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Connections between Campos-Bolanos and Murofushi–Sugeno Representations of a Fuzzy Measure

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Abstract: Nonadditivity of a fuzzy measure, as an indicator of defectiveness, makes a fuzzy mea-sure less useful in applications compared to additive, probabilistic measures. In order to neutralize this indicator of defectiveness to some degree, it is important to study the representations of fuzzy measures, including, in particular, additive, probabilistic representations. In this paper, we discuss a couple of probability representations of a fuzzy measure: the Campos-Bolanos representation (CBR) and the Murofushi–Sugeno representation (MSR). The CBR is mainly represented by the Associated Probability Class (APC). The APC is well studied and the aspects of its use can be found in many interesting studies. This is especially true for the environment of interactive attributes in their identification and multi-attribute group decision-making (MAGDM) models, related to the attributes' Shapley values and interaction indexes. The MSR is a less-used tool in practice today. The main motivation of the research presented here was to explore the connections between these two representations, which will help increase the usability of the MSR in practice in the future. In the MSR, we constructed the nonequivalent representation class (NERC) of a fuzzy measure. This probabilistic new representation is somewhat similar to the APC in the CBR environment. The proposition on the existence of the MSR induced by the CBR was proven. The presented formula of the APC by the NERC was obtained. The duality property of fuzzy measures for the CBR is well studied with respect to fuzzy measures—Choquet second-order dual capacities. Significant properties were proven for the representation of a monotone expectation (ME) under the NERC conditions: as is known, the necessary and sufficient conditions for the existence of the second-order Choquet dual capacities are proven in the terms of the APC of a CBR and ME. After establishing the links between the APC of a CBR and the NERC of a MSR, we proved the same in the case of the MSR. A recursive connection formula between the interaction indexes, Shapley values, and the probability distribution of the NERC of a two-order additive fuzzy measure was obtained in the environment of a general MAGDM. A new distance concept was introduced for all fuzzy measures' classes defined in finite sets in terms of the NERC. The distance between two fuzzy measures was defined as the distance between their NERCs. This distance is equivalent to the distance defined on the same class under the conditions of the APC of a CBR. The correctness proposition on the extension of the distance between fuzzy measures for the NERC was preserved: distances between any two fuzzy measures and between their dual fuzzy measures also coincided in the CBR as the MSR. After parameterization, the calculation formula of the new distance was obtained. An illustrative example was considered in order to easily present the obtained results. The connection schemes between the CBR and MSR and the sequential scheme of key facts and results obtained are presented at the end of this work.

Keywords: fuzzy measure; Choquet capacities of order two; associated probabilities; probability representation of a fuzzy measure; distance; monotone expectation

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1. Introduction

Often in the expert knowledge-based models of complex processes or events, researchers use nonadditive but monotone estimators instead of additive ones [1–8]. This occurs because of the complex nature of the entity of study and the deficiency of statistical data, or its absence. One such monotone estimator is a fuzzy measure (the same as a monotone measure [6] or a capacity [5]). The requirement of additivity of classical measures implies the supposition that split sets are noninteractive regarding the measured property. This is very limiting in some application contexts. For instance, we analyze the work of workers whose purpose is to produce certain type of products. The workers are split into groups $G_1, G_2, \ldots, G_n$, and suppose that $\nu(G_i)$ is the number of products produced by group G_i, $i = 1, \ldots, n$. One may easily suppose that for any two groups G_i, G_j, any of the following may be true:

a. $\nu(G_i \cup G_j) = \nu(G_i) + \nu(G_j)$ when groups G_i and G_j work disjointedly;
b. $\nu(G_i \cup G_j) > \nu(G_i) + \nu(G_j)$ when the groups work together and their collaboration is effective;
c. $\nu(G_i \cup G_j) < \nu(G_i) + \nu(G_j)$ when the groups work together and their collaboration is ineffective.

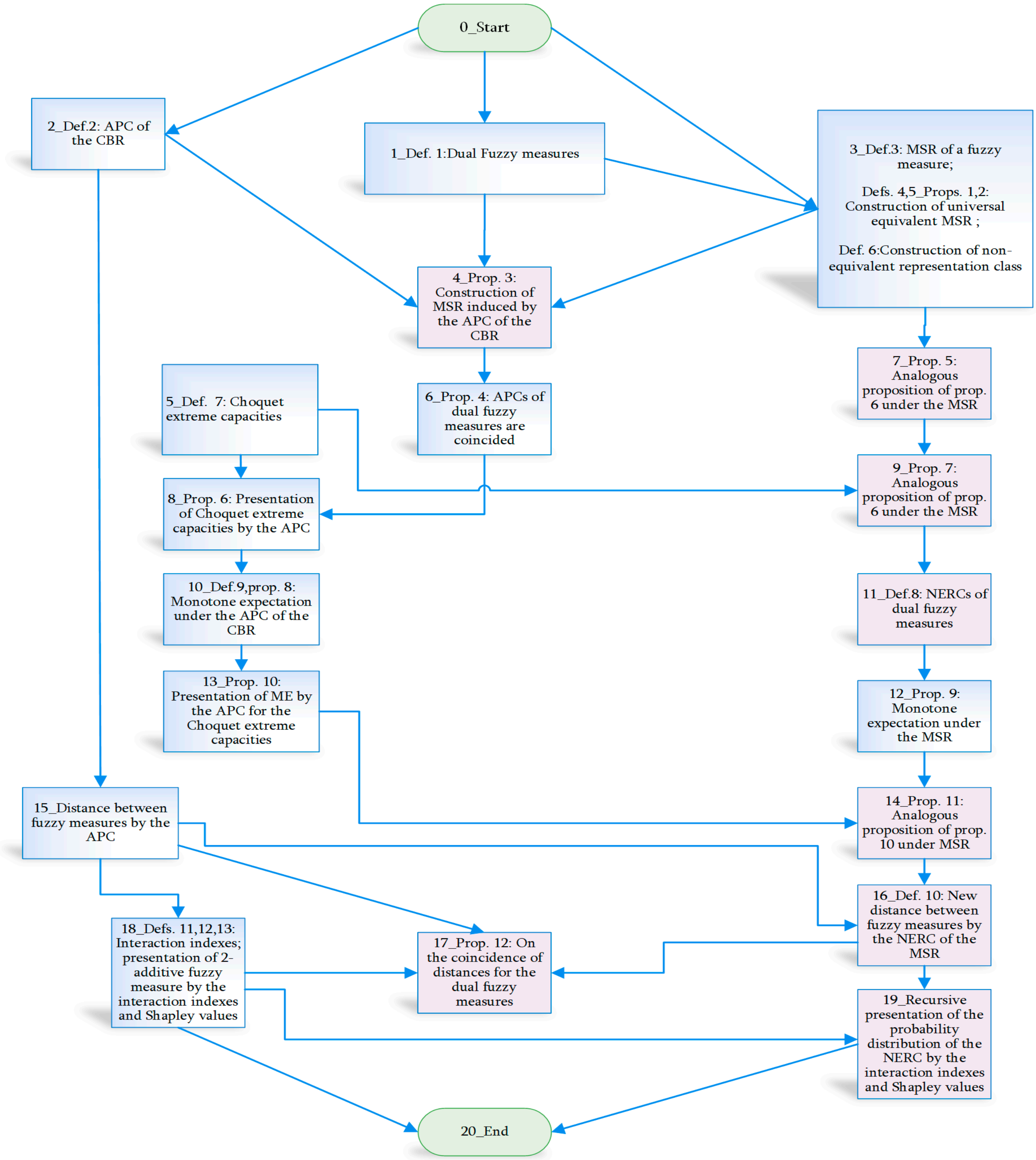
Clearly, (b) and (c) cases are due to the way the groups work together as they interact with each other. In these cases, the productivity rate ν. is nonadditive but naturally fulfills the monotonicity condition with respect to inclusion.

However, the use of fuzzy measure in practical models is associated with a number of difficulties. In [4], we read "The flexibility of non-additive (fuzzy) measures when modelling interaction comes at a significant cost: the exponential number of coalitions whose contributions need to be quantified. This gives rise to two problems: their interpretation and elicitation. If a fuzzy measure based model is to be understood by domain experts, the large number of capacity values need to be combined into some sort of characteristic indices, such as the overall importance of an input in all coalitions, or the overall interaction of a pair of inputs. On the other hand, if a fuzzy measure is to be pecified, either by the experts or by machine learning techniques, it has to be done through a few desirability criteria and in a computationally efficient way".

However, there are also other difficulties when using a fuzzy measure. A fuzzy measure is distinguished by many important properties from the additive measure, probability, but it should also be noted that nonadditivity limits the applicability in the practice of a fuzzy measure. This is why authors have continued to research the additive representations of fuzzy measures. These can be nonadditivity indexes [9–12] or the degrees of defectiveness of nonadditivity [2], importance values, interaction indexes [7,13–16], and other parameters. We especially highlight the probabilistic representations of a fuzzy measure [8,17–19] as the main object of our research. Therefore, the authors of this work have continued to study probabilistic representations of fuzzy measures [20–30], which have received new perspectives for their use.

