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Article

Liouville-Type Results for a Two-Dimensional Stretching Eyring–Powell Fluid Flowing along the z-Axis

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Abstract: The purpose of this study is to establish Liouville-type results for a three-dimensional incompressible, unsteady flow described by the Eyring–Powell fluid equations. The fluid is studied in a plane Ω_p while it moves along the z-axis. Therefore the main functions to analyze are given by $u(x,y,z,t)$ and $v(x,y,z,t)$, belonging to Ω_p. The results are obtained for globally bounded initial data as well as their corresponding derivatives, and the variations in velocity along the z-axis belong to the space L^2 and BMO. Under such conditions, Liouville-type results are obtained and extended to L^p, $p > 2$.

Keywords: Liouville-type results; Eyring–Powell fluid; three-dimensional flow; unsteady flow

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1. Introduction

In many situations, the fluid principles typically considered to model substances such as water, air or oil are supported by the Newtonian fluid concept. Nonetheless, in certain applications, a non-Newtonian description is required to further characterize the prescribed behavior. As an example, the mathematical modeling of muds and sludges in mines can be described by non-Newtonian fluid approaches. In addition, there exist other applications, such as biomedical flows or lubrication, where non-Newtonian fluids constitute the baseline modeling. The particular rheological properties of non-Newtonian flows lead, in some cases, to the categorization of such fluids through the use of dedicated theories. This is the case of the so-called Eyring–Powell fluid. Numerous studies have been conducted for this particular fluid under flowing conditions (see [1–10] for some dedicated articles). More particularly, the Eyring–Powell fluid has been considered in this area in some recent studies. In [11], the authors propose a computational procedure based on a shooting method approach to obtain solutions in a three-dimensional Eyring–Powell fluid over a stretching sheet. In [12], an Eyring–Powell fluid is considered to act over an exponentially stretching surface where heat and mass transfer processes are modeled by nanoparticles. Additional studies devoted to exploring solutions under analytical and numerical exercises exist; nonetheless, little has been explored in relation to the regularity of solutions of three-dimensional Eyring–Powell fluids.

In relation to Liouville-type results, it shall be noted that there exists an extensive literature devoted to developing regularity criteria for Navier–Stokes equations and, more particularly, devoted to Liouville-type results for the 3D stationary fluid described by Navier–Stokes equations. As an example of this, refer to the Magnetohydrodynamics (MHD) and Hall-MHD equations treated in [13–18]. On the contrary, some further literature is required to explore the Liouville-type results for a 3D Eyring–Powell fluid, particularly when flowing along the z-axis while stretching on a plane.

Motivated by the described facts, the objective of the presented analysis is to establish Liouville-type results for an Eyring–Powell fluid over a plane with stretching velocities and flowing along the positive z-axis.

The fluid under analysis is considered to be incompressible and electrically conductive in the presence of an applied magnetic field B_0. Consider the classical Cartesian coordinate system such that the Ω_p- plane corresponds to a fluid sheet and the flow invades the space $z \geq 0$. The surface stretching velocities along the $x-$ and $y-$directions are $u_{z=0}(x) = ax$ and $v_{z=0}(y) = ay$, respectively. Note that the idea is to study the fluid kinematics and the associated Liouville results in the Ω_p plane through the fluid motion along the z-axis; therefore, the main functions to analyze are given by $u(x,y,z,t)$ and $v(x,y,z,t)$. The velocity components, continuity principles and the governing equations of an Eyring–Powell fluid along the spatial domain $\Omega = \Omega_p \times [0,\infty)$ are

$$\mathbf{V} = [u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)] \qquad \text{and} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \left(\nu + \frac{1}{\beta d_1 \rho_f}\right)\frac{\partial^2 u}{\partial z^2} - \frac{1}{2\beta d_1^3 \rho_f}\left(\frac{\partial u}{\partial z}\right)^2 \frac{\partial^2 u}{\partial z^2} - \frac{\sigma B_0^2}{\rho_f} u, \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \left(\nu + \frac{1}{\beta d_1 \rho_f}\right)\frac{\partial^2 v}{\partial z^2} - \frac{1}{2\beta d_1^3 \rho_f}\left(\frac{\partial v}{\partial z}\right)^2 \frac{\partial^2 v}{\partial z^2} - \frac{\sigma B_0^2}{\rho_f} v. \tag{3}$$

The boundary conditions reflect the existence of stretching velocity fields in the plane $z = 0$. In addition, it is assumed that the fluid is at rest for any dimensional variable at ∞.

$$\begin{aligned} u &= ax, \quad v = ay, \quad w = 0 \quad for \quad \Omega_p \equiv z = 0, \\ u &\to 0, \quad v \to 0 \quad \text{as} \quad z \to \infty, \\ u &= 0, \quad v = 0, \quad w = 0 \quad \text{if} \quad (x,y) \to (-\infty, -\infty), \\ u &= 0, \quad v = 0, \quad w = 0 \quad \text{if} \quad (x,y) \to (\infty, \infty). \end{aligned}$$

In addition, we use the following initial conditions:

$$u(x,y,z,0) = u_0(x,y,z) \qquad \text{and} \qquad v(x,y,z,0) = v_0(x,y,z), \tag{4}$$

where u, v, w are the first, second and third components of the velocity, respectively, while ν is the kinematic viscosity (defined as the ratio of the dynamic viscosity μ and the fluid density ρ_f). Note that β and d_1 are two fluid parameters and σ is related to the charge distributions along the fluid.

Here, we assume that the shear stress is zero at $z = 0$ considering the case $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$. Additionally, $u = v \sim 0$ for $t \to \infty$.

2. Preliminaries

In this section, we introduce some notation and collect some preliminary results required to support the introduced analysis. Firstly, the usual Lebesgue space of real-valued functions is $L^p(Q),\ Q = \Omega \times (0,\infty)$ with the norm $\|\cdot\|_{L^p}$.

$$\|f\|_{L^p} = \left\{ \begin{array}{l} \left(\int\limits_Q |f(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty \\ ess\ \sup_{x \in Q} |f(x)|, \quad p = \infty. \end{array} \right\}.$$

In addition, consider the Bounded Mean Oscillation (BMO) homogeneous space in $B_r = \{(x,y,z,t) \in R^3 \times [0,\infty);\ (x^2+y^2+z^2)^{1/2} < r,\ t < r\}$ with the following norm (see [19]).

$$\|g\|_{BMO} = \sup_{r>0}\left(\frac{1}{|B_r(x)|}\right)\int_{B_r(x)}\left|g(y)-\left(\frac{1}{|B_r(y)|}\right)\int_{B_r(y)} g(z)\right|dy.$$

Now, the following lemma is supported by a result in reference [20].

Lemma 1. *Let $1 < b < a < \infty$. Then,*

$$\|u_1\|_{L^a} \le \|u_1\|_{BMO}^{1-\frac{b}{a}}\|u_1\|_{L^b}^{\frac{b}{a}}.$$

The following lemma is required as well to support the analysis to come and is based on the anisotropic inequality for Sobolev spaces.

Lemma 2. *Let $f, g, h \in C_c^\infty(R^3)$*

$$\iiint_\Omega |\, fgh \,|\, dxdydz \le \overline{C}\|f\|_{L^q}^{\frac{\alpha-1}{\alpha}}\left\|\frac{\partial f}{\partial x}\right\|_{L^s}^{\frac{1}{\alpha}}\|g\|_{L^2}^{\frac{\alpha-2}{\alpha}}\left\|\frac{\partial g}{\partial y}\right\|_{L^2}^{\frac{1}{\alpha}}\|h\|_{L^2},$$

where $\alpha > 2$, $1 \le q, s < \infty$, $\frac{\alpha-1}{q} + \frac{1}{s} = 1$, and $\overline{C}$ is constant.

