73380	0.80855
		Bias	0.06352	−0.00884	−0.07433	0.00475	−0.00931
		MSE	0.03542	0.22032	0.13827	0.00131	0.00643
	90	MLE	0.84168	3.98337	2.96389	0.73189	0.81021
		Bias	0.04168	−0.01664	−0.03611	0.00285	−0.00765
		MSE	0.02752	0.20084	0.15640	0.00134	0.00786

Table 2. The ML estimates, bias, and MSE of the parameters α, $\beta, \lambda, S(x_0)$ and $h(x_0)$ for true parameter values ($\alpha = 3$, $\beta = 0.8$, $\lambda = 3$).

n	r		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{S}(x_0)$	$\hat{h}(x_0)$
30	**24**	MLE	3.28695	0.88241	3.08763	0.87765	0.05352
		Bias	0.28695	0.08241	0.08763	0.00161	−0.00000
		MSE	0.96247	0.04896	0.18824	0.00214	0.00020
	27	MLE	3.20203	0.87913	3.07017	0.87230	0.05497
		Bias	0.20203	0.07913	0.07017	−0.00374	0.00144
		MSE	0.86952	0.04667	0.15728	0.00203	0.00017
50	40	MLE	3.17675	0.86190	3.02640	0.87315	0.05506
		Bias	0.17675	0.06191	0.02631	−0.00288	0.00154
		MSE	0.52161	0.02995	0.10801	0.00132	0.00012
	45	MLE	3.09316	0.84395	3.04172	0.87372	0.05457
		Bias	0.09316	0.04395	0.04172	−0.00232	0.00104
		MSE	0.33268	0.02347	0.08888	0.00102	0.00009
100	**80**	MLE	3.09505	0.83487	3.02198	0.87445	0.05438
		Bias	0.09504	0.03487	0.02198	−0.00159	0.00086
		MSE	0.25649	0.01383	0.07476	0.00077	0.00006
	90	MLE	3.09843	0.82840	3.02111	0.87805	0.05333
		Bias	0.09843	0.02839	0.02111	0.00201	−0.00020
		MSE	0.14981	0.01212	0.05592	0.00046	0.00003

Table 3. The ML estimates, bias, and MSE of the parameters α, β, λ, $S(x_0)$ and $h(x_0)$ for true parameter values ($\alpha = 3$, $\beta = 8$, $\lambda = 2$).

n	r		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{S}(x_0)$	$\hat{h}(x_0)$
30	**24**	MLE	3.76231	8.04103	1.83068	0.22896	1.62124
		Bias	0.76231	0.04103	−0.16932	0.00425	0.01296
		MSE	2.04744	0.26625	0.17399	0.00279	0.02465
	27	MLE	3.71919	7.99749	1.84219	0.23080	1.61052
		Bias	0.71919	−0.00251	−0.15782	0.00609	0.00224
		MSE	2.04687	0.29352	0.16868	0.00289	0.02392
50	**40**	MLE	3.26032	7.98509	1.93754	0.22686	1.60902
		Bias	0.26032	−0.01491	−0.06246	0.00215	0.00074
		MSE	0.59433	0.21877	0.07295	0.00142	0.01337
	45	MLE	3.27549	7.95998	1.93706	0.22960	1.59955
		Bias	0.27549	−0.04002	−0.06294	0.00489	−0.00873
		MSE	0.61335	0.20042	0.07451	0.00132	0.01260
100	**80**	MLE	3.17247	7.97533	1.97010	0.23002	1.59641
		Bias	0.17247	−0.02467	−0.02990	0.00531	−0.01187
		MSE	0.41601	0.17126	0.05337	0.00072	0.00807
	90	MLE	3.18983	7.98382	1.95748	0.22795	1.60322
		Bias	0.18983	−0.01618	−0.04253	0.00324	−0.00506
		MSE	0.39486	0.16581	0.04962	0.00068	0.00795

Bayes estimates of the parameters α, β, λ, $S(x_0)$, and $h(x_0)$ could be found using the following Algorithm 6.

Algorithm 6 Bayesian method of the parameters α, β, λ, $S(x_0)$, and $h(x_0)$ based on type-II censored samples.

1. For given true parameter values selected as (α, β, λ), generate a random sample of size n from Equation (5).
2. Arrange Step 1 in ascending order to obtain $X_{1:n} < X_{2:n} < X_{3:n} \cdots < X_{n:n}$.
3. Obtain the censored sample according to the censoring percentage.
4. For given values of the hyperparameters parameters a, b, c, d, f, ν, compute the Bayes estimates of the parameters α, β, λ via the standard Bayes technique for the first three cases using the ***NIntegrate*** function under the SE, LINEX, and GE loss functions as shown in Table 4.
5. For given values of the hyperparameters parameters a, b, c, d, f, ν, compute the Bayes estimates of the parameters α, β, λ via the importance sampling technique for all cases under the SE, LINEX, and GE loss functions, as shown in Table 4.
6. Compute the Bayes estimates of $S(x_0)$ and $h(x_0)$ for the four cases from (6) and (7), respectively, using the estimates in the previous steps.
7. Repeat Steps 1–6 1000 times.
8. Calculate the mean, bias, and MSE for each estimate.

The simulation study was carried out with the true value of parameter ($\alpha = 0.8$, $\beta = 4$, $\lambda = 3$) and different sample sizes n = 30, 50, and 100 for all four cases. Furthermore, we considered three different values of the LINEX shape parameter $(\tau = 0.001, \tau = 2, \tau = 5)$ and three values of the GE shape parameter ($q = -1, q = 3, q = -3$). The Bayes estimates for first three cases were derived based on the standard Bayes and importance sampling techniques. For the last case, this was obtained via the importance sampling technique. The values of the hyperparameters for the standard Bayes technique are $(a = 2, b = 4, c = 5, d = 2, f = 2, \nu = 5)$. Further, the values of the hyperparameters for the importance sampling technique are $(a = 80,\ b = 0.1, c = 14, d = 2, f = 30, \nu = 0.001)$ with a sample size of $N = 1000$. For given $S(x_0) = 0.72904$ and $h(x_0) = 0.81786$ at $x_0 = 1$, the ML and Bayes estimates of the $S(x_0)$ and $h(x_0)$ were computed. The four cases were calculated as follows.

Table 4. The four cases for calculating the Bayes estimators via the standard Bayes and importance sampling techniques.

	Case 1 When α Is Unknown	**Case 2 When β Is Unknown**
standard Bayes technique	The Bayes estimates of $\varphi(\alpha)$ under SE, LINEX, and GE are obtained by computing Equations (25), (27) and (29).	The Bayes estimates of $\varphi(\beta)$ are obtained numerically under the SE, LINEX, and GE loss functions by evaluating (38), (39) and (40), respectively.
importance sampling technique	According to Algorithm 1, the Bayes estimates of $\varphi(\alpha)$ are obtained under the SE, LINEX, and GE loss functions by computing Equations (31), (32) and (33), respectively.	Based on Algorithm 2, the Bayes estimates of $\varphi(\beta)$ are obtained numerically by computing Equations (42)–(44) under three loss functions.
	Case 3 When λ Is Unknown	**Case 4 When β and λ Are Unknown**
standard Bayes technique	The Bayes estimates of $\varphi(\lambda)$ are obtained under the SE, LINEX, and GE loss functions by evaluating (49), (50) and (51), respectively.	—
importance sampling technique	Based on Algorithm 3, the Bayes estimates of $\varphi(\lambda)$ are obtained numerically by computing Equations (53)–(55) under three loss functions.	The Bayes estimates of $\varphi(\beta, \lambda)$ are obtained numerically according to Algorithm 4 under the SE, LINEX, and GE loss functions by computing Equations (70), (71) and (72), respectively.

Tables 5–11 show the Bayes and ML estimates of the parameters.

The results of the two estimation methods from Tables 5–11 are summarized as follows:

1. From Tables 5–11, the MSEs of the ML estimates and Bayes estimates of α, β, λ, $S(x_0)$, and $h(x_0)$ decrease as the sample size increases;
2. From Tables 5–11, the mean, bias, and MSE values for the Bayes estimates under the SE loss function, the LINEX loss function with ($\tau = 0.001$), and the GE loss function with ($q = -1$) are very similar;
3. For Case 1, the Bayes estimates via the standard Bayes technique of α, $S(x_0)$, and $h(x_0)$ perform the estimates better than the ML estimates under the different loss functions, as shown in Table 5. According to Table 6, we note that the ML estimates give better values than the Bayes estimates via the importance sampling technique;
4. From Table 5, the Bayes estimates under the GE loss function ($q = -3$) are considered the best estimates of $S(x_0)$;
5. From Table 7, the Bayes estimates of β, $S(x_0)$, and $h(x_0)$ via the standard Bayes technique perform the best based on MSEs and biases at $n = 30$. Furthermore, when $n = 50, 100$, the ML estimates of β, $S(x_0)$, and $h(x_0)$ perform the estimates better than the Bayes estimates;
6. When β is unknown, the Bayes estimates of β via the importance sampling technique perform the best at $n = 30$, under the LINEX loss function ($\tau = 5$). For $n = 50$, 100, the ML estimates of β give the best estimates. Furthermore, based on the MSEs and biases, the ML estimates of $S(x_0)$ and $h(x_0)$ give the best estimates (see Table 8);
7. Based on the MSEs, the ML estimates of λ, $S(x_0)$, and $h(x_0)$ perform the estimates better than the Bayes estimates based on the two techniques (see Tables 9 and 10);
8. From Table 11, the ML estimates of λ, $S(x_0)$, and $h(x_0)$ perform the best based on the smallest MSEs. Besides, the Bayes estimates of β via the importance sampling technique perform the best under the LINEX loss function ($\tau = 5$).

Table 5. The ML and Bayesian estimates of the unknown parameters α, $S(x_0)$, and $h(x_0)$ via the standard Bayes technique.

