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Minty Variational Principle for Nonsmooth Interval-Valued Vector Optimization Problems on Hadamard Manifolds

Savin Treanţă [1,*], Priyanka Mishra [2] and Balendu Bhooshan Upadhyay [2]

[1] Department of Applied Mathematics, University Politehnica of Bucharest, 060042 Bucharest, Romania
[2] Department of Mathematics, Indian Institute of Technology Patna, Patna 801103, Bihar, India; priyanka.iitp14@gmail.com (P.M.); bhooshan@iitp.ac.in (B.B.U.)
* Correspondence: savin.treanta@upb.ro

Abstract: This article deals with the classes of approximate Minty- and Stampacchia-type vector variational inequalities on Hadamard manifolds and a class of nonsmooth interval-valued vector optimization problems. By using the Clarke subdifferentials, we define a new class of functions on Hadamard manifolds, namely, the geodesic LU-approximately convex functions. Under geodesic LU-approximate convexity hypothesis, we derive the relationship between the solutions of these approximate vector variational inequalities and nonsmooth interval-valued vector optimization problems. This paper extends and generalizes some existing results in the literature.

Keywords: Clarke subdifferentials; geodesic LU-approximately convex functions; Hadamard manifolds

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1. Introduction

In traditional mathematical programming problems, the coefficients are usually always considered as deterministic values. However, in many real-world optimization problems, this assumption is not satisfied. Since the coefficients of a programming problem are either subject to errors of measurements and estimators or vary with market fluctuations, it is therefore always difficult to obtain exact data. In order to solve optimization problems, three different approaches are employed, namely, the stochastic optimization problem, deterministic optimization problem, and interval-valued optimization problem. In interval-valued optimization, the coefficients of the objective and constraint functions are compact intervals. For recent development and updated surveys of interval-valued optimization, we refer to the refs. [1–9]. The assumption or specification of probabilistic distribution (as in stochastic programming) or possible distribution (as in fuzzy programming) is not required for interval programming. Antczak [10] derived optimality and duality conditions for the nonsmooth interval-valued vector optimization problems.

Convexity is very restrictive notion for the solution of several real-world problems, for instance, mathematical economics. Luc et al. [11] defined the class of ϵ-convex functions in order to generalize the notion of convexity. The class of approximately convex functions was introduced by Ngai et al. [12] using ϵ-convexity. Daniilidis and Georgiev [13] established that a locally Lipschitz function is approximately convex if, and only if its Clarke's subdifferential is a submonotone operator. Ngai and Penot [12] derived several characterizations for approximate convex functions in terms of the generalized subdifferential. Amini-Harandi and Farajzadeh [14] extended and refined the results of Daniilidis and Georgiev [13] from Banach spaces to locally convex spaces.

Giannessi [15,16] introduced the vector versions of Minty [17] and Stampacchia [18] variational inequalities for finite dimensional Euclidean spaces. Since then, many researchers studied vector variational inequalities and their generalizations arduously as an efficient tool to find optimal solutions of vector optimization problems (see, for instance, the refs. [19–24] and the references cited therein). Németh [25] defined the notion of variational inequalities on Hadamard manifolds. Barani [26] proposed the concept of strong monotonicity for set-valued mappings and some notions of strong convexity for locally

Lipschitz functions on Hadamard manifolds. Chen and Huang [27] derived the relationship between convex vector optimization problems and vector variational inequalities using the Clarke subdifferential and proved certain existence theorems. Recently, Chen and Fang [28] established the relationship between Minty and Stampacchia vector variational inequalities and nonsmooth vector optimization problems under pseudoconvexity assumptions. Upadhyay and Mishra [29] studied the equivalence among approximate vector variational inequalities and interval-valued vector optimization problems involving approximate LU-pseudoconvex functions. For other ideas on this topic, the reader can consult Ceng et al. [30].

Pareto optimal solutions or efficient solutions have been extensively used in vector optimization problems. Due to the complexity of vector optimization problems, many researchers have been studying several variants of efficient solutions in recent years (see, for instance, the refs. [31–35] and the references cited therein). For the vector optimization problem, Loridan [36] introduced the notion of ϵ-efficient solutions. Mishra and Laha [37] introduced the concept of approximate efficient solution for vector optimization problems using approximately star-shaped functions. The characterization and applications of approximate efficient solutions of vector optimization problems have been studied by several authors (see, for instance, the refs. [33,36–38] and the references cited therein).

The paper is organized as follows: In Section 2, we provide a few definitions and preliminaries. We consider approximate Stampacchia and Minty vector variational inequalities in Section 3 and derive a relationship between the approximate efficient solutions of nonsmooth interval-valued vector optimization problems on Hadamard manifolds using LU-approximately convex functions. The results are summarized in Section 4.

2. Definition and Preliminaries

Let $\mathbb{R}^p$ be the p-dimensional Euclidean space, $\mathbb{R}^p_+$ be the non-negative orthant of $\mathbb{R}^p$ and 0 be the origin of the non-negative orthant. Let $\mathrm{int}(\mathbb{R}^p_+)$ be the positive orthant of $\mathbb{R}^p$. For $y, z \in \mathbb{R}^p$, the following notions for equality and inequalities will be used throughout the sequel:

(i) $z = y, \iff z_j = y_j, \forall j = 1, \dots, p;$
(ii) $z < y, \iff z_j < y_j, \forall j = 1, \dots, p;$
(iii) $z \leqq y, \iff z_j \leq y_j, \forall j = 1, \dots, p;$
(iv) $z \leq y, \iff z_j \leq y_j, \forall j = 1, \dots, p, j \neq l$, and $z_l < y_l$ for some l.

Now, we recall the notions of interval analysis from Moore [39,40]. Let $C = [c^L, c^U]$ denote a closed interval, where c^L and c^U denote the lower and upper bounds of C, respectively. Let $\mathcal{I}$ be the class of all closed intervals in $\mathbb{R}$. For $C = [c^L, c^U]$, $D = [d^L, d^U] \in \mathcal{I}$, we have

(i) $C + D = \{c + d : c \in C \text{ and } d \in D\} = [c^L + d^L, c^U + d^U];$
(ii) $-C = \{-c : c \in C\} = [-c^U, -c^L];$
(iii) $C \times D = \{cd : c \in C \text{ and } d \in D\} = [\min_{cd}, \max_{cd}],$

where $\min_{cd} = \min\{c^L d^L, c^L d^U, c^U d^L, c^U d^U\}$ and $\max_{cd} = \max\{c^L d^L, c^L d^U, c^U d^L, c^U d^U\}$.

Additionally, we have

$$C - D = C + (-D) = [c^L - d^U, c^U - d^L],$$

$$\alpha C = \{\alpha c : c \in C\} = \begin{cases} [\alpha c^L, \alpha c^U], & \alpha \geq 0, \\ [\alpha c^U, \alpha c^L], & \alpha < 0, \end{cases}$$

where $\alpha \in \mathbb{R}$. $c \in \mathbb{R}$ can be represented as the closed interval $C_c = [c, c]$.

Let $C = [c^L, c^U]$, $D = [d^L, d^U] \in \mathcal{I}$. Then, we define

1. $C \preceq_{LU} D \iff c^L \leq d^L$ and $c^U \leq d^U$,
2. $C \prec_{LU} D \iff C \preceq_{LU} D$ and $C \neq D$, that is, one of the following is satisfied:
   (a) $c^L < d^L$ and $c^U < d^U$; or

(b) $c^L \leq d^L$ and $c^U < d^U$; or
(c) $c^L < d^L$ and $c^U \leq d^U$.

Remark 1. *The intervals* $C = [c^L, c^U]$, $D = [d^L, d^U] \in \mathcal{I}$ *are comparable if and only if* $C \preceq_{LU} D$ *or* $C \succeq_{LU} D$.

Let $C_1, \ldots, C_p \in \mathcal{I}$ be closed intervals and $\mathbf{C} = (C_1, \ldots, C_p)$ denotes an interval-valued vector. For the interval-valued vectors $\mathbf{C}$ and $\mathbf{D}$, such that C_j and D_j are comparable for each $j = 1, \ldots, p$, we have

1. $\mathbf{C} \preceq_{LU} \mathbf{D}$ if and only if $C_j \preceq_{LU} D_j$ for all $j = 1, \ldots, p$;
2. $\mathbf{C} \prec_{LU} \mathbf{D}$ if and only if $C_j \preceq_{LU} D_j$ for all $j = 1, \ldots, p$, $j \neq l$ and $C_l \prec_{LU} D_l$ for some l.