Now, consider the following definition of a cut-off function:

Definition 1. *Let $B_R\{(x,t) \in R^3 \times [0,T); |x_i| < r,\ t < r,\ 0 < r < R\}$, and let $\phi \in C_0^\infty(R^3)$ be a cut-off function such that $\phi = 1$ in B_1 and $\phi = 0$ outside B_2. Consider, then, the following function:*

$$\phi_R(x,t) = \phi\left(\frac{x_1}{R}, \frac{x_2}{R}, \frac{x_3}{R}, \frac{t}{R}\right),$$

satisfying

$$\left\|\nabla^k\phi_R\right\|_{L^\infty} \le \frac{C}{R^k} \text{ and } \left\|\frac{\partial^k\phi_R}{\partial t^k}\right\|_{L^\infty} \le \frac{C}{R^k}.$$

3. Statements of Results

In this paper, we established the Liouville-type results for three-dimensional Erying–Powell fluid flow equations and show that the solutions are bounded. This baseline idea is inspired by the work of Zhouyn Li and Penchen Niu in [21]. They established the Liouville-type theorems for the stationary Hall-MHD equations in R^3. The results are stated as follows:

Theorem 1. *Consider the following bounds:*

$\iiint_\Omega |u_0(x,y,z)|^2 dxdydz < \infty$ *and* $\iiint_\Omega |v_0(x,y,z)|^2 dxdydz < \infty$.

In addition, consider $\left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}\right) \in L^2((0,\infty), BMO)$.

Then, for $R \to \infty$, the solutions $u(x,y,z,t)$ and $v(x,y,z,t)$ are bounded under same norm as u_0, v_0 on $\Omega \times [0,\infty]$.

Theorem 2. *Consider the following bounds:*

$\iiint_\Omega \left|\frac{\partial^2 u_0(x,y,z)}{\partial z^2}\right|^2 dxdydz < \infty$ *and* $\iiint_\Omega \left|\frac{\partial^2 v_0(x,y,z)}{\partial z^2}\right|^2 dxdydz < \infty$.

In addition, consider $\left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}, \frac{\partial^2 u}{\partial z^2}, \frac{\partial^2 v}{\partial z^2}\right) \in L^2((0,\infty), BMO)$.

Then, as $R \to \infty$, the solutions $u(x,y,z,t)$ and $v(x,y,z,t)$ are bounded under same norm as u_0, v_0 on $\Omega \times [0,\infty]$.

Theorem 3. *Consider the following bounds*
$\iiint\limits_{\Omega} |u_0(x,y,z)|^p dxdydz < \infty$ *and* $\iiint\limits_{\Omega} |v_0(x,y,z)|^p dxdydz < \infty$.
In addition, consider $\left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}\right) \in L^2((0,\infty), BMO)$.
Then, for $R \to \infty$*, the solutions* $u(x,y,z,t)$ *and* $v(x,y,t)$ *are bounded on* $\Omega \times [0,\infty]$*, considering the* L^p*,* $p > 2$ *norm.*

Proof of Theorem 1. First, multiply expression (2) by $(u\,\phi_R)$. Afterwards, consider the integration of the second term with respect to x, the integration of the third term with respect to y and the fourth and onward terms with respect to z:

$$\int\limits_{B_{2R}} \frac{\partial u}{\partial t} u\phi_R dxdydzdt - \frac{1}{3} \int\limits_{B_{2R}\setminus B_R} u^3 \frac{\partial \phi_R}{\partial x} dxdydzdt - \frac{1}{2} \int\limits_{B_{2R}\setminus B_R} \frac{\partial v}{\partial y} \phi_R u^2 dxdydzdt \tag{5}$$

$$-\frac{1}{2} \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial y} v u^2 dxdydzdt - \frac{1}{2} \int\limits_{B_{2R}\setminus B_R} \frac{\partial w}{\partial z} \phi_R u^2 dxdydzdt - \frac{1}{2} \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial z} w u^2 dxdydzdt$$

$$= \left(\nu + \frac{1}{\beta d_1 \rho_f}\right) \left[- \int\limits_{B_{2R}\setminus B_R} u \frac{\partial u}{\partial z} \frac{\partial \phi_R}{\partial z} dxdydzdt - \int\limits_{B_{2R}\setminus B_R} \phi_R \left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt \right]$$

$$- \frac{1}{2\beta d_1^3 \rho_f} I_1 - \frac{\sigma B_0^2}{\rho_f} \int\limits_{B_{2R}} u^2 \phi_R dxdydzdt.$$

where

$$I_1 = \int\limits_{B_{2R}} \left(\frac{\partial u}{\partial z}\right)^2 \frac{\partial^2 u}{\partial z^2} u\phi_R dxdydzdt.$$

Integrating I_1, we obtain

$$\begin{aligned} I_1 &= -\frac{1}{3} \int\limits_{B_{2R}\setminus B_R} \frac{\partial}{\partial z}(u\phi_R) \left(\frac{\partial u}{\partial z}\right)^3 dxdydzdt \\ &= -\frac{1}{3} \int\limits_{B_{2R}\setminus B_R} \phi_R \left(\frac{\partial u}{\partial z}\right)^4 dxdydzdt - \frac{1}{3} \int\limits_{B_{2R}\setminus B_R} u \left(\frac{\partial u}{\partial z}\right)^3 \frac{\partial \phi_R}{\partial z} dxdydzdt. \end{aligned}$$

Now consider the third and fifth terms of (5)

$$\begin{aligned} &-\frac{1}{2} \int\limits_{B_{2R}\setminus B_R} \frac{\partial v}{\partial y} \phi_R u^2 dxdydzdt - \frac{1}{2} \int\limits_{B_{2R}\setminus B_R} \frac{\partial w}{\partial z} \phi_R u^2 dxdydzdt \\ &= -\frac{1}{2} \int\limits_{B_{2R}\setminus B_R} \phi_R u^2 \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dxdydzdt = \frac{1}{2} \int\limits_{B_{2R}\setminus B_R} \phi_R u^2 \frac{\partial u}{\partial x} dxdydzdt, \end{aligned}$$

where we have used expression (1). After integration on the right-hand side, we obtain

$$\begin{aligned} &-\frac{1}{2} \int\limits_{B_{2R}\setminus B_R} \frac{\partial v}{\partial y} \phi_R u^2 dxdydzdt - \frac{1}{2} \int\limits_{B_{2R}\setminus B_R} \frac{\partial w}{\partial z} \phi_R u^2 dxdydzdt \\ &= \frac{1}{6} \left(u^3 \phi_R\right)_{\partial\left(B_{2R}\setminus B_R\right)} - \frac{1}{6} \int\limits_{B_{2R}\setminus B_R} u^3 \frac{\partial \phi_R}{\partial x} dxdydzdt. \end{aligned}$$

As $r >> 1$, it is possible to state that $u^3\phi_R = 0$ over $\partial\left(B_{2R}\backslash B_R\right)$. Then, the following holds:

$$-\frac{1}{2}\int\limits_{B_{2R}\backslash B_R}\frac{\partial v}{\partial y}\phi_R u^2 dxdydzdt-\frac{1}{2}\int\limits_{B_{2R}\backslash B_R}\frac{\partial w}{\partial z}\phi_R u^2 dxdydzdt=-\frac{1}{6}\int\limits_{B_{2R}\backslash B_R}u^3\frac{\partial \phi_R}{\partial x}dxdydzdt.$$

Utilizing the values of these integrals and I_1 in (5), we have

$$\begin{aligned}
&\int\limits_{B_{2R}}\frac{\partial u}{\partial t}u\phi_R dxdydzdt-\frac{1}{6}\int\limits_{B_{2R}\backslash B_R}\frac{\partial \phi_R}{\partial x}u^3 dxdydzdt\\
&-\frac{1}{2}\int\limits_{B_{2R}\backslash B_R}\frac{\partial \phi}{\partial y}vu^2 dxdydzdt-\frac{1}{2}\int\limits_{B_{2R}\backslash B_R}\frac{\partial \phi}{\partial z}wu^2 dxdydzdt\\
&=\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\left\{-\int\limits_{B_{2R}\backslash B_R}\frac{\partial \phi}{\partial z}u\frac{\partial u}{\partial z}dxdydzdt-\int\limits_{B_{2R}\backslash B_R}\phi_R\left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt\right\}\\
&-\frac{1}{2\beta d_1^3\rho_f}\left\{-\frac{1}{3}\int\limits_{B_{2R}\backslash B_R}\phi_R\left(\frac{\partial u}{\partial z}\right)^4 dxdydzdt-\frac{1}{3}\int\limits_{B_{2R}\backslash B_R}\frac{\partial \phi}{\partial z}u\left(\frac{\partial u}{\partial z}\right)^3 dxdydzdt\right\}\\
&-\frac{\sigma B_0^2}{\rho_f}\int\limits_{B_{2R}}u^2\phi_R dxdydzdt,
\end{aligned}$$