Let us briefly overview the studies on fuzzy measure representations. One direction is probabilistic representations. In [17], a monotone expectation was defined as the Choquet finite integral. The theorems of its existence were proven with respect to the fuzzy measure's associated probabilities class. In [18], the authors introduced the notion of distance on fuzzy measures defined on finite sets as the distance between their associated probability classes. An application of this distance to measure the uncertainty and specificity of fuzzy measures was presented. In [19], the MSR of a fuzzy measure was defined. Basic propositions on the existence of specific fuzzy measures were proven in terms of the MSR. Some theorems of Murofushi and Sugeno [19] regarding the representation of fuzzy measures and also the Choquet integral were given in [8]. It was revealed that, if there exists a certain dependence between two measurable functions, then the Choquet integral turns out to be additive for those functions. Furthermore, this work discussed null sets regarding fuzzy measures and also fuzzy measures given on a class distinct from a σ-algebra. Two probability representations of a finite fuzzy measure—the CBR and MSR from the point of view of their applicability in practical MAGDM models—were analyzed in [24]. In [31], the Murofushi–Sugeno-type new representation-interpreter was constructed for the concrete class of finite fuzzy measures. The universal interpreter of a monotone (fuzzy) measure in the probability representation of Murofushi–Sugeno under the Choquet integral environment and second-order dual capacities were considered in [30]. The next direction is the study of the fuzzy measure nonadditivity indexes related to the use of a fuzzy measure in interactive MAGDM models, where some interactions between attributes are observed in the decision-making process. The nonadditivity index is a competent indicator that reflects the degree of interaction of the attributes. In [12], this index is used to indicate the range of advantages with respect to the alternatives of a decision-maker. In [9], the nonadditivity index for replacement of the Shapley concurrent interaction index and construction of an undated MAGDM-based decision scheme was introduced. In the work, a scheme was developed to calculate the nonadditivity index. A decision support algorithm was also built, which generates a dominance relationship to obtain the optimal ranking of alternatives. Some important properties of the nonadditivity index were discussed in [11]. A capacity identification algorithm based on the notion of the nonadditivity index was presented. The algorithm was formulated with linear constraints that represent the explicit or implicit advantages of the decision-maker over the alternatives. The model presents a linear programming problem to identify the optimal capacity. On the basis of the nonadditivity index [10], a capacity identification simulation algorithm was constructed, taking into account the given interactions of the attributes. Attempts at fewer computations of the new algorithm in the generation process compared to other similar algorithms ware established. We single out one more direction of the research—the additivity defectiveness of the capacity [2]. In this paper, the authors introduced the concept of capacity defectiveness, which is the degree of capacity nonadditivity. For certain capacity classes, the defectiveness coefficient was calculated or approximated. On the basis of the defectiveness index, an optimal approximation approach for fuzzy integrals was developed in the case of the replacement of the fuzzy measure with the classical measure. Particularly noteworthy are the problems of the identification of fuzzy measures (capacities) in the context of their representations, such as the interaction indexes of MAGDM attributes and the importance values (Shapley values). In [14], we find some representations of a fuzzy measure such as Mobius transformation, interaction indexes of attributes, and importance values (Shapley values). K-order additive fuzzy measures' representations were considered. In [15], it was shown that each discrete fuzzy measure represents a certain k-order additive fuzzy measure. The paper presented alternative ways to represent capacity by interaction indexes of attributes and Shapley values. Based on Heuristic Least Mean Squares, the learning algorithm for identification of k-maxitive measures was designed in [32]. In [33], the authors considered the structure and qualitative properties of a specific type of fuzzy measure that can be applied to the model of interactive MAGDM. Having constructed the general form of a nonadditive set function, the authors utilized the interaction coefficient, Möbius representation, and dual measure regarding the proposed measure. In [34], the generalized interaction index, gindex, was introduced. Its calculation needs vast resources in both time and memory. In order to reduce the complexity issues, the authors presented algorithms for the calculation of the gindex for k-maxitive measures. In [35], a new visualization scheme for a better understanding of the inner workings of a fuzzy measure was given. In [1], a joint Choquet integral-fuzzy measures (CI-FM) operator that utilizes the interactions between elements of information was studied. In [36], the authors, for identification of a fuzzy measure (capacity), used a hesitant fuzzy linguistic term set in describing the interactivity between attributes. In [16], the approaches proposed in the literature presented today were discussed, as well as their capabilities for capacity identification issues. Their advantages and disadvantages were discussed. Finally, we note the studies that connect capacity identification problems with the representations of the probabilities associated with them and with attribute interaction indices in the context of decision-making models with the new aggregation operators built there. In [24], in the environment of intuitionistic MAGDM, the new aggregation weighted operators were constructed that consider pairwise interactions of attributes. The new aggregates are based on an associated probabilistic class of a fuzzy measure. In [26], new aggregation operators reflecting the interactions between all attribute combinations in fuzzy MAGDM models were built. In [27], the authors created a Choquet integral-based new aggregation operators' family under q-rung orthopair fuzzy sets information. New operators consider all pairwise interactions between attributes. Based on the Shapley entropy maximum principle, a two-order additive fuzzy measure was created in the environment of a pair of interaction indexes and importance values of attributes of a MAGDM model. Approximately similar results were obtained in [28] under q-rung picture linguistic sets information. Associated intuitionistic fuzzy Choquet averaging (As-IFCA) and geometric (As-IFCG) aggregation operators were constructed in [23]. The As-IFPA (As-IFPG) operator coincided with the Intuitionistic Fuzzy Probabilistic Averaging (Intuitionistic Fuzzy Probabilistic Geometric) operator when a probability measure was used instead of a fuzzy measure in aggregations. Pair interaction indexes were used for the identification of a two-order additive fuzzy measure. Some variants of the newly constructed operators were utilized in the decision-making problem on certain fiscal policies. In [25], the Associated Immediate Probability Intuitionistic Fuzzy Order Weighted Averaging and Geometric operators were constructed. Associated probability distributions in the role of uncertainty measure were used. The new aggregation operators were considered in the intuitionistic fuzzy environment and their certain properties were given. In [29], a new method of possibilistic discrimination analysis was developed for the creation of positive and negative discrimination measures used for each alternative applicant's specific attribute. The gained discrimination pair reproduces the interaction of attributes in an intuitionistic fuzzy environment. For the intuitionistic fuzzy assessments, the new intuitionistic aggregation operators, the AsP-IFOWA and AsP-IFOWG, were introduced and considered. These operators represent the extensions of the Choquet finite integral and Yager's OWA operators.

The main object of our study was two probabilistic representations of a fuzzy measure defined on finite sets—the MSR [8,19] and CBR [17,18]. The motivation for the research consisted of the following. It should be noted that the use of the CBR for practical purposes was presented to a higher degree than the MSR. This is related to the more "flexible" structure of the CBR and the ability to easily embed its APC in problems such as fuzzy measure identification [13–16] and MAGDM modeling of the Shapley values and interaction indices of interacting attributes of the environment [9,22,26–28]. The main motivation for this study was to establish links between these two probabilistic representations and to obtain evidence for representations of the Choquet integral in the MSR environment as the Choquet integral is the main aggregation tool in MAGDM modeling, which considers the interactions between attributes. Clearly, the results obtained would help in the future to increase the usability of the MSR apparatus in practice, and particularly in MAGDM modeling. Basic definitions and main propositions on the CBR and MSR are presented in Section 2. The connection between the CBR and MSR is studied and schematically and compositionally presented in Section 3. In Section 4, one important class of fuzzy measures—Choquet second-order capacities—is studied in the MSR environment. Propositions on the existence of Choquet second-order capacities in the MSR are proven analogously to those proven in the CBR. Important properties of the monotone expectation (ME) [4] in the MSR environment are also proven in Section 4. In Section 5, a new definition of distance on the class of fuzzy measures in the MSR environment is introduced. It is proven that the new distance is equivalent to the corresponding definition in the CBR. By parameterization of the new distance, its calculation formula is obtained from a practical point of view. The correctness principle between two fuzzy measures at corresponding dual measures is preserved. In Section 6, the linear recursive formula between the attributes' interaction indexes and probability distribution of the NERC of a two-order additive fuzzy measure is constructed. In Section 7, the example of a two-element set is considered, where some of the main results obtained are shown in a simple way. The last section presents the main results of the research and the prospects of future research in the direction of the problems discussed in the article. The sequential scheme of key facts and obtained results is presented by the Scheme 1. New results are highlighted in pink in the scheme.



Scheme 1. Sequential scheme of key facts and obtained results.

2. Preliminary Concepts

Take a finite set $Y = \{y_1, y_2, \ldots, y_n\}$ and let $(Y, \mathscr{B}(Y), \nu)$ be a fuzzy measure space [4,17,37].

Definition 1. Ref. [17]: $\nu,\ \nu^* : \mathscr{B}(Y) \to [0;1]$ *are dual fuzzy measures if* $\forall C \in \mathscr{B}(Y)$:

$$\nu^*(C) = 1 - \nu(\overline{C}),\ \overline{C} = Y \backslash C.$$

Let S_n denote the permutation group of all-natural numbers from 1 to n.

Definition 2. Ref. [17]: *$\forall\sigma \in S_n$, the probability distribution*

$$\{\psi_\sigma(y_{\sigma(1)}), \psi_\sigma(y_{\sigma(2)}), \ldots, \psi_\sigma(y_{\sigma(n)})\}$$

on Y is called an associated probability (AP) of a fuzzy measure ν, if

$$\begin{cases} \psi_\sigma(y_{\sigma(1)}) = \nu(\{y_{\sigma(1)}\}), \\ \psi_\sigma(y_{\sigma(2)}) = \nu(\{y_{\sigma(1)}, y_{\sigma(2)}\}) - \nu(\{y_{\sigma(1)}\}), \\ \ldots\ldots\ldots\ldots\ldots \\ \psi_\sigma(y_{\sigma(l)}) = \nu(\{y_{\sigma(1)}, \ldots, y_{\sigma(l)}\}) - \nu(\{y_{\sigma(1)}, \ldots, y_{\sigma(l-1)}\}), \\ \ldots\ldots\ldots\ldots\ldots \\ \psi_\sigma(y_{\sigma(n)}) = \nu(\{y_{\sigma(1)}, \ldots, y_{\sigma(n)}\}) - \nu(\{y_{\sigma(1)}, \ldots, y_{\sigma(n-1)}\}), \end{cases} \tag{1}$$

and a set of all APs $\{\psi_\sigma(y_{\sigma(1)}), \psi_\sigma(y_{\sigma(2)}), \ldots, \psi_\sigma(y_{\sigma(n)})\}_{\sigma\in S_n}$ is called an associated probabilities class (APC) of a fuzzy measure ν.

It is well known [17] that $\forall C \in \mathscr{B}(Y)\ \exists\sigma = \sigma_C \in S_n$ such that

$$\nu(C) = \psi_{\sigma_C}(C) \tag{2}$$

The APC uniquely defines a fuzzy measure ν on $\mathscr{B}(Y)$ [4], and we call it the CBR of a fuzzy measure ν.

Let $(\Theta,\mathscr{B}(\Theta),\pi)$ be some finite probability measure space, where Θ is a finite set of some definite "indexes" [19]. Let $\zeta : \mathscr{B}(Y) \to \mathscr{B}(\Theta)$ be a 0–1 order-preserving homomorphism such that $\nu = \pi \circ \zeta$, i.e., $\zeta(\emptyset)=\emptyset$, $\zeta(Y) = \Theta$; if $A, B \in\mathscr{B}(Y)$ and $A \subset B$, then $\zeta(A) \subset \zeta(B)$ and $\forall G \in \mathscr{B}(Y)$:

$$\nu(G) = \pi(\zeta(G))$$

Definition 3. Ref. [19]: *$(\Theta,\mathscr{B}(\Theta),\zeta,\pi)$ is called the Murofushi–Sugeno representation (MSR) of a fuzzy measure ν if $\forall G \in \mathscr{B}(Y)$:*

$$\nu(G) = \pi(\zeta(G)) = \sum_{\theta\in\zeta(G)} \pi_\theta \equiv \sum_{\theta\in\zeta(G)} \pi(\theta) \tag{3}$$

It is clear that the $(\Theta,\mathscr{B}(\Theta))$ space is not unique. We construct the MSR $(\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y)$ such that an arbitrary MSR $(\Theta, \mathscr{B}(\Theta), \zeta, \pi)$ is an equivalent representation of $(\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y$, where Θ_Y and ζ_Y do not depend on π and ζ (see Definition 5).