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
30	24	α	Mean	0.81865	0.78855	0.78855	0.77828	0.76692	0.78855	0.75448	0.82343
			Bias	0.01865	−0.01148	−0.01148	−0.02172	−0.03308	−0.01148	−0.04552	0.02343
			MSE	0.01887	0.01353	0.01353	0.01362	0.01343	0.01353	0.01516	0.01556
		$S(x_0)$	Mean	0.73107	0.71139	0.71139	0.70988	0.70922	0.71139	0.70186	0.72454
			Bias	0.00203	−0.01766	−0.01766	−0.01916	−0.01982	−0.01766	−0.02718	−0.00450
			MSE	0.00426	0.00362	0.00362	0.00384	0.00401	0.00362	0.00443	0.00340
		$h(x_0)$	Mean	0.80731	0.84351	0.84351	0.82278	0.78846	0.84351	0.79688	0.84694
			Bias	−0.01055	0.02566	0.02566	0.00492	−0.02940	0.02566	−0.02098	0.02908
			MSE	0.01848	0.01440	0.01440	0.01443	0.01570	0.01440	0.01665	0.01468
	27	α	Mean	0.81917	0.79216	0.79216	0.78044	0.76701	0.79216	0.75675	0.82325
			Bias	0.01917	−0.00784	−0.00784	−0.01956	−0.03299	−0.00784	−0.04325	0.02325
			MSE	0.01834	0.01359	0.01359	0.01385	0.01359	0.01359	0.01528	0.01574
		$S(x_0)$	Mean	0.73155	0.71345	0.71345	0.71092	0.70921	0.71345	0.70296	0.72447
			Bias	0.00250	−0.01559	−0.01559	−0.01813	−0.01983	−0.01559	−0.02608	−0.00457
			MSE	0.00415	0.00355	0.00355	0.00386	0.00398	0.00355	0.00443	0.00336
		$h(x_0)$	Mean	0.80652	0.83954	0.83954	0.82074	0.78878	0.83954	0.79485	0.84697
			Bias	−0.01133	0.02168	0.02168	0.00288	−0.02908	0.02168	−0.02300	0.02911
			MSE	0.01797	0.01425	0.01425	0.01469	0.01562	0.01425	0.01704	0.01467
50	40	α	Mean	0.81429	0.80057	0.80057	0.77973	0.78116	0.80057	0.76534	0.81616
			Bias	0.01429	0.00057	0.00057	−0.02028	−0.01884	0.00057	−0.03466	0.01616
			MSE	0.01037	0.00853	0.00853	0.00917	0.00878	0.00853	0.00994	0.00976
		$S(x_0)$	Mean	0.73215	0.72202	0.72202	0.71340	0.71759	0.72202	0.70869	0.72696
			Bias	0.00311	−0.00702	−0.00702	−0.01564	−0.01146	−0.00702	−0.02035	−0.00209
			MSE	0.00241	0.00218	0.00218	0.00258	0.00246	0.00218	0.00282	0.00223
		$h(x_0)$	Mean	0.80789	0.82577	0.82577	0.82830	0.79768	0.82577	0.81316	0.83430
			Bias	−0.00997	0.00791	0.00791	0.01045	−0.02017	0.00791	−0.00469	0.01644
			MSE	0.01043	0.00897	0.00897	0.00974	0.01006	0.00897	0.01035	0.00950
	45	α	Mean	0.81224	0.80057	0.80057	0.78936	0.78158	0.80057	0.77503	0.81643
			Bias	0.01224	0.00057	0.00057	−0.01064	−0.01842	0.00057	−0.02497	0.01643
			MSE	0.01046	0.00853	0.00853	0.00919	0.00860	0.00853	0.00969	0.00958
		$S(x_0)$	Mean	0.73105	0.72202	0.72202	0.71835	0.71788	0.72202	0.71375	0.72721
			Bias	0.00201	−0.00702	−0.00702	−0.01069	−0.01116	−0.00702	−0.01529	−0.00184
			MSE	0.00249	0.00218	0.00218	0.00243	0.00238	0.00218	0.00263	0.00216
		$h(x_0)$	Mean	0.81007	0.82577	0.82577	0.81829	0.79734	0.82577	0.80287	0.83380
			Bias	−0.00779	0.00791	0.00791	0.00044	−0.02052	0.00791	−0.01498	0.01594
			MSE	0.01066	0.00897	0.00897	0.00972	0.00982	0.00897	0.01066	0.00925

Table 5. *Cont.*

n	r	Parameters		ML Estimates	Bayes Estimates SE	LINEX $\tau = 0.001$	$\tau = 2$	$\tau = 5$	GE $q = -1$	$q = 3$	$q = -3$
100	80	α	Mean	0.80413	0.79939	0.79939	0.79528	0.78855	0.79939	0.78803	0.80636
			Bias	0.00413	−0.00061	−0.00061	−0.00472	−0.01145	−0.00061	−0.01197	0.00636
			MSE	0.00506	0.00468	0.00468	0.00454	0.00476	0.00468	0.00466	0.00496
		$S(x_0)$	Mean	0.72907	0.72486	0.72486	0.72400	0.72220	0.72486	0.72176	0.72698
			Bias	0.00003	−0.00419	−0.00419	−0.00504	−0.00684	−0.00419	−0.00728	−0.00207
			MSE	0.00126	0.00119	0.00119	0.00118	0.00128	0.00119	0.00123	0.00121
		$h(x_0)$	Mean	0.81597	0.82298	0.82298	0.81716	0.80936	0.82298	0.80948	0.82824
			Bias	−0.00189	0.00512	0.00512	−0.00079	−0.00849	0.00512	−0.00837	0.01038
			MSE	0.00532	0.00497	0.00497	0.00485	0.00523	0.00497	0.00509	0.00514
	90	α	Mean	0.80925	0.79939	0.79939	0.78986	0.79195	0.79939	0.78264	0.80978
			Bias	0.00925	−0.00061	−0.00061	−0.01014	−0.00805	−0.00061	−0.01736	0.00981
			MSE	0.00558	0.00468	0.00468	0.00443	0.00428	0.00468	0.00463	0.00458
		$S(x_0)$	Mean	0.73148	0.72486	0.72486	0.72129	0.72414	0.72486	0.71903	0.72887
			Bias	0.00244	−0.00419	−0.00419	−0.00775	−0.00490	−0.00419	−0.01001	−0.00017
			MSE	0.00134	0.00119	0.00119	0.00117	0.00115	0.00119	0.00123	0.00110
		$h(x_0)$	Mean	0.81088	0.82298	0.82298	0.82279	0.80565	0.82298	0.81525	0.82449
			Bias	−0.00698	0.00512	0.00512	0.00494	−0.01221	0.00512	−0.00261	0.00663
			MSE	0.00574	0.00497	0.00497	0.00468	0.00486	0.00497	0.00483	0.00464

Table 6. The ML and Bayesian estimates of the unknown parameters α, $S(x_0)$, and $h(x_0)$ via the importance sampling technique.

n	r	Parameters		ML Estimates	Bayes Estimates SE	LINEX $\tau = 0.001$	$\tau = 2$	$\tau = 5$	GE $q = -1$	$q = 3$	$q = -3$
30	24	α	Mean	0.81865	1.67661	1.67661	1.67045	1.67654	1.67661	1.67018	1.68018
			Bias	0.01865	0.87661	0.87661	0.87045	0.87654	0.87661	0.87018	0.88018
			MSE	0.01887	0.80599	0.80599	0.79622	0.80872	0.80599	0.79580	0.81512
		$S(x_0)$	Mean	0.73107	0.95051	0.95051	0.94989	0.95050	0.95051	0.94988	0.95056
			Bias	0.00203	0.22147	0.22147	0.22085	0.22146	0.22147	0.22083	0.22152
			MSE	0.00426	0.04944	0.04944	0.04917	0.04946	0.04944	0.04917	0.04948
		$h(x_0)$	Mean	0.80731	0.23678	0.23678	0.23875	0.23595	0.23678	0.23778	0.23714
			Bias	−0.01055	−0.58108	−0.58108	−0.57910	−0.58190	−0.58108	−0.58008	−0.58072
			MSE	0.01848	0.34284	0.34284	0.34064	0.34411	0.34284	0.34175	0.34278
	27	α	Mean	0.81917	1.59218	1.59218	1.58224	1.58460	1.59218	1.58198	1.58762
			Bias	0.01917	0.79218	0.79218	0.78224	0.78460	0.79218	0.78198	0.78726
			MSE	0.01834	0.66092	0.66092	0.64575	0.652221	0.66092	0.64539	0.65639