which implies that

$$\begin{aligned}
\int\limits_{B_{2R}}\frac{\partial u}{\partial t}u\phi_R dxdydzdt&=\frac{1}{6}\int\limits_{B_{2R}\backslash B_R}\frac{\partial \phi}{\partial x}u^3 dxdydzdt-\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\int\limits_{B_{2R}\backslash B_R}\phi_R\left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt\\
&+\frac{1}{6\beta d_1^3\rho_f}\int\limits_{B_{2R}\backslash B_R}\phi_R\left(\frac{\partial u}{\partial z}\right)^4 dxdydzdt+\frac{1}{2}\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\int\limits_{B_{2R}\backslash B_R}u^2\frac{\partial^2\phi}{\partial z^2}dxdydzdt\\
&+\frac{1}{6\beta d_1^3\rho_f}\int\limits_{B_{2R}\backslash B_R}u\left(\frac{\partial u}{\partial z}\right)^3\frac{\partial \phi}{\partial z}dxdydzdt+\frac{1}{2}\int\limits_{B_{2R}\backslash B_R}\frac{\partial \phi}{\partial y}vu^2 dxdydzdt\\
&+\frac{1}{2}\int\limits_{B_{2R}\backslash B_R}\frac{\partial \phi}{\partial z}wu^2 dxdydzdt-\frac{\sigma B_0^2}{\rho_f}\int\limits_{B_{2R}}u^2\phi_R dxdydzdt.
\end{aligned}$$

Upon integration with regard to t, the following holds

$$
\begin{aligned}
&\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\int\limits_{B_{2R}\setminus B_R}\phi_R\left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt+\frac{\sigma B_0^2}{\rho_f}\int\limits_{B_{2R}}u^2\phi_R dxdydzdt\\
= \;& -\iiint\limits_{\Omega}|u_0(x,y,z)|^2dxdydz+\frac{1}{2}\int\limits_{B_{2R}\setminus B_R}u^2\frac{\partial\phi_R}{\partial t}dxdydzdt+\frac{1}{6}\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi}{\partial x}u^3 dxdydzdt\\
&+\frac{1}{6\beta d_1^3\rho_f}\int\limits_{B_{2R}\setminus B_R}\phi_R\left(\frac{\partial u}{\partial z}\right)^4 dxdydzdt+\frac{1}{2}\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\int\limits_{B_{2R}\setminus B_R}u^2\frac{\partial^2\phi}{\partial z^2}dxdydzdt\\
&+\frac{1}{6\beta d_1^3\rho_f}\int\limits_{B_{2R}\setminus B_R}u\left(\frac{\partial u}{\partial z}\right)^3\frac{\partial\phi}{\partial z}dxdydzdt+\frac{1}{2}\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi}{\partial y}vu^2dxdydzdt\\
&+\frac{1}{2}\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi}{\partial z}wu^2dxdydzdt\\
\le \;& -\iiint\limits_{\Omega}|u_0(x,y,z)|^2dxdydz+\left\|\frac{\partial\phi}{\partial t}\right\|_{L^\infty}\|u\|^2_{L^2(B_{2R}\setminus B_R)}+\frac{1}{6}\left\|\frac{\partial\phi}{\partial x}\right\|_{L^\infty}\|u\|^3_{L^3(B_{2R}\setminus B_R)}\\
&+\frac{1}{2}\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\left\|\frac{\partial^2\phi}{\partial z^2}\right\|_{L^\infty}\|u\|^2_{L^2(B_{2R}\setminus B_R)}+\frac{1}{6\beta d_1^3\rho_f}\left\|\frac{\partial u}{\partial z}\right\|^4_{L^4(B_{2R}\setminus B_R)}\\
&+\frac{1}{6\beta d_1^3\rho_f}\left\|\frac{\partial\phi}{\partial z}\right\|_{L^\infty}\|u\|_{L^2}\left\|\frac{\partial u}{\partial z}\right\|^3_{L^6(B_{2R}\setminus B_R)}+\frac{1}{2}\left\|\frac{\partial\phi}{\partial y}\right\|_{L^\infty}\|v\|_{L^2}\|u\|^2_{L^4(B_{2R}\setminus B_R)}\\
&+\frac{1}{2}\left\|\frac{\partial\phi}{\partial z}\right\|_{L^\infty}\|w\|_{L^2}\|u\|^2_{L^4(B_{2R}\setminus B_R)},
\end{aligned}
$$

where we used Holder's inequality, since

$$
\begin{aligned}
\left\|\frac{\partial\phi_R}{\partial x}\right\|_{L^\infty} &\le \|\nabla\phi_R\|_{L^\infty},\left\|\frac{\partial\phi_R}{\partial y}\right\|_{L^\infty}\le\|\nabla\phi_R\|_{L^\infty}\\
\left\|\frac{\partial\phi_R}{\partial z}\right\|_{L^\infty} &\le \|\nabla\phi_R\|_{L^\infty}\text{ and }\left\|\frac{\partial\phi}{\partial t}\right\|_{L^\infty}\le\frac{C}{R},
\end{aligned}
$$

after using the above inequalities and Lemma 1, we obtain

$$
\begin{aligned}
&\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\left\|\frac{\partial u}{\partial z}\right\|^2_{L^2(B_{2R})}+\frac{\sigma B_0^2}{\rho_f}\|u\|^2_{L^2(B_{2R})}\\
&\le-\iiint\limits_{\Omega}|u_0(x,y,z)|^2dxdydz+\frac{C}{R}\|u\|^2_{L^2(B_{2R}\setminus B_R)}+\frac{C}{6R}\|u\|^3_{L^3(B_{2R}\setminus B_R)}\\
&+\frac{C}{2R}\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\|u\|^2_{L^2(B_{2R}\setminus B_R)}+\frac{1}{6\beta d_1^3\rho_f}\left\|\frac{\partial u}{\partial z}\right\|^2_{L^2(B_{2R}\setminus B_R)}\left\|\frac{\partial u}{\partial z}\right\|^2_{BMO}\\
&+\frac{C}{6R\beta d_1^3\rho_f}\|u\|_{L^2}\left\|\frac{\partial u}{\partial z}\right\|^3_{L^6(B_{2R}\setminus B_R)}+\frac{C}{2R}\|v\|_{L^2}\|u\|^2_{L^4(B_{2R}\setminus B_R)}\\
&+\frac{C}{2R}\|w\|_{L^2}\|u\|^2_{L^4B(_{2R}\setminus B_R)}.
\end{aligned}
$$

Since $\frac{\partial u}{\partial z} \in L^2((0,\infty), BMO)$, then

$$\left(\nu + \frac{1}{\beta d_1 \rho_f}\right)\left\|\frac{\partial u}{\partial z}\right\|^2_{L^2(B_{2R})} + \frac{\sigma B_0^2}{\rho_f}\|u\|^2_{L^2(B_{2R})} \leq \iiint\limits_{\Omega} |u_0(x,y,z)|^2 dxdydz + \frac{C}{R}\|u\|^2_{L^2(B_{2R}\setminus B_R)}$$
$$+\frac{C}{6R}\|u\|^3_{L^3(B_{2R}\setminus B_R)} + \frac{C}{2R}\left(\nu + \frac{1}{\beta d_1 \rho_f}\right)\|u\|^2_{L^2(B_{2R}\setminus B_R)} + \frac{k}{6\beta d_1^3 \rho_f}\left\|\frac{\partial u}{\partial z}\right\|^2_{L^2(B_{2R}\setminus B_R)}$$
$$+\frac{C}{6R\beta d_1^3 \rho_f}\|u\|_{L^2}\left\|\frac{\partial u}{\partial z}\right\|^3_{L^6(B_{2R}\setminus B_R)} + \frac{C}{2R}\|v\|_{L^2}\|u\|^2_{L^4(B_{2R}\setminus B_R)}$$
$$+\frac{C}{2R}\|w\|_{L^2}\|u\|^2_{L^4(B_{2R}\setminus B_R)}.$$

Taking $R \to \infty$, we obtain

$$\begin{aligned}\frac{\sigma B_0^2}{\rho_f}\|u\|^2_{L^2} + \left(\nu + \frac{1}{\beta d_1 \rho_f} - \frac{k}{6\beta d_1^3 \rho_f}\right)\left\|\frac{\partial u}{\partial z}\right\|^2_{L^2} dt &\leq -\iiint\limits_{\Omega} |u_0(x,y,z)|^2 dxdydz \\ &\leq \iiint\limits_{\Omega} |u_0(x,y,z)|^2 dxdydz.\end{aligned}$$

As

$$\left(\nu + \frac{1}{\beta d_1 \rho_f} - \frac{k}{6\beta d_1^3 \rho_f}\right) \geq 0, \text{ and } \frac{\sigma B_0^2}{\rho_f} \geq 0,$$

therefore

$$\left\|\frac{\partial u}{\partial z}\right\|^2_{L^2} \leq \iiint\limits_{\Omega} |u_0(x,y,z)|^2 dxdydz \text{ or } \|u\|^2_{L^2} \leq \iiint\limits_{\Omega} |u_0(x,y,z)|^2 dxdydz,$$

which implies that u is bounded.