Definition 4. Ref. [19]: *Let Θ_Y denote the set of all semi-filters in $\mathscr{B}(Y)$, where the semi-filter Se in $\mathscr{B}(Y)$ is a subset from $\mathscr{B}(Y)$ with the properties: $Y \in Se$, $\emptyset \notin Se$; if $C \in Se$ and $C \subset B$, then $B \in Se$.*

Let ζ_Y be a mapping from $\mathscr{B}(Y)$ to $\mathscr{B}(\Theta_Y)$ by the equality

$$\zeta_Y(C) = \{Se \in \Theta_Y / C \in Se\},\ \forall C \in\mathscr{B}(Y). \tag{4}$$

Obviously, ζ_Y is a $0-1$ order-preserving homomorphism and $\forall C \in\mathscr{B}(Y)$.

$$\nu(C) = \pi_Y(\zeta_Y(C)). \tag{5}$$

Now, we construct the probability measure π_Y:

Definition 5. Ref. [19]: *Suppose that $\forall y_i \in Y$, $\nu(\{y_i\}) > 0$. MSRs $(\Theta_Y, \mathscr{B}(\Theta), \zeta, \pi)$ and $(\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y)$ are called equivalent if $\exists M : \mathscr{B}(Y) \to \mathscr{B}(\Theta_Y)$ such that*

$$M(\zeta(C)) = \zeta_Y(C), \forall C \in \mathscr{B}(Y),\ \pi(E) = \pi_Y(M(E)),\ \forall E \in \mathscr{B}(\Theta). \tag{6}$$

Note that constraint $\forall y_i \in Y, \nu(\{y_i\}) > 0$ is natural from an application and practical point of view.

Proposition 1. *Ref. [19]: For every MSR $(\Theta, \mathscr{B}(\Theta), \zeta, \pi)$, there exists its equivalent representation $(\Theta_Y), \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y)$ and*

$$\pi_Y(C) = \pi(\omega^{-1}(C)), \forall C \in \mathscr{B}(\Theta_Y), \tag{7}$$

where $\omega : \Theta \to \mathscr{B}(\Theta_Y)$ is such that $\forall \theta \in \Theta$:

$$\omega(\theta) = \{B \in \mathscr{B}(Y)/\theta \in \zeta(B)\}. \tag{8}$$

It is simple to prove that $\mathscr{B}(\Theta_Y)$ are semi-filters. Notice that in $(\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y)$, $(\Theta_Y, \mathscr{B}(\Theta_Y))$ and ζ_Y do not depend on the fuzzy measure ν.

Proposition 2. Ref. [19]: *For every fuzzy measure ν: $\mathscr{B}(Y) \to [0;1]$, there exists a probability measure $\pi_Y : \mathscr{B}(Y) \to [0;1]$ and MSR $(\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y)$, that $\forall C \in \mathscr{B}(Y)$:*

$$\nu(C) = \pi_Y(\zeta_Y(C)) \tag{9}$$

From Propositions 1-2, we easily show that for the construction of the representation $(\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y)$ of a fuzzy measure ν, it is sufficient to construct probability measure π_Y. Note that if $(\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi'_Y)$ and $(\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi'_Y)$ are two MSRs of ν, then $\forall C \in \mathscr{B}(Y)$:

$$\nu(C) = \pi'_Y(\zeta_Y(C)) = \pi''_Y(\zeta_Y(C)) \tag{10}$$

i.e., projections of the probability measures π'_Y and π''_Y on the sets $\zeta_Y(C) \in \mathscr{B}(\Theta_Y)$ coincide. Then, from $[0;1]^{\mathscr{B}(\Theta_Y)}$, we may select the class of probability measures of nonequivalent representations of the fuzzy measure ν

$$L_Y^\nu = \{\pi_Y \in [0;1]^{\mathscr{B}(\Theta_Y)} / \forall C \in \mathscr{B}(Y), \nu(C) = \pi_Y(\zeta_Y(C))\}. \tag{11}$$

Definition 6. $\{\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y\}_{\pi_Y \in L_Y^\nu}$ *representations are called the nonequivalent representations class (NERC) of the fuzzy measure ν.*

Notice that the NERC completely describes the fuzzy measure ν and analogously describes APC $\{\psi_\sigma\}_{\sigma \in S_n}$ in the CBR.

3. Connection between Campos-Bolanos and Murofushi–Sugeno Representations

It is clear that $\forall \sigma \in S_n, i = 1, 2, \ldots, n$,

$$\psi_\sigma(y_{\sigma(i)}) = \pi(\zeta(\{y_{\sigma(1)}, \ldots, y_{\sigma(i)}\})) - \pi(\zeta(\{y_{\sigma(1)}, \ldots, y_{\sigma(i-1)}\})), \tag{12}$$

where $\pi(\zeta(\{y_{\sigma(0)}\})) \equiv 0$, i.e., if MSR $(\Theta, \mathscr{B}(\Theta), \zeta, \pi)$ of the fuzzy measure ν is known, then from (12), we obtain APC $\{\psi_\sigma\}_{\sigma \in S_n}$ of the CBR. On the contrary, take APC $\{\psi_\sigma\}_{\sigma \in S_n}$ of the CBR of the fuzzy measure ν. Then, we construct MSR $(\Theta, \mathscr{B}(\Theta), \zeta, \pi)$ induced from $\{\psi_\sigma\}_{\sigma \in S_n}$.

Proposition 3. *Given APC $\{\psi_\sigma\}_{\sigma \in S_n}$ of the fuzzy measure ν, then there exists MSR $(\Theta, \mathscr{B}(\Theta), \zeta, \pi)$ that is induced by the CBR.*

Proof. Let us denote the set

$$\Theta = \{\psi_\sigma(C) > 0/\sigma \in S_n, C \in \mathscr{B}(Y)\} \tag{13}$$

and consider probability measure π on $\mathscr{B}(\Theta) : \forall \psi_\sigma(C) > 0$,

$$\pi(\{\psi_\sigma(C)\}) = \psi_\sigma(C) - \max_{\Theta_C^\sigma} \psi_\beta(D), \tag{14}$$

where

$$\Theta_C^\sigma = \{\psi_\beta(D)/\beta \in S_n, D \in \mathscr{B}(Y), \psi_\beta(D) < \psi_\sigma(C)\}.$$

It is not difficult to prove that π is a probability measure on $\mathscr{B}(Y)$. Then, $\forall C \in \mathscr{B}(Y)$:

$$\pi(\zeta(C)) = \sum_{\psi_\sigma(D) \in \eta(C)} \pi(\{\psi_\sigma(D)\}), \tag{15}$$

where $\zeta(C)$ is defined as

$$\zeta(C) = \{\psi_\sigma(D) > 0/\sigma \in S_n, D \in \mathscr{B}(Y), \psi_\sigma(D) \le \nu(C)\}. \tag{16}$$

With regard to (2) and (15), after elementary simplification, we receive

$$\pi(\zeta(C)) = \sum_{\substack{\psi_\sigma(D) \le \psi_\sigma(C) \\ \sigma \in S_n,}} \{\psi_\sigma(D) - \max_{\Theta_D^\sigma} \psi_\beta(G)\} = \psi_{\sigma_C}(C) \equiv \nu(C), \tag{17}$$

where $\Theta_D^\sigma = \{\psi_\beta(G)/\beta \in S_n, G \in \mathscr{B}(Y), \psi_\beta(G) < \psi_\sigma(D)$. $\square$

The constructed MSR is called an induced representation by APC $-\{\psi_\sigma\}_{\sigma \in S_n}$. Figure 1 presents the schematic connection between Campos-Bolanos and Murofushi–Sugeno representations. From Figure 2, it follows that $\forall G \in \mathscr{B}(Y)$:

$$\begin{aligned} &\xi \circ \rho(G) = \psi_{\sigma_C}(G) = \nu(G), \\ &\pi \circ \zeta(G) = \pi(\zeta(G)) = \nu(G), \\ &\pi \circ \mu \circ \rho(G) = \pi(\mu(\rho(G))) = \nu(G). \end{aligned} \tag{18}$$

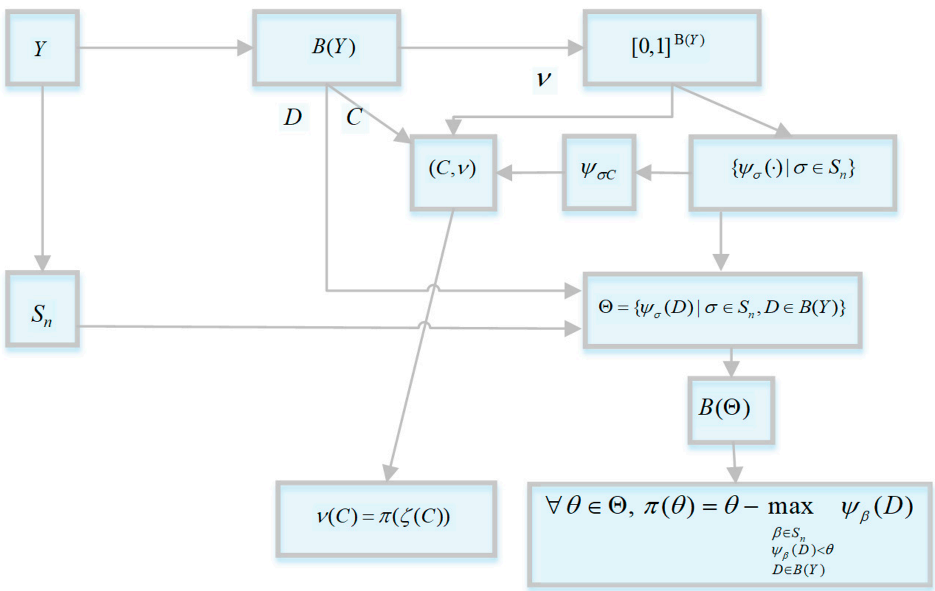


Figure 1. The schematic connection between Campos-Bolanos and Murofushi–Sugeno representations.