Table 6. *Cont.*

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
30	27	$S(x_0)$	Mean	0.73155	0.94133	0.94133	0.94017	0.94019	0.94133	0.94015	0.94024
			Bias	0.00250	0.21228	0.21228	0.21112	0.21114	0.21228	0.21110	0.21120
			MSE	0.00415	0.04553	0.04553	0.04505	0.04514	0.04553	0.04504	0.04516
		$h(x_0)$	Mean	0.80652	0.26985	0.26985	0.27369	0.27273	0.26985	0.27290	0.27374
			Bias	−0.01133	−0.54800	−0.54800	−0.54417	−0.54512	−0.54800	−0.54496	−0.54412
			MSE	0.01797	0.30597	0.30597	0.30183	0.30362	0.30597	0.30268	0.30258
50	40	α	Mean	0.81429	1.02888	1.02888	1.02199	1.02842	1.02888	1.02189	1.02878
			Bias	0.01429	0.22888	0.22888	0.22199	0.22842	0.22888	0.22189	0.22879
			MSE	0.01037	0.06184	0.06184	0.05813	0.06164	0.06184	0.05809	0.06180
		$S(x_0)$	Mean	0.73215	0.82255	0.82255	0.82034	0.82241	0.82255	0.82032	0.82245
			Bias	0.00311	0.09350	0.09350	0.09130	0.09337	0.09350	0.09128	0.09342
			MSE	0.00241	0.00982	0.00982	0.00938	0.00984	0.00982	0.00937	0.00985
		$h(x_0)$	Mean	0.80789	0.60854	0.60854	0.61387	0.60854	0.60854	0.61373	0.60884
			Bias	−0.00997	−0.20932	−0.20932	−0.20399	−0.20932	−0.20932	−0.20413	−0.20902
			MSE	0.01043	0.05011	0.05011	0.04761	0.05025	0.05011	0.04767	0.05014
	45	α	Mean	0.81224	1.01755	1.01755	1.01760	1.01795	1.01755	1.01750	1.01824
			Bias	0.01224	0.21755	0.21755	0.21760	0.21795	0.21755	0.21750	0.21824
			MSE	0.01046	0.05601	0.05601	0.05580	0.05697	0.05601	0.05577	0.05709
		$S(x_0)$	Mean	0.73105	0.81880	0.81880	0.81891	0.81874	0.81880	0.81889	0.81878
			Bias	0.00201	0.08976	0.08976	0.08986	0.08970	0.08976	0.08984	0.08974
			MSE	0.00249	0.00909	0.00909	0.00909	0.00920	0.00909	0.00909	0.00920
		$h(x_0)$	Mean	0.81007	0.61768	0.61768	0.61741	0.61736	0.61768	0.61728	0.61761
			Bias	−0.00779	−0.20017	−0.20017	−0.20045	−0.20050	−0.20017	−0.20058	−0.20025
			MSE	0.01066	0.04604	0.04604	0.04599	0.04675	0.04604	0.04603	0.04666
100	80	α	Mean	0.80413	0.64656	0.64656	0.64757	0.64685	0.64656	0.64756	0.64688
			Bias	0.00413	−0.15344	−0.15344	−0.15243	−0.15315	−0.15344	−0.15245	−0.15312
			MSE	0.00506	0.02546	0.02546	0.02526	0.02531	0.02546	0.02527	0.02530
		$S(x_0)$	Mean	0.72907	0.64000	0.64000	0.64060	0.64022	0.64000	0.64059	0.64024
			Bias	0.00003	−0.08904	−0.08904	−0.08844	−0.08882	−0.08904	−0.08845	−0.08880
			MSE	0.00126	0.00872	0.00872	0.00865	0.00865	0.00872	0.00865	0.00865
		$h(x_0)$	Mean	0.81597	0.98945	0.98945	0.98829	0.98903	0.98945	0.98828	0.98906
			Bias	−0.00189	0.17159	0.17159	0.17043	0.17117	0.17159	0.17042	0.17120
			MSE	0.00532	0.03213	0.03213	0.03187	0.03189	0.03213	0.03187	0.03190
	90	α	Mean	0.80925	0.65127	0.65127	0.65130	0.64958	0.65127	0.65128	0.64961
			Bias	0.00925	−0.14873	−0.14873	−0.14871	−0.15042	−0.14873	−0.14872	−0.15039
			MSE	0.00558	0.02400	0.02400	0.02392	0.02447	0.02400	0.02392	0.02446

Table 6. *Cont.*

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
100	90	$S(x_0)$	Mean	0.73148	0.64305	0.64305	0.64310	0.64198	0.64305	0.64309	0.64199
			Bias	0.00244	−0.08560	−0.08560	−0.08594	−0.08706	−0.08560	−0.08595	−0.08705
			MSE	0.00134	0.00815	0.00815	0.00812	0.00834	0.00815	0.00812	0.00833
		$h(x_0)$	Mean	0.81088	0.98385	0.98385	0.98377	0.98579	0.98385	0.98376	0.98583
			Bias	−0.00698	0.16599	0.16599	0.16592	0.16794	0.16599	0.16591	0.16797
			MSE	0.00574	0.03015	0.03015	0.03003	0.03078	0.03015	0.03003	0.03079

Table 7. The ML and Bayesian estimates of the unknown parameters β, $S(x_0)$, and $h(x_0)$ via the standard Bayes technique.

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
30	24	β	Mean	4.00137	3.63379	3.63379	3.21065	2.75023	3.63379	3.37739	3.84007
			Bias	0.00137	−0.36621	−0.36621	−0.78935	−1.24977	−0.36621	−0.62261	−0.15993
			MSE	0.52494	0.47324	0.47324	0.82890	1.66793	0.47324	0.68487	0.38669
		$S(x_0)$	Mean	0.73031	0.74993	0.74993	0.74633	0.74334	0.74993	0.74383	0.74885
			Bias	0.00126	0.02089	0.02089	0.01729	0.01429	0.02089	0.01479	0.01981
			MSE	0.00130	0.00130	0.00130	0.00121	0.00109	0.00130	0.00117	0.00121
		$h(x_0)$	Mean	0.81692	0.74893	0.74893	0.73874	0.71619	0.74893	0.70815	0.78359
			Bias	−0.00094	−0.06892	−0.06892	−0.07911	−0.10167	−0.06892	−0.10971	−0.03426
			MSE	0.01729	0.01604	0.01604	0.01690	0.01902	0.01604	0.02225	0.01299
	27	β	Mean	3.95886	3.73469	3.73469	3.24211	2.82172	3.73469	3.39740	3.83642
			Bias	−0.04114	−0.26531	−0.26531	−0.75789	−1.17828	−0.26531	−0.60260	−0.16358
			MSE	0.43395	0.41495	0.41495	0.77379	1.50751	0.41495	0.64107	0.39905
		$S(x_0)$	Mean	0.73219	0.74465	0.74465	0.74681	0.74303	0.74465	0.74458	0.74802
			Bias	0.00315	0.01661	0.01661	0.01777	0.01398	0.01661	0.01554	0.01897
			MSE	0.00108	0.00111	0.00111	0.00116	0.00111	0.00111	0.00112	0.00121
		$h(x_0)$	Mean	0.80941	0.76751	0.76751	0.73881	0.72206	0.76751	0.71138	0.78332
			Bias	−0.00845	−0.05034	−0.05034	−0.07904	−0.09580	−0.05034	−0.10648	−0.03453
			MSE	0.01432	0.01395	0.01395	0.01616	0.01838	0.01395	0.02089	0.01338
50	40	β	Mean	3.99325	3.80054	3.80054	3.44623	3.07134	3.80054	3.58543	3.90027
			Bias	−0.00675	−0.19947	−0.19947	−0.55368	−0.92866	−0.19947	−0.41457	−0.00973
			MSE	0.31444	0.34557	0.34557	0.49296	0.99045	0.34557	0.41568	0.32374
		$S(x_0)$	Mean	0.73017	0.74084	0.74084	0.74046	0.73840	0.74084	0.73876	074216
			Bias	0.00113	0.01179	0.01179	0.01142	0.00935	0.01179	0.00971	0.01311
			MSE	0.00077	0.00090	0.00090	0.00082	0.00085	0.00090	0.00080	0.00090

Table 7. *Cont.*

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
50	40	$h(x_0)$	Mean	0.81593	0.77997	0.77997	0.76505	0.74927	0.77997	0.74497	0.79609
			Bias	−0.00193	−0.03789	−0.03789	−0.05281	−0.06858	−0.03789	−0.07289	−0.02177
			MSE	0.01034	0.01152	0.01152	0.01122	0.01302	0.01152	0.01355	0.01074
	45	β	Mean	4.00362	3.77522	3.77522	3.47241	3.14740	3.77522	3.59934	3.91113
			Bias	0.00362	−0.22479	−0.22479	−0.52759	−0.85260	−0.22479	−0.40066	−0.08887
			MSE	0.29272	0.32260	0.32260	0.54804	0.85118	0.32260	0.38937	0.29091
		$S(x_0)$	Mean	0.72960	0.74192	0.74192	0.74083	0.73739	0.74192	0.73932	0.74077
			Bias	0.00056	0.01288	0.01288	0.01179	0.00084	0.01288	0.01028	0.01173
			MSE	0.00072	0.00085	0.00085	0.00078	0.00076	0.00085	0.00076	0.00079
		$h(x_0)$	Mean	0.81786	0.77552	0.77552	0.76505	0.75623	0.77552	0.74719	0.79844
			Bias	2.80599×10^{-6}	−0.04233	−0.04233	−0.05281	−0.06163	−0.04233	−0.07067	−0.01942
			MSE	0.00961	0.01080	0.01080	0.01067	0.01145	0.01080	0.01271	0.00963
100	80	β	Mean	4.02104	3.89432	3.89432	3.68636	3.41183	3.89432	3.77820	3.92103
			Bias	0.02104	−0.10568	−0.10568	−0.31364	−0.58817	−0.10568	−0.22180	−0.0790
			MSE	0.18577	0.20450	0.20450	0.25076	0.46617	0.20450	0.22767	0.21109
		$S(x_0)$	Mean	0.72847	0.73536	0.73536	0.73507	0.73545	0.73536	0.73412	0.73752
			Bias	−0.00058	0.00632	0.00632	0.00602	0.00640	0.00632	0.00508	0.00847
			MSE	0.00045	0.00051	0.00051	0.00051	0.00054	0.00051	0.00050	0.00056
		$h(x_0)$	Mean	0.82126	0.79773	0.79773	0.78971	0.77531	0.79773	0.77891	0.80144
			Bias	0.00340	−0.02012	−0.02012	−0.02814	−0.04255	−0.02012	−0.03895	−0.01641
			MSE	0.00609	0.00676	0.00676	0.00681	0.00779	0.00676	0.00746	0.00699
	90	β	Mean	4.02520	3.90955	3.90955	3.72222	3.47851	3.90955	3.80560	3.94136
			Bias	0.02520	−0.09045	−0.09045	−0.27778	−0.52149	−0.09045	−0.19440	−0.05864
			MSE	0.17038	0.17502	0.17502	0.20935	0.38175	0.17502	0.19019	0.18087
		$S(x_0)$	Mean	0.72822	0.73447	0.73447	0.73424	0.73414	0.73447	0.73340	0.73599
			Bias	−0.00082	0.00542	0.00542	0.00520	0.00509	0.00542	0.00436	0.00695
			MSE	0.00042	0.000439	0.000439	0.00042	0.00046	0.000439	0.00042	0.00047
		$h(x_0)$	Mean	0.82205	0.80062	0.80062	0.79333	0.78199	0.80062	0.78377	0.80538
			Bias	0.00419	−0.01724	−0.01724	−0.02453	−0.03586	−0.01724	−0.03409	−0.01248
			MSE	0.00559	0.00578	0.00578	0.00571	0.00653	0.00578	0.00623	0.00597

Table 8. The ML and Bayesian estimates of the unknown parameters β, $S(x_0)$, and $h(x_0)$ via the importance sampling technique.

