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A Study on the Experimental Design for the Lifetime Performance Index of Rayleigh Lifetime Distribution under Progressive Type I Interval Censoring

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Abstract: With the rapid development of technology, improving product life performance has become a very important issue in recent decades. The lifetime performance index is used in this research for the assessment of the lifetime performance of products following the Rayleigh distribution. Based on the hypothesis testing procedure with this index, using the maximum likelihood estimator as a testing statistic, the sampling design is determined and the related values are tabulated for practical use to reach the given power level and minimize the total experimental cost under progressive type I interval censoring. When the inspection interval length is fixed and the number of inspection intervals is not fixed, the required number of inspection intervals and sample size with the minimum total cost are determined and tabulated. When the termination time is not fixed, the required number of inspection intervals, sample size and equal interval length reaching the minimum total cost are determined and tabulated. Lastly, a practical example is given to illustrate the use of this sampling design for the testing procedure to determine whether the process is capable.

Keywords: censored sample; Rayleigh distribution; maximum likelihood estimator; lifetime performance indices; testing algorithmic procedure; sampling design

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1. Introduction

The process capability index C_L proposed by Montgomery [1] is frequently used to measure the larger-the-better quality characteristics such as lifetime, mpg, tensile strength, durability, etc. For measuring the lifetime of products, this index is called the lifetime performance index. In many cases, the experimenters can only observe censored data. Two censoring types, namely type I censoring and type II censoring, are frequently considered. Type I censoring occurs if the life test of n subjects stops at a predetermined time and the number of observations is random. Type II censoring occurs if the life test stops when a predetermined number of failure times is observed. Progressive censoring has the property of allowing the removal of units at some time points that may not be the final termination point. More inferences about progressive censored data can be seen in Balakrishnan and Aggarwala [2] and Aggarwala [3]. For a progressive type II right censored sample, Wu et al. [4] assessed the lifetime performance index of products with an exponential distribution. Wu et al. [5] proposed a Bayesian test of the lifetime performance index for exponential products based on a progressive type II right censored sample. Lee et al. [6] proposed a decision procedure for the performance assessment of the lifetime index of products for the Gompertz distribution under progressive type II right censoring. Lee et al. [7] assessed the lifetime performance index of Rayleigh products based on Bayesian estimation using progressive type II right censored samples. Wu et al. [8] implemented a testing procedure for the lifetime performance index of Burr XII products with a progressive type II right censored sample. For progressive type I interval censored data from the Gompertz lifetime distribution, a testing procedure for the lifetime performance index was proposed by Wu and Hsieh [9] based on a progressive type I interval censored sample. Based on this testing procedure, a reliability sampling design was developed by Wu et al. [10] for products following Gompertz distribution. For products following Weibull lifetime distribution, Wu and Lin [11] proposed a hypothesis testing procedure for the lifetime performance index using progressive type I interval censored data. Wu et al. [12] investigated the sampling design for the testing of the lifetime performance index from Weibull distribution. This study focuses on the Rayleigh distribution, which is an asymmetric probability distribution. Wu et al. [13] proposed a testing procedure to test whether the lifetime performance index meets the desired target for products following Rayleigh distribution under progressive type I interval censoring, and the related testing procedure is summarized in Section 2. The progressive type I interval censoring scheme is depicted as follows: there are n products that are subjected to a life test at time 0. Let $(t_1, \ldots, t_m)$ and $(p_1, \ldots, p_m)$ be the predetermined inspection times and the pre-specified removal percentages of the remaining survival units at time t_i, $i = 1, \ldots, m$, where $p_m = 1$ and $t_m = T$ is the termination time for the experiment. At the ith inspection time interval (t_{i-1}, t_i), the number of failure units X_i follows a binomial distribution bin $(n - \sum_{j=1}^{i-1} X_j - \sum_{j=1}^{i-1} R_j, q_i)$, where $q_i = \dfrac{F_U(t_i) - F_U(t_{i-1})}{1 - F_U(t_{i-1})}$ and $F_U(x)$ is the cumulative distribution function for the lifetime variable U. Furthermore, R_i units are randomly removed from the remaining $n - \sum_{j=1}^{i} X_j - \sum_{j=1}^{i-1} R_j$ survival units and R_i follows a binomial distribution bin $(n - \sum_{j=1}^{i} X_j - \sum_{j=1}^{i-1} R_j, p_i)$, $i = 1, \ldots, m$. Then, the observed number of failure units $(X_1, \ldots, X_m)$ for m inspection intervals is the progressive type I interval censored sample under a progressive censoring scheme $(R_1, \ldots, R_m)$ with removal probabilities $p_1, \ldots, p_m$. There is an increasing relationship between the lifetime performance index and the conforming rate. If the desired conforming rate is specified, the desired lifetime performance index c_0 is determined. The null hypothesis $H_0 : C_L \leq c_0$ vs. the alternative hypothesis $H_a : C_L > c_0$ are set up. Using the maximum likelihood estimator of the lifetime performance index, the testing procedure for Rayleigh lifetime products under a level of significance α was developed by Wu et al. [13]. The power function has also been obtained. Further details of the methodologies for the testing procedure are given in Section 2. Based on the testing procedure proposed in Wu et al. [13], we conduct a study on the experimental design for the lifetime performance index from Rayleigh distributed products based on progressive type I interval censored data. The optimal sampling design consists of two parts and is given in Section 3. The first part is to determine the minimum number of inspection intervals yielding the minimum total cost under the pre-specified power level and level of significance for a fixed total experimental time, which is described in Section 3.1. The second part considers the case when the interval time of the experiment is not fixed. Under this consideration, the minimum number of inspection intervals and the equal interval length reaching the minimum total cost are determined and tabulated in Section 3.2. In Section 3.3, a practical example is given to illustrate the application of the optimal sampling design for the testing procedure to test whether the process is capable. Finally, the conclusions are presented in Section 4.

2. Methodology for Testing the Lifetime Performance Index

The lifetime U of a product following Rayleigh distribution has a probability density function (pdf) and a cumulative distribution function (cdf) as follows:

$$f_U(u) = \frac{u}{\lambda^2} \exp\{-\frac{u^2}{2\lambda^2}\},\ u \geq 0, \lambda > 0 \tag{1}$$

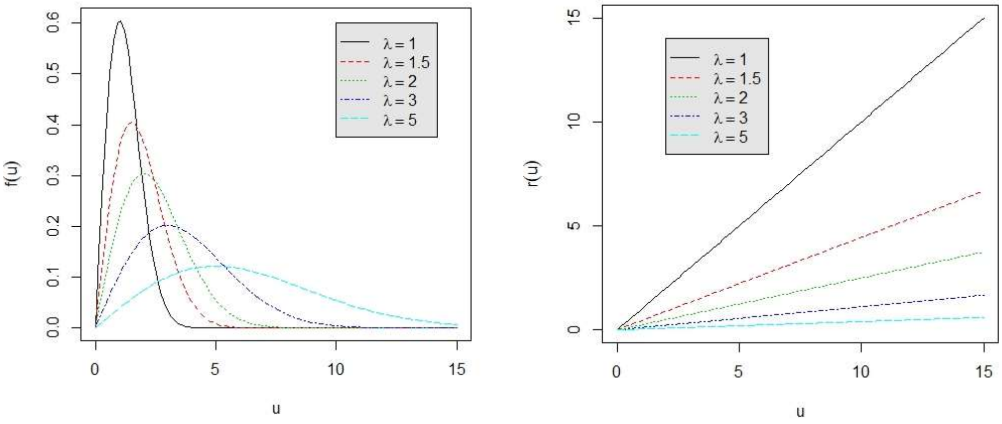
and

$$F_U(u) = 1 - \exp\{-\frac{u^2}{2\lambda^2}\},\ u \geq 0, \lambda > 0. \tag{2}$$

The failure rate function is defined as

$$r_U(u) = \frac{f_U(u)}{1 - F_U(u)} = \frac{u}{\lambda^2}. \tag{3}$$

The curves for the pdf and failure rate function under λ = 1, 1.5, 2, 3, 5 are displayed in Figure 1a,b, respectively.