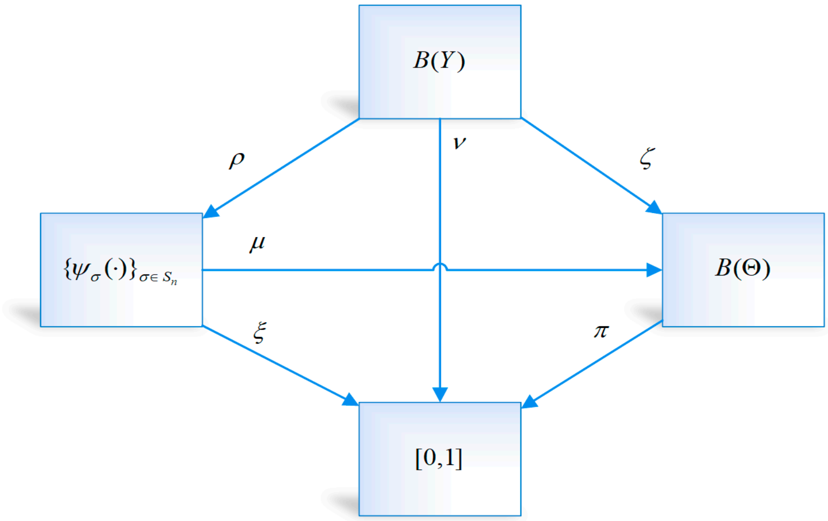


Figure 2. The compositional connection between Campos-Bolanos and Murofushi–Sugeno representations.

4. Choquet's Capacities of Order Two in Murofushi–Sugeno Representations

In this section, propositions on the existence of Choquet extreme capacities in the environment of the MSR are proven.

Proposition 4. Ref. [17]: *Take two fuzzy measures ν and ν^* on $\mathscr{B}(Y)$. ν and ν^* are dual fuzzy measures if and only if their APCs coincide: $\{\psi_\sigma(\cdot)\}_{\sigma\in S_n} = \{\psi^*_\sigma(\cdot)\}_{\sigma\in S_n}$, and for $\forall\sigma \in S_n$ and its dual permutation $\sigma^* \in S_n(\forall i = 1,2,\ldots,n : \sigma(i) = \sigma^*(n-i+1))$, the following equalities hold: $\psi_\sigma(y_{\sigma(i)}) = \psi_{\sigma^*}(y_{\sigma(i)}), i = 1,2,\ldots,n$.* Let

$$\{\Theta_X,\mathscr{B}(\Theta_Y),\zeta_Y,\pi_Y\}_{\pi_Y} \in L^\nu_Y,\ \{\Theta_X,\mathscr{B}(\Theta_Y),\zeta_Y,\pi^*_Y\}_{\pi^*_Y} \in L^{\nu^*}_Y \tag{19}$$

be NERCs to the fuzzy measures ν and ν^*, respectively.

Proposition 5. *Take two fuzzy measures ν and ν^* on $\mathscr{B}(Y)$. ν and ν^* are dual fuzzy measures if and only if, for any pair of probability measures $\forall\pi_Y \in L^\nu_Y$, $\forall\pi^*_Y \in L^{\nu^*}_Y$, and $\forall G \in\mathscr{B}(Y)$:*

$$\pi_Y(\zeta_Y(G)) = \pi^*_Y(\overline{\zeta_Y(\overline{G})}) \tag{20}$$

Necessity: Let ν and ν^* be dual fuzzy measures, i.e., $\forall G \in\mathscr{B}(Y)$: $\nu(G) = 1-\nu^*(\overline{G})$; let $(\Theta,\mathscr{B}(\Theta),\zeta,\pi)$ and $(\Theta^*,\mathscr{B}(\Theta^*),\zeta^*,\pi^*)$ be any MSR of ν and ν^*, respectively. Then, $\pi(\zeta(G)) = \pi^*(\overline{\zeta(\overline{G})})$. From Proposition 1, there exist probability measures $\pi^0_Y \in L^\nu_Y, \pi^{*0}_Y \in L^{\nu^*}_Y$ such that $\forall G \in\mathscr{B}(Y)$:

$$\pi^0_Y(\zeta_Y(G)) = \pi^0_Y(\omega^{-1}(\zeta(G))) = \pi^0_Y(\zeta(G)) = \pi^{*0}(\overline{\zeta^*(\overline{G})}) = \\ \pi^{*0}_Y((\omega^*)^{-1}(\overline{\zeta^*(\overline{G})})) == \pi^{*0}_Y(\overline{\zeta_Y(\overline{G})}).$$

As π^0,π^{*0} are any probability measures of the MSR, then $\forall\pi_Y \in L^\nu_Y,\forall\pi^*_Y \in L^{\nu^*}_Y$ measures $\exists\pi \sim \pi_X$ and $\exists\pi^* \sim \pi^*_Y$, the probability measures for which (20) is true.

Sufficiency: Let (20) be true for arbitrary probability measures $\pi_Y \in L^\nu_Y,\forall\pi^*_Y \in L^{\nu^*}_Y$. Take two MSRs: $(\Theta,\mathscr{B}(\Theta),\zeta,\pi)$ and $(\Theta^*,\mathscr{B}(\Theta^*),\zeta^*,\pi^*)$ corresponding to ν and ν^*, respectively, and their equivalent representations $(\Theta_Y,\mathscr{B}(\Theta_Y),\zeta_Y,\pi_Y)$ $(\Theta_Y,\mathscr{B}(\Theta_Y),\zeta_Y,\pi^*_Y)$ such that $\forall G \in\mathscr{B}\,(Y)$:

$$\nu(G) = \pi(\zeta(G)) = \pi_Y(\omega^{-1}(\zeta(G))) = \pi_Y(\zeta_Y(G)) = \pi^*_Y(\overline{\zeta_Y(\overline{G})}) = \\ \pi^*((\tau^*)^{-1}(\overline{\zeta_Y(\overline{G})})) = \pi^*(\overline{\zeta^*(\overline{G})}) = 1-\pi^*(\zeta^*(\overline{G})) = 1-\nu^*(\overline{G}).$$

Notice that if ν is an auto-dual fuzzy measure $((\nu)^* = \nu)$, i.e., $\forall G \in \mathscr{B}(Y)\ \nu(G) = 1 - \nu(\overline{G})$, but ν is not a probability measure, then (20) will be changed by

$$\pi_Y(\zeta_Y(G)) = \pi_Y(\overline{\zeta_Y(\overline{G})}). \tag{21}$$

If ν is a probability measure, then $\forall G, B \in \mathscr{B}\ (Y), G \cap B = \varnothing$

$$\pi_Y(\zeta_Y(G) \cup \zeta_Y(B)) = \pi_Y(\zeta_Y(G \cup B)) \tag{22}$$

Definition 7. Refs. [5,17]: *Take dual fuzzy measures ν, ν^* on $\mathscr{B}\ (Y)$. ν and ν^* are called Choquet lower and upper capacities of order two, respectively, if $\forall G, B \in \mathscr{B}\ (Y)$:*

$$\begin{gathered} \nu(G \cup B) + \nu(G \cap B) \geq \nu(G) + \nu(B), \\ \nu^*(G \cup B) + \nu^*(G \cap B) \leq \nu^*(G) + \nu^*(B). \end{gathered} \tag{23}$$

Proposition 6. Ref. [17]: *Take dual fuzzy measures ν, ν^* on $\mathscr{B}\ (Y)$. ν and ν^* are Choquet lower and upper capacities of order two, respectively, if and only if $\forall C \in \mathscr{B}(Y)$*

$$\nu(C) = \min_{\sigma \in S_n} \psi_\sigma(C), \quad \nu^*(C) = \max_{\sigma \in S_n} \psi_\sigma(C),$$

where $\{\psi_\sigma(\cdot)\}_{\sigma \in S_n}$ is the APC of ν and ν^* (note that $\{\psi_\sigma(\cdot)\}_{\sigma \in S_n} = \{\psi^*_\sigma(\cdot)\}_{\sigma \in S_n}$).

We have an almost similar proposition for the MSR:

Proposition 7. *Take dual fuzzy measures ν, ν^* on $\mathscr{B}\ (Y)$. ν and ν^* are Choquet lower and upper capacities of order two, respectively, if and only if, for arbitrary probability measures $\pi_Y \in L^\nu_Y$, $\pi^*_Y \in L^{\nu^*}_Y$, and $\forall G, B \in \mathscr{B}\ (Y)$,*

$$\begin{gathered} \pi_Y(\zeta_Y(G \cup B)) \geq \pi_Y(\zeta_Y(G) \cup \zeta_Y(B)), \\ \pi^*_Y(\zeta_Y(G \cap B)) \leq \pi^*_Y(\zeta_Y(G) \cap \zeta_Y(B)). \end{gathered} \tag{24}$$

Necessity: Take dual fuzzy measures ν, ν^* and let ν and ν^* be Choquet lower and upper capacities of order two, respectively, and also take any MSR of ν and ν^*, $(\Theta_Y,, \mathscr{B}(\Theta_Y), \zeta, \pi_Y)$ and $(\Theta_Y, \mathscr{B}\ (\Theta_Y), \zeta, \pi^*_Y)$, respectively. From (23), we have

$$\pi_Y(\zeta_Y(G \cup B)) + \pi_Y(\zeta_Y(G \cap B)) \geq \pi_Y(\zeta_Y(G)) + \pi_Y(\zeta_Y(B)). \tag{25}$$

We know that

$$\begin{gathered} \zeta_Y(G \cap B) \subset \zeta_Y(G), \zeta_Y(G \cap B) \subset \zeta_Y(B), \zeta_Y(G \cap B) \subset \zeta_Y(G) \cap \zeta_Y(B), \\ \pi_Y(\zeta_Y(G \cap B)) \leq \pi_Y(\zeta_Y(G) \cap \zeta_Y(B)) \end{gathered}$$

and from (25), we obtain

$$\begin{gathered} \pi_Y(\zeta_Y(G) \cup \zeta_Y(B)) = \pi_Y(\zeta_Y(G)) + \pi_Y(\zeta_Y(B)) - \pi_Y(\zeta_Y(G) \cap \zeta_Y(B)) \leq \\ \leq \pi_Y(\zeta_Y(G \cup B)) + \pi_Y(\zeta_Y(G \cap B)) - \pi_Y(\zeta_Y(G) \cap \zeta_Y(B)) \leq \\ \leq \pi_Y(\zeta_Y(G \cup B)). \end{gathered}$$

Analogously, we have proven the second inequality of (24).