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
30	24	β	Mean	4.00137	5.01528	5.01528	4.35481	3.70618	5.01528	4.68073	5.06025
			Bias	0.00137	1.01528	1.01528	0.35481	−0.29382	1.01528	0.68073	1.06025
			MSE	0.52494	1.73584	1.73584	0.48266	0.29270	1.73584	1.00883	1.82667
		$S(x_0)$	Mean	0.73031	0.68394	0.68394	0.68497	0.68418	0.68394	0.68241	0.68929
			Bias	0.00126	−0.04510	−0.04510	−0.04408	−0.04487	−0.04510	−0.04663	−0.03975
			MSE	0.00130	0.00341	0.00341	0.00318	0.00337	0.00341	0.00346	0.00285
		$h(x_0)$	Mean	0.81692	0.99770	0.99770	0.96702	0.93335	0.99770	0.94390	1.00351
			Bias	−0.00094	0.17984	0.17984	0.14916	0.11550	0.17984	0.12605	0.18565
			MSE	0.01729	0.05445	0.05445	0.03999	0.02975	0.05445	0.03339	0.05635
	27	β	Mean	3.95886	4.88051	4.88051	4.21714	3.68235	4.88051	4.41388	4.57499
			Bias	−0.04114	0.88051	0.88051	0.21714	−0.31765	0.88051	0.41388	0.57499
			MSE	0.43395	1.42262	1.42262	0.46346	0.30434	1.42262	0.75868	0.92222
		$S(x_0)$	Mean	0.73219	0.68924	0.68924	0.70414	0.7041 8	0.68924	0.70320	0.70628
			Bias	0.00315	−0.03980	−0.03980	−0.02491	−0.02486	−0.03980	−0.05840	−0.02276
			MSE	0.00108	0.00288	0.00288	0.00196	0.00187	0.00288	0.00203	0.00173
		$h(x_0)$	Mean	0.80941	0.97433	0.97433	0.90614	0.888818	0.97433	0.89388	0.91906
			Bias	−0.00845	0.15648	0.15648	0.08829	0.07033	0.15648	0.07602	0.10120
			MSE	0.01432	0.04485	0.04485	0.02710	0.02106	0.04485	0.02472	0.02903
50	40	β	Mean	3.99325	4.93070	4.93070	4.29403	3.90463	4.93070	4.44350	4.63432
			Bias	−0.00675	0.93070	0.93070	0.29403	−0.09537	0.93070	0.44350	0.63432
			MSE	0.31444	1.43752	1.43752	0.42635	0.23429	1.43752	0.63256	0.95327
		$S(x_0)$	Mean	0.73017	0.68672	0.68672	0.70382	0.70097	0.68672	0.70307	0.70263
			Bias	0.00113	−0.04233	−0.04233	−0.02522	−0.02807	−0.04233	−0.02597	−0.02642
			MSE	0.00077	0.00293	0.00293	0.00164	0.00193	0.00293	0.00169	0.00182
		$h(x_0)$	Mean	0.81593	0.98346	0.98346	0.90816	0.90575	0.98346	0.89885	0.93003
			Bias	−0.00193	0.16560	0.16560	0.09031	0.08789	0.16560	0.08099	0.11218
			MSE	0.01034	0.04540	0.04540	0.02248	0.02319	0.04540	0.02064	0.03003
	45	β	Mean	4.00362	4.30424	4.30424	4.22921	3.85349	4.30424	4.27484	4.11707
			Bias	0.00362	0.30424	0.30424	0.22921	−0.14651	0.30424	0.27484	0.11707
			MSE	0.29272	0.53169	0.53169	0.44464	0.29407	0.53169	0.49591	0.41340
		$S(x_0)$	Mean	0.72960	0.71534	0.71534	0.71516	0.68614	0.72460	0.71494	0.72503
			Bias	0.00056	−0.01370	−0.01370	−0.01388	−0.00444	−0.01370	−0.01409	−0.00402
			MSE	0.00072	0.00118	0.00118	0.00116	0.00096	0.00118	0.00117	0.00096
		$h(x_0)$	Mean	0.81786	0.87184	0.87184	0.87012	0.83195	0.87184	0.86723	0.83792
			Bias	2.80599×10^{-6}	0.05398	0.05398	0.05226	0.01409	0.05398	0.04937	0.02006
			MSE	0.00961	0.01708	0.01708	0.01636	0.01297	0.01708	0.01605	0.01342

Table 8. *Cont.*

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
100	80	β	Mean	4.02104	4.52275	4.52275	4.31180	3.97709	4.52275	4.33079	4.07013
			Bias	0.02104	0.52275	0.52275	0.31180	−0.02291	0.52275	0.33079	0.07013
			MSE	0.18577	0.61202	0.61202	0.40122	0.22290	0.61202	0.42078	0.25983
		$S(x_0)$	Mean	0.72847	0.70462	0.70462	0.71301	0.72635	0.70462	0.71291	0.72651
			Bias	−0.00058	−0.02442	−0.02442	−0.01603	−0.00270	−0.02442	−0.01613	−0.00253
			MSE	0.00045	0.00133	0.00133	0.00095	0.00061	0.00133	0.00095	0.00061
		$h(x_0)$	Mean	0.82126	0.91137	0.91137	0.87845	0.82762	0.91137	0.87722	0.82990
			Bias	0.00340	0.09352	0.09352	0.06059	0.00976	0.09352	0.05936	0.01204
			MSE	0.00609	0.01957	0.01957	0.01368	0.00837	0.01957	0.01354	0.00845
	90	β	Mean	4.02520	3.71260	3.71260	3.72549	3.85324	3.71260	3.72811	3.87665
			Bias	0.02520	−0.28740	−0.28740	−0.27451	−0.14676	−0.28740	−0.27189	−0.12335
			MSE	0.17038	0.26820	0.26820	0.29151	0.23423	0.26820	0.29072	0.23222
		$S(x_0)$	Mean	0.72822	0.74401	0.74401	0.74315	0.73580	0.74401	0.74313	0.73586
			Bias	−0.00082	0.01497	0.01497	0.01410	0.00676	0.01497	0.01408	0.00681
			MSE	0.00042	0.00070	0.00070	0.00075	0.00058	0.00070	0.00075	0.00058
		$h(x_0)$	Mean	0.82205	0.76509	0.76509	0.76820	0.79421	0.76509	0.76792	0.79492
			Bias	0.00419	−0.05276	−0.05276	−0.04966	−0.02365	−0.05276	−0.04994	−0.02294
			MSE	0.00559	0.00896	0.00896	0.00966	0.00771	0.00896	0.00969	0.00768

Table 9. The ML and Bayesian estimates of the unknown parameters λ, $S(x_0)$, and $h(x_0)$ via the standard Bayes technique.

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
30	24	λ	Mean	3.02350	2.88990	2.88990	2.80094	2.68837	2.88990	2.82593	2.92667
			Bias	0.02350	−0.11010	−0.11010	−0.19906	−0.31163	−0.11010	−0.17407	−0.07333
			MSE	0.10261	0.12187	0.12187	0.13665	0.18271	0.12187	0.13498	0.11895
		$S(x_0)$	Mean	0.72733	0.69350	0.69350	0.68927	0.68403	0.69350	0.68035	0.70038
			Bias	−0.00171	−0.03554	−0.03554	−0.03977	−0.04501	−0.03554	−0.04870	−0.02867
			MSE	0.00387	0.00605	0.00605	0.00639	0.00700	0.00605	0.00759	0.00534
		$h(x_0)$	Mean	0.89150	0.89845	0.89845	0.87606	0.84129	0.89845	0.84459	0.92167
			Bias	0.00164	0.08059	0.08059	0.05820	0.02349	0.08059	0.02673	0.10381
			MSE	0.02237	0.03386	0.03386	0.02930	0.02509	0.03386	0.02845	0.03753
	27	λ	Mean	3.03297	2.77639	2.77639	2.68512	2.58070	2.77639	2.70513	2.79645
			Bias	0.03297	−0.22361	−0.22361	−0.31488	−0.41930	−0.22361	−0.29487	−0.20355
			MSE	0.10538	0.12345	0.12345	0.16758	0.23514	0.12345	0.16040	0.11978