Similarly, multiplying by $(v\phi_R)$ in expression (3) and applying the same procedure, it is concluded on the boundness of v as well. □

Proof of Theorem 2. By multiplying expression (2) by $\left(-\frac{\partial^2 u}{\partial z^2}\phi_R\right)$ and upon integration by parts, the following holds:

$$\begin{aligned}&\frac{1}{2}\int\limits_{B_{2R}\setminus B_R} \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial z^2}\right)^2 \phi_R dxdydzdt + \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial t}\frac{\partial u}{\partial z}\frac{\partial \phi_R}{\partial z} dxdydzdt + I_3 \\ = \quad & -\left(\nu + \frac{1}{\beta d_1 \rho_f}\right)\int\limits_{B_{2R}}\left(\frac{\partial^2 u}{\partial z^2}\right)^2 \phi_R\, dxdydzdt + \frac{1}{2\beta d_1^3 \rho_f}\int\limits_{B_{2R}\setminus B_R}\left(\frac{\partial u}{\partial z}\right)^2\left(\frac{\partial^2 u}{\partial z^2}\right)^2 \phi_R\, dxdydzdt \\ & -\frac{\sigma B_0^2}{\rho_f}\int\limits_{B_{2R}\setminus B_R}\left(u\frac{\partial \phi_R}{\partial z}\frac{\partial u}{\partial z} + \phi_R\left(\frac{\partial u}{\partial z}\right)^2\right) dxdydzdt,\end{aligned} \tag{6}$$

where

$$I_3 = -\int\limits_{B_{2R}} \frac{\partial^2 u}{\partial z^2}\phi_R(V\cdot\nabla u) dxdydzdt$$

$$
\begin{aligned}
&= \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial z}\frac{\partial \phi_R}{\partial z}(V\cdot\nabla u)dxdydzdt + \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial z}\frac{\partial}{\partial z}(V\cdot\nabla u)\phi_R dxdydzdt \\
&= \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial z}\frac{\partial \phi_R}{\partial z}(V\cdot\nabla u)dxdydzdt + \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial z}\frac{\partial V}{\partial z}\cdot\nabla u\,\phi_R dxdydzdt \\
&\quad + \int\limits_{B_{2R}\setminus B_R} V\cdot\frac{\partial u}{\partial z}\frac{\partial \nabla u}{\partial z}\phi_R dxdydzdt.
\end{aligned}
$$

By applying integration by parts in the last term and using expression (1), we obtain

$$
\begin{aligned}
I_3 &= \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial z}\frac{\partial \phi_R}{\partial z}(V\cdot\nabla u)dxdydzdt + \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial z}\frac{\partial V}{\partial z}\cdot\nabla u\phi dxdydzdt \\
&\quad -\frac{1}{2}\int\limits_{B_{2R}\setminus B_R} \nabla\phi_R\cdot V\left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt.
\end{aligned}
$$

After the substitution of the obtained expression for I_3 into (6), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{B_{2R}\setminus B_R} \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial z^2}\right)^2\phi_R dxdydzdt + \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial t}\frac{\partial u}{\partial z}\frac{\partial \phi_R}{\partial z}dxdydzdt \\
&\quad + \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial z}\frac{\partial \phi_R}{\partial z}(V\cdot\nabla u)dxdydzdt \\
&+ \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial z}\frac{\partial V}{\partial z}\cdot\nabla u\phi_R dxdydzdt - \frac{1}{2}\int\limits_{B_{2R}\setminus B_R} \nabla\phi_R\cdot V\left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt \\
&= -\left(\nu + \frac{1}{\beta d_1\rho_f}\right)\int\limits_{B_{2R}\setminus B_R} \left(\frac{\partial^2 u}{\partial z^2}\right)^2\phi_R\, dxdydzdt \\
&\quad + \frac{1}{2\beta d_1^3\rho_f}\int\limits_{B_{2R}\setminus B_R} \left(\frac{\partial u}{\partial z}\right)^2\left(\frac{\partial^2 u}{\partial z^2}\right)^2\phi_R\, dxdydzdt \\
&-\frac{\sigma B_0^2}{\rho_f}\int\limits_{B_{2R}\setminus B_R} u\frac{\partial \phi_R}{\partial z}\frac{\partial u}{\partial z}\,dxdydzdt - \frac{\sigma B_0^2}{\rho_f}\int\limits_{B_{2R}\setminus B_R} \phi_R\left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt.
\end{aligned}
$$

After using Young's inequality on the second term on the right-hand side and arranging

$$
\begin{aligned}
&\frac{\sigma B_0^2}{\rho_f}\int\limits_{B_{2R}\setminus B_R} \phi_R\left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt + \left(\nu + \frac{1}{\beta d_1\rho_f}\right)\int\limits_{B_{2R}\setminus B_R} \left(\frac{\partial^2 u}{\partial z^2}\right)^2\phi_R\, dxdydzdt \\
&\le -\frac{1}{2}\int\limits_{B_{2R}\setminus B_R} \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial z^2}\right)^2\phi_R dxdydzdt - \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial t}\frac{\partial u}{\partial z}\frac{\partial \phi_R}{\partial z}dxdydzdt
\end{aligned}
$$

$$-\int_{B_{2R}\setminus B_R} \frac{\partial u}{\partial z}\frac{\partial \phi_R}{\partial z}(V\cdot\nabla u)dxdydzdt + I_4$$
$$+\frac{1}{2}\int_{B_{2R}\setminus B_R} \nabla\phi_R\cdot V\left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt + \frac{1}{4\beta d_1^3\rho_f}\int_{B_{2R}\setminus B_R}\left(\frac{\partial u}{\partial z}\right)^4\phi_R\, dxdydzdt$$
$$+\frac{1}{4\beta d_1^3\rho_f}\int_{B_{2R}\setminus B_R}\left(\frac{\partial^2 u}{\partial z^2}\right)^4\phi_R\, dxdydzdt - \frac{\sigma B_0^2}{\rho_f}\int_{B_{2R}\setminus B_R} u\frac{\partial\phi_R}{\partial z}\frac{\partial u}{\partial z}\, dxdydzdt, \tag{7}$$

where

$$I_4 = \int_{B_{2R}\setminus B_R}\frac{\partial u}{\partial z}\frac{\partial V}{\partial z}\cdot\nabla u\phi_R dxdydzdt$$

$$= \int_{B_{2R}\setminus B_R}\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial z}\right)^2\phi_R dxdydzdt + \int_{B_{2R}\setminus B_R}\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\phi_R dxdydzdt + \int_{B_{2R}\setminus B_R}\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\phi_R dxdydzdt.$$

By applying Equation (1) to the third term on the right-hand side

$$I_4 = -\int_{B_{2R}\setminus B_R}\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial z}\right)^2\phi_R dxdydzdt + \int_{B_{2R}\setminus B_R}\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\phi_R dxdydzdt.$$