The function $\Phi : \mathbb{R}^n \to \mathcal{I}$ is said to be an interval-valued function, where $\Phi(z) = [\Phi^L(z), \Phi^U(z)]$ and Φ^L, $\Phi^U : \mathbb{R}^n \to \mathbb{R}$ are real valued functions satisfying $\Phi^L(z) \leq \Phi^U(z)$, for all $z \in \mathbb{R}^n$.

The following notions of Riemannian manifolds are from [26,28].

Let H be a connected manifold with finite dimension m. For $z \in H$, $T_z H$ denotes the tangent space of H at z and $TH = \cup_{z \in H} T_z H$ denotes the tangent bundle of H. H is a Riemannian manifold endowed with a Riemannian metric $\langle ., . \rangle_p$ on the tangent space $T_z H$ with an associated norm denoted by $\|.\|_p$. Given a piecewise differentiable curve $\Omega : [a, b] \to H$ joining $\Omega(a) = p$ to $\Omega(b) = q$, the length of Ω is defined by

$$L(\Omega) := \int_a^b \|\Omega'(\mu)\|_p d\mu.$$

For any $p, q \in H$, the Riemannian distance between p and q is defined by $d(p, q) := \inf_\Omega L(\Omega)$, is the infimum over all piecewise differentiable curve joining p and q. This distance function d induces the original topology on H. On every Riemannian manifold, there exists exactly one covariant derivation called a Levi–Civita connection denoted by ∇. We also recall that a geodesic is a C^∞ smooth path Ω whose tangent is parallel along the path Ω, that is, Ω satisfies the equation

$$\nabla_{\frac{d\Omega(\mu)}{d\mu}} \frac{d\Omega(\mu)}{d\mu} = 0.$$

It is known that a Levi–Civita connection ∇ induces an isometry $P^{t_2}_{\mu_1, \Omega} : T_{\Omega(\mu_1)} H \to T_{\Omega(\mu_2)} H$, the so-called parallel translation along Ω from $\Omega(\mu_1)$ to $\Omega(\mu_2)$. Any path Ω joining p and q in H such that $L(\Omega) = d(p, q)$ is a geodesic, and it is called a minimal geodesic. If H is complete, then any points in H can be joined by a minimal geodesic.

In the following, let us suppose that H is complete. The exponential map $exp_z : T_z H \to H$ at z is defined by $exp_z v = \Omega_v(1, z)$, for every $v \in T_z H$, where $\Omega(.) = \Omega_v(., z)$ is the geodesic starting at z with velocity v, that is, $\Omega(0) = z$ and $\Omega'(0) = v$. It is easy to see that $exp_z(\mu v) = \Omega_v(\mu, z)$, for each real number μ. We note that the map exp_z is differentiable on $T_z H$, for every $z \in H$.

A simply connected complete Riemannian manifold with nonpositive sectional curvature is called a Hadamard manifold. If H is a Hadamard manifold, then $exp_z : T_p H \to H$ is a diffeomorphism for every $p \in H$, and if $z, y \in H$, then there exists a unique minimal geodesic joining z and y.

Now, we recall the following notions of nonsmooth analysis from Barani [26] and Hosseini and Pouryayevali [41].

Definition 1. *A nonempty subset* Γ *of* H *is said to be a geodesic convex set if for all* $z, y \in \Gamma$, *the geodesic joining* z *to* y *is contained in* Γ.

Definition 2. *Let function $\Phi : H \to]-\infty, \infty]$ be a proper function. The function Φ is said to be Lipschitz near $\bar{y} \in H$, if there exists a positive constant $L_{\bar{y}}$, and $\delta_z > 0$, such that*

$$|\Phi(z) - \Phi(y)| \leq L_{\bar{y}} d(z, y), \ \forall z, y \in B(\bar{y}, \delta_z),$$

where $B(\bar{y}, \delta_z) := \{z \in H : d(\bar{y}, z) < \delta_z\}$. Moreover, the function Φ is locally Lipschitz on H, if it is Lipschitz near $\bar{y}$, for any $\bar{y} \in H$.

From now onwards, let $\Gamma \subseteq H$ be an open geodesic convex subset of H and $\Phi : \Gamma \to \mathbb{R}$ be a locally Lipschitz function on Γ.

Definition 3. *The Clarke generalized directional derivative $\Phi^\circ(z; \nu)$ of Φ at $z \in \Gamma$, in the direction of a vector $\nu \in T_z H$, is defined as*

$$\Phi^\circ(z; \nu) := \limsup_{\substack{y \to z \\ \mu \downarrow 0}} \frac{\Phi o \phi^{-1}((\phi(y) + \mu d\phi(z)(\nu)) - \Phi o \phi^{-1}(\phi(y))}{\mu},$$

where (ϕ, U) is a chart at z. Indeed, $\Phi^\circ(z; \nu) = (\Phi o \phi^{-1})^\circ(\phi(z); d\phi(z)(\nu))$.

Definition 4. *The Clarke generalized subdifferential of Φ at $z \in \Gamma$, denoted by $\partial_c \Phi(z)$, is the subset of $T_z H^*$ defined by*

$$\partial_c \Phi(z) := \{\xi \in T_z H^* : \Phi^\circ(z; \nu) \geq \langle \xi, \nu \rangle, \forall \nu \in T_z H\}.$$

Definition 5. *(Lebourg Mean Value Theorem [26]) Let $z, y \in H$ and $\Omega : [0,1] \to H$ be a smooth path joining z and y. Let Φ be a locally Lipschitz function on $\Omega(\mu)$ for all $\mu \in [0,1]$. Then, there exist $0 < \mu_0 < 1$ and $\xi \in \partial_c \Phi(\Omega(\mu_0))$, such that*

$$\Phi(y) - \Phi(z) = \langle \xi, \Omega'(\mu_0) \rangle.$$

Now, we consider the following notions of geodesic approximate convexity and geodesic approximate monotonicity on Hadamard manifolds.

Let $A : H \to 2^{TH}$ be a multi-valued map such that $Az \subseteq T_z H$, for each $z \in H$, and the domain $D(A)$ of A is defined by

$$D(A) =: \{z \in H : A(z) \neq \emptyset\}.$$

Definition 6. *The function Φ is said to be geodesic approximately convex at $\bar{y} \in \Gamma$, if for any $\alpha > 0$, there exists $\delta > 0$, such that for each $z, y \in B(\bar{y}, \delta) \cap \Gamma$, we have*

$$\Phi(\Omega(\mu)) \leq \mu \Phi(y) + (1 - \mu)\Phi(z) + \alpha \mu (1 - \mu) \left\| \exp_z^{-1} y \right\|, \ \forall \mu \in [0, 1].$$

Definition 7. *Let $A : H \to 2^{TH}$ be a multi-valued map. Then A is said to be geodesic submonotone at $\bar{y} \in H$, if for every $\alpha > 0$, there exists $\delta \geq 0$, such that for every $z, y \in B(\bar{y}, \delta) \cap D(A)$, and for every $\xi \in A(z), \eta \in A(y)$, one has*

$$\left\langle P^0_{1,\Omega} \xi - \eta, \exp_y^{-1} z \right\rangle \geq -\alpha \left\| \exp_y^{-1} z \right\|,$$

where $\Omega(\mu) := \exp_y(\mu \exp_y^{-1} z), \mu \in [0, 1]$.

For locally Lipschitz geodesic approximately convex functions, we have the following characterization.

Theorem 1. *The function Φ is geodesic approximately convex at $\bar{y} \in \Gamma$ if and only if for every $\alpha > 0$, there exists $\delta > 0$, such that for any $z, y \in B(\bar{y}, \delta) \cap \Gamma$ and $\xi \in \partial_c \Phi(z)$, one has*

$$\Phi(y) - \Phi(z) \geq \left\langle \xi, exp_z^{-1} y \right\rangle - \alpha \left\| exp_z^{-1} y \right\|. \tag{1}$$

The following theorem establishes the relationship between a geodesic approximately convex function and geodesic submonotonicity of its Clarke subdifferential on Hadamard manifolds.