(**a**) (**b**)

Figure 1. (**a**) The curve of pdf for Rayleigh distribution; (**b**) the curve of failure rate function for Rayleigh distribution.

Let $Y = U^2$. Then, this new random variable Y has a one-parameter exponential distribution and its pdf and cdf are given as follows:

$$f_Y(y) = \frac{1}{2\lambda^2} \exp\{-\frac{y}{2\lambda^2}\},\ y \geq 0, \lambda > 0 \tag{4}$$

and

$$F_Y(y) = 1 - \exp\{-\frac{y}{2\lambda^2}\},\ y \geq 0, \lambda > 0, \tag{5}$$

where λ^2 is the shape parameter and the failure rate function is defined as

$$r_Y(y) = \frac{f_y(y)}{1 - F_y(y)} = \frac{1}{2\lambda^2}, \tag{6}$$

where λ^2 is the scale parameter.

From Montgomery [1], the lifetime performance index is defined as:

$$C_L = \frac{\mu - L}{\sigma}, \tag{7}$$

where μ denotes the process mean, σ represents the process standard deviation, and L is the known lower specification limit. Suppose that the lower specification limit for U is LU. Then, the lower specification limit for Y is $L = LU^2$.

The mean and standard deviation of Y are $\mu = 2\lambda^2$ and $\sigma = 2\lambda^2$. Substituting μ and σ by this information, we then have $C_L = \frac{\mu - L}{\sigma} = \frac{2\lambda^2 - L}{2\lambda^2} = 1 - \frac{L}{2\lambda^2}$.

The conforming rate is obtained as

$$P_r = P(U \geq L_U) = P(Y \geq L) = \exp\left(-\frac{L}{2\lambda^2}\right) = \exp(C_L - 1), \quad -\infty < C_L < 1.$$

The conforming rate is an increasing function of the lifetime performance index C_L. If the experimenter desires the conforming rate to be 0.8607, the desired lifetime performance index should be c_0 = 0.85.

For a progressive type I interval censoring sample $(X_1, \ldots, X_m)$ under a progressive censoring scheme $(R_1, \ldots, R_m)$ with removal probabilities $p_1, \ldots, p_m$, the likelihood function is

$$L(\lambda) = \prod_{i=1}^{m} \left(e^{-\frac{t_{i-1}^2 x_i}{2\lambda^2}} - e^{-\frac{t_i^2 x_i}{2\lambda^2}} \right) e^{-\frac{t_i^2 R_i}{2\lambda^2}}.$$

Wu et al. [13] obtained the maximum likelihood estimator (MLE) for λ as the numerical solution of the following log-likelihood equation:

$$\begin{aligned} \tfrac{d}{d\lambda} \ln L(\lambda) &= \sum_{i=1}^{m} \left(-x_i \frac{\exp\{-\frac{t_i^2 - t_{i-1}^2}{2\lambda^2}\}}{1-\exp\{-\frac{t_i^2 - t_{i-1}^2}{2\lambda^2}\}} \left(\frac{t_i^2 - t_{i-1}^2}{\lambda^2} \right) + \frac{t_{i-1}^2 x_i + t_1^2 R_i}{\lambda^3} \right) \\ &= -\tfrac{1}{\lambda^3} \sum_{i=1}^{m} \left(x_i \left(t_i^2 - t_{i-1}^2 \right) \left(\exp\{-\frac{t_i^2 - t_{i-1}^2}{2\lambda^2}\} - 1 \right)^2 - \left(t_{i-1}^2 x_i + t_1^2 R_i \right) \right). \end{aligned} \tag{8}$$

Its asymptotic variance is the reciprocal of the following Fisher's information:

$$\begin{aligned} I(\lambda) = -E[\frac{d^2 \ln L(\lambda)}{d\lambda^2}] = \frac{n}{\lambda^2} \sum_{i=1}^{m} \{ \frac{1-q_i}{q_i} [4 \ln^2(1-q_i) + 6 q_i \ln(1-q_i)] \\ + \frac{3(t_{i-1}^2 q_i + t_i^2 p_i (1-q_i))}{\lambda^2} \} \cdot \prod_{j=1}^{i-1} (1-p_j)(1-q_j); \end{aligned} \tag{9}$$

where $q_i = 1 - \exp\left(-\frac{t_i^2 - t_{i-1}^2}{2\lambda^2} \right)$.

Then, we have $\hat{\lambda} \underset{n\to\infty}{\overset{d}{\to}} \mathrm{N}\left(\lambda, I^{-1}(\lambda)\right)$.

For equal interval lengths, $t_i - t_{i-1} = t$ and $t_i = it$, $i = 1, \ldots, m$ are replaced in Equations (8) and (9).

By the property of the invariance of MLE, the MLE of C_L can be obtained as

$$\hat{C}_L = 1 - \frac{L}{2\hat{\lambda}} . \hat{C}_L = 1 - \frac{L}{2\hat{\lambda}^2} \tag{10}$$

Let c_0 be the desired level of the lifetime performance index to make the process capable. Then, we wish to test $H_0 : C_L \le c_0$ (the process is not capable) vs. $H_a : C_L > c_0$ (the process is capable) using the MLE of C_L, $\hat{C}_L = 1 - \frac{L}{2\hat{\lambda}^2}$. as the test statistic, and the critical region for this test is $\left\{ \hat{C}_L \middle| \hat{C}_L > C_L^0 \right\}$. From Wu et al. [9], the critical value is found to be $C_L^0 = 1 - \frac{L}{2\left(\lambda_0 + Z_\alpha \sqrt{I^{-1}(\lambda_0)}\right)^2}$, under the level of significance α, where $\lambda_0 = \sqrt{\frac{L}{2(1-c_0)}}$ and Z_α represents the right-tailed α percentile of standard normal distribution. We will conclude that the alternative hypothesis is supported if $\hat{C}_L > C_L^0$ is satisfied.

Moreover, the power $h(c_1)$ of this statistical test at the point of $C_L = c_1 > c_0$ is

$$h(c_1) = P\left(\hat{C}_L > C_L^0 \middle| c_1 = 1 - \frac{L}{\lambda_1^2} \right) = \Phi\left(\frac{\lambda_0 - \lambda_1 + Z_{1-\alpha}\sqrt{I^{-1}(\lambda_0)}}{\sqrt{I^{-1}(\lambda_1)}} \right), \tag{11}$$

where $\Phi(\cdot)$ is the cdf for the standard normal distribution, $\lambda_0 = \sqrt{\frac{L}{2(1-c_0)}}$ and $\lambda_1 = \sqrt{\frac{L}{2(1-c_1)}}$.