Table 9. *Cont.*

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
30	27	$h(x_0)$	Mean	0.81531	0.95272	0.95272	0.93646	0.90221	0.95272	0.90794	0.98287
			Bias	−0.00255	0.13486	0.13486	0.11861	0.08435	0.13486	0.09009	0.16501
			MSE	0.02285	0.03888	0.03888	0.03525	0.02713	0.03888	0.03085	0.04856
		$S(x_0)$	Mean	0.72906	0.67094	0.67094	0.66374	0.65772	0.67094	0.65409	0.67485
			Bias	0.000016	−0.05811	−0.05811	−0.06530	−0.07132	−0.05811	−0.07495	−0.05420
			MSE	0.00395	0.00706	0.00706	0.00822	0.00911	0.00706	0.00989	0.00664
50	40	λ	Mean	3.00595	3.04594	3.04594	2.97749	2.90249	3.04594	2.99763	3.07076
			Bias	0.00595	0.04594	0.04594	−0.02251	−0.09751	0.04594	−0.00237	0.07076
			MSE	0.05281	0.09092	0.09092	0.08649	0.08430	0.09092	0.09053	0.09503
		$S(x_0)$	Mean	0.72688	0.72875	0.72875	0.72480	0.72355	0.72875	0.72022	0.73277
			Bias	−0.00217	−0.00029	−0.00029	−0.00424	−0.00550	−0.00029	−0.00882	0.00373
			MSE	0.00211	0.00325	0.00325	0.00347	0.00334	0.00325	0.00370	0.00309
		$h(x_0)$	Mean	0.82174	0.81501	0.81501	0.80523	0.78035	0.81501	0.78425	0.82975
			Bias	0.00389	−0.00284	−0.00284	−0.01262	−0.03751	−0.00284	−0.03361	0.01189
			MSE	0.01209	0.01898	0.01898	0.01940	0.01872	0.01898	0.02117	0.01862
	45	λ	Mean	3.00590	2.90753	2.90753	2.84820	2.78888	2.90753	2.86444	2.94098
			Bias	0.00590	−0.09247	−0.09247	−0.15180	−0.21112	−0.09247	−0.13556	−0.05902
			MSE	0.05150	0.06803	0.06803	0.07886	0.09791	0.06803	0.07695	0.06776
		$S(x_0)$	Mean	0.72696	0.70264	0.70264	0.69916	0.69909	0.70264	0.69415	0.70881
			Bias	−0.00208	−0.02641	−0.02641	−0.02989	−0.02995	−0.02641	−0.03490	−0.02023
			MSE	0.00202	0.00329	0.00329	0.00353	0.00370	0.00329	0.00398	0.00304
		$h(x_0)$	Mean	0.82158	0.87826	0.87826	0.86676	0.83816	0.87826	0.84777	0.88764
			Bias	0.00372	0.06040	0.06040	0.04891	0.02031	0.06040	0.02992	0.06979
			MSE	0.01161	0.01839	0.01839	0.01686	0.01500	0.01839	0.01602	0.02017
100	80	λ	Mean	2.99586	3.14547	3.14547	3.11569	3.06687	3.14547	3.12795	3.16113
			Bias	−0.00414	0.14547	0.14547	0.11569	0.06687	0.14547	0.12795	0.16113
			MSE	0.01098	0.09093	0.09093	0.08384	0.06509	0.09093	0.08881	0.09265
		$S(x_0)$	Mean	0.72752	0.75051	0.75051	0.74998	0.74866	0.75051	0.74797	0.75305
			Bias	−0.00152	0.02147	0.02147	0.02093	0.01962	0.02147	0.01892	0.02400
			MSE	0.00044	0.00261	0.00261	0.00273	0.00254	0.00261	0.00271	0.00264
		$h(x_0)$	Mean	0.82123	0.76348	0.76348	0.75526	0.74468	0.76348	0.74405	0.76983
			Bias	0.00337	−0.05438	−0.05438	−0.06560	−0.07318	−0.05438	−0.07380	−0.04803
			MSE	0.00250	0.01584	0.01584	0.01727	0.01738	0.01584	0.01905	0.01483
	90	λ	Mean	2.99828	2.98179	2.98179	2.95965	2.92042	2.98179	2.96936	3.00405
			Bias	−0.00172	−0.01821	−0.01821	−0.04035	−0.07958	−0.01821	−0.03064	0.00405
			MSE	0.01196	0.03735	0.03735	0.03685	0.03998	0.03735	0.03708	0.03716

Table 9. *Cont.*

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
100	90	$S(x_0)$	Mean	0.72794	0.72114	0.72114	0.72153	0.72051	0.72114	0.71928	0.72524
			Bias	−0.00111	−0.00791	−0.00791	−0.00752	−0.00853	−0.00791	−0.00977	−0.00380
			MSE	0.00048	0.00152	0.00152	0.00150	0.00154	0.00152	0.00158	0.00143
		$h(x_0)$	Mean	0.82020	0.83515	0.83515	0.82441	0.81232	0.83515	0.81433	0.83768
			Bias	0.00235	0.01730	0.01730	0.00656	−0.00554	0.01730	−0.00353	0.01982
			MSE	0.00273	0.00870	0.00870	0.00819	0.00805	0.00870	0.00835	0.00864

Table 10. The ML and Bayesian estimates of the unknown parameters λ, $S(x_0)$, and $h(x_0)$ via the importance sampling technique.

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
30	24	λ	Mean	3.02350	4.10736	4.10736	4.03328	3.81324	4.10736	4.10185	4.16576
			Bias	0.02350	1.10736	1.10736	1.03328	0.81324	1.10736	1.10185	1.16576
			MSE	0.10261	1.49058	1.49058	1.34756	0.86803	1.49058	1.53082	1.65682
		$S(x_0)$	Mean	0.72733	0.87067	0.87067	0.87277	0.86982	0.87067	0.87101	0.87427
			Bias	−0.00171	0.14163	0.14163	0.14373	0.14077	0.14163	0.14196	0.14523
			MSE	0.00387	0.02235	0.02235	0.02328	0.02232	0.02235	0.02288	0.02339
		$h(x_0)$	Mean	0.89150	0.44980	0.44980	0.43213	0.42517	0.44980	0.41085	0.46252
			Bias	0.00164	−0.36805	−0.36805	−0.38573	−0.39268	−0.36805	−0.40701	−0.35534
			MSE	0.02237	0.15292	0.15292	0.16807	0.17119	0.15292	0.18540	0.14449
	27	λ	Mean	3.03297	3.89131	3.89131	3.78047	3.60496	3.89131	3.83332	3.90260
			Bias	0.03297	0.89131	0.89131	0.78047	0.60496	0.89131	0.83332	0.90260
			MSE	0.10538	1.02343	1.02343	0.81888	0.50581	1.02343	0.92742	1.02966
		$S(x_0)$	Mean	0.72906	0.84946	0.84946	0.84693	0.84566	0.84946	0.84481	0.85056
			Bias	0.000016	0.12042	0.12042	0.11789	0.11662	0.12042	0.11577	0.12151
			MSE	0.00395	0.01709	0.01709	0.01681	0.01616	0.01709	0.01641	0.01718
		$h(x_0)$	Mean	0.81531	0.50806	0.50806	0.50268	0.49126	0.50806	0.48351	0.52755
			Bias	−0.00255	−0.30980	−0.30980	−0.31518	−0.32660	−0.30980	−0.33434	−0.29030
			MSE	0.02285	0.11475	0.11475	0.11952	0.12364	0.11475	0.13273	0.10155
50	40	λ	Mean	3.00595	3.35995	3.35995	3.35887	3.34464	3.35995	3.36414	3.38831
			Bias	0.00595	0.35995	0.35995	0.35887	0.34464	0.35995	0.36414	0.38831
			MSE	0.05281	0.25250	0.25250	0.25440	0.23530	0.25250	0.25959	0.27492
		$S(x_0)$	Mean	0.72688	0.78564	0.78564	0.78693	0.78888	0.78564	0.78639	0.79005
			Bias	−0.00217	0.05660	0.05660	0.05789	0.05983	0.05660	0.05734	0.06100
			MSE	0.00211	0.00618	0.00618	0.00632	0.00637	0.00618	0.00628	0.00649

Table 10. *Cont.*

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
50	40	$h(x_0)$	Mean	0.82174	0.06763	0.06763	0.67058	0.66243	0.06763	0.66794	0.66911
			Bias	0.00389	−0.14155	−0.14155	−0.14728	−0.15543	−0.14155	−0.14991	−0.14874
			MSE	0.01209	0.03864	0.03864	0.04022	0.04160	0.03864	0.04106	0.03969
	45	λ	Mean	3.00590	3.26007	3.26007	3.28537	3.25634	3.26007	3.29045	3.29795
			Bias	0.00590	0.26007	0.26007	0.28537	0.25634	0.26007	0.29045	0.29795
			MSE	0.05150	0.16148	0.16148	0.18213	0.16170	0.16148	0.18621	0.19122
		$S(x_0)$	Mean	0.72696	0.77077	0.77077	0.77626	0.77529	0.77077	0.77567	0.77656
			Bias	−0.00208	0.04173	0.04173	0.04722	0.04624	0.04173	0.04662	0.04751
			MSE	0.00202	0.00427	0.00427	0.00487	0.00484	0.00427	0.00484	0.00493
		$h(x_0)$	Mean	0.82158	0.71391	0.71391	0.69751	0.69632	0.71391	0.69481	0.70328
			Bias	0.00372	−0.10395	−0.10395	−0.12035	−0.12153	−0.10395	−0.12305	−0.11458
			MSE	0.01161	0.02630	0.02630	0.03071	0.03116	0.02630	0.03143	0.02963
100	80	λ	Mean	2.99586	2.88566	2.88566	2.88583	2.88962	2.88566	2.88604	2.89182
			Bias	−0.00414	−0.11434	−0.11434	−0.11417	−0.11038	−0.11434	−0.11396	−0.10818
			MSE	0.01098	0.05951	0.05951	0.05756	0.05676	0.05951	0.05753	0.05624
		$S(x_0)$	Mean	0.72752	0.70238	0.70238	0.70264	0.70369	0.70238	0.70258	0.70382
			Bias	−0.00152	−0.02666	−0.02666	−0.02640	−0.02535	−0.02666	−0.02646	−0.02522
			MSE	0.00044	0.00277	0.00277	0.00272	0.00260	0.00277	0.00273	0.00259
		$h(x_0)$	Mean	0.82123	0.88026	0.88026	0.87943	0.87657	0.88026	0.87924	0.87720
			Bias	0.00337	0.06240	0.06240	0.06158	0.05871	0.06240	0.06138	0.05934
			MSE	0.00250	0.01551	0.01551	0.01519	0.01449	0.01551	0.01516	0.01460
	90	λ	Mean	2.99828	2.80776	2.80776	2.81487	2.80215	2.80776	2.81506	2.80426
			Bias	−0.00172	−0.19224	−0.19224	−0.18513	−0.19785	−0.19224	−0.18494	−0.19574
			MSE	0.01196	0.07459	0.07459	0.07104	0.07827	0.07459	0.07098	0.07743
		$S(x_0)$	Mean	0.72794	0.68584	0.68584	0.68761	0.68481	0.68584	0.68754	0.68495
			Bias	−0.00111	−0.04320	−0.04320	−0.04143	−0.04424	−0.04320	−0.04151	−0.04409
			MSE	0.00048	0.00373	0.00373	0.00352	0.00392	0.00373	0.00353	0.00390
		$h(x_0)$	Mean	0.82020	0.91954	0.91954	0.91512	0.92128	0.91954	0.91493	0.92194
			Bias	0.00235	0.10168	0.10168	0.09726	0.10342	0.10168	0.09707	0.10408
			MSE	0.00273	0.02070	0.02070	0.01947	0.02156	0.02070	0.01943	0.02172

Table 11. The ML and Bayesian estimates of the unknown parameters β, λ, $S(x_0)$, and $h(x_0)$ via the importance sampling technique.