Theorem 2. *The function Φ is geodesic approximately convex at $\bar{y} \in \Gamma$ if and only if $\partial_c \Phi$ is geodesic submonotone at $\bar{y} \in \Gamma$.*

Definition 8. *A function $\Phi : \Gamma \to \mathcal{I}$ is said to be locally Lipschitz on Γ, if the real valued functions Φ^L and Φ^U are locally Lipschitz on Γ.*

Definition 9. *An interval-valued function $\Phi : \Gamma \to \mathcal{I}$ is geodesic LU-approximately convex at $\bar{y} \in \Gamma$ if the real-valued functions Φ^L and Φ^U are geodesic approximately convex at $\bar{y} \in \Gamma$.*

Consider the following nonsmooth interval-valued vector optimization problem:

$$\begin{array}{lll} \text{(NIVOP)} & \min & \mathbf{\Phi}(z), \\ & \text{subject to} & z \in \Gamma, \end{array}$$

where $\mathbf{\Phi}(z) = (\Phi_1(z), \ldots, \Phi_m(z))$ such that for each $j \in J := \{1, \ldots, m\}$, $\Phi_j(z) = [\Phi_j^L(z), \Phi_j^U(z)] : \Gamma \to \mathcal{I}$ is a locally Lipschitz interval-valued function.

Definition 10. *A point $\bar{y} \in \Gamma$ is said to be an approximate efficient solution of (NIVOP):*

(LUAES)$_1$, if for each $\alpha > 0$ sufficiently small, there does not exist $\delta > 0$ such that

$$\mathbf{\Phi}(z) \prec_{LU} \mathbf{\Phi}(\bar{y}) + \alpha \| exp_{\bar{y}}^{-1} z \| e, \ \forall z \in B(\bar{y}; \delta) \cap \Gamma, \ z \neq \bar{y};$$

(LUAES)$_2$, if for each $\alpha > 0$ sufficiently small, there exists $\delta > 0$ such that

$$\mathbf{\Phi}(z) \not\prec_{LU} \mathbf{\Phi}(\bar{y}) + \alpha \| exp_{\bar{y}}^{-1} z \| e, \ \forall z \in B(\bar{y}; \delta) \cap \Gamma;$$

(LUAES)$_3$, if for each $\alpha > 0$, there exists $\delta > 0$ such that

$$\mathbf{\Phi}(z) \not\prec_{LU} \mathbf{\Phi}(\bar{y}) - \alpha \| exp_{\bar{y}}^{-1} z \| e, \ \forall z \in B(\bar{y}; \delta) \cap \Gamma,$$

where $e = \underbrace{(1, \ldots, 1)}_{m \text{ times}}$.

Remark 2. *If $H = \mathbb{R}^m$, then $exp_z^{-1} y = y - z$. In this particular case, the notions of (LUAES)$_1$, (LUAES)$_2$ and (LUAES)$_3$ reduce to (ALUES)$_1$, (ALUES)$_2$ and (ALUES)$_3$, respectively, as considered by Upadhyay and Mishra [29].*

Now, we consider the following approximate Minty and Stampacchia vector variational inequalities which will be used in the sequel:

(AMVVI)$_1$ To find $\bar{y} \in \Gamma$ such that, for each $\alpha > 0$ sufficiently small, there does not exist $\delta > 0$ such that, for any $\xi_j^L \in \partial_c \Phi_j^L(z)$ and $\xi_j^U \in \partial_c \Phi_j^U(z)$, $j \in J$, one has

$$\left(\left\langle \xi_1^L, exp_z^{-1} \bar{y} \right\rangle, \ldots, \left\langle \xi_m^L, exp_z^{-1} \bar{y} \right\rangle \right) \geq \alpha \left\| exp_z^{-1} \bar{y} \right\| e,$$
$$\left(\left\langle \xi_1^U, exp_z^{-1} \bar{y} \right\rangle, \ldots, \left\langle \xi_m^U, exp_z^{-1} \bar{y} \right\rangle \right) \geq \alpha \left\| exp_z^{-1} \bar{y} \right\| e, \ \forall z \in B(\bar{y}; \delta) \cap \Gamma, \ z \neq \bar{y};$$

(AMVVI)$_2$ To find $\bar{y} \in \Gamma$ such that, for each sufficiently small $\alpha > 0$, there exists $\delta > 0$ such that, for any $\xi_j^L \in \partial_c \Phi_j^L(z)$ and $\xi_j^U \in \partial_c \Phi_j^U(z)$, $j \in J$, one has

$$\left(\left\langle \xi_1^L, exp_z^{-1}\bar{y}\right\rangle, \ldots, \left\langle \xi_m^L, exp_z^{-1}\bar{y}\right\rangle\right) \ngeq \alpha \left\| exp_z^{-1}\bar{y} \right\| e,$$
$$\left(\left\langle \xi_1^U, exp_z^{-1}\bar{y}\right\rangle, \ldots, \left\langle \xi_m^U, exp_z^{-1}\bar{y}\right\rangle\right) \ngeq \alpha \left\| exp_z^{-1}\bar{y} \right\| e, \ \forall z \in B(\bar{y};\delta) \cap \Gamma, \ z \neq \bar{y};$$

(AMVVI)$_3$ To find $\bar{y} \in \Gamma$ such that, for each $\alpha > 0$, there exists $\delta > 0$ such that, for any $\xi_j^L \in \partial_c \Phi_j^L(z)$ and $\xi_j^U \in \partial_c \Phi_j^U(z)$, $j \in J$, one has

$$\left(\left\langle \xi_1^L, exp_z^{-1}\bar{y}\right\rangle, \ldots, \left\langle \xi_m^L, exp_z^{-1}\bar{y}\right\rangle\right) \ngeq -\alpha \left\| exp_z^{-1}\bar{y} \right\| e,$$
$$\left(\left\langle \xi_1^U, exp_z^{-1}\bar{y}\right\rangle, \ldots, \left\langle \xi_m^U, exp_z^{-1}\bar{y}\right\rangle\right) \ngeq -\alpha \left\| exp_z^{-1}\bar{y} \right\| e, \ \forall z \in B(\bar{y};\delta) \cap \Gamma, \ z \neq \bar{y};$$

(ASVVI)$_1$ To find $\bar{y} \in \Gamma$ such that, for each sufficiently small $\alpha > 0$, there exist $z \in \Gamma$, $z \neq \bar{y}$, $\zeta_j^L \in \partial_c \Phi_j^L(\bar{y})$ and $\zeta_j^U \in \partial_c \Phi_j^U(\bar{y})$, $j \in J$, such that

$$\left(\left\langle \zeta_1^L, exp_{\bar{y}}^{-1}z\right\rangle, \ldots, \left\langle \zeta_m^L, exp_{\bar{y}}^{-1}z\right\rangle\right) \nleq \alpha \left\| exp_{\bar{y}}^{-1}z \right\| e,$$
$$\left(\left\langle \zeta_1^U, exp_{\bar{y}}^{-1}z\right\rangle, \ldots, \left\langle \zeta_m^U, exp_{\bar{y}}^{-1}z\right\rangle\right) \nleq \alpha \left\| exp_{\bar{y}}^{-1}z \right\| e;$$

(ASVVI)$_2$ To find $\bar{y} \in \Gamma$ such that, for each sufficiently small $\alpha > 0$ and for all $z \in \Gamma$, $\zeta_j^L \in \partial_c \Phi_j^L(\bar{y})$ and $\zeta_j^U \in \partial_c \Phi_j^U(\bar{y})$, $j \in J$, one has

$$\left(\left\langle \zeta_1^L, exp_{\bar{y}}^{-1}z\right\rangle, \ldots, \left\langle \zeta_m^L, exp_{\bar{y}}^{-1}z\right\rangle\right) \nleq \alpha \left\| exp_{\bar{y}}^{-1}z \right\| e,$$
$$\left(\left\langle \zeta_1^U, exp_{\bar{y}}^{-1}z\right\rangle, \ldots, \left\langle \zeta_m^U, exp_{\bar{y}}^{-1}z\right\rangle\right) \nleq \alpha \left\| exp_{\bar{y}}^{-1}z \right\| e;$$

(ASVVI)$_3$ To find $\bar{y} \in \Gamma$ such that, for each $\alpha > 0$, there exists $\delta > 0$ such that, for all $z \in B(\bar{y};\delta) \cap \Gamma$, $\zeta_j^L \in \partial_c \Phi_j^L(\bar{y})$ and $\zeta_j^U \in \partial_c \Phi_j^U(\bar{y})$, $j \in J$, one has

$$\left(\left\langle \zeta_1^L, exp_{\bar{y}}^{-1}z\right\rangle, \ldots, \left\langle \zeta_m^L, exp_{\bar{y}}^{-1}z\right\rangle\right) \nleq -\alpha \left\| exp_{\bar{y}}^{-1}z \right\| e,$$
$$\left(\left\langle \zeta_1^U, exp_{\bar{y}}^{-1}z\right\rangle, \ldots, \left\langle \zeta_m^U, exp_{\bar{y}}^{-1}z\right\rangle\right) \nleq -\alpha \left\| exp_{\bar{y}}^{-1}z \right\| e,$$

where $e = \underbrace{(1, \ldots, 1)}_{m \ times}$.