3. Sampling Design

In this section, the reliability sampling design is investigated under different setups and considerations. As described in Section 3.1, when the termination time of the experiment T is fixed, the required sample size and minimal number of inspection intervals are determined so that the given power of the level α testing procedure can be reached and the total cost of the experiment can be minimized. In Section 3.2, when the termination time of the experiment T is fixed, the required sample size, minimal number of inspection intervals and the equal lengths of the interval are determined so that the given power of the level α testing procedure can be reached and the total cost of the experiment can be minimized. Let

$$g(\lambda) = I^{-1}(\lambda)/n = \lambda^2 \left[\sum_{i=1}^{m} \{ \frac{1-q_i}{q_i} [4\ln^2(1-q_i) + 6q_i \ln(1-q_i)] + \frac{3(t_{i-1}^2 q_i + t_i^2 p_i(1-q_i))}{\lambda^2} \} \cdot \prod_{j=1}^{i-1} (1-p_j)(1-q_j) \right]^{-1}$$

The function $g(\lambda)$ is independent of the sample size n. In terms of $g(\lambda)$, the power function can be rewritten as $h(c_1) = \Phi\left(\frac{(\lambda_0 - \lambda_1)\sqrt{n} + Z_{1-\alpha}\sqrt{g(\lambda_0)}}{\sqrt{g(\lambda_1)}} \right)$.

In order to attain the pre-specified power $1-\beta$ or the probability of type II error β at c_1 under the level of significance α, we set the above power function to be $1-\beta$, and then the sample size is determined as

$$n = ceiling\left(\frac{Z_\beta \sqrt{g(\lambda_1)} + Z_\alpha \sqrt{g(\lambda_0)}}{\lambda_0 - \lambda_1} \right)^2, \tag{12}$$

where ceiling(x) is a ceiling function mapping x to the smallest integer greater than or equal to x.

3.1. The Optimal m and n for Fixed T

The smaller the number of inspection intervals m, the more convenient it is for experimenters to collect the progressive type I interval sample. Let m_0 be the upper limit of m specified by the experimenter such that $m \leq m_0$. The default value of m_0 is 20 if the experimenter does not specify it in advance. The sample size n is a function of m. In this section, we wish to determine the optimal (m,n) to yield the minimum total cost incurred during the progressive type I interval censoring procedure. Similar to Huang and Wu [14], we consider the following costs:

1. Installation cost C_a—the cost of installing all test units;
2. Sample cost C_s—the cost per test unit;
3. Inspection cost C_I—the cost of using the inspection equipment;
4. Operation cost C_o—the cost consisting of the personnel cost, depreciation of test equipment and so on. It is proportional to the length of the experimental time period.

Integrating all these costs, we have the total cost of

$$TC(m,n) = C_a + nC_s + m\,C_I + T\,C_o, \tag{13}$$

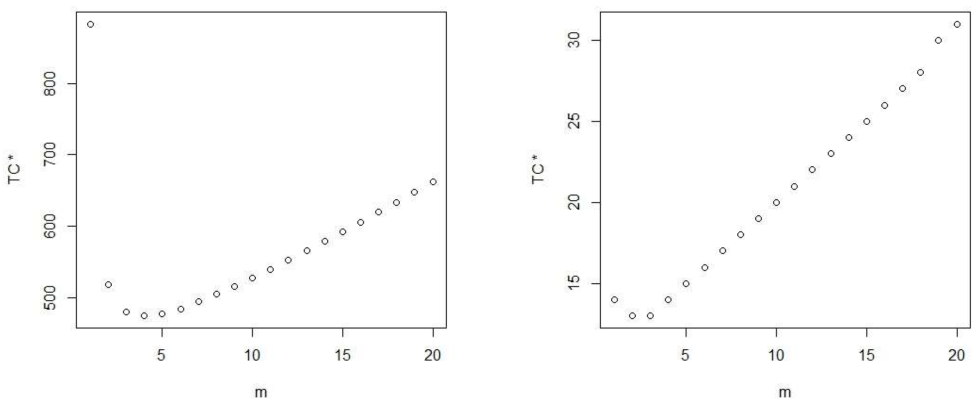
where n is given in Equation (12).

The Algorithm 1 using the numeration method to search the optimal (m,n) is given as follows:

Algorithm 1: Search the optimal (m,n)

Step 1: Give the pre-assigned values of $m = 1$, c_0, c_1, α, β, p, T, L and m_0 (the default value is 20) and the four costs $C_a = aC_o$, $C_s = bC_o$, $C_I = cC_o$, C_o.
Step 2: Compute the sample size n in Equation (12) and then compute the total cost $TC(m,n)$ in Equation (13).
Step 3: If $m < m_0$, then $m = m + 1$ and go to Step 2; otherwise, go to Step 4.
Step 4: For an array of total costs, the optimal solution of m value denoted by m^* is the m value with the minimum total cost. Then, the sample size n^* in Equation (12) can be computed.

Once the optimal values of m^* and n^* are determined, the critical value of $C_L^0 = 1 - \dfrac{L}{2\left(\lambda_0 + Z_\alpha\sqrt{I^{-1}(\lambda_0)}\right)^2}$ can be obtained as well. Consider $c_0 = 0.85$, $C_o = 1$ and $a = b = c = 1$. For testing $H_0 : C_L \leq 0.85$ when $\beta = 0.25$, $\alpha = 0.05$, $p = 0.05$, $c_1 = 0.90$, $m_0 = 20$, $L = 0.05$, $T = 1.0$, we plot the total cost versus $m = 1{:}m_0$ in Figure 2a. From this figure, we can see that the minimum total cost is incurred when $m = 3$, with the total cost of 43. For a different setup of parameters, $\beta = 0.15$, $\alpha = 0.05$, $p = 0.01$, $c_1 = 0.95$, another total cost curve with $m = 1{:}m_0$ is given in Figure 2b. From this figure, we can see that the curve is a concave upwards curve and the minimum total cost occurs when $m = 2$, with the total cost of 13.



(**a**) (**b**)

Figure 2. (**a**): Total cost curve at $m = 1{:}m_0$ for the first case; (**b**) total cost curve at $m = 1{:}m_{0.}$ for the second case.

The minimum suggested inspection intervals m^* and the corresponding sample size n^* to attain the minimum total cost TC (m^*,n^*) under the constraint of $m < 20$ for testing $H_0 : C_L \leq 0.85$ are tabulated in Tables 1 and 2 for $c_1 = 0.875$, 0.90 and $c_1 = 0.925$, 0.95, respectively, under the condition of $\alpha = 0.01$, 0.05, 0.1, $\beta = 0.25$, 0.20, 0.15, $L = 0.05$, $T = 1.0$, $p = 0.05$, 0.075, 0.1. The corresponding critical values are also tabulated in these two tables.

For example, suppose that the users wish to conduct the level 0.05 hypothesis testing of $H_0 : C_L \leq 0.85$ under the power of 0.85 at $c_1 = 0.90$, $p = 0.05$ and $\mathrm{m}_0 = 20$. According to Table 1, the minimum required sample size is 53 and the minimum number of inspection intervals is 3. The minimum total cost can also be found as $TC = 58$ and the critical value is 0.881256.