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
30	24	β	Mean	4.16607	5.28285	5.28285	5.06967	4.52576	5.28285	5.20078	5.04745
			Bias	0.16607	1.28285	1.28285	1.06967	0.52576	1.28285	1.20078	1.04745
			MSE	1.67464	1.72315	1.72315	1.19408	0.29629	1.72315	1.50711	1.15652
		λ	Mean	3.03499	4.26197	4.26197	4.17048	3.95053	4.26197	4.21649	4.20813
			Bias	0.03499	1.26197	1.26197	1.17048	0.95053	1.26197	1.21649	1.20813
			MSE	0.25231	1.75037	1.75037	1.52003	1.01733	1.75037	1.64189	1.61824
		$S(x_0)$	Mean	0.73095	0.86271	0.86271	0.86064	0.85785	0.86271	0.85912	0.86186
			Bias	0.00191	0.13367	0.13367	0.13160	0.12881	0.13367	0.13008	0.13281
			MSE	0.00381	0.01949	0.01949	0.01909	0.01831	0.01949	0.01876	0.01924
		$h(x_0)$	Mean	0.80548	0.50812	0.50812	0.50185	0.48638	0.50812	0.47848	0.52651
			Bias	−0.01238	−0.30973	−0.30973	−0.31601	−0.33147	−0.30973	−0.33937	−0.29135
			MSE	0.02586	0.11072	0.11072	0.11496	0.12278	0.11072	0.13028	0.09937
	27	β	Mean	3.93086	5.20625	5.20625	5.02841	4.50613	5.20625	5.13947	4.96516
			Bias	−0.06914	1.20625	1.20625	1.02841	0.50613	1.20625	1.13947	0.96516
			MSE	0.99679	1.54481	1.54481	1.12231	0.28407	1.54481	1.37923	1.00406
		λ	Mean	2.96485	4.21493	4.21493	4.12865	3.96655	4.21493	4.16967	4.20868
			Bias	−0.03516	1.21493	1.21493	1.12865	0.96656	1.21493	1.16967	1.20868
			MSE	0.22540	1.65026	1.65026	1.41292	1.04495	1.65026	1.51704	1.61560
		$S(x_0)$	Mean	0.72508	0.85882	0.85882	0.85756	0.86006	0.85882	0.85608	0.86378
			Bias	−0.00396	0.12978	0.12978	0.12852	0.13101	0.12978	0.12704	0.13473
			MSE	0.00415	0.01867	0.01867	0.01820	0.01877	0.01867	0.01787	0.01966
		$h(x_0)$	Mean	0.80782	0.51779	0.51779	0.51001	0.48025	0.51779	0.48721	0.51821
			Bias	−0.01003	−0.30007	−0.30007	−0.30785	−0.33761	−0.30007	−0.29964	−0.28406
			MSE	0.02341	0.10651	0.10651	0.10894	0.12634	0.10651	0.12354	0.10339
50	40	β	Mean	4.05159	4.66134	4.66134	4.58093	4.3328	4.66134	4.63320	4.69727
			Bias	0.05159	0.66134	0.66134	0.58093	0.43329	0.66134	0.63320	0.69727
			MSE	0.93907	0.54113	0.54113	0.43373	0.26693	0.54113	0.50220	0.59661
		λ	Mean	2.9930	3.44872	3.44872	3.43808	3.42955	3.44872	3.44368	3.47011
			Bias	−0.00699	0.44872	0.44872	0.43808	0.42955	0.44872	0.44368	0.47011
			MSE	0.15451	0.28342	0.28342	0.27496	0.26221	0.28342	0.28092	0.30397
		$S(x_0)$	Mean	0.72659	0.77715	0.77715	0.77642	0.77788	0.77715	0.77535	0.78020
			Bias	−0.00328	0.04811	0.04811	0.04738	0.04883	0.04811	0.04630	0.05115
			MSE	0.00259	0.00461	0.00461	0.00459	0.00466	0.00461	0.00453	0.00484
		$h(x_0)$	Mean	0.81581	0.73021	0.73021	0.72526	0.71216	0.73021	0.71607	0.73270
			Bias	−0.00205	−0.08764	−0.08764	−0.09259	−0.10570	−0.08764	−0.10178	−0.08516
			MSE	0.01664	0.02475	0.02475	0.02555	0.02724	0.02475	0.02738	0.02423

Table 11. *Cont.*

n	r	Parameters		ML Estimates	Bayes Estimates						
					SE	LINEX			GE		
						$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
50	45	β	Mean	4.09112	4.45987	4.45987	4.39046	4.29615	4.45987	4.42271	4.47300
			Bias	0.09112	0.45987	0.45987	0.39046	0.29615	0.45987	0.42271	0.47300
			MSE	0.85313	0.30271	0.30271	0.24252	0.16649	0.30271	0.27075	0.31575
		λ	Mean	3.01325	3.43271	3.43271	3.40664	3.37651	3.43271	3.41345	3.42173
			Bias	0.01532	0.43271	0.43271	0.40664	0.37651	0.43271	0.41345	0.42173
			MSE	0.15142	0.27312	0.27312	0.25098	0.22452	0.27312	0.25766	0.26411
		$S(x_0)$	Mean	0.72828	0.78136	0.78136	0.77934	0.77736	0.78136	0.77816	0.77985
			Bias	−0.00076	0.05231	0.05231	0.05030	0.04832	0.05231	0.04911	0.05081
			MSE	0.00263	0.00503	0.00503	0.00489	0.00463	0.00503	0.00481	0.00481
		$h(x_0)$	Mean	0.81454	0.71017	0.71017	0.70742	0.70328	0.71017	0.69774	0.72367
			Bias	−0.00332	−0.10768	−0.10768	−0.11044	−0.11458	−0.10768	−0.12011	−0.09419
			MSE	0.01549	0.02820	0.02820	0.02879	0.02832	0.02820	0.03108	0.02514
100	80	β	Mean	4.05071	4.59825	4.59825	4.56338	3.78855	4.59825	4.58227	3.86137
			Bias	0.05070	0.59825	0.59825	0.56338	−0.21145	0.59825	0.58227	−0.13863
			MSE	0.72617	0.47415	0.47415	0.42489	0.12062	0.47415	0.44698	0.09656
		λ	Mean	2.99159	3.15642	3.15642	3.13728	3.10475	3.15642	3.13801	3.11545
			Bias	−0.00841	0.15642	0.15642	0.13728	0.10475	0.15642	0.13801	0.11545
			MSE	0.09857	0.06558	0.06558	0.05925	0.05418	0.06558	0.05947	0.05666
		$S(x_0)$	Mean	0.72703	0.72996	0.72996	0.72630	0.75405	0.72996	0.72584	0.75520
			Bias	−0.00201	0.00092	0.00092	−0.00274	0.02501	0.00092	−0.00320	0.026153
			MSE	0.00134	0.00169	0.00169	0.00181	0.00225	0.00169	0.00182	0.00228
		$h(x_0)$	Mean	0.81773	0.85016	0.85016	0.85649	0.74084	0.85016	0.85334	0.75007
			Bias	−0.00012	0.03231	0.03231	0.03864	−0.07702	0.03231	0.03549	−0.06778
			MSE	0.00853	0.01369	0.01369	0.01488	0.01634	0.01369	0.01459	0.01537
	90	β	Mean	4.13316	4.12649	4.12649	4.11049	4.10047	4.12649	4.11853	4.14749
			Bias	0.13316	0.12649	0.12649	0.11049	0.10047	0.12649	0.11853	0.14749
			MSE	0.63163	0.09053	0.09053	0.08638	0.08943	0.09053	0.08811	0.10049
		λ	Mean	3.03671	3.14705	3.14705	3.13727	3.12428	3.14705	3.13863	3.13364
			Bias	0.03671	0.14705	0.14705	0.13727	0.12428	0.14705	0.13863	0.13364
			MSE	0.08624	0.06907	0.06907	0.06479	0.05953	0.06907	0.06532	0.06197
		$S(x_0)$	Mean	0.73130	0.74822	0.74822	0.74685	0.74436	0.74822	0.74640	0.74528
			Bias	0.00225	0.01918	0.01918	0.01781	0.01532	0.01918	0.01735	0.01624
			MSE	0.00123	0.00207	0.00207	0.00204	0.00188	0.00207	0.00203	0.00189
		$h(x_0)$	Mean	0.81449	0.77804	0.77804	0.77874	0.78228	0.77804	0.77556	0.78958
			Bias	−0.00398	−0.03982	−0.03982	−0.03911	−0.03558	−0.03982	−0.04230	−0.02828
			MSE	0.00769	0.01314	0.01314	0.01315	0.01228	0.01314	0.01342	0.01209