Remark 3. *If $H = \mathbb{R}^m$, then $exp_z^{-1}y = y - z$. In this particular case, the notions of (AMVVI)$_1$, (AMVVI)$_2$ and (AMVVI)$_3$ reduce to (AMVI)$_1$, (AMVI)$_2$ and (AMVI)$_3$, respectively, as considered by Upadhyay and Mishra [29].*

Remark 4. *In $H = \mathbb{R}^m$, then $exp_z^{-1}y = y - z$. In this particular case, the notions of (ASVVI)$_1$, (ASVVI)$_2$ and (ASVVI)$_3$ reduce to (ASVI)$_1$, (ASVI)$_2$ and (ASVI)$_3$, respectively, as considered by Upadhyay and Mishra [29].*

3. Relationship among (NIVOP), (AMVVI) and (ASVVI)

In this section, we derive some equivalence relations between the nonsmooth interval-valued vector optimization problem (NIVOP) and approximate vector variational inequalities (AMVVI)$_1$, (AMVVI)$_2$, (AMVVI)$_3$, (ASVVI)$_1$, (ASVVI)$_2$ and (ASVVI)$_3$ under geodesic LU-approximate convexity.

Theorem 3. *For each $j \in J$, let Φ_j be geodesic LU-approximately convex at $\bar{y} \in \Gamma$. Then, the following statements hold:*

(a) *If $\bar{y}$ is a (LUAES)$_1$ of the (NIVOP), then $\bar{y}$ is a solution of the (AMVVI)$_1$;*
(b) *If $\bar{y}$ is a (LUAES)$_2$ of the (NIVOP), then $\bar{y}$ is a solution of the (AMVVI)$_2$;*
(c) *If $\bar{y}$ is a solution of the (AMVVI)$_3$, then $\bar{y}$ is a (LUAES)$_3$ of the (NIVOP).*

Proof. (a) Assume that $\bar{y}$ is a (LUAES)$_1$ of (NIVOP), but it is not a solution of (AMVVI)$_1$. Then, for some sufficiently small $\alpha > 0$, there exists $\bar{\delta} > 0$, such that for any $z \in B(\bar{y};\bar{\delta}) \cap \Gamma$, $\xi_j^L \in \partial_c \Phi_j^L(z)$ and $\xi_j^U \in \partial_c \Phi_j^U(z)$, $j \in J$, one has

$$\begin{aligned}
&(\langle \xi_1^L, exp_z^{-1}\bar{y}\rangle, \ldots, \langle \xi_m^L, exp_z^{-1}\bar{y}\rangle) \geq \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|e,\\
&(\langle \xi_1^U, exp_z^{-1}\bar{y}\rangle, \ldots, \langle \xi_m^U, exp_z^{-1}\bar{y}\rangle) \geq \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|e,
\end{aligned}$$

that is,

$$\begin{aligned}
&\left\langle \xi_j^L, exp_z^{-1}\bar{y}\right\rangle - \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\| \geq 0\\
&\left\langle \xi_j^U, exp_z^{-1}\bar{y}\right\rangle - \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\| \geq 0, \ \forall j \in J, \ j \neq k,\\
&\text{and}\\
&\langle \xi_k^L, exp_z^{-1}\bar{y}\rangle - \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\| > 0\\
&\langle \xi_k^U, exp_z^{-1}\bar{y}\rangle - \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\| > 0, \ \text{for some } k \in J.
\end{aligned} \tag{2}$$

Since each Φ_j, $j \in J$ is geodesic LU-approximately convex at $\bar{y} \in \Gamma$, it follows that, for any $\alpha > 0$, there exists $\hat{\delta} > 0$ such that, for every $z \in B(\bar{y};\hat{\delta}) \cap \Gamma$, $\xi_j^L \in \partial_c \Phi_j^L(z)$ and $\xi_j^U \in \partial_c \Phi_j^U(z)$, $j \in J$, we get

$$\begin{aligned}
&\Phi_j^L(\bar{y}) - \Phi_j^L(z) \geq \langle \xi_j^L, exp_z^{-1}\bar{y}\rangle - \alpha\|exp_z^{-1}\bar{y}\|,\\
&\Phi_j^U(\bar{y}) - \Phi_j^U(z) \geq \langle \xi_j^U, exp_z^{-1}\bar{y}\rangle - \alpha\|exp_z^{-1}\bar{y}\|, \ \forall j \in J.
\end{aligned} \tag{3}$$

By setting $\tilde{\delta} := \min\{\bar{\delta}, \hat{\delta}\}$, from (2) and (3), it follows that for some sufficiently small $\alpha > 0$, there exists $\tilde{\delta} > 0$ such that, for each $z \in B(\bar{y};\tilde{\delta}) \cap \Gamma$, we obtain

$$\mathbf{\Phi}(z) - \mathbf{\Phi}(\bar{y}) \prec_{LU} \frac{\alpha}{2}\|exp_z^{-1}\bar{y}\|e,$$

and this contradicts that $\bar{y}$ is an (LUAES)$_1$ of (NIVOP).

(b) On the contrary, suppose that $\bar{y}$ does not solve (AMVVI)$_2$. Then, there exists sufficiently small $\alpha > 0$, such that for all $\delta > 0$, there exists $z \in B(\bar{y};\bar{\delta}) \cap \Gamma$, $\xi_j^L \in \partial_c \Phi_j^L(z)$ and $\xi_j^U \in \partial_c \Phi_j^U(z)$, $j \in J$, we have

$$\begin{aligned}
&(\langle \xi_1^L, exp_z^{-1}\bar{y}\rangle, \ldots, \langle \xi_m^L, exp_z^{-1}\bar{y}\rangle) \geq \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|e,\\
&(\langle \xi_1^U, exp_z^{-1}\bar{y}\rangle, \ldots, \langle \xi_m^U, exp_z^{-1}\bar{y}\rangle) \geq \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|e,
\end{aligned} \tag{4}$$

that is,

$$\begin{aligned}
&\langle \xi_j^L, exp_z^{-1}\bar{y}\rangle - \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\| \geq 0,\\
&\langle \xi_j^U, exp_z^{-1}\bar{y}\rangle - \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\| \geq 0, \ \forall j \in J, \ j \neq k,\\
&\text{and}\\
&\langle \xi_k^L, exp_z^{-1}\bar{y}\rangle - \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\| > 0,\\
&\langle \xi_k^U, exp_z^{-1}\bar{y}\rangle - \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\| > 0, \ \text{for some } k \in J.
\end{aligned} \tag{5}$$

Since each Φ_j, $j \in J$ is geodesic LU-approximately convex at $\bar{y} \in \Gamma$, it follows that, for any $\alpha > 0$, there exists $\hat{\delta} > 0$ such that, for every $z \in B(\bar{y};\hat{\delta}) \cap \Gamma$, $\xi_j^L \in \partial_c \Phi_j^L(z)$ and $\xi_j^U \in \partial_c \Phi_j^U(z)$, $j \in J$, one has

$$\begin{aligned} \Phi_j^L(\bar{y}) - \Phi_j^L(z) &\geq \langle \xi_j^L, exp_z^{-1}\bar{y} \rangle - \alpha \| exp_z^{-1}\bar{y} \|, \\ \Phi_j^U(\bar{y}) - \Phi_j^U(z) &\geq \langle \xi_j^U, exp_z^{-1}\bar{y} \rangle - \alpha \| exp_z^{-1}\bar{y} \|, \ \forall j \in J. \end{aligned} \tag{6}$$

By setting $\tilde{\delta} := \min\{\bar{\delta}, \hat{\delta}\}$, from (5) and (6), it follows that for some sufficiently small $\alpha > 0$, and for all $\tilde{\delta} > 0$, there exists $z \in B(\bar{y};\tilde{\delta}) \cap \Gamma$, such that

$$\mathbf{\Phi}(z) - \mathbf{\Phi}(\bar{y}) \prec_{LU} \frac{\alpha}{2} \| exp_z^{-1}\bar{y} \| e,$$

which is in contradiction to the fact that the $\bar{y}$ is an $(\text{LUAES})_2$ of (NIVOP).