Sufficiency: Suppose, for the arbitrary pair of probability measures $\pi_Y \in L^\nu_Y, \pi^*_Y \in L^{\nu^*}_Y$, the inequality (24) is valid; let $(\theta^*, \mathscr{B}(\theta^*), \zeta^*, \pi^*)$ be any MSR of the fuzzy measure ν^*. If $\pi^{*0}_Y \in L^{\nu*}_Y$ is equivalent to π^*, then we have $\forall G, B \in \mathscr{B}(Y)$:

$$\begin{gathered} \nu^*(G \cup B) = \pi^*(\zeta^*(G \cup B)) = \pi^*((\omega^*)^{-1}(\zeta^{*0}_Y(G \cup B))) = \pi^{*0}_Y(\zeta^{*0}_Y(G \cup B)) \leq \\ \leq \pi^{*0}_Y(\zeta^{*0}_Y(G) \cup \zeta^{*0}_Y(B)) = \pi^{*0}_Y(\zeta^{*0}_Y(G)) + \pi^{*0}_Y(\zeta^{*0}_Y(B)) - \pi^{*0}_Y(\zeta^{*0}_Y(G) \cap \zeta^{*0}_Y(B)) \leq \\ \leq \pi^{*0}_Y(\zeta^{*0}_Y(G)) + \pi^{*0}_Y(\zeta^{*0}_Y(B)) - \pi^{*0}_Y(\zeta^{*0}_Y(G \cap B)) = \pi^{*0}_Y(\omega^*(\zeta^*(G))) + \\ + \pi^{*0}_Y(\omega^*(\zeta^*(G \cap B))) = \pi^*(\zeta^*(G)) + \pi^*(\zeta^*(B)) - \pi^*(\zeta^*(G \cap B)) = \\ = \nu^*(G) + \nu^*(G) - \nu^*(G \cap B), \end{gathered}$$

i.e., ν^* is the Choquet upper capacity of order two. Analogously, we have proven the first inequality of (23).

Definition 8. *NERCs L_Y^ν and $L_Y^{\nu*}$ of the pair of dual fuzzy measures ν and ν^* are called nonequivalent probability representations' dual classes.*

It can be trivially proven that a fuzzy measure $\nu : \mathscr{B}(Y) \to [0;1]$ is a probability measure if and only if $L_Y^\nu \cap L_Y^{\nu*} \neq \varnothing$.

Definition 9. Ref. [17]: *Take some fuzzy subset W on Y and let χ_W be its membership function $\chi_W : Y \to [0;1]$. A monotone expectation (ME) of χ_W with respect to the fuzzy measure ν is defined as the Choquet integral:*

$$ME_\nu(\chi_W) = \int_0^1 \nu(\chi_F(y) \geq \alpha)d\alpha \equiv (Ch)\int_Y \chi_W d\nu, \tag{26}$$

where $d\alpha$ is the Lebesgue measure on [0;1].

If $\sigma \in S_n$ is such a permutation that

$$\chi_W(y_{\sigma(1)}) \leq \chi_W(y_{\sigma(2)}) \leq \cdots \leq \chi_W(y_{\sigma(n)})$$

and $A_i \equiv \{y_{\sigma(i)}, y_{\sigma(i+1)}, \ldots, y_{\sigma(n)}\}, i = 1, 2, \ldots, n$ are nested subsets of Y, then ME may be presented as

$$ME_\nu(\chi_W) = \sum_{i=1}^{n} \chi_W(y_{\sigma(i)})\{\nu(A_i) - \nu(A_{i+1})\}$$

where $\nu(A_{n+1}) \equiv 0$. The Choquet integral has been studied well in the CBR [17,18] under different fuzzy environments [20,22,24,26–28]:

Proposition 8. Ref. [17]: *Take a fuzzy measure ν and its APC $\{\psi_\sigma\}_{\sigma \in S_n}$. Then, there exists the permutation $\sigma_0 \in S_n$ for which*

$$ME_\nu(\chi_W) = E_{\psi_{\sigma_0}}(\chi_W) = \int_Y \chi_W d\psi_{\sigma_0} \tag{27}$$

More exactly, a ME is represented as a mathematical expectation of χ_W with respect to the probability measure ψ_{σ_0}. The analogous proposition for the MSR can be proven:

Proposition 9. Ref. [19]: *Take the fuzzy measure ν and its MSR $(\Theta, \mathscr{B}(\Theta), \zeta, \pi)$. Then, there exists the nonnegative function $\hat{\chi}_W$ on Θ that*

$$ME_\nu(\chi_W) = (Ch)\int_Y \chi_W d\nu = \int_\Theta \hat{\chi}_W d\pi = E_\pi(\hat{\chi}_W) \tag{28}$$

where $\forall\theta \in \Theta$:

$$\hat{\chi}_W(\theta) = \sup\{\alpha/\theta \in \zeta(\{y/\chi_W(y) \geq \alpha, 0 \leq \alpha \leq 1\})\} \tag{29}$$

More exactly, the Choquet integral is represented as a Lebesgue integral on Θ with respect to the probability measure π.

Proposition 10. *Ref. [17]: Take dual fuzzy measures ν, ν^*. ν and ν^* are Choquet lower and upper capacities of order two if and only if*

$$ME_\nu(\chi_W) = \min_{\sigma \in S_n} E_{\psi_\sigma}(\chi_W), ME_{\nu*}(\chi_W) = \max_{\sigma \in S_n} E_{\psi_\sigma}(\chi_W) \tag{30}$$

Now, we prove the analogous proposition for the MSR:

Proposition 11. *Take two fuzzy subsets W_1 and W_2 on Y with compatibility functions χ_{W_1}, χ_{W_2} and dual fuzzy measures ν, ν^*. ν and ν^* are Choquet lower and upper capacities of order two, respectively, if and only if*

$$\begin{gathered} ME_\nu(\chi_{W_1} + \chi_{W_2}) \geq ME_\nu(\chi_{W_1}) + ME_\nu(\chi_{W_2}), \\ ME_{\nu*}(\chi_{W_1} + \chi_{W_2}) \leq ME_{\nu*}(\chi_{W_1}) + ME_{\nu*}(\chi_{W_2}). \end{gathered} \tag{31}$$

Necessity: Let ν, ν^* be Choquet dual capacities of order two. Let Proposition 7 and Equation (24) be true. Using the properties of the supremum's function and a mathematical expectation, we have

$$\begin{gathered} ME_\nu(\chi_{W_1} + \chi_{W_2}) = E_\pi(\sup\{\alpha/\theta \in \zeta(\{y/\chi_{W_1}(y) + \chi_{W_2}(y) \geq \alpha\}), 0 \leq \alpha \leq 1\}) \geq \\ E_\pi(\sup\{\alpha/\theta \in \zeta(\{y/\chi_{W_1}(y) \geq \tfrac{\alpha}{2}\} \cup \{y/\chi_{W_2}(y) \geq \tfrac{\alpha}{2}\})\}) \geq \\ \geq E_\pi(\sup\{\alpha/\theta \in \zeta(\{y|\chi_{W_1}(y) \geq \tfrac{\alpha}{2}\}) \cup \{y/\chi_{W_2}(y) \geq \tfrac{\alpha}{2}\})\}) \geq \\ \geq E_\pi(\sup\{\tfrac{\alpha}{2}/\theta \in \zeta(\{y/\chi_{W_1}(y) \geq \tfrac{\alpha}{2}\})\} + \sup\{\tfrac{\alpha}{2}/\theta \in \zeta(\{y/\chi_{W_2}(y) \geq \tfrac{\alpha}{2}\})\}) = \\ E_\pi(\sup\{\alpha\prime/\theta \in \zeta(\{y/\chi_{W_1}(y) \geq \alpha\prime\})\}) + E_\pi(\sup\{\alpha\prime/\theta \in \zeta(\{y/\chi_{W_2}(y) \geq \alpha'\})\}) = \\ = ME_\nu(\chi_{W_1}) + ME_\nu(\chi_{W_2}). \end{gathered}$$

We have analogously proven the second inequality of (31).

Sufficiency: Let the inequalities (31) be valid $\forall C, B \in \mathscr{B}(Y)$. If $\chi_{W_1} \equiv I_C, \chi_{W_2} \equiv I_B$, where I_C and I_B are indicator sets of C and B, respectively. We have

$$ME_\nu(I_C + I_B) \geq ME_\nu(I_C) + ME_\nu(I_B)$$

It is simple to show that

$$ME_\nu(I_{C\cup B} + I_{C\cap B}) = ME_\nu(I_{C\cup B}) + ME_\nu(I_{C\cap B})$$

Note that $\forall y \in Y : I_C(y) + I_B(y) = I_{C\cup B}(y) + I_{C\cap B}(y)$, and from the property of a monotone expectation $ME_\nu(I_C + I_B) = ME_\nu(I_{C\cup B} + I_{C\cap B})$, we have

$$\begin{gathered} \nu(C) + \nu(B) = ME_\nu(I_C) + ME_\nu(I_B) \leq ME_\nu(I_C + I_B) = ME_\nu(I_{C\cup B} + I_{C\cap B}) = \\ = ME_\nu(I_{C\cup B}) + ME_\nu(I_{C\cap B}) = \nu(C \cup B) + \nu(C \cap B). \end{gathered}$$

i.e., ν is the Choquet lower capacity of order two. Therefore, ν^* will be an upper capacity of order two, which follows from the second inequality of (31).

5. Distance on Fuzzy Measures' Space in MSR

Sometimes, in the practical examples, distances on the space of fuzzy measures are defined through distances between their APCs [18]:

$$D(\nu_1; \nu_2) = D(\{\psi^1_\sigma(\cdot)\}_{\sigma\in S_n}; \{\psi^2_\sigma(\cdot)\}_{\sigma\in S_n}) \tag{32}$$

where $\nu_1, \nu_2 \in [0;1]^{\mathscr{B}(Y)}$ are fuzzy measures; $\{\psi^1_\sigma(\cdot)\}_{\sigma\in S_n}$, $\{\psi^2_\sigma(\cdot)\}_{\sigma\in S_n}$ are APCs of ν_1, ν_2, respectively. Consider the distance D_2:

$$D_2^2(\nu_1, \nu_2) = \sum_{\sigma\in S_n} \sum_{i=1}^{n} \left(\psi^{(1)}_\sigma(y_{\sigma(i)}) - \psi^{(2)}_\sigma(y_{\sigma(i)})\right)^2. \tag{33}$$

Practically, the distance D_2 between fuzzy measures ν_1 and ν_2 is reduced to the distances between probability measures [18].

