mathematics

Article

The Natural Approaches of Shafer-Fink Inequality for Inverse Sine Function

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Abstract: In this paper, we obtain some new natural approaches of Shafer-Fink inequality for arc sine function and the square of arc sine function by using the power series expansions of certain functions, which generalize and strengthen those in the existing literature.

Keywords: sharp double inequality of Shafer-Fink; arc sine function; the square of arc sine function

Citation: Zhu, L. The Natural Approaches of Shafer-Fink Inequality for Inverse Sine Function. *Mathematics* **2022**, *10*, 647. https://doi.org/10.3390/math10040647

Academic Editor: Ioannis K. Argyros

Received: 11 February 2022
Accepted: 18 February 2022
Published: 19 February 2022

1. Introduction

Fink [1] (or see [2]) shown a upper bound for inverse sine function, and obtained the following result, which is called Shafer-Fink inequality:

$$\frac{3x}{2+\sqrt{1-x^2}} \leq \arcsin x \leq \frac{\pi x}{2+\sqrt{1-x^2}},\ 0 \leq x \leq 1. \tag{1}$$

Some new proof and various improvements of Shafer-Fink inequality can be found in [3–13]. In [14], Bercu obtained the generalizations and refinements of Shafer-Fink inequality as follows.

Proposition 1 ([14] Theorem 1)**.** *For every real number $0 \leq x \leq 1$, the following two-sided inequality holds:*

$$\frac{x^5}{180}+\frac{x^7}{189} \leq \arcsin x - \frac{3x}{2+\sqrt{1-x^2}} \leq \frac{\pi-3}{2}. \tag{2}$$

Proposition 2 ([14] Theorem 2)**.** *For every $0 \leq x \leq 1$, we have:*

$$\frac{x^5/60+11x^7/840}{2+\sqrt{1-x^2}} \leq \arcsin x - \frac{3x}{2+\sqrt{1-x^2}}. \tag{3}$$

Maleševic, Rašajski and Lutovac [15] gave a lower bound for the function $\arcsin x$ as follows.

Proposition 3 ([15] Theorem 2)**.** *If $m \in \mathbb{N}$ and $m \geq 2$, then*

$$\frac{3x+\sum_{n=2}^{m} E(n)x^{2n+1}}{2+\sqrt{1-x^2}} \leq \arcsin x \tag{4}$$

for every $x \in [0,1]$, where

$$E(n) = \frac{n(2n-1)!}{(2n+1)2^{2n-2}(n!)^2} - \frac{2n\cdot 2^{2n-2}((n-1)!)^2}{(2n+1)!},\ n \in \mathbb{N},\ n \geq 2. \tag{5}$$

At this point, it is necessary for us to recall the results of Zhu [7] :

Proposition 4 ([7] Theorem 6)**.** *Let $0 < x < 1$. Then,* *(1) when $p \geq 1$ or $p < 0$, the double inequality*

$$\frac{3x^p}{2+\left(\sqrt{1-x^2}\right)^p} \leq (\arcsin x)^p \leq \frac{(\pi x)^p}{2^p+(\pi^p-2^p)\left(\sqrt{1-x^2}\right)^p} \tag{6}$$

holds;

(2) when $0 \leq p \leq 4/5$, the double inequality (6) is reversed.

Inspired by the above approximation inequalities, we consider the asymptotic expansions of the functions $\left(2+\sqrt{1-x^2}\right)\arcsin x$ and $(3-x^2)(\arcsin x)^2$ to establish some new bilateral approximation of Shafer-Fink inequality, and give some deeper conclusions drawn for $\arcsin x$ and $(\arcsin x)^2$.

Theorem 1. *Let $|x| \leq 1$, $\{a_n\}_{n\geq 0}$ be defined by*

$$a_0 = 3;\ a_n = \left[\frac{(2n-1)!!}{2^{n-1}n!} - \frac{2^{n-1}n!}{n(2n-1)!!}\right]\frac{1}{2n+1},\ n \geq 1, \tag{7}$$

$m \in \mathbb{N}$, $m \geq 2$, $\alpha_m = a_m$ and $\beta_m = \pi - \sum_{n=0}^{m-1} a_n$. Then,

(i) when $0 \leq x \leq 1$, the double inequality

$$\frac{\sum_{n=0}^{m-1} a_n x^{2n+1} + \alpha_m x^{2m+1}}{2+\sqrt{1-x^2}} \leq \arcsin x \leq \frac{\sum_{n=0}^{m-1} a_n x^{2n+1} + \beta_m x^{2m+1}}{2+\sqrt{1-x^2}} \tag{8}$$

holds with the best constants α_m and β_m;

(ii) when $-1 \leq x \leq 0$, the double inequality

$$\frac{\sum_{n=0}^{m-1} a_n x^{2n+1} + \beta_m x^{2m+1}}{2+\sqrt{1-x^2}} \leq \arcsin x \leq \frac{\sum_{n=0}^{m-1} a_n x^{2n+1} + \alpha_m x^{2m+1}}{2+\sqrt{1-x^2}} \tag{9}$$

holds with the best constants β_m and α_m.