According to Tables 1 and 2, the optimal number of inspection intervals m is nonincreasing when c_1 is increasing and the range of m is 1~4. In Figure 3, we plot the minimum total cost TC^* vs. c_1 for $\alpha = 0.01$, 0.05, 0.1 at $\beta = 0.25$ and $p = 0.05$. In Figure 4, we plot the minimum total cost TC^* vs. c_1 for $1 - \beta = 0.75$, 0.80, 0.85 at $\alpha = 0.1$ and $p = 0.05$. In Figure 5, we plot the minimum total cost TC^* vs. c_1 for $p = 0.05$, 0.075, 0.1 at $\alpha = 0.1$ and $\beta = 0.25$. From Figure 3, we have the finding that the minimum total cost TC^* decreases when the level of significance increases. From Figure 4, we find that the minimum total cost TC^* increases when the test power increases. From Figure 5, we find that the minimum total cost TC^* increases when the removal probability increases. Overall, the minimum total cost TC^* becomes smaller when the value of c_1 becomes larger.

Table 1. The optimal (m^*,n^*), total cost TC and critical value for c_1 = 0.875, 0.90, α = 0.01, 0.05, 0.1, β = 0.25, 0.20, 0.15 and p = 0.05, 0.075, 0.1 under m_0 = 20, L = 0.05 and c_0 = 0.85.

c_1			0.875				0.90			
α	β	p	m^*m^*	n^*n^*	$TCTC$	C_L^0 C_L^0	m^*m^*	n^*n^*	$TCTC$	C_L^0 C_L^0
0.01	0.25	0.050	4	309	315	0.869497	3	61	66	0.889173
		0.075	3	319	324	0.869534	3	62	67	0.889269
		0.100	3	327	332	0.869515	3	64	69	0.889127
	0.20	0.050	4	347	353	0.868502	3	69	74	0.887289
		0.075	3	359	364	0.868519	3	71	76	0.887197
		0.100	3	368	373	0.868502	3	72	77	0.887327
	0.15	0.050	4	395	401	0.867445	3	79	84	0.885299
		0.075	3	408	413	0.867474	3	81	86	0.885262
		0.100	3	418	423	0.867462	3	83	88	0.885237
0.05	0.25	0.050	4	187	193	0.867883	3	38	43	0.885858
		0.075	3	194	199	0.867877	2	40	44	0.886332
		0.100	3	198	203	0.867895	2	41	45	0.886105
	0.20	0.050	4	217	223	0.86671	3	44	49	0.883769
		0.075	3	225	230	0.866709	3	45	50	0.883769
		0.100	3	230	235	0.866714	3	46	51	0.883778
	0.15	0.050	4	255	261	0.865517	3	53	58	0.881256
		0.075	3	264	269	0.865527	3	54	59	0.881305
		0.100	3	270	275	0.865528	3	55	60	0.881362
0.10	0.25	0.050	3	136	141	0.866571	2	29	33	0.883662
		0.075	3	139	144	0.866575	3	28	33	0.883426
		0.100	3	143	148	0.866532	2	30	34	0.883447
	0.20	0.050	3	162	167	0.865291	3	33	38	0.880926
		0.075	3	166	171	0.865277	2	35	39	0.881249
		0.100	3	170	175	0.865269	2	36	40	0.881003
	0.15	0.050	4	193	199	0.864012	3	41	46	0.878209
		0.075	3	200	205	0.864015	2	43	47	0.878647
		0.100	3	205	210	0.864002	2	44	48	0.878479

Table 2. The optimal (m^*,n^*), total cost TC and critical value for c_1 = 0.925, 0.95, α = 0.01, 0.05, 0.1, β = 0.25, 0.20, 0.15 and p = 0.05, 0.075, 0.1 under m_0 = 20, L = 0.05 and c_0 = 0.85.

c_1			0.925				0.95			
α	β	p	m^*	n^*	TC	C_L^0	m^*	n^*	TC	C_L^0
0.01	0.25	0.050	2	21	25	0.909725	2	9	13	0.927812
		0.075	2	22	26	0.908972	2	9	13	0.928015
		0.100	2	22	26	0.909158	2	9	13	0.92822
	0.20	0.050	3	23	28	0.906423	2	10	14	0.925489

Table 2. *Cont.*

c_1			0.925				0.95			
α	β	p	m^*	n^*	TC	C_L^0	m^*	n^*	TC	C_L^0
		0.075	2	25	29	0.906429	2	10	14	0.92569
		0.100	2	25	29	0.906611	2	10	14	0.925894
	0.15	0.050	3	27	32	0.903317	2	12	16	0.921503
		0.075	2	29	33	0.90355	2	12	16	0.921702
		0.100	2	29	33	0.903728	2	12	16	0.921903
0.05	0.25	0.050	2	14	18	0.904044	2	6	10	0.921499
		0.075	2	14	18	0.904221	2	6	10	0.921699
		0.100	2	14	18	0.904399	2	6	10	0.9219
	0.20	0.050	2	16	20	0.901514	2	7	11	0.918176
		0.075	2	16	20	0.901686	2	7	11	0.918373
		0.100	2	17	21	0.900729	2	7	11	0.918571
	0.15	0.050	2	20	24	0.897447	2	9	13	0.912881
		0.075	2	20	24	0.89761	2	9	13	0.913071
		0.100	2	20	24	0.897775	2	9	13	0.913263
0.10	0.25	0.050	2	10	14	0.900971	1	6	9	0.921343
		0.075	2	10	14	0.901141	1	6	9	0.921343
		0.100	2	11	15	0.899556	1	6	9	0.921343
	0.20	0.050	2	13	17	0.896243	1	7	10	0.918022
		0.075	2	13	17	0.896404	1	7	10	0.918022
		0.100	2	13	17	0.896566	1	7	10	0.918022
	0.15	0.050	2	15	19	0.893789	2	7	11	0.907841
		0.075	2	16	20	0.892863	2	7	11	0.908024
		0.100	2	16	20	0.893017	2	7	11	0.908208

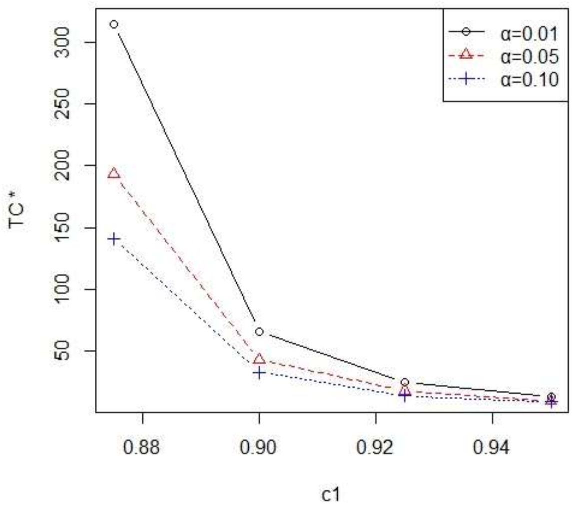


Figure 3. Minimum total cost curve for α = 0.01, 0.05, 0.1 at β = 0.25 and p = 0.05.