(c) Suppose that $\bar{y} \in \Gamma$ is a solution of $(\text{AMVVI})_3$ but not a $(\text{LUAES})_3$ of (NIVOP); then, for some $\alpha > 0$ and for each $\delta > 0$, there exists $z \in B(\bar{y}, \delta) \cap \Gamma$, such that

$$\mathbf{\Phi}(z) - \mathbf{\Phi}(\bar{y}) \prec_{LU} -\frac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \| e,$$

that is,

$$\begin{aligned} &\Phi_j(z) - \Phi_j(\bar{y}) \preceq_{LU} -\tfrac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \|, \ \forall j \in J, \ j \neq k, \\ &\Phi_k(z) - \Phi_k(\bar{y}) \prec_{LU} -\tfrac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \|, \ \text{for some } k \in J. \end{aligned} \tag{7}$$

From (7) it follows that

$$\begin{aligned} &\Phi_j^L(z) - \Phi_j^L(\bar{y}) < \tfrac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \|, \\ &\Phi_j^U(z) - \Phi_j^U(\bar{y}) < \tfrac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \|, \ \forall j \in J. \end{aligned} \tag{8}$$

Since each Φ_j, $j \in J$ is a geodesic approximately convex function at $\bar{y} \in \Gamma$, therefore for any $\alpha > 0$, there exists $\delta > 0$, such that for every $z \in B(\bar{y}, \bar{\delta}) \cap \Gamma$, we have

$$\begin{aligned} &\Phi_j^L(\Omega(\mu)) \leq \mu \Phi_j^L(z) + (1-\mu)\Phi_j^L(\bar{y}) + \tfrac{\alpha}{2}\mu(1-\mu) \| exp_{\bar{y}}^{-1} z \|, \\ &\Phi_j^U(\Omega(\mu)) \leq \mu \Phi_j^U(z) + (1-\mu)\Phi_j^U(\bar{y}) + \tfrac{\alpha}{2}\mu(1-\mu) \| exp_{\bar{y}}^{-1} z \|, \ \forall \mu \in [0,1], j \in J, \end{aligned}$$

that is,

$$\begin{aligned} &\Phi_j^L(\Omega(\mu)) - \Phi_j^L(\bar{y}) \leq \mu[\Phi_j^L(z) - \Phi_j^L(\bar{y}) + \tfrac{\alpha}{2}(1-\mu) \| exp_{\bar{y}}^{-1} z \|], \\ &\Phi_j^U(\Omega(\mu)) - \Phi_j^U(\bar{y}) \leq \mu[\Phi_j^U(z) - \Phi_j^U(\bar{y}) + \tfrac{\alpha}{2}(1-\mu) \| exp_{\bar{y}}^{-1} z \|], \ \forall j \in J, \end{aligned} \tag{9}$$

where $\Omega(\mu) = exp_{\bar{y}}(\mu exp_{\bar{y}}^{-1} z)$ is a geodesic joining $\bar{y}$ to z. Define $\eta : [0,1] \to H$ as

$$\eta(s) = \Omega(s\mu), \quad \forall s \in [0,1].$$

By Lebourg mean value theorem, there exists $l_j, l_j^* \in (0,\mu)$, $\xi_j^L \in \partial_c \Phi_j^L(\eta(l_j))$ and $\xi_j^U \in \partial_c \Phi_j^U(\eta(l_j^*))$, such that

$$\begin{aligned} \Phi_j^L(\Omega(\mu)) - \Phi_j^L(\bar{y}) &= \langle \xi_j^L, \eta'(a_j) \rangle = \mu \langle \xi_j^L, \Omega'(a_j) \rangle, \\ \Phi_j^U(\Omega(\mu)) - \Phi_j^U(\bar{y}) &= \langle \xi_j^U, \eta'(a_j^*) \rangle = \mu \langle \xi_j^U, \Omega'(a_j^*) \rangle, \ \forall j \in J, \end{aligned} \tag{10}$$

where $a_j = l_j \mu < \mu$ and $a_j^* = l_j^* \mu < \mu$.

From (9) and (10), we get

$$\begin{aligned}&\langle \xi_j^L, \Omega'(a_j)\rangle \le \Phi_j^L(z) - \Phi_j^L(\bar{y}) + \tfrac{\alpha}{2}(1-\mu)\|exp_{\bar{y}}^{-1}z\|,\\ &\langle \xi_j^U, \Omega'(a_j^*)\rangle \le \Phi_j^U(z) - \Phi_j^U(\bar{y}) + \tfrac{\alpha}{2}(1-\mu)\|exp_{\bar{y}}^{-1}z\|,\ \forall j \in J.\end{aligned} \tag{11}$$

From (8) and (11), it follows that

$$\begin{aligned}&\langle \xi_j^L, \Omega'(a_j)\rangle < \alpha(1-\tfrac{\mu}{2})\|exp_{\bar{y}}^{-1}z\|,\\ &\langle \xi_j^U, \Omega'(a_j^*)\rangle < \alpha(1-\tfrac{\mu}{2})\|exp_{\bar{y}}^{-1}z\|,\end{aligned} \tag{12}$$

where $\xi_j^L \in \partial_c\Phi_j^L(\Omega(a_j))$ and $\xi_j^U \in \partial_c\Phi_j^U(\Omega(a_j^*)), j \in J$.
Choosing $\mu_0 < \min\{a_1, \dots, a_m, a_1^*, \dots, a_m^*\}$ we have

$$exp_{\Omega(a_j)}^{-1}\Omega(\mu_0) = (\mu_0 - a_j)\Omega'(a_j) = \frac{a_j - \mu_0}{a_j} exp_{\Omega(a_j)}^{-1}\bar{y},\ \forall j \in J, \tag{13}$$

$$exp_{\Omega(a_j^*)}^{-1}\Omega(\mu_0) = (\mu_0 - a_j^*)\Omega'(a_j^*) = \frac{a_j^* - \mu_0}{a_j^*} exp_{\Omega(a_j^*)}^{-1}\bar{y},\ \forall j \in J, \tag{14}$$

$$exp_{\Omega(\mu_0)}^{-1}\Omega(a_j) = (a_j - \mu_0)\Omega'(\mu_0) = \frac{\mu_0 - a_j}{\mu_0} exp_{\Omega(\mu_0)}^{-1}\bar{y},\ \forall j \in J, \tag{15}$$

$$exp_{\Omega(\mu_0)}^{-1}\Omega(a_j^*) = (a_j^* - \mu_0)\Omega'(\mu_0) = \frac{\mu_0 - a_j^*}{\mu_0} exp_{\Omega(\mu_0)}^{-1}\bar{y},\ \forall j \in J, \tag{16}$$

and

$$\|exp_{\bar{y}}^{-1}z\| = \|exp_z^{-1}\bar{y}\| = \frac{1}{\mu_0}\|exp_{\Omega(\mu_0)}^{-1}\bar{y}\|. \tag{17}$$

From (12)–(14) we get

$$\begin{aligned}&\langle \xi_j^L, exp_{\Omega(a_j)}^{-1}\Omega(\mu_0)\rangle < \alpha(\mu_0 - a_j)(1-\tfrac{\mu}{2})\|exp_{\bar{y}}^{-1}z\|,\\ &\langle \xi_j^U, exp_{\Omega(a_j^*)}^{-1}\Omega(\mu_0)\rangle < \alpha(\mu_0 - a_j^*)(1-\tfrac{\mu}{2})\|exp_{\bar{y}}^{-1}z\|,\ \forall j \in J.\end{aligned} \tag{18}$$