Theorem 2. *Let $|x| \leq 1$, $\{b_n\}_{n\geq 1}$ be defined by*

$$b_1 = 3;\ b_n = \frac{(n-2)(4n-3)(n-2)!2^{n-2}}{n(n-1)(2n-1)!!} \geq 0,\ n \geq 2, \tag{10}$$

$m \in \mathbb{N}$, $m \geq 3$, $\lambda_m = b_m$ and $\mu_m = \pi^2/2 - \sum_{n=0}^{m-1} b_n$. Then, the double inequality

$$\frac{\sum_{n=1}^{m-1} b_n x^{2n} + \lambda_m x^{2m}}{3-x^2} \leq (\arcsin x)^2 \leq \frac{\sum_{n=1}^{m-1} b_n x^{2n} + \mu_m x^{2m}}{3-x^2} \tag{11}$$

holds with the best constants λ_m and μ_m.

2. Lemmas

This article needs the following two lemmas.

Lemma 1 ([16–20]). *For $|x| < 1$,*

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{(2x)^{2n-1}}{n\binom{2n}{n}}. \tag{12}$$

Integrating the functions on both sides of the inequality (12) from 0 to x, we have

Lemma 2 ([18,20])**.** *For* $|x| < 1$,

$$(\arcsin x)^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n^2\binom{2n}{n}}. \tag{13}$$

3. Proof of Theorem 1

Since (8) and (9) hold for $x = 0$, we assume that $0 < |x| \leq 1$ to discuss problems below. Let

$$f(x) = \left(2 + \sqrt{1 - x^2}\right)\arcsin x. \tag{14}$$

Then, when $0 < |x| < 1$, by Lemma 1 we have

$$\begin{aligned} f'(x) &= \frac{2}{\sqrt{1-x^2}} + 1 - x\frac{\arcsin x}{\sqrt{1-x^2}} \\ &= 2\left[1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}x^{2n}\right] + 1 - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n\binom{2n}{n}} \\ &= 3 + \sum_{n=1}^{\infty}\left[\frac{2(2n-1)!!}{2^n n!} - \frac{2^{2n-1}}{n\binom{2n}{n}}\right]x^{2n}. \end{aligned} \tag{15}$$

Integrating (15) from 0 to x, we have

$$\begin{aligned} f(x) &= \int_0^x f'(t)dt = 3x + \sum_{n=1}^{\infty}\left[\frac{2(2n-1)!!}{2^n n!} - \frac{2^{2n-1}}{n\binom{2n}{n}}\right]\frac{1}{2n+1}x^{2n+1} \\ &= 3x + \sum_{n=1}^{\infty}\left[\frac{2(2n-1)!!}{2^n n!} - \frac{2^{2n-1}}{n\frac{2^n(2n-1)!!}{n!}}\right]\frac{1}{2n+1}x^{2n+1} \\ &= 3x + \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^{n-1}n!} - \frac{2^{n-1}n!}{n(2n-1)!!}\right]\frac{1}{2n+1}x^{2n+1} \\ : &= \sum_{n=0}^{\infty}a_n x^{2n+1}, \end{aligned} \tag{16}$$

where a_n is defined by (7). Clearly, it is easy to prove $a_n \geq 0$ for $n \geq 1$, among them, $a_1 = 0$.

Now, we go into the following even function

$$\begin{aligned} F(x) &= \frac{\left(2 + \sqrt{1-x^2}\right)\arcsin x - \sum_{n=0}^{m-1}a_n x^{2n+1}}{x^{2m+1}} \\ &= \frac{f(x) - \sum_{n=0}^{m-1}a_n x^{2n+1}}{x^{2m+1}} = \frac{\sum_{n=m}^{\infty}a_n x^{2n+1}}{x^{2m+1}} \\ &= \sum_{n=m}^{\infty}a_n x^{2(n-m)} = a_m + \sum_{n=m+1}^{\infty}a_n x^{2(n-m)}, \end{aligned}$$

which is decreasing on $[-1, 0)$ and increasing on $(0, 1]$. Since

$$F(0^{\pm}) = a_m := \alpha_m, F(\pm 1) = \pi - \sum_{n=0}^{m-1}a_n := \beta_m,$$