Take NERCs $\{(\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y)_{\pi_Y\in L_Y^{\nu_i}}\}, i = 1,2$ of fuzzy measures ν_1 and ν_2, respectively. Let us introduce a new distance between fuzzy measures ν_1, ν_2:

Definition 10. *A distance between fuzzy measures is defined as the distance between NERCs $L_Y^{\nu_1}$ and $L_Y^{\nu_2}$:*

$$D^2(\nu_1,\nu_2) = D^2(L_Y^{\nu_1},L_Y^{\nu_2}) = \inf_{\pi_X^{(1)}\in L_Y^{\nu_1},\pi_Y^{(2)}\in L_Y^{\nu_2}} D^2(\pi_Y^{(1)},\pi_Y^{(2)}) \tag{34}$$

The main goal of this section is to parameterize the distance D from a calculation point of view. Let Θ be a given "specific" set [19]:

$$\Theta = \{\pi_{\overline{0}},\pi_{\overline{1}},\ldots,\pi_{\overline{n}},\pi_{\overline{12}},\ldots,\pi_{\overline{n-1,n}},\pi_{\overline{123}},\ldots,\pi_{\overline{n-2,n-1,n}},\ldots,\pi_{\overline{12\ldots n}}\} \tag{35}$$

and let ζ be constructed as

$$\zeta(\{y_{j_1},y_{j_2},\ldots,y_{j_k}\}) = \{\overline{0},\overline{j}_1,\overline{j}_2,\ldots,\overline{j}_k,\overline{j_1j_2},\ldots,\overline{j_{k-1}j_k},\ldots,\overline{j_1,\ldots,j_{k-1},j_k}\}$$

where expression $\overline{\text{ab}}$ on a and b is a "concatenation" operation of numbers a and b. Then, $\forall C = \{y_{j_1},\ldots,y_{j_k}\} \in \mathscr{B}\,(Y)$:

$$\nu(C) = \sum_{\theta\in\zeta(C)} \pi_\theta = \pi_{\overline{0}} + \pi_{\overline{j_1}} + \pi_{\overline{j_2}} + \ldots + \pi_{\overline{j_k}} + \pi_{\overline{j_1j_2}} + \ldots + \pi_{\overline{j_{k-1}j_k}} + \pi_{\overline{j_1j_2\cdots j_k}} \tag{36}$$

We have to present (36) as the sum of "parts" of $y_{j_i} \in Y$, the elements' indexes of which are constructed by all subsets $B \subset C$, where $y_{j_i} \in B$. If $\nu : \mathscr{B}\,(Y) \to [0;1]$ is a known fuzzy measure, then the scheme of finding parameters of the probability π is as follows:

$$\begin{gathered}
\nu_1 \equiv \nu(\{y_1\}) = \pi_{\overline{0}} + \pi_{\overline{1}},\\
\ldots\ldots\ldots\ldots\\
\nu_n \equiv \nu(\{y_n\}) = \pi_{\overline{0}} + \pi_{\overline{n}},\\
\nu_{12} \equiv \nu(\{y_1,y_2\}) = \pi_{\overline{0}} + \pi_{\overline{1}} + \pi_{\overline{2}} + \pi_{\overline{12}},\\
\ldots\ldots\ldots\ldots\\
\nu_{\overline{n-1,n}} \equiv \nu(\{y_{n-1},y_n\}) = \pi_{\overline{0}} + \pi_{\overline{n-1}} + \pi_{\overline{n}} + \pi_{\overline{n-1,n}},
\end{gathered} \tag{37}$$

$$\begin{gathered}
\nu_{\overline{123}} \equiv \nu(\{y_1,y_2,y_3\}) = \pi_{\overline{0}} + \pi_{\overline{1}} + \pi_2 + \pi_{\overline{3}} + \pi_{\overline{12}} + \pi_{\overline{13}} + \pi_{\overline{23}} + \pi_{\overline{123}},\\
\ldots\ldots\ldots\ldots\\
1 \equiv \nu(\{y_1,y_2,\ldots,y_n\}) = \pi_{\overline{0}} + \pi_{\overline{1}} + \ldots + \pi_{\overline{n}} + \pi_{\overline{12}} + \ldots + \pi_{\overline{n-1,n}} + \pi_{\overline{123}} + \ldots + \pi_{\overline{12\ldots n}}.
\end{gathered}$$

We have $2^n - 1$ equations with 2^n unknown parameters of the probability π. One parameter is free and, for convenience, let it be π_0. Then,

$$\begin{gathered}
\pi_1 = \nu_{\overline{1}} - \pi_{\overline{0}},\\
\pi_2 = \nu_{\overline{2}} - \pi_{\overline{0}},\\
\ldots\ldots\ldots\\
\pi_n = \nu_{\overline{n}} - \pi_{\overline{0}},\\
\pi_{\overline{12}} = \nu_{\overline{12}} - \pi_{\overline{1}} - \pi_{\overline{2}} - \pi_{\overline{0}},\\
\ldots\ldots\ldots\ldots\ldots\ldots\\
\pi_{\overline{n-1,n}} = \nu_{\overline{n-1,n}} - \pi_{\overline{n-1}} - \pi_{\overline{n}} - \pi_{\overline{0}},\\
\pi_{\overline{123}} = \nu_{\overline{123}} - \pi_{\overline{1}} - \pi_{\overline{2}} - \pi_{\overline{3}} - \pi_{\overline{12}} - \pi_{\overline{13}} - \pi_{\overline{23}} - \pi_{\overline{0}},\\
\ldots\ldots\ldots\ldots\ldots\ldots\\
\pi_{\overline{12\ldots n}} = \nu_{\overline{12\ldots n}} - \pi_{\overline{1}} - \pi_{\overline{2}} - \pi_{\overline{3}} - \ldots - \pi_{\overline{23\ldots n}} - \pi_{\overline{0}},
\end{gathered} \tag{38}$$

It is clear that $\forall C \in \mathscr{B}\,(Y)$:

$$\nu(C) = \pi(\zeta(C)) \equiv f(\pi_{\overline{0}}) \tag{39}$$

where f is a linear function of $\pi_{\overline{0}}$. Let $0 \le N_0^- \le \pi_{\overline{0}} \le N_0^+ \le 1$. Let ν_1 and ν_2 be fuzzy measures on $\mathscr{B}(Y)$. We know that $\forall C \in \mathscr{B}(Y)$ $\pi_Y(\zeta_Y(C)) = \pi(\omega^{-1}(C))$ and the value of π_Y is also a linear function of parameter π_0. Let $\pi_Y(\zeta_Y(C)) = \pi_0 + \widetilde{\pi}(C)$, where $\widetilde{\pi}(C)$

is known as a certain expression of parameters and numbers $\pi_{\overline{1}}, \ldots, \pi_{\overline{12\ldots n}}$, which may be calculated by (36). Let $D \equiv D_2$ be the distance between probability measures. By transformation (34), we obtain:

$$D_2^2(\nu_1, \nu_2) = \inf_{\pi_Y^{(1)} \in L_Y^{(2)}, \pi_Y^{(2)} \in L_Y^{(2)}} D_2^2(\pi_Y^{(1)}, \pi_Y^{(2)}) \overset{def}{=} \inf_{\pi_Y^{(1)} \in L_Y^{(2)}, \pi_Y^{(2)} \in L_Y^{(2)}} \sum_{i=1}^{n} (\pi_Y^{(1)}(y_i) - \pi_Y^{(2)}(y_i))^2.$$

Then,

$$D_2^2(\nu_1, \nu_2) = \inf_{\substack{N_1^- \le \pi_0^{(1)} \le N_1^+ \\ N_2^- \le \pi_0^{(2)} \le N_2^+}} D_2^2(\pi_0^{(1)} + \widetilde{\pi}_1(\cdot), \pi_0^{(2)} + \widetilde{\pi}_2(\cdot)) =$$

$$= \inf_{\substack{N_1^- \le \pi_0^{(1)} \le N_1^+ \\ N_2^- \le \pi_0^{(2)} \le N_2^+}} \sum_{i=1}^{n} ((\pi_0^{(1)} - \pi_0^{(2)}) + (\widetilde{\pi}_1(y_i) - \widetilde{\pi}_2(y_i)))^2 == \inf_{\substack{N_1^- \le \pi_0^{(1)} \le N_1^+ \\ N_2^- \le \pi_0^{(2)} \le N_2^+}} \{n(\pi_0^{(1)} - \pi_0^{(2)})^2 +$$

$$2(\pi_0^{(1)} - \pi_0^{(2)}) \sum_{i=1}^{n} ((\widetilde{\pi}_1(y_i) - \widetilde{\pi}_2(y_i)) + \sum_{i=1}^{n} (\widetilde{\pi}_1(y_i) - \widetilde{\pi}_2(y_i))^2\}.$$

Denote

$$\pi_0^{(1)} - \pi_0^{(2)} \equiv \pi_0, \ \sum_{i=1}^{n} (\widetilde{\pi}_1(y_i) - \widetilde{\pi}_2(y_i)) \equiv b, \ \sum_{i=1}^{n} (\widetilde{\pi}_1(y_i) - \widetilde{\pi}_2(y_i))^2 \equiv c$$

Then,

$$D_2^2(\nu_1, \nu_2) = \inf_{N_1^- - N_2^+ \le \pi_0 \le N_1^+ - N_2^-} \{n\pi_0^2 + 2b\pi_0 + c\} =$$

$$= \begin{cases} \min\{n(N^-)^2 + 2bN^- + c : \quad n(N^+)^2 + 2bN^+ + c\}, & if - \frac{b}{n} \notin [N^-; N^+] \\ \frac{nc - b^2}{n}, & if - \frac{b}{n} \in [N^-; N^+], \end{cases} \tag{40}$$

where $N^- \equiv N_1^- - N_2^+$, $N^+ = N_1^+ - N_2^-$.

Analogously [18], we prove the proposition on the correctness of the definition of a distance between fuzzy measures.

Proposition 12. *If ν_1 and $\nu_2$are any fuzzy measures on $\mathscr{B}(Y)$, then*

$$D_2(\nu_1, \nu_2) = D_2(\nu_1^*, \nu_2^*). \tag{41}$$

Therefore, from an information analysis point of view, a fuzzy measure and its dual fuzzy measure contain the same information but codified in a different way.

6. Connections between Two-Order Additive Fuzzy Measure and Interaction Indexes of Attributes in the MSR Environment

A fuzzy measure, as a basic uncertainty index of subjective information measurement in the decision-making process, in some MAGDM represents flexibly a certain kind of interaction among the decision attributes and can vary from redundancy (negative interaction) to synergy (positive interaction). In this section, connections between the interaction indexes and the probability distribution of the NERC of a two-order additive fuzzy measure are constructed.

Definition 11. [6]: *Let ν be a set function (not necessarily capacity) on some MAGDM attributes' set $Y = \{y_1, \ldots, y_n\}$. The Mobius representation of ν is a set function $m_\nu : 2^S \to R^1$ defined by*

$$m_\nu(G) = \sum_{B \subset G} (-1)^{|G \backslash B|} \nu(B) \ \forall G \subset Y, \tag{42}$$

where $|G|$ is the cardinality of the set G, $R^1 = (-\infty, +\infty)$.