Since each Φ_j^L and $\Phi_j^U, j \in J$ are geodesic approximately convex at $\bar{y} \in \Gamma$, it follows that $\partial_c\Phi_j^L$ and $\partial_c\Phi_j^U$, $j \in J$ is geodesic submonotone at $\bar{y}$. Hence, for any $\alpha > 0$, there exists $\delta > 0$, such that for all $z \in B(\bar{y}, \delta) \cap \Gamma$, we have

$$\begin{aligned}&\langle P_{a_j,\Omega}^{\mu_0}\zeta_j^L - \xi_j^L, exp_{\Omega(a_j)}^{-1}\Omega(\mu_0)\rangle \ge -\alpha\|exp_{\Omega(a_j)}^{-1}\Omega(\mu_0)\|,\\ &\langle P_{a_j^*,\Omega}^{\mu_0}\zeta_j^U - \xi_j^U, exp_{\Omega(a_j^*)}^{-1}\Omega(\mu_0)\rangle \ge -\alpha\|exp_{\Omega(a_j^*)}^{-1}\Omega(\mu_0)\|,\end{aligned} \tag{19}$$

for all $\xi_j^L \in \partial_c\Phi_j^L(\Omega(a_j))$ and $\zeta_j^L \in \partial_c\Phi_j^L(\Omega(\mu_0))$, $\xi_j^U \in \partial_c\Phi_j^U(\Omega(a_j^*))$ and $\zeta_j^U \in \partial_c\Phi_j^U(\Omega(\mu_0))$, $j \in J$.
Therefore, from (17)–(19), we have

$$\left\langle P^{a_j}_{\mu_0,\Omega}(P^{\mu_0}_{a_j,\Omega}\zeta^L_j), P^{a_j}_{\mu_0,\Omega}\exp^{-1}_{\Omega_{(a_j)}}\Omega(\mu_0)\right\rangle > \frac{\alpha}{\mu_0}(\mu_0-a_j)(1-\frac{\mu}{2})\left\|exp^{-1}_{\Omega(\mu_0)}\bar{y}\right\| - \alpha\left\|\exp^{-1}_{\Omega(a_j)}\Omega(\mu_0)\right\|,$$
$$\left\langle P^{a^*_j}_{\mu_0,\Omega}(P^{\mu_0}_{a^*_j,\Omega}\zeta^U_j), P^{a^*_j}_{\mu_0,\Omega}\exp^{-1}_{\Omega_{(a^*_j)}}\Omega(\mu_0)\right\rangle > \frac{\alpha}{\mu_0}(\mu_0-a^*_j)(1-\frac{\mu}{2})\left\|exp^{-1}_{\Omega(\mu_0)}\bar{y}\right\| - \alpha\left\|\exp^{-1}_{\Omega(a^*_j)}\Omega(\mu_0)\right\|,$$

for all $\zeta^L_j \in \partial_c\Phi^L_j(\Omega(\mu_0))$ and $\zeta^U_j \in \partial_c\Phi^U_j(\Omega(\mu_0))$, $j \in J$,
or equivalently,

$$\begin{aligned}\left\langle \zeta^L_j, -\exp^{-1}_{\Omega_{(\mu_0)}}\Omega(a_j)\right\rangle &> \tfrac{\alpha}{\mu_0}(\mu_0-a_j)(1-\tfrac{\mu}{2})\left\|exp^{-1}_{\Omega(\mu_0)}\bar{y}\right\| - \alpha\left\|\exp^{-1}_{\Omega(a_j)}\Omega(\mu_0)\right\|,\\ \left\langle \zeta^U_j, -\exp^{-1}_{\Omega_{(\mu_0)}}\Omega(a^*_j)\right\rangle &> \tfrac{\alpha}{\mu_0}(\mu_0-a^*_j)(1-\tfrac{\mu}{2})\left\|exp^{-1}_{\Omega(\mu_0)}\bar{y}\right\| - \alpha\left\|\exp^{-1}_{\Omega(a^*_j)}\Omega(\mu_0)\right\|,\end{aligned} \tag{20}$$

for all $j \in J$.
In view of (17), Equation (20) reduces to

$$\begin{aligned}\left\langle \zeta^L_j, \exp^{-1}_{\Omega_{(\mu_0)}}\bar{y}\right\rangle &> -\alpha(2-\tfrac{\mu}{2})\left\|exp^{-1}_{\Omega(\mu_0)}\bar{y}\right\|,\\ \left\langle \zeta^U_j, \exp^{-1}_{\Omega_{(\mu_0)}}\bar{y}\right\rangle &> -\alpha(2-\tfrac{\mu}{2})\left\|exp^{-1}_{\Omega(\mu_0)}\bar{y}\right\|,\ \forall j \in J.\end{aligned} \tag{21}$$

This contradicts the assumption that $\bar{y}$ is a solution of $(\mathrm{AMVVI})_3$. The proof is complete. □

To illustrate the significance of Theorem 3, we consider the following interval-valued vector optimization problem on a Hadamard manifold.

Example 1.

$$(P) \qquad min\, \mathbf{\Phi}(z) = (\Phi_1(z), \Phi_2(z)),$$
$$\text{subject to } z \in \Gamma \subseteq H,$$

where $\Phi_1, \Phi_2 : \Gamma \to \mathcal{I}$ *are interval-valued functions defined on* $\Gamma = \{y : y = e^{\mu}, \mu \in [-2, 0]\}$ *and* $H = \{z \in \mathbb{R} : z > 0\}$ *is the Riemannian manifold with Riemannian metric* $\langle u, v\rangle = g(z)uv$ *with* $g : H \to]0, \infty[$ *and sectional curvature* $\kappa = 0$. *It is clear that the set* Γ *is a geodesic convex set.*

The tangent plane at any point $z \in H$, *denoted by* T_zH, *equals* $\mathbb{R}$. *The Riemannian distance function* $d : H \times H \to \mathbb{R}$ *is given by*

$$d(z, y) = \left\|exp^{-1}_z y\right\| = \left|ln\frac{z}{y}\right|.$$

The geodesic curve $\Omega : \mathbb{R} \to H$ *starting from* $\Omega(0) = z$ *and with a tangent unit vector* $\Omega' = w \in T_zH$ *of* Ω *at the starting point* z *is given by*

$$\Omega(\mu) = exp_z(\mu w) = ze^{(\frac{w}{z})\mu}.$$

The inverse of an exponential map for any $z, y \in H$ *is given by*

$$exp^{-1}_z y = z\, ln\,(\frac{y}{z}).$$

Consider the functions $\Phi_1^L, \Phi_1^U, \Phi_2^L, \Phi_2^U : \Gamma \to \mathbb{R}$ *are given by*

$$\Phi_1^L(z) = \begin{cases} z - \frac{5}{4}, & z \geq \frac{1}{2} \\ z^2 - 2z, & z < \frac{1}{2} \end{cases} \quad \text{and} \quad \Phi_1^U(z) = \begin{cases} 2z - 1, & z \geq \frac{1}{2} \\ 2z^2 - z, & z < \frac{1}{2} \end{cases}.$$

$$\Phi_2^L(z) = \begin{cases} z^2 + z, & z \geq \frac{1}{2} \\ -z + \frac{5}{4}, & z < \frac{1}{2} \end{cases} \quad \text{and} \quad \Phi_2^U(z) = \begin{cases} z^2 + 2z, & z \geq \frac{1}{2} \\ -z + \frac{7}{4}, & z < \frac{1}{2} \end{cases}.$$

It is clear that the functions Φ_1 *and* Φ_2 *are locally Lipschitz functions on* Γ. *The subdifferentials of* Φ_1^L, Φ_1^U, Φ_2^L *and* Φ_2^U *are given by*

$$\partial_c\Phi_1^L(z) = \begin{cases} z^2, & z > \frac{1}{2} \\ [-\frac{1}{4}, \frac{1}{4}], & z = \frac{1}{2} \\ 2z^3 - 2z^2, & z < \frac{1}{2} \end{cases} \quad \text{and} \quad \partial_c\Phi_1^U(z) = \begin{cases} 2z^2, & z > \frac{1}{2} \\ [\frac{1}{4}, \frac{1}{2}], & z = \frac{1}{2} \\ 4z^3 - z^2, & z < \frac{1}{2} \end{cases}.$$

$$\partial_c\Phi_2^L(z) = \begin{cases} 2z^3 + z^2, & z > \frac{1}{2} \\ [-\frac{1}{4}, \frac{1}{2}], & z = \frac{1}{2} \\ -z^2, & z < \frac{1}{2} \end{cases} \quad \text{and} \quad \partial_c\Phi_1^U(z) = \begin{cases} 2z^3 + 2z^2, & z > \frac{1}{2} \\ [-\frac{1}{4}, \frac{3}{4}], & z = \frac{1}{2} \\ -z^2, & z < \frac{1}{2} \end{cases}.$$