4. Application

The performance of the BIED based on type-II censored samples is illustrated through two real datasets. The BIED model was compared with other lifetime models, such as the inverse exponential distribution (IED) as a special case of the BIED, introduced by Lin et al. [14], the inverse Weibull distribution (IWD) introduced by Keller and Kamath [15], the Weibull inverted exponential distribution (WIED) defined by [16], the Weibull exponential distribution (WED) (see [17]), and the odd Fréchet inverse exponential distribution (OFIED) introduced by Alrajhi [18]. Moreover, the model selection criteria were considered, which included the Akaike information criterion (AIC), log-likelihood (ℓ), Bayesian information criterion (BIC), consistent Akaike information criterion ($CAIC$), and Hannan–Quinn information criterion ($HQIC$). The smallest values of the AIC, BIC, $CAIC$, and $HQIC$, and the highest ℓ value determine the best-fit model for the data. For more details about these criteria and their uses, see [19,20].

$$AIC = -2\ell(\hat{\Theta}) + 2p$$

$$BIC = -2\ell(\hat{\Theta}) + p\log n$$

$$CAIC = AIC + \frac{2p(p+1)}{n-p-1}$$

$$HQIC = -2\ell(\hat{\Theta}) + 2p\log(\log n)$$

where $\ell(\hat{\Theta})$ denotes the log-likelihood function, p is the number of parameters, and n is the sample size:

- Aluminum coupons' cut:

The following data consisting of 102 observations were used by Birnbaum and Saunders [21] and correspond to the fatigue life of 6061-T6 aluminum coupons in cycles ($\times 10^{-3}$) with the maximum pressure of 26,000 psi. These coupons were cut parallel to the direction of rolling and oscillated at 18 cycles per second.

233, 258, 268, 276, 290, 310, 312, 315, 318, 321, 321, 329, 335, 336, 338, 338, 342, 342, 342, 344, 349, 350, 350, 351, 351, 352, 352, 356, 358, 358, 360, 362, 363, 366, 367, 370, 370, 372, 372, 374, 375, 376, 379, 379, 380, 382, 389, 389, 395, 396, 400, 400, 400, 403, 404, 406, 408, 408, 410, 412, 414, 416, 416, 416, 420, 422, 423, 426, 428, 432, 432, 433, 433, 437, 438, 439, 439, 443, 445, 445, 452, 456, 456, 460, 464, 466, 468, 470, 470, 473, 474, 476, 476, 486, 488, 489, 490, 491, 503, 517, 540, 560.

Descriptive statistics of these data are presented in Table 12.

Table 12. Descriptive statistics of the fatigue life of 6061-T6 aluminum coupons' cut data.

Measure	Value	Measure	Value
n	102	**Minimum**	223
Maximum	560	**Mean**	397.88
Q1	352	**Q3**	439
Median	400	**Skewness**	−0.003
Kurtosis	2.85	**Variance**	3884.30
Standard deviation	62.32		

Based on the descriptive statistics in Table 12, we observed that the skewness value was close to zero; thus, the distribution of the fatigue life of the 6061-T6 aluminum coupons' cut dataset was approximately normal, while the variance was 3884.30, which indicate high variability in the dataset.

According to Figure 1, we can note that there were no outliers in the fatigue life of the 6061-T6 aluminum coupons' cut data. The ML estimates and Bayesian estimates via the standard Bayes technique of the BIED parameters for the fatigue life of the 6061-T6 aluminum coupons' cut data are presented in Table 13 at two censoring percentages of 80% and 90%.

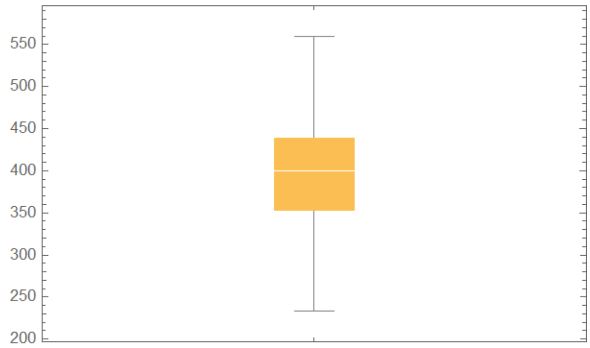


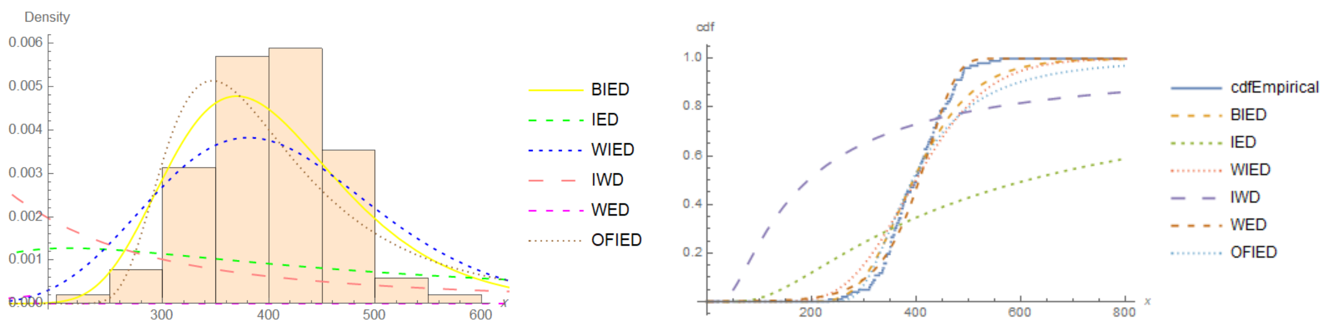
Figure 1. Boxplot for the fatigue life of the 6061-T6 aluminum coupons' cut data.

Table 13. ML and Bayesian estimates of the unknown parameters β and λ via the standard Bayes technique for the fatigue life of the 6061-T6 aluminum coupons' cut data.

Model	r	Parameters	ML Estimates	Bayes Estimates						
				SE	LINEX			GE		
					$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
BIED	82	λ	411.2780	3.8423	3.8423	3.1272	2.4938	3.8423	3.3474	4.0774
		β	1.5256	0.1721	0.1721	0.1717	0.1710	0.1721	0.1672	0.1745
	92	λ	432.7910	4.3285	4.3285	3.5184	2.8021	4.3285	3.8318	4.5658
		β	1.83122	0.1968	0.1968	0.1963	0.1955	0.1968	0.1917	0.1993

The ML estimates of the model parameters and the performance of the BIED against other models for the fatigue life of the 6061-T6 aluminum coupons' cut data are presented in Table 14 at two censoring percentages of 80% and 90%.

In Table 14, the values of the AIC, BIC, CAIC, and HQIC show that the BIED was the best model for analyzing the fatigue life of the 6061-T6 aluminum coupons' cut data. Furthermore, we can consider that the WED is a good alternative model for these data. The estimated PDF and estimated CDF of the models for the fatigue life of the 6061-T6 aluminum coupons' cut data at two censoring percentages of 80% and 90% are shown in Figures 2 and 3.

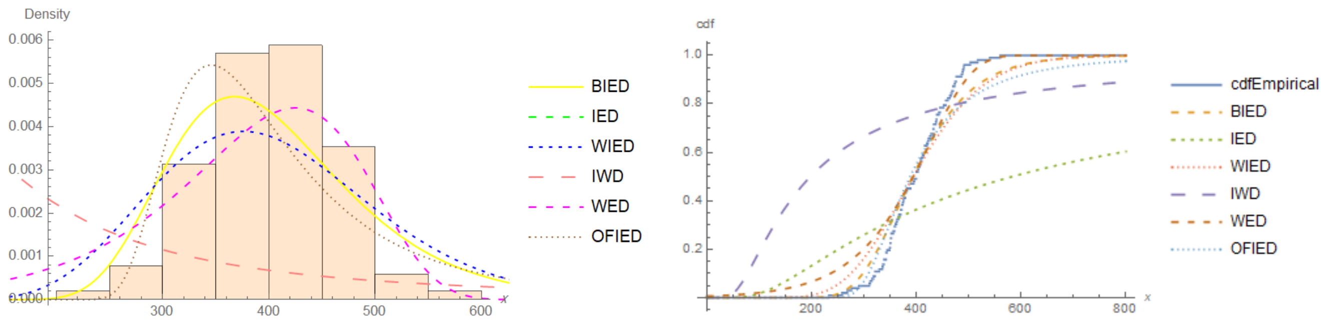


(**a**) Estimated PDFs. (**b**) Estimated CDFs.

Figure 2. Plots of the estimated PDF and estimated CDF of the models for the fatigue life of the 6061-T6 aluminum coupons' cut data for $r = 82$.

Table 14. ML estimates of the model parameters and the statistics of the AIC, BIC, CAIC, HQIC, and ℓ for the fatigue life of the 6061-T6 aluminum coupons' cut data.

Models	r	ML Estimates			ℓ	AIC	BIC	CAIC	HQIC
		α	β	λ					
BIED	82	44.7839	20.9497	150.0270	**−151.342**	**308.684**	**316.559**	**308.929**	**311.873**
	92	43.2797	20.0114	148.1150	**−172.933**	**351.867**	**359.742**	**352.112**	**355.056**
IED	82	—	—	424.3190	−248.584	499.168	501.793	499.208	500.231
	92	—	—	404.397	−288.006	578.013	580.638	578.053	579.076
WIED	82	52.3513	0.6796	2540.0300	−159.233	324.467	332.342	324.712	327.655
	92	53.5041	0.6438	2660.4100	−180.559	367.117	374.992	367.362	370.306
IWD	82	138.1880	1.0897	—	−293.464	590.927	596.177	591.049	593.053
	92	148.2970	1.2775	—	−321.958	647.916	653.166	648.037	650.042
WED	82	0.0004	0.0047	3.9449	−151.652	309.303	317.178	309.548	312.492
	92	0.0064	0.0106	1.1240	−179.223	364.447	372.321	364.691	367.635
OFIED	82	3.5426	—	252.0410	−157.345	318.689	323.939	318.810	320.815
	92	3.7253	—	250.2300	−180.356	364.713	369.963	364.834	366.839



(**a**) Estimated PDFs. (**b**) Estimated CDFs.