Introducing the obtained value of I_4 in (7)

$$\frac{\sigma B_0^2}{\rho_f}\int_{B_{2R}\setminus B_R}\phi_R\left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt + \left(\nu + \frac{1}{\beta d_1\rho_f}\right)\int_{B_{2R}\setminus B_R}\left(\frac{\partial^2 u}{\partial z^2}\right)^2\phi_R\, dxdydzdt$$
$$\leq -\frac{1}{2}\int_{B_{2R}\setminus B_R}\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial z^2}\right)^2\phi_R dxdydzdt - \int_{B_{2R}\setminus B_R}\frac{\partial u}{\partial t}\frac{\partial u}{\partial z}\frac{\partial\phi_R}{\partial z}dxdydzdt$$
$$-\int_{B_{2R}\setminus B_R}\frac{\partial u}{\partial z}\frac{\partial\phi_R}{\partial z}(V\cdot\nabla u)dxdydzdt - \int_{B_{2R}\setminus B_R}\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial z}\right)^2\phi_R dxdydzdt$$
$$+\int_{B_{2R}\setminus B_R}\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\phi_R dxdydzdt$$
$$+\frac{1}{2}\int_{B_{2R}\setminus B_R}\nabla\phi_R\cdot V\left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt + \frac{1}{4\beta d_1^3\rho_f}\int_{B_{2R}\setminus B_R}\left(\frac{\partial u}{\partial z}\right)^4\phi_R\, dxdydzdt$$
$$+\frac{1}{4\beta d_1^3\rho_f}\int_{B_{2R}\setminus B_R}\left(\frac{\partial^2 u}{\partial z^2}\right)^4\phi_R\, dxdydzdt - \frac{\sigma B_0^2}{\rho_f}\int_{B_{2R}\setminus B_R} u\frac{\partial\phi_R}{\partial z}\frac{\partial u}{\partial z}\, dxdydzdt$$

Integrating the first term on the right-hand side with regard to t

$$\frac{\sigma B_0^2}{\rho_f}\int_{B_{2R}\setminus B_R}\phi_R\left(\frac{\partial u}{\partial z}\right)^2 dxdydzdt + \left(\nu + \frac{1}{\beta d_1\rho_f}\right)\int_{B_{2R}\setminus B_R}\left(\frac{\partial^2 u}{\partial z^2}\right)^2\phi_R\, dxdydzdt$$

$$
\begin{aligned}
&\leq -\iiint\limits_{\Omega} \left| \frac{\partial^2 u_0(x,y,z)}{\partial z^2} \right|^2 dxdydz + \frac{1}{2} \int\limits_{B_{2R}\backslash B_R} \left(\frac{\partial^2 u}{\partial z^2} \right)^2 \frac{\partial \phi_R}{\partial t} dxdydzdt \\
&- \int\limits_{B_{2R}\backslash B_R} \frac{\partial u}{\partial t} \frac{\partial u}{\partial z} \frac{\partial \phi_R}{\partial z} dxdydzdt - \int\limits_{B_{2R}\backslash B_R} \frac{\partial u}{\partial z} \frac{\partial \phi_R}{\partial z} (V \cdot \nabla u) dxdydzdt + I_5 + I_6 \\
&+ \frac{1}{2} \int\limits_{B_{2R}\backslash B_R} \nabla \phi_R \cdot V \left(\frac{\partial u}{\partial z} \right)^2 dxdydzdt + \frac{1}{4\beta d_1^3 \rho_f} \int\limits_{B_{2R}\backslash B_R} \left(\frac{\partial u}{\partial z} \right)^4 \phi_R \, dxdydzdt \\
&+ \frac{1}{4\beta d_1^3 \rho_f} \int\limits_{B_{2R}\backslash B_R} \left(\frac{\partial^2 u}{\partial z^2} \right)^4 \phi_R \, dxdydzdt - \frac{\sigma B_0^2}{\rho_f} \int\limits_{B_{2R}\backslash B_R} u \frac{\partial \phi_R}{\partial z} \frac{\partial u}{\partial z} \, dxdydzdt, \qquad (8)
\end{aligned}
$$

where

$$I_5 = -\int\limits_{B_{2R}\backslash B_R} \frac{\partial u}{\partial x} \left(\frac{\partial u}{\partial z} \right)^2 \phi_R dxdydzdt,$$

and

$$I_6 = \int\limits_{B_{2R}\backslash B_R} \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} \phi_R dxdydzdt.$$

Using Lemma 2 on I_5 and I_6, we obtain

$$
\begin{aligned}
I_5 &\leq \overline{K_3} \left\| \frac{\partial u}{\partial z} \right\|^2_{L^4(B_{2R}\backslash B_R)} \left\| \frac{\partial u}{\partial x} \right\|^{\frac{3}{4}}_{L^4(B_{2R}\backslash B_R)} \left\| \frac{\partial^2 u}{\partial x^2} \right\|^{\frac{1}{2}}_{L^4(B_{2R}\backslash B_R)} \|\phi_R\|^{\frac{1}{2}}_{L^2} \left\| \frac{\partial \phi_R}{\partial y} \right\|^{\frac{1}{4}}_{L^2(B_{2R}\backslash B_R)} \\
&\leq \in \left\| \frac{\partial u}{\partial z} \right\|^4_{L^4(B_{2R}\backslash B_R)} + \overline{K_4} \left\| \frac{\partial u}{\partial x} \right\|^{\frac{3}{2}}_{L^4(B_{2R}\backslash B_R)} \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^4(B_{2R}\backslash B_R)} \|\phi_R\|_{L^2} \left\| \frac{\partial \phi_R}{\partial y} \right\|_{L^2(B_{2R}\backslash B_R)}
\end{aligned}
$$

$$
\begin{aligned}
I_6 &\leq \overline{K_5} \left\| \frac{\partial u}{\partial z} \right\|_{L^2(B_{2R}\backslash B_R)} \left\| \left(\frac{\partial u}{\partial x} \right)^2 \right\|^{\frac{3}{4}}_{L^2(B_{2R}\backslash B_R)} \left\| \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right)^2 \right\|^{\frac{1}{2}}_{L^2(B_{2R}\backslash B_R)} \|\phi_R\|^{\frac{1}{2}}_{L^2} \left\| \frac{\partial \phi_R}{\partial y} \right\|^{\frac{1}{2}}_{L^2(B_{2R}\backslash B_R)} \\
&\leq \in \left\| \frac{\partial u}{\partial z} \right\|^2_{L^2(B_{2R}\backslash B_R)} + \overline{K_6} \left\| \left(\frac{\partial u}{\partial x} \right)^2 \right\|^{\frac{3}{2}}_{L^4(B_{2R}\backslash B_R)} \left\| \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right)^2 \right\|_{L^4(B_{2R}\backslash B_R)} \|\phi_R\|^{1}_{L^2} \left\| \frac{\partial \phi_R}{\partial y} \right\|_{L^2(B_{2R}\backslash B_R)}
\end{aligned}
$$

From Holder's inequality, Equation (8) becomes

$$
\begin{aligned}
&\left(\frac{\sigma B_0^2}{\rho_f} - \in \right) \left\| \frac{\partial u}{\partial z} \right\|^2_{L^2(B_{2R}\backslash B_R)} + \left(\nu + \frac{1}{\beta d_1 \rho_f} \right) \left\| \frac{\partial^2 u}{\partial z^2} \right\|^2_{L^2(B_{2R}\backslash B_R)} \\
&\leq -\iiint\limits_{\Omega} \left| \frac{\partial^2 u_0(x,y,z)}{\partial z^2} \right|^2 dxdydz + \left\| \frac{\partial \phi_R}{\partial z} \right\|_{L^\infty(B_{2R}\backslash B_R)} \left\| \frac{\partial u}{\partial t} \right\|_{L^2(B_{2R}\backslash B_R)} \left\| \frac{\partial u}{\partial z} \right\|_{L^2(B_{2R}\backslash B_R)} \\
&+ \overline{K_4} \left\| \frac{\partial u}{\partial x} \right\|^{\frac{3}{2}}_{L^4(B_{2R}\backslash B_R)} \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^4(B_{2R}\backslash B_R)} . \|\phi_R\|_{L^2(B_{2R}\backslash B_R)} \left\| \frac{\partial \phi_R}{\partial y} \right\|_{L^2} \\
&+ \overline{K_6} \left\| \left(\frac{\partial u}{\partial x} \right)^2 \right\|^{\frac{3}{2}}_{L^4(B_{2R}\backslash B_R)} \left\| \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right)^2 \right\|_{L^4(B_{2R}\backslash B_R)} . \|\phi_R\|_{L^2(B_{2R}\backslash B_R)} \left\| \frac{\partial \phi_R}{\partial y} \right\|_{L^2} \\
&+ \frac{1}{2} \|V\|_{L^2(B_{2R}\backslash B_R)} \left\| \frac{\partial u}{\partial z} \right\|^2_{L^4(B_{2R}\backslash B_R)} \|\nabla \phi_R\|_{L^\infty} + \left(\frac{1}{4\beta d_1^3 \rho_f} + \in \right) \left\| \frac{\partial u}{\partial z} \right\|^4_{L^4(B_{2R}\backslash B_R)}
\end{aligned}
$$