It is known [6] that any capacity (fuzzy measure), more generally, any set function, ν can be uniquely expressed in terms of its Mobius representation by

$$\nu(G) = \sum_{B \subset G} m_\nu(B) \quad \forall G \subset Y. \tag{43}$$

Definition 12. [15]: *Take some fuzzy measure ν on Y. ν is said be $k, k \in \{1, 2, \ldots, n\}$ additive, if its Mobius representation satisfies $m_\nu(G) = 0$ for all $G \subset Y$ such that $G > k$ and there exists at least one subset $G \subset Y$ such that $m_\nu(G) \neq 0$.*

In the following, we consider the two-order additive fuzzy measure [15].

Definition 13. [15]: *Take some fuzzy measure ν on Y. (1) The overall importance value of an attribute $y_i \in Y$ is called its Shapley value*

$$I_i = \sum_{G \subset Y \setminus \{y_i\}} [(|Y| - |G| - 1)!/(|Y|!)] \cdot [\nu(G \cup \{y_i\}) - \nu(G)], \tag{44}$$

(2) The interaction index of two attributes $y_i, y_j \in Y,\ i \neq j$ is defined by

$$\begin{aligned} I_{ij} = & \sum_{G \subset Y \setminus \{y_i, y_j\}} [(|Y| - |G| - 2)!/(|Y|! - 1)] \cdot [\nu(G \cup \{y_i, y_j\}) \\ & - \nu(G \cup \{y_i\}) - \nu(G \cup \{y_j\}) + \nu(G)]. \end{aligned} \tag{45}$$

In [27], the linear connection between a two-order additive fuzzy measure, some attributes importance values, and interaction indexes was shown. $\forall \sigma = \{\sigma(1), \ldots, \sigma(n)\} \in S_n$, $l = 1, \ldots, n$,

$$\nu(\{y_{\sigma(1)}, \ldots, y_{\sigma(l)}\}) = \sum_{q=1}^{l} I_{\sigma(q)} - \frac{1}{2} \cdot \sum_{j=1}^{l} \sum_{q \in N_{\sigma(l)}} I_{\sigma(j)q} \tag{46}$$

where $N_{\sigma(l)}$ denotes an index subset $N_{\sigma(l)} = \{1, \ldots, n\} \setminus \{\sigma(1), \ldots, \sigma(l)\}$. Using the definition of the APC of a fuzzy measure, the connections between associated probabilities, importance values, and interaction indexes of attributes were obtained in [27]:

$$\psi_\sigma(y_{\sigma(l)}) = I_{\sigma(l)} + (1/2) \cdot \sum_{q=1}^{l-1} I_{\sigma(l)\sigma(q)} - (1/2) \cdot \sum_{q=l+1}^{n} I_{\sigma(l)\,\sigma(q)}, \tag{47}$$

It was mentioned in [27] that "if $l = 1$, then the second addend is zero, and if $l = n$, then the third addend is zero. Representation of the associated probability (47) has an interesting interpretation in terms of the representation of interaction between attributes. In (47) in the positive role are involved corresponding interaction indexes of $\{y_{\sigma(1)}\}$, $\{y_{\sigma(1)}, y_{\sigma(2)}\}$, $\ldots$, $\{y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(l-1)}\}$ structure, while in negative role are involved relevant interaction indexes of $\{y_{\sigma(l+1)}\}$, $\ldots$, $\{y_{\sigma(l+1)}, \ldots, y_{\sigma(n)}\}$ structure. Therefore, $n \times n!$ probabilities are constructed with $n\,(n+1)/2$ interaction indexes $J = \{I_{ij}\},\ i \neq j,\ I_{ij} = I_{ji}$ and n values of overall importance values ($I = \{I_i\},\ i = 1, \ldots, n$)".

From Definition 2 (Formula (1)), we have the linear connection between the probability of the APC of a two-order additive fuzzy measure, pair-wise interaction indexes of attributesand Shapley values:

$$\psi_\sigma(y_{\sigma(l)}) = \nu(\{y_{\sigma(1)}, \ldots, y_{\sigma(l)}\}) - \nu(\{y_{\sigma(1)}, \ldots, y_{\sigma(l-1)}\}),\ l = 1, \ldots, n \tag{48}$$

From Section 3, for every probability distribution $\{\pi_Y\}_{\pi_Y \in L_Y^\nu}$ of *NERC* of the fuzzy measure $\nu - \{\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y\}_{\pi_Y \in L_Y^\nu}$, we have: $\forall \sigma \in S_n, i = 1, 2, \ldots, n$,

$$\psi_\sigma(y_{\sigma(l)}) = \pi_Y(\zeta_Y(\{y_{\sigma(1)}, \ldots, y_{\sigma(l)}\})) - \pi_Y(\zeta_Y(\{y_{\sigma(1)}, \ldots, y_{\sigma(l-1)}\})),\ l = 1, \ldots, n \tag{49}$$

Comparing Formulas (47)–(49), we obtain the recursive calculation connection for the *NERC* probability π_Y on attributes' set $Y = \{y_1, \ldots, y_n\}$ by the composition $\zeta_Y \circ \pi_Y$:

$$\begin{aligned} &\pi_Y(\zeta_Y(\{y_{\sigma(1)}, \ldots, y_{\sigma(l)}\})) = \pi_Y(\zeta_Y(\{y_{\sigma(1)}, \ldots, y_{\sigma(l-1)}\})) + \\ &+I_{\sigma(l-1)} + (1/2) \cdot \sum_{q=1}^{l-2} I_{\sigma(l-1)\,\sigma(q)} - (1/2) \cdot \sum_{q=l}^{n} I_{\sigma(l-1)\,\sigma(q)}, \\ &l = 2, \ldots, n;\ \ \pi_Y(\zeta_Y(\{y_{\sigma(0)}\})) \equiv 0;\ \ \pi_Y(\zeta_Y(\{y_{\sigma(1)}\})) \equiv \psi_\sigma(y_{\sigma(1)}). \end{aligned} \tag{50}$$

7. An Illustrative Example

For a clearer representation of the results obtained above, we consider an example when $Y = \{y_1, y_2\}$ with the following semi-filters:

$$Se_1 = \{\{y_1\}, Y\},\ Se_2 = \{\{y_2\}, Y\},\ Se_3 = \{Y\}.$$

Then,

$$\Theta_Y = \{Se_1, Se_2, Se_3\},\ \mathscr{B}(\Theta_Y) = \{\varnothing, Se_1, Se_2, Se_3, \Theta_Y\}; \forall C \in \mathscr{B}(Y):$$

$$\zeta_Y(C) = \{Se \in \theta_Y / C \in Se\};$$

$$\begin{gathered} \zeta_Y(\{y_1\}) = \{Se_1\} = \{\{y_1\}, Y\},\ \zeta_Y(\{y_2\}) = \{Se_2\} = \{\{y_2\}, Y\}, \\ \zeta_Y(Y) = \{Se_1, Se_2, Se_3\} = \theta_Y,\ \zeta(\varnothing) = \varnothing. \end{gathered}$$

In addition, ζ_Y is a 0–1 order-preserving homomorphism.
Let $\nu_1 = \nu(\{y_1\}), \nu_2 = \nu(\{y_2\}), \nu_1^* = \nu^*(\{y_1\}), \nu_2^* = \nu^*(\{y_2\})$;
Let $(\theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y)$ be the MSR of ν. Then,

$$\begin{aligned} &\nu_1 = \pi_Y(\zeta_Y(\{y_1\})) = \pi_Y(\{Se_1\}) = 1 - \nu_2^*, \\ &\nu_2 = \pi_Y(\zeta_Y(\{y_2\})) = \pi_Y(\{Se_2\}) = 1 - \nu_1^*,\ \pi(\Theta_Y) \equiv 1 \\ &\pi_Y(\{Se_1 \cup Se_2\}) = \pi_Y(\{Y\}) = 1 - \nu_1 - \nu_2, \end{aligned}$$

This representation is schematically shown in Figure 3. From (35), we have MSR

$$(\theta, \mathscr{B}(\Theta), \zeta, \pi),\ \text{where } \theta = \{\overline{0}, \overline{1}, \overline{2}, \overline{12}\},\ \text{and}$$

$$\begin{gathered} \zeta(\{y_1\}) = \{\overline{0}, \overline{1}\} \Rightarrow \nu_1 = \nu(\{y_1\}) = \pi_{\overline{0}} + \pi_{\overline{1}}; \\ \zeta(\{y_2\}) = \{\overline{0}, \overline{2}\} \Rightarrow \nu_2 = \nu(\{y_2\}) = \pi_{\overline{0}} + \pi_{\overline{2}}; \\ \zeta(\{Y\}) = \{\overline{0}, \overline{1}, \overline{2}, \overline{12}\} \Rightarrow 1 = \nu(Y) = \pi_{\overline{0}} + \pi_{\overline{1}} + \pi_{\overline{2}} + \pi_{\overline{12}}. \end{gathered}$$

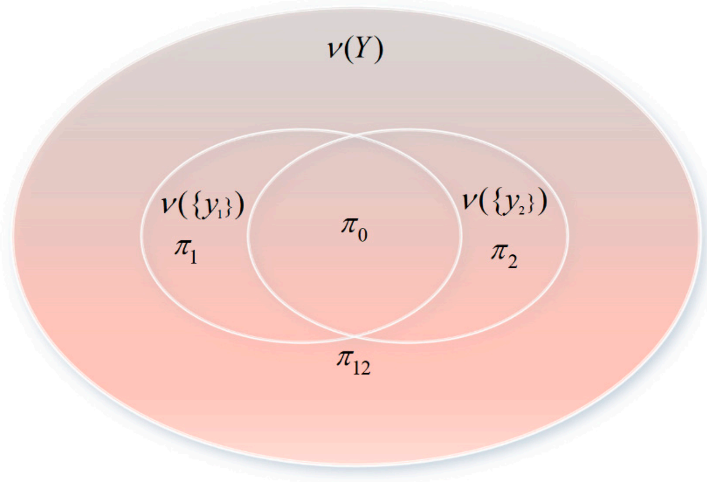


Figure 3. The schematic connection between the fuzzy measure ν and parameters of probability π.

We obtain the following system of linear equations:

$$\begin{cases} \nu_1 = \pi_{\bar{0}} + \pi_{\bar{1}}, \\ \nu_2 = \pi_{\bar{0}} + \pi_{\bar{2}}, \\ 1 = \pi_{\bar{0}} + \pi_{\bar{1}} + \pi_{\bar{2}} + \pi_{\overline{12}}. \end{cases}$$

We have three equations with four unknowns. We have to calculate $D_2^2(\nu_1, \nu_2)$, where ν_1, ν_2 are two fuzzy measures on $\mathscr{B}(Y)$. We obtain

$$D_2^2(\nu_1,\nu_2) = \inf_{-1\le\lambda_0\le 1}\{4\lambda_0^2 - 4(\tilde{\nu}_1+\tilde{\nu}_2)\lambda_0 + [\tilde{\nu}_1^2 + \tilde{\nu}_2^2 + (\tilde{\nu}_1^2+\tilde{\nu}_2^2)^2]\} = \tilde{\nu}_1^2 + \tilde{\nu}_2^2$$

where $\pi_{0(\min)} = \frac{\tilde{\nu}_1+\tilde{\nu}_2}{2}$, $\tilde{\nu}_1 \equiv \nu_1^{(1)} - \nu_1^{(2)}$, $\tilde{\nu}_2 \equiv \nu_2^{(1)} - \nu_2^{(2)}$.