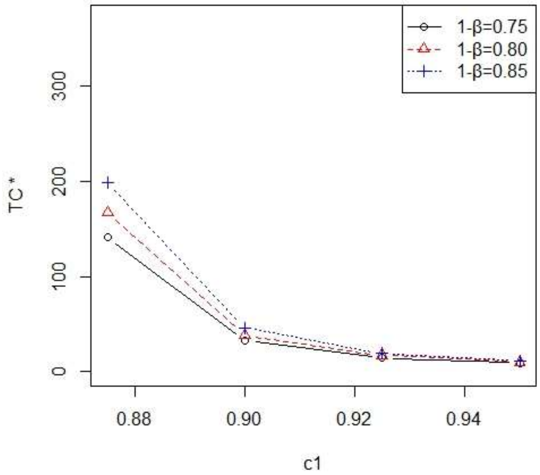


Figure 4. Minimum total cost curve for $1 - \beta = 0.75, 0.80, 0.85$ at $\alpha = 0.1$ and $p = 0.05$.

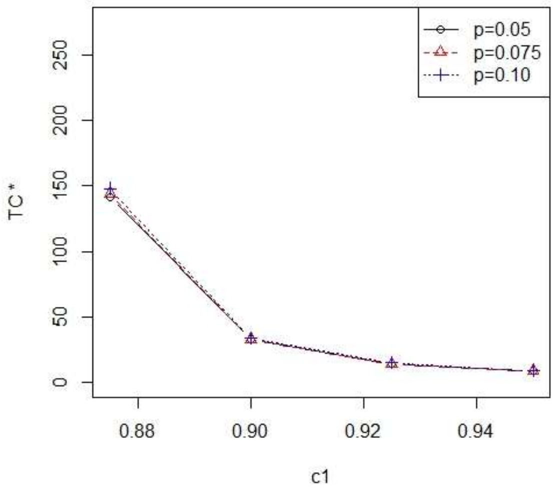


Figure 5. Minimum total cost curve for $p = 0.05, 0.075, 0.1$ at $\alpha = 0.1$ and $\beta = 0.25$.

3.2. The Optimal m, t and n When the Interval Time of the Experiment Is Not Fixed

In this subsection, the interval time of experiment t is not fixed. The inspection time t for each inspection interval is random. We wish to determine the optimal (m,t,n) to yield the minimum total cost incurred during the type I interval censoring procedure. The total cost is

$$TC(m,t,n) = C_a + nC_s + m\,C_I + mt\,C_o, \tag{14}$$

where n is given in Equation (12).

The Algorithm 2 using the numeration method to search the optimal (m,t,n) is given as follows:

Algorithm 2: Search the optimal (m,t,n)

Step 1: Give the pre-assigned values of $m = 1$, c_0, c_1, α, β and p, L and m_0 (the default value is 20) and the four costs $C_a = aC_o$, $C_s = bC_o$, $C_I = cC_o$, C_o.

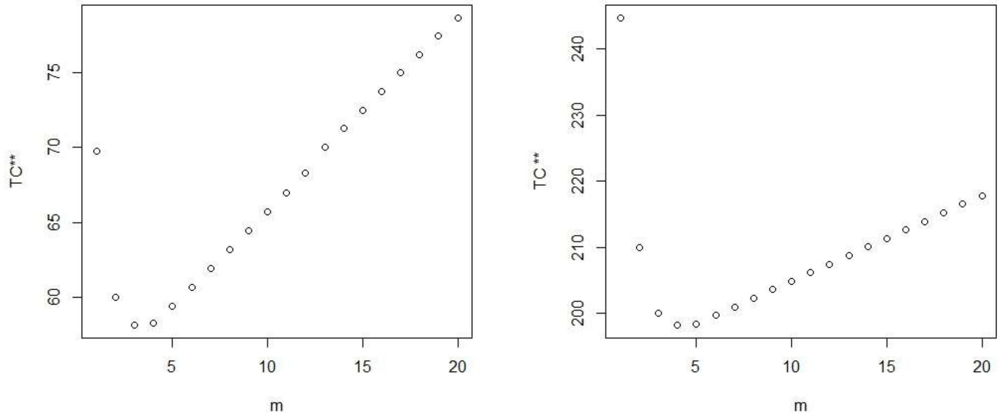
Step 2: Find the optimal value of t^* such that $TC(m,t,n)$ in Equation (14) is minimized (the optimize function in R is used to find t^*). Use $t = t^*$ to compute the sample size n in Equation (12) and compute the corresponsong total cost $TC(m,t^*,n)$ in Equation (14).

Step 3: If $m < m_0$, then $m = m + 1$ and go to Step 2; otherwise, go to Step 4.

Step 4: For an array of total costs, we determine the optimal value of m denoted by m^* such that $TC(m^*,t^*,n) = TC^{**} = \min_{m \le m_0} TC(m,t^*,n)$ is reached. With $m = m^*$ and $t = t^*$, the related sample size n^* in Equation (12) can be computed.

Once the optimal values of m^*, n^* and t^* are determined, the critical value of $C_L^0 = 1 - \frac{L}{2\left(\lambda_0 + Z_\alpha \sqrt{I^{-1}(\lambda_0)}\right)^2}$ can be determined as well.

Consider c_0 = 0.85, C_o = 1 and a = b = c = 1. When β = 0.15, α = 0.05, p = 0.1, c_1 = 0.9, m_0 = 20, L = 0.05, T = 1.0, we plot m = 1:m_0 against its corresponding total cost in Figure 6a. We can see that it is a concave upwards curve and the minimum total cost occurs when m = 3, with the total cost of 58.14254. For a different setup of parameters, β = 0.25, α = 0.05, p = 0.1, c_1 = 0.875, another total cost curve with m = 1:m_0 is given in Figure 6b. We can see that this is a concave upwards curve and the minimum total cost is incurred when m = 4, with the total cost of 198.2394. For other combinations of setups, we also find similar patterns.



(**a**) (**b**)

Figure 6. (a) Total cost curve at m = 1:m_0 for the first case; (b) Total cost curve at m = 1:$m_{0.}$ for the second case.

The minimum suggested inspection intervals m^*, inspection interval time length t^* and sample size n^* to yield the minimum total cost $TC(m^*,t^*,n^*)$ denoted by TC^{**} under the constraint of $m < m_0$, where m_0 = 20 for testing $H_0 : C_L \leq 0.85$, are tabulated in Tables 3 and 4 at α = 0.01, 0.05, 0.1, β = 0.25, 0.20, 0.15, L = 0.05, T = 1.0, p = 0.05, 0.075, 0.1 for c_1 = 0.875, 0.90 and c_1 = 0.925, 0.95, respectively. The corresponding critical values are also tabulated.