$$+\frac{1}{4\beta d_1^3\rho_f}\left\|\frac{\partial^2 u}{\partial z^2}\right\|^4_{L^4(B_{2R}\backslash B_R)}+\left\|\frac{\partial u}{\partial z}\right\|_{L^2(B_{2R}\backslash B_R)}\left\|\frac{\partial \phi_R}{\partial z}\right\|_{L^\infty}\|(V\cdot\nabla u)\|_{L^2(B_{2R}\backslash B_R)}$$

$$+\frac{\sigma B_0^2}{\rho_f}\left\|\frac{\partial \phi_R}{\partial z}\right\|_{L^\infty}\left\|\frac{\partial u}{\partial z}\right\|_{L^2(B_{2R}\backslash B_R)}\|u\|_{L^2(B_{2R}\backslash B_R)},$$

$$\leq -\iiint_\Omega\left|\frac{\partial^2 u_0(x,y,z)}{\partial z^2}\right|^2 dxdydz+\left\|\frac{\partial \phi_R}{\partial z}\right\|_{L^\infty}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(B_{2R}\backslash B_R)}\left\|\frac{\partial u}{\partial z}\right\|_{L^2(B_{2R}\backslash B_R)}$$

$$+\overline{K_4}\left\|\frac{\partial u}{\partial x}\right\|^{\frac{3}{2}}_{L^4(B_{2R}\backslash B_R)}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^4(B_{2R}\backslash B_R)}\left\|\frac{\partial \phi_R}{\partial y}\right\|_{L^2}$$

$$+\overline{K_6}\left\|\left(\frac{\partial u}{\partial x}\right)^2\right\|^{\frac{3}{2}}_{L^4(B_{2R}\backslash B_R)}\left\|\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2\right\|_{L^4(B_{2R}\backslash B_R)}\left\|\frac{\partial \phi_R}{\partial y}\right\|_{L^2}$$

$$+\frac{1}{2}\|V\|_{L^2(B_{2R}\backslash B_R)}\left\|\frac{\partial u}{\partial z}\right\|^2_{L^4(B_{2R}\backslash B_R)}\|\nabla\phi_R\|_{L^\infty}+\left(\frac{1}{4\beta d_1^3\rho_f}+\in\right)\left\|\frac{\partial u}{\partial z}\right\|^2_{L^2(B_{2R}\backslash B_R)}\left\|\frac{\partial u}{\partial z}\right\|^2_{BMO}$$

$$+\frac{1}{4\beta d_1^3\rho_f}\left\|\frac{\partial^2 u}{\partial z^2}\right\|^2_{L^2(B_{2R}\backslash B_R)}\left\|\frac{\partial^2 u}{\partial z^2}\right\|^2_{BMO}+\left\|\frac{\partial u}{\partial z}\right\|_{L^2(B_{2R}\backslash B_R)}\left\|\frac{\partial \phi_R}{\partial z}\right\|_{L^\infty(B_{2R}\backslash B_R)}\|(V\cdot\nabla u)\|_{L^2(B_{2R}\backslash B_R)}$$

$$+\frac{\sigma B_0^2}{\rho_f}\left\|\frac{\partial \phi_R}{\partial z}\right\|_{L^\infty(B_{2R}\backslash B_R)}\left\|\frac{\partial u}{\partial z}\right\|_{L^2(B_{2R}\backslash B_R)}\|u\|_{L^2(B_{2R}\backslash B_R)},$$

where we used Lemma 1, since

$$\begin{aligned}\left\|\frac{\partial \phi_R}{\partial x}\right\|_{L^\infty} &\leq \|\nabla\phi_R\|_{L^\infty},\left\|\frac{\partial \phi_R}{\partial y}\right\|_{L^\infty}\leq\|\nabla\phi_R\|_{L^\infty}\\ \left\|\frac{\partial \phi_R}{\partial z}\right\|_{L^\infty} &\leq \|\nabla\phi_R\|_{L^\infty},\ \|\nabla\phi_R\|_{L^\infty}\leq\frac{C}{R}\text{ and }\left\|\frac{\partial \phi_R}{\partial t}\right\|_{L^\infty}\leq\frac{C}{R}\end{aligned}$$

and $\left(\frac{\partial u}{\partial z},\frac{\partial^2 u}{\partial z^2}\right)\in L^2((0,\infty),BMO)$. Therefore,

$$\begin{aligned}&\left(\frac{\sigma B_0^2}{\rho_f}-\in-K_7\right)\left\|\frac{\partial u}{\partial z}\right\|^2_{L^2(B_{2R}\backslash B_R)}+(\nu-K_8)\left\|\frac{\partial^2 u}{\partial z^2}\right\|^2_{L^2(B_{2R}\backslash B_R)}\\ \leq\ &-\iiint_\Omega\left|\frac{\partial^2 u_0(x,y,z)}{\partial z^2}\right|^2 dxdydz+\frac{C\overline{K_4}}{R}\left\|\frac{\partial u}{\partial x}\right\|^{\frac{3}{2}}_{L^4(B_{2R}\backslash B_R)}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^4(B_{2R}\backslash B_R)}\\ &+\frac{C\overline{K_6}}{R}\left\|\left(\frac{\partial u}{\partial x}\right)^2\right\|^{\frac{3}{2}}_{L^4(B_{2R}\backslash B_R)}\left\|\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2\right\|_{L^4(B_{2R}\backslash B_R)}\\ &+\frac{C}{R}\left\|\frac{\partial u}{\partial z}\right\|_{L^2}\|(V\cdot\nabla u)\|_{L^2(B_{2R}\backslash B_R)}+\frac{\sigma B_0^2}{\rho_f}\frac{C}{R}\left\|\frac{\partial u}{\partial z}\right\|_{L^2}\|u\|_{L^2(B_{2R}\backslash B_R)},\end{aligned}$$

Taking $R \to \infty$, we obtain

$$\begin{aligned}\left(\frac{\sigma B_0^2}{\rho_f} - \in - K_7\right)\left\|\frac{\partial u}{\partial z}\right\|_{L^2}^2 + (\nu - K_8)\left\|\frac{\partial^2 u}{\partial z^2}\right\|_{L^2}^2 &\leq -\iiint\limits_{\Omega}\left|\frac{\partial^2 u_0(x,y,z)}{\partial z^2}\right|^2 dxdydz \\ &\leq \iiint\limits_{\Omega}\left|\frac{\partial^2 u_0(x,y,z)}{\partial z^2}\right|^2 dxdydz.\end{aligned}$$

As

$$\left(\frac{\sigma B_0^2}{\rho_f} - \in - K_7\right) \geq 0 \text{ and } (\nu - K_8) \geq 0\,,$$

therefore

$$\left\|\frac{\partial u}{\partial z}\right\|_{L^2}^2 \leq \iiint\limits_{\Omega}\left|\frac{\partial^2 u_0(x,y,z)}{\partial z^2}\right|^2 dxdydz \text{ and } \left\|\frac{\partial^2 u}{\partial z^2}\right\|_{L^2}^2 \leq \iiint\limits_{\Omega}\left|\frac{\partial^2 u_0(x,y,z)}{\partial z^2}\right|^2 dxdydz.$$

Considering Poincare's inequality,

$$K_8\|u\|_{L^2} \leq \left\|\frac{\partial u}{\partial z}\right\|_{L^2}^2 \leq \iiint\limits_{\Omega}\left|\frac{\partial^2 u_0(x,y,z)}{\partial z^2}\right|^2 dxdydz,$$

which implies that u is bounded.