We can verify that the function Φ_1 *is geodesic LU-approximately convex at* $\bar{y} = \frac{1}{2}$, *as for each* $\alpha > 0$, *we can get* $0 < \delta_1 < \frac{-1+\sqrt{1+1.5\alpha}}{3}$ *such that*

$$\Phi_1^L(z) - \Phi_1^L(\bar{y}) \geq \langle \xi_1^L, exp_{\bar{y}}^{-1}z \rangle - \alpha \| exp_{\bar{y}}^{-1}z \|, \; \forall \xi_1^L \in \partial_c\Phi_1^L(\bar{y}),$$
$$\Phi_1^U(z) - \Phi_1^U(\bar{y}) \geq \langle \xi_1^U, exp_{\bar{y}}^{-1}z \rangle - \alpha \| exp_{\bar{y}}^{-1}z \|, \; \forall \xi_1^U \in \partial_c\Phi_1^U(\bar{y}).$$

Similarly, the function Φ_2 *is geodesic LU-approximately convex at* $\bar{y} = \frac{1}{2}$, *as for each* $\alpha > 0$ *we can get* $\delta_2 \in (0, 1)$ *such that*

$$\Phi_2^L(z) - \Phi_2^L(\bar{y}) \geq \langle \xi_2^L, exp_{\bar{y}}^{-1}z \rangle - \alpha \| exp_{\bar{y}}^{-1}z \|, \; \forall \xi_2^L \in \partial_c\Phi_2^L(\bar{y}),$$
$$\Phi_2^U(z) - \Phi_2^U(\bar{y}) \geq \langle \xi_2^U, exp_{\bar{y}}^{-1}z \rangle - \alpha \| exp_{\bar{y}}^{-1}z \|, \; \forall \xi_2^U \in \partial_c\Phi_2^U(\bar{y}).$$

Moreover, $\bar{y} = \frac{1}{2}$ *is an (LUAES)*$_1$ *of the problem (P). Since, for each sufficiently small* $\alpha > 0$, *there does not exist any* $\delta > 0$, *such that for all* $z \in B(\bar{y}, \delta) \cap \Gamma$, *we have*

$$\mathbf{\Phi}(z) - \mathbf{\Phi}(\bar{y}) \not\prec_{LU} \alpha \| exp_{\bar{y}}^{-1}z \| e.$$

Similarly, $\bar{y} = \frac{1}{2}$ *is an (LUAES)*$_2$ *of the problem (P). Since, for each sufficiently small* $\alpha > 0$, *for all* $\delta > 0$, *there does not exist any* $z \in B(\bar{y}, \delta) \cap \Gamma$, *such that*

$$\mathbf{\Phi}(z) - \mathbf{\Phi}(\bar{y}) \prec_{LU} \alpha \| exp_{\bar{y}}^{-1}z \| e.$$

Furthermore, $\bar{y} = \frac{1}{2}$ *solves (AMVVI)*$_1$. *Since, for any sufficiently small* $\alpha > 0$, *there does not exist any* $\delta > 0$, *such that for all* $z \in B(\bar{y}, \delta) \cap \Gamma$, *we have*

$$(\langle \xi_1^L, exp_z^{-1}\bar{y} \rangle, \langle \xi_2^L, exp_z^{-1}\bar{y} \rangle) \ngeq \alpha \| exp_z^{-1}\bar{y} \| e,$$
$$(\langle \xi_1^U, exp_z^{-1}\bar{y} \rangle, \langle \xi_2^U, exp_z^{-1}\bar{y} \rangle) \ngeq \alpha \| exp_z^{-1}\bar{y} \| e,$$

for all $\xi_j^L \in \partial_c\Phi_j^L(z)$ *and* $\xi_j^U \in \partial_c\Phi_j^U(z)$, $j = 1, 2$.

Similarly, $\bar{y} = \frac{1}{2}$ *solves (AMVVI)*$_2$. *Since, for each* $\delta > 0$ *and sufficiently small* $\alpha > 0$, *there does not exist any* $z \in B(\bar{y}, \delta) \cap \Gamma$, *such that*

$$(\langle \xi_1^L, exp_z^{-1}\bar{y} \rangle, \langle \xi_2^L, exp_z^{-1}\bar{y} \rangle) \geq \alpha \| exp_z^{-1}\bar{y} \| e,$$
$$(\langle \xi_1^U, exp_z^{-1}\bar{y} \rangle, \langle \xi_2^U, exp_z^{-1}\bar{y} \rangle) \geq \alpha \| exp_z^{-1}\bar{y} \| e,$$

for all $\xi_j^L \in \partial_c \Phi_j^L(z)$ *and* $\xi_j^U \in \partial_c \Phi_j^U(z)$, $j = 1, 2$.

Theorem 4. *For each* $j \in J$, *let* Φ_j *be geodesic LU-approximately convex at* $\bar{y} \in \Gamma$. *Then, the following statements hold:*

1. *If for some* $k \in J$, Φ_k^L *and* Φ_k^U *are strictly geodesic approximately convex at* $\bar{y} \in \Gamma$ *and* $\bar{y}$ *is a solution for the (ASVVI)*$_1$, *then* $\bar{y}$ *is an (LUAES)*$_1$ *of the (NIVOP);*
2. *If* $\bar{y}$ *is a solution the (ASVVI)*$_2$, *then* $\bar{y}$ *is an (LUAES)*$_2$ *of the (NIVOP);*
3. *If* $\bar{y}$ *is a solution of the (ASVVI)*$_3$, *then* $\bar{y}$ *is an (LUAES)*$_3$ *of the (NIVOP).*

Proof. 1. Let $\bar{y}$ be a solution of (ASVVI)$_1$ and suppose, to the contrary, that it is not an (LUAES)$_1$. There exist $\alpha > 0$ and $\delta > 0$, such that

$$\boldsymbol{\Phi}(z) - \boldsymbol{\Phi}(\bar{y}) \prec_{LU} \frac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \| e, \ \forall z \in B(\bar{y}; \delta) \cap \Gamma, \ z \neq \bar{y}. \tag{22}$$

Since each Φ_j, $j \in J$ is geodesic LU-approximately convex, then for any $\alpha > 0$, there exist $\delta' > 0$, such that

$$\begin{aligned} \langle \zeta_j^L, exp_{\bar{y}}^{-1} z \rangle - \tfrac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \| &\leq \Phi_j^L(z) - \Phi_j^L(\bar{y}), \\ \langle \zeta_j^U, exp_{\bar{y}}^{-1} z \rangle - \tfrac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \| &\leq \Phi_j^U(z) - \Phi_j^U(\bar{y}), \end{aligned} \tag{23}$$

for all $z \in B(\bar{y}, \delta') \cap \Gamma$ and $\zeta_j^L \in \partial_c \Phi_j^L(\bar{y})$, $\zeta_j^U \in \partial_c \Phi_j^U(\bar{y})$, $j \in J$, $j \neq k$ and for some $k \in J$, Φ_k^L and Φ_k^U are strictly geodesic LU-approximately convex functions. Then for any $\alpha > 0$, there exist $\delta'' > 0$, such that

$$\begin{aligned} \langle \zeta_k^L, exp_{\bar{y}}^{-1} z \rangle - \tfrac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \| &< \Phi_k^L(z) - \Phi_k^L(\bar{y}), \\ \langle \zeta_k^U, exp_{\bar{y}}^{-1} z \rangle - \tfrac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \| &< \Phi_k^U(z) - \Phi_k^U(\bar{y}), \end{aligned} \tag{24}$$

for each $z \in B(\bar{y}, \delta'')$, $\zeta_k^L \in \partial_c \Phi_k^L(\bar{y})$ and $\zeta_k^U \in \partial_c \Phi_k^U(\bar{y})$.
Setting $\delta^* = \min\{\delta, \delta', \delta''\}$, from (22)–(24), it follows that

$$\begin{aligned} (\langle \zeta_1^L, exp_{\bar{y}}^{-1} z \rangle, \ldots, \langle \zeta_m^L, exp_{\bar{y}}^{-1} z \rangle) &\leq \alpha \| exp_{\bar{y}}^{-1} z \| e, \\ (\langle \zeta_1^U, exp_{\bar{y}}^{-1} z \rangle, \ldots, \langle \zeta_m^U, exp_{\bar{y}}^{-1} z \rangle) &\leq \alpha \| exp_{\bar{y}}^{-1} z \| e, \end{aligned} \tag{25}$$

for each $z \in B(\bar{y}, \delta^*)$, $z \neq \bar{y}$, which is a contradiction.