Table 3. The optimal (m^*,t^*,n^*), total cost TC and critical value for c_1 = 0.875, 0.90, α = 0.01, 0.05, 0.1, β = 0.25, 0.20, 0.15 and p = 0.05, 0.075, 0.1 under m_0 = 20, L = 0.05 and c_0 = 0.85.

c_1			**0.875**					**0.90**				
α	β	p	m^*	t^*	n^*	TC^{**}	C_L^0	m^*	t^*	n^*	TC^{**}	C_L^0
0.01	0.15	0.050	5	0.26	297	304.285	0.8696	4	0.28	58	64.127	0.8895
		0.075	5	0.27	307	314.352	0.8696	3	0.35	61	66.052	0.8894
		0.100	4	0.31	317	323.256	0.8696	3	0.36	62	67.093	0.8895
	0.20	0.050	6	0.24	332	340.464	0.8686	4	0.28	66	72.118	0.8875
		0.075	5	0.27	345	352.343	0.8686	4	0.32	67	73.293	0.8876
		0.100	4	0.32	356	362.263	0.8686	3	0.39	70	75.163	0.8876
	0.25	0.050	6	0.25	377	385.484	0.8675	4	0.3	75	81.219	0.8856
		0.075	5	0.28	391	398.416	0.8675	4	0.32	77	83.271	0.8856
		0.100	4	0.32	404	410.288	0.8675	4	0.33	79	85.317	0.8856
0.05	0.15	0.050	5	0.25	180	187.25	0.8679	3	0.34	37	42.018	0.8862
		0.075	4	0.3	187	193.2	0.8679	3	0.34	38	43.006	0.8861
		0.100	4	0.31	192	198.239	0.8679	3	0.38	38	43.142	0.8864
	0.20	0.050	5	0.25	209	216.236	0.8668	3	0.36	43	48.084	0.8840
		0.075	5	0.28	215	222.415	0.8668	3	0.36	44	49.079	0.8840
		0.100	4	0.31	223	229.223	0.8668	3	0.36	45	50.08	0.8840

Table 3. *Cont.*

c_1			0.875					0.90				
α	β	p	m^*	t^*	n^*	TC^{**}	C_L^0	m^*	t^*	n^*	TC^{**}	C_L^0
	0.25	0.050	5	0.25	245	252.254	0.8656	3	0.37	51	56.098	0.8817
		0.075	4	0.31	254	260.244	0.8656	3	0.37	52	57.12	0.8817
		0.100	4	0.32	261	267.264	0.8656	3	0.38	53	58.143	0.8817
0.10	0.15	0.050	5	0.26	129	136.314	0.8666	3	0.34	27	32.019	0.8836
		0.075	4	0.32	134	140.282	0.8666	3	0.33	28	32.978	0.8834
		0.100	4	0.31	138	144.24	0.8666	3	0.36	28	33.067	0.8836
	0.20	0.050	6	0.24	153	161.421	0.8653	3	0.38	32	37.139	0.8813
		0.075	4	0.3	160	166.195	0.8653	3	0.36	33	38.069	0.8811
		0.100	4	0.31	164	170.26	0.8653	3	0.35	34	39.042	0.8810
	0.25	0.050	5	0.25	185	192.275	0.8641	3	0.38	39	44.132	0.8788
		0.075	4	0.31	192	198.245	0.8641	3	0.37	40	45.104	0.8787
		0.100	4	0.32	197	203.298	0.8641	3	0.36	41	46.093	0.8786

Table 4. The optimal (m^*,t^*,n^*), total cost TC and critical value for c_1 = 0.925, 0.95, α = 0.01, 0.05, 0.1, β = 0.25, 0.20, 0.15 and p = 0.05, 0.075, 0.1 under m_0 = 20, L = 0.05 and c_0 = 0.85.

c_1			0.925					0.95				
α	β	p	m^*	t^*	n^*	TC^{**}	C_L^0	m^*	t^*	n^*	TC^{**}	C_L^0
0.01	0.15	0.050	2	0.48	21	24.958	0.9097	2	0.52	8	12.047	0.9306
		0.075	3	0.35	20	25.043	0.9095	2	0.55	8	12.108	0.9310
		0.100	3	0.39	20	25.166	0.9099	2	0.61	8	12.22	0.9317
	0.20	0.050	3	0.33	23	27.988	0.9064	2	0.46	10	13.919	0.9254
		0.075	3	0.36	23	28.065	0.9067	2	0.47	10	13.943	0.9256
		0.100	3	0.4	23	28.192	0.9072	2	0.48	10	13.97	0.9259
	0.25	0.050	3	0.37	26	31.111	0.9040	2	0.61	11	15.213	0.9245
		0.075	3	0.35	27	32.054	0.9036	2	0.48	12	15.954	0.9216
		0.100	3	0.38	27	32.151	0.9040	2	0.49	12	15.978	0.9219
0.05	0.15	0.050	2	0.42	14	17.843	0.9042	1	0.74	7	9.736	0.9220
		0.075	2	0.43	14	17.866	0.9044	1	0.74	7	9.736	0.9220
		0.100	2	0.45	14	17.892	0.9044	1	0.74	7	9.736	0.9220
	0.20	0.050	2	0.47	16	19.947	0.9014	2	0.45	7	10.906	0.9181
		0.075	2	0.49	16	19.987	0.9017	2	0.46	7	10.926	0.9183
		0.100	2	0.52	16	20.042	0.9019	2	0.47	7	10.947	0.9185
	0.25	0.050	2	0.51	19	23.014	0.8984	2	0.53	8	12.067	0.9155
		0.075	2	0.54	19	23.083	0.8988	2	0.55	8	12.1	0.9159
		0.100	3	0.42	18	23.256	0.8987	2	0.57	8	12.137	0.9162
0.10	0.15	0.050	2	0.47	10	13.935	0.9009	2	0.54	4	8.085	0.9198
		0.075	2	0.49	10	13.971	0.9011	2	0.56	4	8.126	0.9201
		0.100	2	0.51	10	14.017	0.9013	2	0.59	4	8.173	0.9205
	0.20	0.050	2	0.51	12	16.017	0.8977	2	0.56	5	9.113	0.9152
		0.075	2	0.54	12	16.084	0.8980	2	0.58	5	9.152	0.9156
		0.100	2	0.44	13	16.885	0.8967	2	0.6	5	9.197	0.9159
	0.25	0.050	2	0.49	15	18.99	0.8938	2	0.47	7	10.938	0.9078
		0.075	2	0.52	15	19.032	0.8940	2	0.48	7	10.957	0.9080
		0.100	2	0.55	15	19.09	0.8943	2	0.49	7	10.977	0.9082

For example, suppose that the user wishes to conduct the level 0.05 hypothesis testing of $H_0 : C_L \leq 0.85$ under power of 0.75 at c_1 = 0.875, p = 0.05 and m_0 = 20. According to Table 3, the minimum required sample size is 245, the minimum number of inspection intervals is 5 and the inspection interval time length is 0.25. The total cost is calculated as TC = 252.254 and the critical value is 0.8656.

For any other setup of testing procedure, a software program is provided by the authors for users to input L, T, m, c_0, c_1, α, β to output the minimum required sample size n; or input L, T, c_0, c_1, α, β to output the minimum suggested number of inspections m; or input L, m, c_0, c_1, α, β to output the minimum recommended number of inspections m and the equal length of time t for each inspection interval.