Similarly, by multiplying by $\left(-\frac{\partial^2 V}{\partial z^2}\phi_R\right)$ in (3) and repeating the same process, it is possible to obtain the boundness of v. $\square$

Proof of Theorem 3. Multiplying the expression (2) by $(|u|^{p-2}u\phi_R)$, with $p > 2$, and upon integration

$$\begin{aligned}&\int\limits_{B_{2R}\setminus B_R}\frac{\partial u}{\partial t}u^{p-1}\phi_R dxdydzdt - \frac{1}{p+1}\int\limits_{B_{2R}\setminus B_R}\frac{\partial \phi_R}{\partial x}u^{p+1}dxdydzdt \\ &-\frac{1}{p}\int\limits_{B_{2R}\setminus B_R}\frac{\partial v}{\partial y}u^p\phi_R dxdydzdt - \frac{1}{p}\int\limits_{B_{2R}\setminus B_R}\frac{\partial \phi_R}{\partial y}vu^p dxdydzdt \\ &-\frac{1}{p}\int\limits_{B_{2R}\setminus B_R}\frac{\partial w}{\partial z}u^p\phi_R dxdydzdt - \frac{1}{p}\int\limits_{B_{2R}\setminus B_R}\frac{\partial \phi_R}{\partial z}wu^p dxdydzdt \\ &= -\left(\nu + \frac{1}{\beta d_1\rho_f}\right)\left[(p-1)\int\limits_{B_{2R}\setminus B_R}u^{p-2}\left(\frac{\partial u}{\partial z}\right)^2\phi_R dxdydzdt + \int\limits_{B_{2R}\setminus B_R}\frac{\partial \phi_R}{\partial z}u^{p-1}\frac{\partial u}{\partial z}dxdydzdt\right] \\ &+\frac{1}{2\beta d_1^3\rho_f}\left[\frac{1}{3}\int\limits_{B_{2R}\setminus B_R}\frac{\partial \phi_R}{\partial z}u^{p-1}\left(\frac{\partial u}{\partial z}\right)^3 dxdydzdt + \frac{(p-1)}{3}\int\limits_{B_{2R}\setminus B_R}\phi_R^{p-2}\left(\frac{\partial u}{\partial z}\right)^4 dxdydzdt\right] \\ &-\frac{\sigma B_0^2}{\rho_f}\int\limits_{B_{2R}\setminus B_R}u^p\phi_R dxdydzdt,\end{aligned}$$

which implies that

$$\int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial t} u^{p-1} \phi_R dxdydzdt = \frac{1}{p} \int\limits_{B_{2R}\setminus B_R} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) u^p \phi_R dxdydzdt$$

$$- \frac{\sigma B_0^2}{\rho_f} \int\limits_{B_{2R}\setminus B_R} u^p \phi_R \, dxdydzdt$$

$$- \left(\nu + \frac{1}{\beta d_1 \rho_f} \right) (p-1) \int\limits_{B_{2R}\setminus B_R} u^{p-2} \left(\frac{\partial u}{\partial z} \right)^2 \phi_R dxdydzdt$$

$$+ \frac{(p-1)}{6\beta d_1^3 \rho_f} \int\limits_{B_{2R}\setminus B_R} \phi_R u^{p-2} \left(\frac{\partial u}{\partial z} \right)^4 dxdydzdt$$

$$+ \frac{1}{p+1} \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial x} u^{p+1} dxdydzdt - \left(\nu + \frac{1}{\beta d_1 \rho_f} \right) \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial z} u^{p-1} \frac{\partial u}{\partial z} dxdydzdt$$

$$+ \frac{1}{6\beta d_1^3 \rho_f} \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial z} u^{p-1} \left(\frac{\partial u}{\partial z} \right)^3 dxdydzdt + \frac{1}{p} \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial y} v u^p dxdydzdt$$

$$+ \frac{1}{p} \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial z} w u^p dxdydzdt.$$

From Equation (1), the following holds

$$\int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial t} u^{p-1} \phi_R dxdydzdt = -\frac{1}{p} \int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial x} u^p \phi_R dxdydzdt - \frac{\sigma B_0^2}{\rho_f} \int\limits_{B_{2R}\setminus B_R} u^p \phi_R \, dxdydzdt$$

$$- \left(\nu + \frac{1}{\beta d_1 \rho_f} \right) (p-1) \int\limits_{B_{2R}\setminus B_R} u^{p-2} \left(\frac{\partial u}{\partial z} \right)^2 \phi_R dxdydzdt$$

$$+ \frac{(p-1)}{6\beta d_1^3 \rho_f} \int\limits_{B_{2R}\setminus B_R} \phi_R^{p-2} \left(\frac{\partial u}{\partial z} \right)^4 dxdydzdt - \left(\nu + \frac{1}{\beta d_1 \rho_f} \right) \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial z} u^{p-1} \frac{\partial u}{\partial z} dxdydzdt$$

$$+ \frac{1}{p+1} \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial x} u^{p+1} dxdydzdt + \frac{1}{p} \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial z} w u^p dxdydzdt$$

$$+ \frac{1}{6\beta d_1^3 \rho_f} \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial z} u^{p-1} \left(\frac{\partial u}{\partial z} \right)^3 dxdydzdt + \frac{1}{p} \int\limits_{B_{2R}\setminus B_R} \frac{\partial \phi_R}{\partial y} v u^p dxdydzdt.$$

By integrating the first term on the right-hand side,

$$\int\limits_{B_{2R}\setminus B_R} \frac{\partial u}{\partial t} u^{p-1} \phi_R dxdydzdt = \frac{1}{p(p+1)} \int\limits_{B_{2R}\setminus B_R} u^{p+1} \frac{\partial \phi_R}{\partial x} dxdydzdt - \frac{\sigma B_0^2}{\rho_f} \int\limits_{B_{2R}\setminus B_R} u^p \phi_R \, dxdydzdt$$

$$- \left(\nu + \frac{1}{\beta d_1 \rho_f} \right) (p-1) \int\limits_{B_{2R}\setminus B_R} u^{p-2} \left(\frac{\partial u}{\partial z} \right)^2 \phi_R dxdydzdt$$

$$+\frac{(p-1)}{6\beta d_1^3\rho_f}\int\limits_{B_{2R}\setminus B_R}\phi_R^{p-2}\left(\frac{\partial u}{\partial z}\right)^4 dxdydzdt$$

$$+\frac{1}{p+1}\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi_R}{\partial x}u^{p+1}dxdydzdt-\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi_R}{\partial z}u^{p-1}\frac{\partial u}{\partial z}dxdydzdt$$

$$+\frac{1}{6\beta d_1^3\rho_f}\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi_R}{\partial z}u^{p-1}\left(\frac{\partial u}{\partial z}\right)^3 dxdydzdt+\frac{1}{p}\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi_R}{\partial y}vu^p dxdydzdt$$

$$+\frac{1}{p}\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi_R}{\partial z}wu^p dxdydzdt.$$

Considering the integration with regard to t, we have

$$\frac{1}{p}\iiint\limits_{\Omega}|u_0(x,y,z)|^p dxdydz-\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi_R}{\partial t}u^p dxdydzdt=\frac{1}{p(p+1)}\int\limits_{B_{2R}\setminus B_R}u^{p+1}\frac{\partial\phi_R}{\partial x}dxdydzdt$$

$$-\frac{\sigma B_0^2}{\rho_f}\int\limits_{B_{2R}\setminus B_R}u^p\phi_R\,dxdydzdt-\left(\nu+\frac{1}{\beta d_1\rho_f}\right)(p-1)\int\limits_{B_{2R}\setminus B_R}u^{p-2}\left(\frac{\partial u}{\partial z}\right)^2\phi_R dxdydzdt$$

$$+\frac{(p-1)}{6\beta d_1^3\rho_f}\int\limits_{B_{2R}\setminus B_R}\phi_R^{p-2}\left(\frac{\partial u}{\partial z}\right)^4 dxdydzdt$$

$$+\frac{1}{p+1}\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi_R}{\partial x}u^{p+1}dxdydzdt-\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi_R}{\partial z}u^{p-1}\frac{\partial u}{\partial z}dxdydzdt$$

$$+\frac{1}{6\beta d_1^3\rho_f}\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi_R}{\partial z}u^{p-1}\left(\frac{\partial u}{\partial z}\right)^3 dxdydzdt+\frac{1}{p}\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi_R}{\partial y}vu^p dxdydzdt$$

$$+\frac{1}{p}\int\limits_{B_{2R}\setminus B_R}\frac{\partial\phi_R}{\partial z}wu^p dxdydzdt.$$