2. Let $\bar{y}$ solves (ASVVI)$_2$. Then, for any $\alpha > 0$, there exist $\hat{\delta} > 0$, such that

$$\begin{aligned} (\langle \zeta_1^L, exp_{\bar{y}}^{-1} z \rangle, \ldots, \langle \zeta_m^L, exp_{\bar{y}}^{-1} z \rangle) &\nleq \alpha \| exp_{\bar{y}}^{-1} z \| e, \\ (\langle \zeta_1^U, exp_{\bar{y}}^{-1} z \rangle, \ldots, \langle \zeta_m^U, exp_{\bar{y}}^{-1} z \rangle) &\nleq \alpha \| exp_{\bar{y}}^{-1} z \| e, \end{aligned} \tag{26}$$

for each $z \in B(\bar{y}, \hat{\delta})$, $\zeta_j^L \in \partial_c \Phi_j^L(\bar{y})$ and $\zeta_j^U \in \partial_c \Phi_j^U(\bar{y})$, $j \in J$.
Since each Φ_j, $j \in J$ is geodesic approximately convex at $\bar{y}$, for any $\alpha > 0$, there exist $\bar{\delta} > 0$ such that

$$\begin{aligned} \Phi_j^L(z) - \Phi_j^L(\bar{y}) &\geq \langle \zeta_j^L, exp_{\bar{y}}^{-1} z \rangle - \tfrac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \|, \\ \Phi_j^U(z) - \Phi_j^U(\bar{y}) &\geq \langle \zeta_j^U, exp_{\bar{y}}^{-1} z \rangle - \tfrac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \|, \end{aligned} \tag{27}$$

for all $z \in B(\bar{y}; \bar{\delta}) \cap \Gamma$, $\zeta_j^L \in \partial_c \Phi_j^L(\bar{y})$ and $\zeta_j^U \in \partial_c \Phi_j^U(\bar{y})$, $j \in J$.
Setting $\tilde{\delta} := \{\hat{\delta}, \bar{\delta}\}$, from (26) and (27), it follows that, for any sufficiently small $\alpha > 0$, there exists $\tilde{\delta} > 0$ such that

$$\boldsymbol{\Phi}(z) - \boldsymbol{\Phi}(\bar{y}) \not\prec_{LU} \frac{\alpha}{2} \| exp_{\bar{y}}^{-1} z \| e, \ \forall z \in B(\bar{y}, \tilde{\delta}).$$

Hence, $\bar{y}$ is an $(LUAES)_2$ of the (NIVOP).

3. First, we will show that if $\bar{y}$ is a solution of $(ASVVI)_3$, then $\bar{y}$ solves $(AMVVI)_3$. Consequently, from Theorem 3, $\bar{y}$ is an $(LUAES)_3$ of (NIVOP). On the contrary, assume that $\bar{y}$ does not solve $(AMVVI)_3$, then for all $\delta > 0$, there exists $z \in B(\bar{y}, \delta)$, $\xi_j^L \in \partial_c \Phi_j^L(z)$ and $\xi_j^U \in \partial_c \Phi_j^U(z)$, $j \in J$, such that

$$\begin{aligned} &(\langle \xi_1^L, exp_z^{-1}\bar{y}\rangle, \ldots, \langle \xi_m^L, exp_z^{-1}\bar{y}\rangle) \geq -\alpha \|exp_z^{-1}\bar{y}\| e, \\ &(\langle \xi_1^U, exp_z^{-1}\bar{y}\rangle, \ldots, \langle \xi_m^U, exp_z^{-1}\bar{y}\rangle) \geq -\alpha \|exp_z^{-1}\bar{y}\| e. \end{aligned} \tag{28}$$

Since each Φ_j, $j \in J$, is geodesic LU-approximate convex, then Φ_j^L and Φ_j^U are geodesic approximate convex functions. Therefore, $\partial_c \Phi_j^L$ and $\partial_c \Phi_j^U$, $j \in J$ are geodesic sub-monotone. For all $z \in B(\bar{y}, \delta)$, $\xi_j^L \in \partial_c \Phi_j^L(z)$, $\xi_j^U \in \partial_c \Phi_j^U(z)$, $\zeta_j^L \in \partial_c \Phi_j^L(\bar{y})$ and $\zeta_j^U \in \partial_c \Phi_j^U(\bar{y})$, $j \in J$, we have

$$\begin{aligned} &\langle P_{0,\Omega}^1 \zeta_j^L - \xi_j^L, exp_z^{-1}\bar{y}\rangle \geq -\tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|, \\ &\langle P_{0,\Omega}^1 \zeta_j^U - \xi_j^U, exp_z^{-1}\bar{y}\rangle \geq -\tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|. \end{aligned} \tag{29}$$

From (28) and (29), we have

$$\begin{aligned} &\langle P_{0,\Omega}^1 \zeta_j^L, exp_z^{-1}\bar{y}\rangle \geq \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|, \\ &\langle P_{0,\Omega}^1 \zeta_j^U, exp_z^{-1}\bar{y}\rangle \geq \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|, \ \forall j \in J,\ j \neq k, \\ &\text{and} \\ &\langle P_{0,\Omega}^1 \zeta_k^L, exp_z^{-1}\bar{y}\rangle > \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|, \\ &\langle P_{0,\Omega}^1 \zeta_k^U, exp_z^{-1}\bar{y}\rangle > \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|, \text{ for some } k \in J, \end{aligned}$$

that is,

$$\begin{aligned} &\langle P_{1,\Omega}^0 (P_{0,\Omega}^1)\zeta_j^L, P_{1,\Omega}^0 exp_z^{-1}\bar{y}\rangle \geq \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|, \\ &\langle P_{1,\Omega}^0 (P_{0,\Omega}^1)\zeta_j^U, P_{1,\Omega}^0 exp_z^{-1}\bar{y}\rangle \geq \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|, \ \forall j \in J,\ j \neq k, \\ &\text{and} \\ &\langle P_{1,\Omega}^0 (P_{0,\Omega}^1)\zeta_k^L, P_{1,\Omega}^0 exp_z^{-1}\bar{y}\rangle > \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|, \\ &\langle P_{1,\Omega}^0 (P_{0,\Omega}^1)\zeta_k^U, P_{1,\Omega}^0 exp_z^{-1}\bar{y}\rangle > \tfrac{\alpha}{2}\|exp_z^{-1}\bar{y}\|, \text{ for some } k \in J. \end{aligned} \tag{30}$$

From (30), it follows that

$$\begin{aligned} &(\langle \zeta_1^L, exp_{\bar{y}}^{-1} z\rangle, \ldots, \langle \zeta_m^L, exp_{\bar{y}}^{-1} z\rangle) \leq -\tfrac{\alpha}{2}\|exp_{\bar{y}}^{-1} z\| e, \\ &(\langle \zeta_1^U, exp_{\bar{y}}^{-1} z\rangle, \ldots, \langle \zeta_m^U, exp_{\bar{y}}^{-1} z\rangle) \leq -\tfrac{\alpha}{2}\|exp_{\bar{y}}^{-1} z\| e, \end{aligned}$$

which contradicts our assumption. This completes the proof. □

4. Conclusions

In this paper, we have considered the classes of approximate Minty and Stampacchia type vector variational inequalities $(AMVVI)_1$, $(AMVVI)_2$, $(AMVVI)_3$, $(ASVVI)_1$, $(ASVVI)_2$ and $(ASVVI)_3$. Under geodesic LU-approximate convexity assumptions, we have proved the equivalence between the solutions of the considered approximate variational inequalities $(AMVVI)_1$, $(AMVVI)_2$, $(AMVVI)_3$, $(ASVVI)_1$, $(ASVVI)_2$ and $(ASVVI)_3$ and approximate efficient solutions $(LUAES)_1$, $(LUAES)_2$, $(LUAES)_3$ of nonsmooth interval-valued vector optimization problem (NIVOP). The results of the paper extended and generalized some earlier results of [19,29,37,42–44].

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