According to Tables 3 and 4, the optimal number of inspection intervals m is nonincreasing when c_1 is increasing and the range of m is 2~6. The optimal length of inspection interval t^* is within 0.24 and 0.32 units of time for $c_1 = 0.875$. The value of t^* is within 0.28 and 0.39 units of time for $c_1 = 0.90$. The value of t^* is within 0.33 and 0.55 units of time for $c_1 = 0.925$. The value of t^* is within 0.45 and 0.74 units of time for $c_1 = 0.95$. In Figure 7, we plot the minimum total cost TC^* vs. c_1 for $\alpha = 0.01, 0.05, 0.1$ at $\beta = 0.25$ and $p = 0.05$. In Figure 8, we plot the minimum total cost TC^* vs. c_1 for $1 - \beta = 0.75, 0.80, 0.85$ at $\alpha = 0.1$ and $p = 0.05$. In Figure 9, we plot the minimum total cost TC^* vs. c_1 for $p = 0.05, 0.075, 0.1$ at $\alpha = 0.1$ and $\beta = 0.25$. From Figure 7, we find that the minimum total cost TC^{**} decreases when the value of α increases. From Figure 8, we find that the minimum total cost TC^{**} increases when the test power $1 - \beta$ increases. From Figure 9, we find that the minimum total cost TC^{**} becomes larger when the removal probability becomes larger. Overall, the minimum total cost TC^{**} becomes smaller when the value of c_1 grows larger.

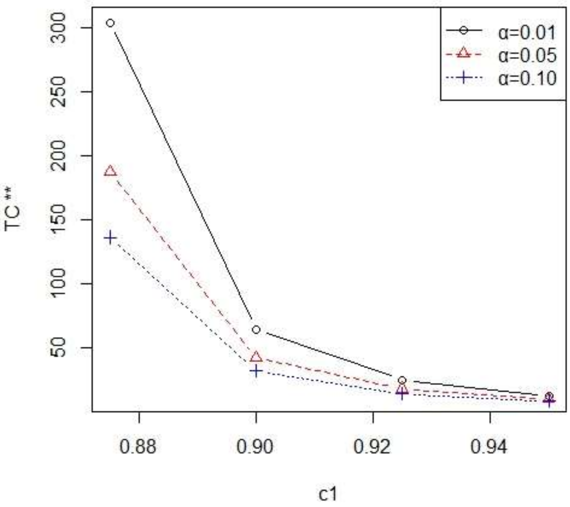


Figure 7. Minimum total cost curve for $\alpha = 0.01, 0.05, 0.1$ at $\beta = 0.25$ and $p = 0.05$.

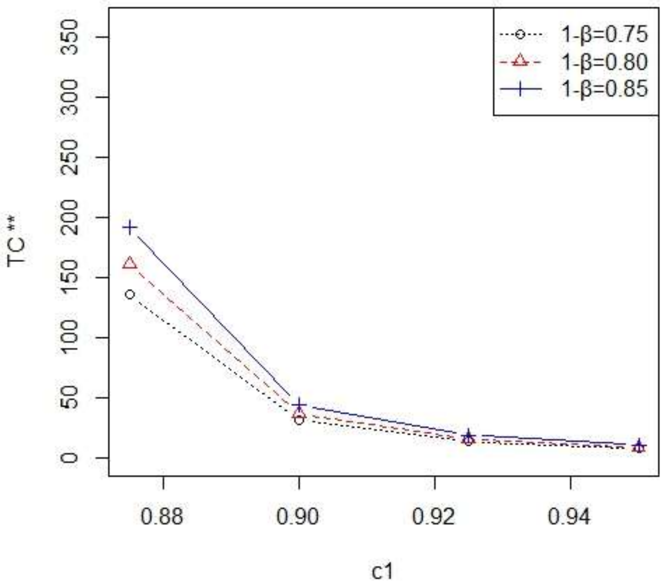


Figure 8. Minimum total cost curve for $1 - \beta = 0.75, 0.80, 0.85$ at $\alpha = 0.1$ and $p = 0.05$.

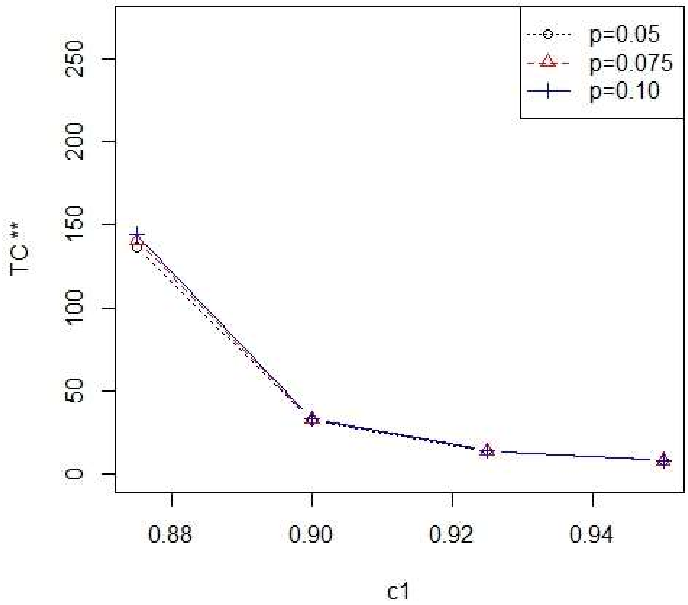


Figure 9. Minimum total cost curve for p = 0.05, 0.075, 0.1 at α = 0.1 and β = 0.25.

3.3. Example

We use the data in Caroni [15] consisting of the failure times (number of cycles in 1000 times) of n = 25 ball bearings in an automatic life test, listed as follows:

0.1788	0.2892	0.3300	0.4152	0.4212	0.4560	0.4848	0.5184	0.5196	0.5412
0.5556	0.6780	0.6780	0.6780	0.6864	0.6864	0.6888	0.8412	0.9312	0.9864
1.0512	1.0584	1.2792	1.2804	1.7340					

In Figure 10, we plot the empirical cumulative distribution function (ecdf) for this data set. The G test based on the Gini statistic (see Gail and Gastwirth [16]) is computed as 0.5052237 with p value 0.9293607 > 0.05. Therefore, we conclude that this data set fits the Rayleigh distribution well. We will now use this example to illustrate the implementation of the previous subsections.

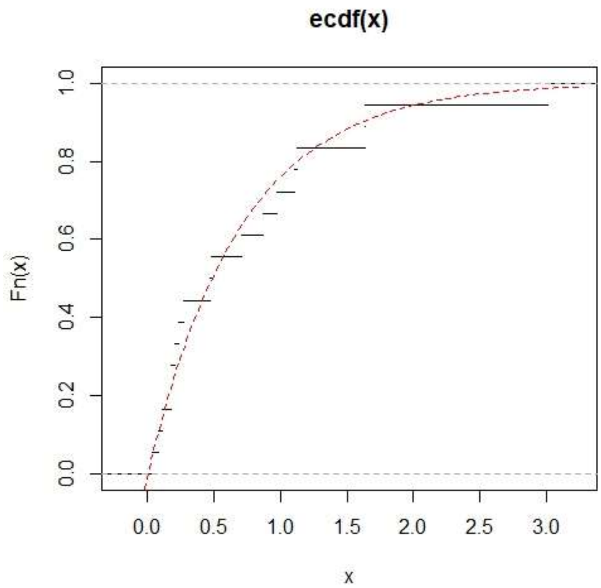


Figure 10. The ecdf for the data set in Caroni [15].

For Section 3.1, we consider the case of α = 0.01, the power 1 − β = 0.85 at c_1 = 0.95 under fixed T = 1.0. From Table 1, we can find that the optimal sampling design is m^* = 2, n^* = 12, yielding the minimum total cost of 20 units under the cost setup of C_o = 1 and a = b = c = 1. The critical value C_L^0 = 0.921702 is also found in this table.