Now, considering Holder's inequality

$$\begin{aligned}
&\frac{\sigma B_0^2}{\rho_f}\|u\|^p_{L^p(B_{2R}\setminus B_R)}+\left(\nu+\frac{1}{\beta d_1\rho_f}\right)(p-1)\|u\|^{p-2}_{L^{2(p-2)}}\left\|\left(\frac{\partial u}{\partial z}\right)\right\|^2_{L^4(B_{2R}\setminus B_R)}\\
&\leq-\frac{1}{p}\iiint\limits_{\Omega}|u_0(x,y,z)|^p dxdydz+\left\|\frac{\partial\phi_R}{\partial t}\right\|_{L^\infty}\|u\|^p_{L^p(B_{2R}\setminus B_R)}\\
&+\frac{1}{p(p+1)}\left\|\frac{\partial\phi_R}{\partial x}\right\|_{L^\infty}\|u\|^{p-1}_{L^{2(p-2)}(B_{2R}\setminus B_R)}
\end{aligned}$$

$$+\frac{(p-1)}{6\beta d_1^3\rho_f}\|u\|^{p-2}_{L^{2(p-2)}}\left\|\left(\frac{\partial u}{\partial z}\right)\right\|^4_{L^8(B_{2R}\setminus B_R)}+\frac{1}{p+1}\left\|\frac{\partial\phi_R}{\partial x}\right\|_{L^\infty}\|u\|^{p+1}_{L^{2(p-1)}(B_{2R}\setminus B_R)}$$

$$+\frac{1}{p}\left\|\frac{\partial\phi_R}{\partial z}\right\|_{L^\infty}\|w\|_{L^2}.\|u\|^p_{L^{2p}(B_{2R}\setminus B_R)}-\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\left\|\frac{\partial\phi_R}{\partial z}\right\|_{L^\infty}\|u\|^{p-1}_{L^{2(p-1)}}\left\|\frac{\partial u}{\partial z}\right\|_{L^2(B_{2R}\setminus B_R)}$$

$$+\frac{1}{6\beta d_1^3\rho_f}\left\|\frac{\partial\phi_R}{\partial z}\right\|_{L^\infty}\|u\|_{L^{2(p-1)}}^{p-1}\left\|\left(\frac{\partial u}{\partial z}\right)\right\|_{L^6(B_{2R}\setminus B_R)}^3+\frac{1}{p}\left\|\frac{\partial\phi_R}{\partial y}\right\|_{L^\infty}\|v\|_{L^2}\|u\|_{L^{2p}(B_{2R}\setminus B_R)}^p. \quad (9)$$

Since

$$\begin{aligned}\left\|\frac{\partial\phi_R}{\partial x}\right\|_{L^\infty} &\le \|\nabla\phi_R\|_{L^\infty}, \left\|\frac{\partial\phi_R}{\partial y}\right\|_{L^\infty}\le\|\nabla\phi_R\|_{L^\infty},\\ \left\|\frac{\partial\phi_R}{\partial z}\right\|_{L^\infty} &\le \|\nabla\phi_R\|_{L^\infty},\ \|\nabla\phi_R\|_{L^\infty}\le\frac{C}{R},\text{ and }\left\|\frac{\partial\phi_R}{\partial t}\right\|_{L^\infty}\le\frac{C}{R}\end{aligned}$$

and using Lemma 1, expression (9) becomes

$$\begin{aligned}&\frac{\sigma B_0^2}{\rho_f}\|u\|_{L^p(B_{2R}\setminus B_R)}^p+\left(\nu+\frac{1}{\beta d_1\rho_f}\right)(p-1)\|u\|_{L^{2(p-2)}}^{p-2}\left\|\left(\frac{\partial u}{\partial z}\right)\right\|_{L^4(B_{2R}\setminus B_R)}^2\\ &\le-\frac{1}{p}\iiint\limits_\Omega|u_0(x,y,z)|^p dxdydz+\frac{C}{R}\|u\|_{L^p(B_{2R}\setminus B_R)}^p+\frac{C}{p(p+1)R}\|u\|_{L^{2(p-2)}(B_{2R}\setminus B_R)}^{p-1}\\ &\quad+\frac{(p-1)}{6\beta d_1^3\rho_f}.\|u\|_{L^{2(p-2)}}^{p-2}\left\|\left(\frac{\partial u}{\partial z}\right)\right\|_{L^4(B_{2R}\setminus B_R)}^2\left\|\left(\frac{\partial u}{\partial z}\right)\right\|_{BMO}^2\\ &\quad+\frac{C}{R(p+1)}\|u\|_{L^{2(p-1)}(B_{2R}\setminus B_R)}^{p+1}+\frac{C}{pR}\|w\|_{L^2}.\|u\|_{L^{2p}(B_{2R}\setminus B_R)}^p\\ &\quad-\left(\nu+\frac{1}{\beta d_1\rho_f}\right)\frac{C}{R}\|u\|_{L^{2(p-1)}}^{p-1}\left\|\frac{\partial u}{\partial z}\right\|_{L^2(B_{2R}\setminus B_R)}\\ &\quad+\frac{C}{6R\beta d_1^3\rho_f}\|u\|_{L^{2(p-1)}}^{p-1}\left\|\left(\frac{\partial u}{\partial z}\right)\right\|_{L^6(B_{2R}\setminus B_R)}^3+\frac{C}{pR}\|v\|_{L^2}.\|u\|_{L^{2p}(B_{2R}\setminus B_R)}^p.\end{aligned}$$

Since $\frac{\partial u}{\partial z}\in L^2((0,\infty),BMO)$ and considering $R\to\infty$,

$$\begin{aligned}&\frac{\sigma B_0^2}{\rho_f}\|u\|_{L^p}^p+(p-1)\left(\nu+\frac{1}{\beta d_1\rho_f}-\frac{K_8}{6\beta d_1^3\rho_f}\right)\|u\|_{L^{2(p-2)}}^{p-2}\left\|\left(\frac{\partial u}{\partial z}\right)\right\|_{L^4}^2\\ &\le-\frac{1}{p}\iiint\limits_\Omega|u_0(x,y,z)|^p dxdydz\le\frac{1}{p}\iiint\limits_\Omega|u_0(x,y,z)|^p dxdydz.\end{aligned}$$

As

$$\left[\left(\nu+\frac{1}{\beta d_1\rho_f}\right)(p-1)-\frac{C_1(p-1)}{6\beta d_1^3\rho_f}\right]\ge0\text{ and }\frac{\sigma B_0}{\rho_f}\ge0,$$

therefore

$$\|u\|_{L^{2(p-2)}}^{p-2}\left\|\left(\frac{\partial u}{\partial z}\right)\right\|_{L^4}^2\le\frac{1}{p}\iiint\limits_\Omega|u_0(x,y,z)|^p dxdydz$$

and

$$\|u\|_{L^p}^p\le\frac{1}{p}\iiint\limits_\Omega|u_0(x,y,z)|^p dxdydz,$$

which implies that u is bounded. Similarly, multiplying by $(|v|^{p-2}v\phi_R)$ in the expression (3) and repeating the same process, we find that v is bounded. □

4. Conclusions

The postulated theorems, Theorems 1–3, have been shown based on the supporting lemmas mentioned along with the proofs. Such theorems have allowed us to comment on the regularity results for an Eyring–Powell fluid with stretching velocities in a plane Ω_p and flowing along the z-axis. The results have been provided for globally bounded initial data as well as their corresponding derivatives and while the variations of velocity along the z-axis belong to the space L^2 and BMO. The regularity results have been given for the general spaces L^2 and L^p, $p > 2$.

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