Figure 3. Plots of the estimated PDF and estimated CDF of the models for the fatigue life of the 6061-T6 aluminum coupons' cut data for $r = 92$.

- Patients suffering from acute myelogenous leukemia:

The following data consisting of 33 observations were studied by Feigl and Zelen [22] and represent the survival times (in weeks) of patients suffering from acute myelogenous leukemia.

1, 1, 2, 3, 3, 3, 4, 4, 4, 4, 5, 7, 8, 16, 16, 17, 22, 22, 26, 30,
39, 43, 56, 56, 65, 65, 65, 100, 108, 121, 134, 143, 156

Descriptive statistics of these data are presented in Table 15.

According to the descriptive statistics in Table 15, we observed that the distribution of the survival times (in weeks) of patients suffering from acute myelogenous leukemia data was positively skewed, while the variance was 2181.17, which indicates high variability in the dataset.

According to Figure 4, we can note that there were no outliers in the survival times (in weeks) of patients suffering from acute myelogenous leukemia data. Additionally, the ML estimates and Bayesian estimates via the standard Bayes technique of the BIED for this dataset are presented in Table 16 at two censoring percentages of 80% and 90%.

The ML estimates of the model parameters and the performance of the BIED against other models for the survival times (in weeks) of patients suffering from acute myelogenous leukemia data are shown in Table 14 at two censoring percentages of 80% and 90%.

Table 15. Descriptive statistics of the survival times (in weeks) of the patients suffering from acute myelogenous leukemia data.

Measure	Value	Measure	Value
n	33	**Minimum**	1
Maximum	156	**Mean**	40.88
Q1	4	**Q3**	65
Median	22	**Skewness**	1.16
Kurtosis	3.12	**Variance**	2181.17
Standard deviation	46.70		

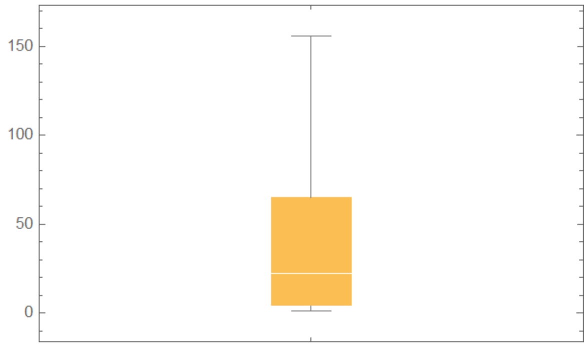


Figure 4. Boxplot for the survival times (in weeks) of patients suffering from acute myelogenous leukemia data.

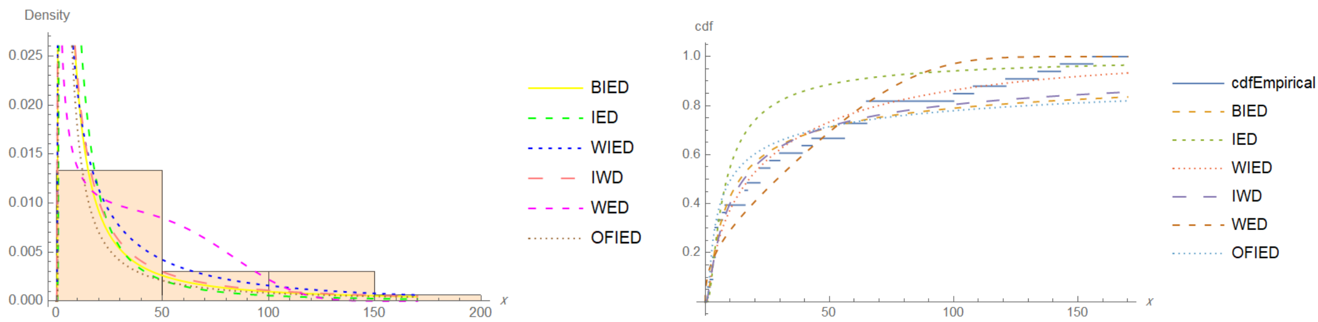
Table 16. ML and Bayesian estimates of the unknown parameters β and λ via the standard Bayes technique for the survival times (in weeks) of patients suffering from acute myelogenous leukemia data.

Model	r	Parameters	ML Estimates	SE	Bayes Estimates					
					LINEX			GE		
					$\tau = 0.001$	$\tau = 2$	$\tau = 5$	$q = -1$	$q = 3$	$q = -3$
BIED	26	λ	4.4313	1.7585	1.7585	1.5146	1.2642	1.7585	1.4025	1.9172
		β	0.4524	0.3622	0.3622	0.3563	0.3481	0.3622	0.3298	0.3786
	30	λ	4.8218	1.9092	1.9092	1.6483	1.3788	1.9092	1.5626	2.0654
		β	0.5142	0.4058	0.4058	0.3993	0.3902	0.4058	0.3737	0.4220

Table 17, based on the values of the AIC, BIC, CAIC, and HQIC, shows that the BIED was the best model for fitting the survival times (in weeks) of patients suffering from acute myelogenous leukemia data. Further, the estimated PDF and estimated CDF of the models for this dataset at two censoring percentages of 80% and 90% are shown in Figures 5 and 6.

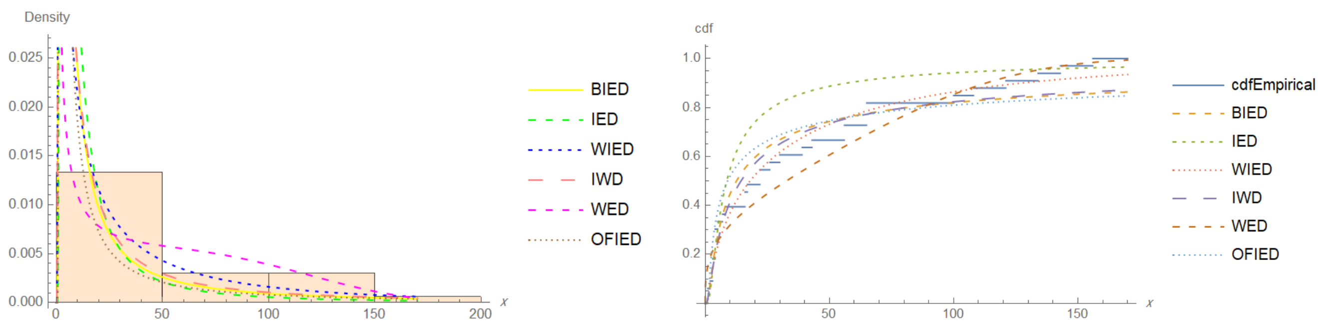
Table 17. ML estimates of the model parameters and the statistics of the AIC, BIC, CAIC, HQIC, and ℓ for the patients suffering from acute myelogenous leukemia data.

Models	r	ML Estimates			ℓ	AIC	BIC	CAIC	HQIC
		α	β	λ					
BIED	26	2.5699	0.4672	1.2091	**−40.404**	**84.808**	**87.801**	**85.208**	**85.815**
	30	5.5948	0.5282	0.5915	**−56.103**	**116.205**	**119.198**	**116.605**	**117.212**
IED	26	—	—	6.0189	−46.888	95.777	97.273	95.906	96.280
	30	—	—	6.0164	−61.848	125.695	127.191	125.824	126.198
WIED	26	0.3738	0.5673	5.1250	−41.701	89.402	93.892	90.230	90.913
	30	0.3535	0.5788	4.9131	−57.983	121.965	126.455	122.793	123.476
IWD	26	8.6392	0.6227	—	−40.672	87.344	91.834	88.172	88.855
	30	8.2490	0.6557	—	−56.731	119.462	123.952	120.29	120.973
WED	26	0.4269	0.4431	0.0478	−44.422	94.844	99.333	95.671	96.354
	30	0.4985	0.3479	0.0390	−61.213	128.426	132.916	129.254	129.937
OFIED	26	0.4202	—	3.6035	−42.007	88.014	91.007	88.414	89.021
	30	0.4600	—	3.3522	−58.573	121.146	124.139	121.546	122.153



(**a**) Estimated PDFs. (**b**) Estimated CDFs.

Figure 5. Plots of the estimated PDF and estimated CDF of the models for the patients suffering from acute myelogenous leukemia data for $r = 26$.



(**a**) Estimated PDFs. (**b**) Estimated CDFs.

Figure 6. Plots of the estimated PDF and estimated CDF of the models for the patients suffering from acute myelogenous leukemia data for $r = 30$.

5. Conclusions

In this article, the maximum likelihood and Bayes estimators of the BIED were derived based on type-II censored samples. The invariance property was used to estimate the survival and hazard functions. Furthermore, in the Bayesian estimation, three loss functions were used with two techniques. The gamma distribution was assumed as a prior distribution for the shape and scale parameters. Besides, it can be concluded that the MSEs of the ML estimates and Bayes estimates for the unknown parameters decreased as the sample size increased. Furthermore, when α was unknown, the Bayes estimates gave better estimates via the standard Bayes technique. The ML estimates gave better estimates than the Bayes estimates using the two techniques when λ was unknown. For the third case, when β was unknown, the ML estimates gave better results than the Bayes estimates as the sample size increased. Likewise, when β and λ were unknown, the Bayes estimates via the importance sampling technique under the LINEX loss function ($\tau = 5$) gave better results than the ML estimates as the sample size increased. Two real datasets were applied, and the BIED provided a better fit than the other compared distributions.

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