We can thus start to perform the testing procedure for $H_0 : C_L \leq 0.85$ as follows:

Step 1: Take a random sample of size n = 12 from the data set. Observe the progressive type I interval censored sample (X_1,X_2) = (3,4) at the pre-set times (t_1,t_2) = (0.5,1.0) with censoring schemes of (R_1,R_2) = (2,3).

Step 2: Obtain the MLE of λ as $\hat{\lambda}$ = 0.6625991 and then obtain the test statistic $\hat{C}_L = 1 - \frac{0.05}{2(0\ .6625991)^2}$ = 0.9430573.

Step 3: Compared with the critical value, we have $\hat{C}_L$ = 0.9557158 > C_L^0 = 0.9042. Thus, we can conclude that the lifetime performance index of the product meets the required level of 0.85.

For Section 3.2, we consider the case of α = 0.05, the power 1 − β = 0.85 at c_1 = 0.925. From Table 4, we can find that the optimal sampling design is m^* = 2, n^* = 14 and t^* = 0.42, yielding the minimum total cost of 17.843 units under the cost setup of C_o = 1 and a = b = c = 1. The corresponding critical value C_L^0 =0.9042 is also found in this table.

We can thus start to perform the testing procedure for $H_0 : C_L \leq 0.85$ as follows:

Step 1: Take a random sample of size n = 14 from the data set. Observe the progressive type I interval censored sample (X_1,X_2) = (1,5) at the pre-set times (t_1,t_2) = (0.42,0.84) with censoring schemes of (R_1,R_2) = (2,6).

Step 2: Obtain the MLE of λ as $\hat{\lambda}$ = 0.7513559 and then obtain the test statistic $\hat{C}_L = 1 - \frac{0.05}{2(0\ .7513559)^2}$ = 0.9557158.

Step 3: Compared with the critical value, we have $\hat{C}_L$ = 0.9557158 > C_L^0 = 0.9042. Therefore, we reach the same conclusion and reject the null hypothesis.

4. Conclusions

Process capability indices are widely applied by manufacturers to evaluate the capability performance of a specific process when the lifetime of a product follows Rayleigh distribution. For many practical cases, a progressive type I interval censored sample is collected instead of a complete sample. We investigate the required minimum number of inspection intervals when the termination time of the experiment is fixed to reach given power and the minimum total cost for a level α test. When the termination time of the experiment is not fixed, the required minimum sample size, number of inspection intervals and the inspection interval time length are determined in this paper to reach given power and the minimum total cost for a level α test under progressive type I interval censoring. In the future, we plan to propose a testing procedure and study the experimental design for other lifetime distributions such as Lomax distribution. Other than the procreative type I interval sampling scheme, our studies can be applied to type II progressively hybrid censored samples.

Author Contributions: Conceptualization, S.-F.W.; methodology, S.-F.W.; software, S.-F.W., T.-H.L., Y.-H.L. and W.-T.C.; validation, T.-H.L., Y.-H.L. and W.-T.C.; formal analysis, S.-F.W.; investigation, S.-F.W., T.-H.L., Y.-H.L. and W.-T.C.; resources, S.-F.W.; data curation, S.-F.W., T.-H.L., Y.-H.L. and W.-T.C.; writing—original draft preparation, S.-F.W. and W.-T.C.; writing—review and editing, S.-F.W.; visualization, T.-H.L., Y.-H.L. and W.-T.C.; supervision, S.-F.W.; project administration, S.-F.W.; funding acquisition, S.-F.W. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: Data are available in a publicly accessible repository The data presented in this study are openly available in Caroni [15].

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Montgomery, D.C. *Introduction to Statistical Quality Control*; John Wiley and Sons Inc.: New York, NY, USA, 1985.
2. Balakrishnan, N.; Aggarwala, R. *Progressive Censoring: Theory, Methods and Applications*; Birkhäuser: Boston, MA, USA, 2000.
3. Aggarwala, R. Progressive interval censoring: Some mathematical results with applications to inference. *Commun. Stat.-Theory Methods* **2001**, *30*, 1921–1935. [CrossRef]
4. Lee, W.C.; Wu, J.W.; Hong, C.W. Assessing the lifetime performance index of products with the exponential distribution under progressively type II right censored samples. *J. Comput. Appl. Math.* **2009**, *231*, 648–656. [CrossRef]
5. Wu, J.W.; Lee, W.C.; Lin, L.S.; Hong, M.L. Bayesian test of lifetime performance index for exponential products based on the progressively type II right censored sample. *J. Quant. Manag.* **2011**, *8*, 57–77.
6. Lee, W.C.; Wu, J.W.; Hong, C.W.; Pan, H.Y.; Hung, W.L. Decision procedure of performance assessment of lifetime index of products for the Gompertz distribution. *Proc. Inst. Mech. Eng. Part B J. Eng. Manuf.* **2010**, *224*, 493–499. [CrossRef]
7. Lee, W.C.; Wu, J.W.; Hong, M.L.; Lin, L.S.; Chan, R.L. Assessing the lifetime performance index of Rayleigh products based on the Bayesian estimation under progressive type II right censored samples. *J. Comput. Appl. Math.* **2011**, *235*, 1676–1688. [CrossRef]
8. Wu, J.W.; Lee, W.C.; Hong, C.W.; Yeh, S.Y. Implementing Lifetime Performance Index of Burr XII Products with Progressively Type II Right Censored Sample. *Int. J. Innov. Comput. Inf. Control.* **2014**, *10*, 671–693.
9. Wu, S.F.; Hsieh, Y.T. The assessment on the lifetime performance index of products with Gompertz distribution based on the progressive type I interval censored sample. *J. Comput. Appl. Math.* **2019**, *351*, 66–76. [CrossRef]
10. Wu, S.F.; Xie, Y.J.; Liao, M.F.; Chang, W.T. Reliability sampling design for the lifetime performance index of Gompertz lifetime distribution under progressive type I interval censoring. *Mathematics* **2021**, *9*, 2109. [CrossRef]
11. Wu, S.F.; Lin, M.J. Computational testing algorithmic procedure of assessment for lifetime performance index of products with weibull distribution under progressive type I interval censoring. *J. Comput. Appl. Math.* **2017**, *311*, 364–374. [CrossRef]
12. Wu, S.F.; Wu, Y.C.; Wu, C.H.; Chang, W.T. Sampling design for the lifetime performance index of Weibull lifetime distribution under progressive type I interval censoring. *Symmetry* **2021**, *13*, 1691.
13. Wu, S.F.; Lin, Y.T.; Chang, W.J.; Chang, C.W.; Lin, C. A computational algorithm for the evaluation on the lifetime performance index of products with Rayleigh distribution under progressive type I interval censoring. *J. Comput. Appl. Math.* **2018**, *328*, 508–519. [CrossRef]
14. Huang, S.R.; Wu, S.J. Reliability sampling plans under progressive type-I interval censoring using cost functions. *IEEE Trans. Reliab.* **2008**, *57*, 445–451. [CrossRef]
15. Caroni, C. The correct "ball bearings" data. *Lifetime Data Anal.* **2002**, *8*, 395–399. [CrossRef] [PubMed]
16. Gail, M.H.; Gastwirth, J.L. A scale-free goodness of fit test for the exponential distribution based on the Gini Statistic. *J. R. Stat. Soc. B* **1978**, *40*, 350–357. [CrossRef]