It is clear that $\tilde{\nu}_1 = \tilde{\nu}_1^*$, $\tilde{\nu}_2 = \tilde{\nu}_2^*$.

Let $\{\psi_\sigma(\cdot)\}_{\sigma\in S_2}$ be the APC of the ν: $\sigma(1)$ $\psi_\sigma(y_{\sigma(2)}) = 1 - \nu(\{y_{\sigma(2)}\})$ (see Figure 4).

σ	$y_{\sigma(1)}$	$y_{\sigma(2)}$
(1,2)	$\psi_1^1 = \psi_1(y_{\sigma(1)})$	$P_1^2 \equiv P_1(x_{\sigma(2)})$
(2,1)	$\psi_2^1 = \psi_2(y_{\sigma(2)})$	$P_2^2 \equiv P_2(x_{\sigma(2)})$

Figure 4. APC of the fuzzy measure.

We have the APC of the ν^* similarly. $\forall C \in \mathscr{B}(Y)$

$$\nu(C) = \pi(\zeta(C)) = \psi_{\sigma_C}(C) \Leftrightarrow \begin{cases} \nu_1 = \pi_{\bar{0}} + \pi_{\bar{1}} = \psi_1^1, \\ \nu_2 = \pi_{\bar{0}} + \pi_{\bar{2}} = \psi_2^1, \\ \pi_{\bar{0}} + \pi_{\bar{1}} + \pi_{\bar{2}} + \pi_{\overline{12}} = \psi_1^1 + \psi_1^2 = \psi_2^1 + \psi_2^2 = 1 \end{cases}$$

and following (13): $\Theta = \{\psi_1^1, \psi_1^2, \psi_2^1, \psi_2^2\}$.

Let $\psi_1^1 < \psi_1^2 < \psi_2^1 < \psi_2^2 < 1$, then $\pi(\psi_1^1) = \psi_1^1$,$\pi(\psi_2^1) = \psi_2^1 - \psi_1^1$, $\pi(\psi_1^2) = \psi_1^2 - \psi_2^1$, $\pi(\psi_2^2) = \psi_2^2 - \psi_1^2$. We construct the MSR $(\theta, \mathscr{B}(\Theta), \zeta, \pi)$. $\forall\theta \in \Theta:$

$$\omega(\theta) = \{C \in \mathscr{B}(Y)/\theta \in \zeta(C)\}$$

and

$$\omega(\psi_1^1) = \{\{y_1\}, Y\} = S_1,\ \omega(\psi_2^1) = \{\{y_2\}, Y\} = Se_2,$$
$$\omega(\psi_1^1) = \{\{y_1\} \cup \{y_2\}\} = Y,\ \omega(\psi_2^2) = Se_1 \cup Se_2.$$

Then, $\forall E \in \mathscr{B}(\theta_Y) : \pi_Y(E) = \pi(\omega^{-1}(E))$ and

$$\pi_Y(Se_1) = \pi(\psi_1^1) = \psi_1^1,\ \pi_Y(Se_2) = \pi(\psi_2^1) = \psi_2^1,$$
$$\pi_Y(Y) = \pi(\psi_1^2) = 1 - \psi_1^1 = \psi_1^2,$$
$$\pi_Y(Se_1 \cup Se_2) = \pi(\psi_2^2) = 1 - \psi_2^1 = \psi_1^2,$$
$$\pi(\Theta_Y) \equiv 1.$$

Assume that ν is a two-order additive fuzzy measure on some MAGDM attribute set Y. Using Formula (50), we can write the connection between ν and pairwise interaction indexes of attributes in the MSR environment. For the permutation $\sigma_1 = (\sigma(1), \sigma(2)) = (1,2)$, we have

$$\pi_Y(\zeta_Y(\{y_1\})) \equiv \psi_{\sigma_1}(y_1),$$

$$\pi_Y(\zeta_Y(\{y_1, .y_2\})) = \pi_Y(\zeta_Y(\{y_1\})) + I_1 - (1/2)I_{12} \equiv 1$$

and for the permutation $\sigma_2 = (\sigma(1), \sigma(2)) = (2,1)$, we have

$$\pi_Y(\zeta_Y(\{y_2\})) \equiv \psi_{\sigma_2}(y_2),$$

$$\pi_Y(\zeta_Y(\{y_1, .y_2\})) = \pi_Y(\zeta_Y(\{y_2\})) + I_2 + (1/2)I_{12} \equiv 1$$

8. Conclusions

The nonadditivity of a fuzzy measure poses problems for its use in demanding problems such as interactive MAGDM. The CBR of a fuzzy measure fits well with some MAGDM models and enhances their reliability, which we cannot say in the case of the MSR of a fuzzy measure. The main motivation of the research was to increase the practical application possibilities of the MSR in various application fields. For this, connections were built between the CBR and MSR. In addition, the representations of the Monotone Expectation (Choquet finite integral) as the most distinctive aggregation operator of the interacting MAGDM models were studied in the MSR environment, as studied in the case of the CBR. In the MSR, we constructed the nonequivalent representation class (NERC) of a fuzzy measure. The proposition on the existence of the MSR induced by the CBR was proven. The presented formula of the APC by the NERC was obtained. The duality of fuzzy measures is an important phenomenon for the presentation of the same but differently codified uncertain expert information. Significant properties were proven for the representation of a monotone expectation (ME) under the NERC conditions: the necessary and sufficient conditions for the existence of the second-order Choquet dual capacities were proven in terms of the NERC of a MSR and ME. A recursive connection formula between the interaction indexes, Shapley values, and the probability distribution of the NERC of a two-order additive fuzzy measure was obtained in the environment of general MAGDM. A new definition of distance on the class of fuzzy measures in the MSR environment was introduced. It was proven that the new distance is equivalent to the corresponding definition in the CBR: distances between any two fuzzy measures and between their dual fuzzy measures coincided as in the CBR and, thus, in the MSR. By parameterization of the new distance, its calculation formula was obtained from a practical point of view. The obtained new results were illustrated by the figures and the Scheme. In addition, for illustration of the obtained results, an illustrative example was presented. Regarding the limitations of the study, it can be said that there are only two possibilistic representations of the fuzzy measures: the CBR and MSR. It is clear from the literature presented in the article that the use of the CBR in practical tasks has significant results, which we cannot say about the MSR, as the latter is quite a complex tool in its use and it is, to some extent, limited. The results of the study presented here, which, to some extent, linked these two presentations, will allow one to increase the use of the MSR. However, future research should confirm the effectiveness of using the latter in some areas, such as cluster analysis tasks. For the future studies of the problems presented here, different fuzzy environments for the aggregation tools in MAGDM by the MSR will be developed. A probability distributions' class of the NERC of the concrete subspaces of fuzzy measures will be constructed and investigated. New results will give us an opportunity to efficiently develop modeling of MAGDM and other problems of complex process investigations under expert evaluations.

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List of Symbols

Y	finite set (universum)
$y, y_i, y_{\sigma(i)}$	elements of Y
$\mathscr{B}(Y)$	set of all subsets of Y
ν	fuzzy measure on $\mathscr{B}(Y)$
ν^*	dual fuzzy measure to ν
$(Y, \mathscr{B}(Y), \nu)$	fuzzy measure space
A, B, C, D, G	subsets of Y
$\overline{C}$	complement of $C(Y \backslash C)$
N	natural number
S_n	permutation group of natural numbers from 1 to n
σ, β	elements of S_n
σ^*	dual permutation of σ
$\psi_\sigma, \psi_\sigma^*$	associated probabilities on $\mathscr{B}(Y)$
$\{\{\psi_\sigma(y_{\sigma(1)}), \psi_\sigma(y_{\sigma(2)}), \ldots, \psi_\sigma(y_{\sigma(n)})\}_{\sigma \in S_n}$	associated probability class (APC) of the fuzzy measure ν
Θ	finite set of some definite "indexes"
θ	element of Θ
$\mathscr{B}(\Theta)$	set of all subsets of Θ
ζ	0–1 order-preserving homomorphism
π, π^*	dual probability measures on $\mathscr{B}(\Theta)$
$(\Theta, \mathscr{B}(\Theta), \zeta, \pi)$	Murofushi–Sugeno representation (MSR) of the fuzzy measure ν
Θ_Y	set of all semi-filters in $\mathscr{B}(Y)$
Se	semi-filter
ζ_Y, ζ_Y^+	0-1 order-preserving homomorphisms
π_Y, π_Y^*, π_Y^+	probability measures on $\mathscr{B}(\Theta_Y)$
M	mapping $\mathscr{B}(Y) \to \mathscr{B}(\Theta_Y)$
Θ_Y^+	subset of Θ_Y
ω	mapping: $\Theta \to \mathscr{B}(\Theta_Y)$
$(\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y)$	equivalent MSR of the fuzzy measure ν.
$L_Y^\nu, L_Y^{\nu^*}$	classes of probability measures of nonequivalent MSRs of dual fuzzy measures ν and ν^*.
$\{\Theta_Y, \mathscr{B}(\Theta_Y), \zeta_Y, \pi_Y\}_{\pi_Y \in L_Y^\nu}$	nonequivalent representation class of the fuzzy measure ν.
Θ_β^σ	subset of APC $\{\psi_\sigma\}_{\sigma \in S_n}$
μ, ξ, ρ	mapping from the composition connection between CBR and MSR
W	fuzzy subset of Y
χ_W	membership function of fuzzy subset W
$ME_\nu(\chi_W)$	monotone expectation of χ_W with respect to the fuzzy measure ν
$(ch)\int\limits_Y \chi_W d\nu$	Choquet's integral on Y
$d\alpha$	Lebesgue measure on [0;1]
K_i	subset of Y
$E_\psi(\chi_W)$	mathematical expectation of χ_W with respect to the probability measure ψ
I_C	indicator of subset C
D, D_2	distances between fuzzy measures
$\overline{ab}$	concatenation operation of numbers a and b
f	linear function
N^-, N^+	lower and upper limit values
a,b, c	constants
I_i	Shapley value
I_{ij}	Interaction index
m_ν	Mobius transformation

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