we have

$$a_m < \frac{\left(2 + \sqrt{1-x^2}\right)\arcsin x - \sum_{n=0}^{m-1}a_n x^{2n+1}}{x^{2m+1}} < \pi - \sum_{n=0}^{m-1}a_n,\ 0 < |x| \leq 1,$$

or

$$a_m x^{2m+1} < \left(2+\sqrt{1-x^2}\right)\arcsin x - \sum_{n=0}^{m-1} a_n x^{2n+1} < \left(\pi - \sum_{n=0}^{m-1} a_n\right) x^{2m+1},\ 0 < x \le 1,$$

$$\left(\pi - \sum_{n=0}^{m-1} a_n\right) x^{2m+1} < \left(2+\sqrt{1-x^2}\right)\arcsin x - \sum_{n=0}^{m-1} a_n x^{2n+1} < a_m x^{2m+1},\ -1 \le x < 0.$$

So the proof of Theorem 1 is complete.

4. Proof of Theorem 2

Let

$$g(x) = \left(3-x^2\right)(\arcsin x)^2. \tag{17}$$

Then, by Lemma 2 and (17),

$$\begin{aligned} g(x) &= \left(3-x^2\right)(\arcsin x)^2 = \left(3-x^2\right)\frac{1}{2}\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2 \frac{2^n(2n-1)!!}{n!}} \\ &= \left(3-x^2\right)\sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)!}{n(2n-1)!!} x^{2n} \\ &= 3x^2 + \sum_{n=2}^{\infty} \frac{(n-2)(4n-3)(n-2)!2^{n-2}}{n(n-1)(2n-1)!!} x^{2n} \\ &= 3x^2 + \sum_{n=3}^{\infty} \frac{(n-2)(4n-3)(n-2)!2^{n-2}}{n(n-1)(2n-1)!!} x^{2n} \\ &: = \sum_{n=1}^{\infty} b_n x^{2n}, \end{aligned}$$

where

$$b_1 = 3;\ b_n = \frac{(n-2)(4n-3)(n-2)!2^{n-2}}{n(n-1)(2n-1)!!} \ge 0, n \ge 2,$$

among them, $b_2 = 0$.

Let

$$G(x) = \frac{(3-x^2)(\arcsin x)^2 - \sum_{n=1}^{m-1} b_n x^{2n}}{x^{2m}} = \frac{g(x) - \sum_{n=1}^{m-1} b_n x^{2n}}{x^{2m}}.$$

Then,

$$G(x) = \frac{\sum_{n=m}^{\infty} b_n x^{2n}}{x^{2m}} = \sum_{n=m}^{\infty} b_n x^{2(n-m)},$$

which is increasing on $(0,1]$.

Since

$$G(0^+) = b_m := \lambda_m,$$

and

$$\begin{aligned} G(1) &= \lim_{x\to 1} \frac{(3-x^2)(\arcsin x)^2 - \sum_{n=1}^{m-1} b_n x^{2n}}{x^{2m}} \\ &= \frac{\pi^2}{2} - \sum_{n=1}^{m-1} b_n := \mu_m, \end{aligned}$$

we have

$$b_m \le \frac{(3-x^2)(\arcsin x)^2 - \sum_{n=1}^{m-1} b_n x^{2n}}{x^{2m}} \le \frac{\pi^2}{2} - \sum_{n=1}^{m-1} b_n,$$

or

$$b_m x^{2m} \le \left(3 - x^2\right)(\arcsin x)^2 - \sum_{n=1}^{m-1} b_n x^{2n} \le \left(\frac{\pi^2}{2} - \sum_{n=1}^{m-1} b_n\right) x^{2m},$$

that is,

$$\sum_{n=1}^{m} b_n x^{2n} \le \left(3 - x^2\right)(\arcsin x)^2 \le \sum_{n=1}^{m-1} b_n x^{2n+1} + \left(\frac{\pi^2}{2} - \sum_{n=1}^{m-1} b_n\right) x^{2m},$$

which implies (11). The proof of Theorem 2 is complete.

5. Corollaries and Remarks

In this section, we draw some new conclusions from Theorems 1 and 2, and compare the results of Theorem 1 with the ones in the literature on the same interval $[0, 1]$.

Remark 1. *The left-hand side inequality of (8) is just the inequality (4) due to $a_n = E(n)$ for $n \ge 0$. Obviously, the expression of a_n in (7) is simpler than $E(n)$ in (5). Most importantly, the method of this paper is simple and direct, and the bilateral sharp inequality is obtained.*

From Theorem 1, we can obtain the following results.

Corollary 1. *Let $0 \le x \le 1$,*

$$\begin{aligned}
A_5(x) &= \frac{1}{60}x^5, \\
B_5(x) &= (\pi - 3)x^5; \\
A_7(x) &= \frac{1}{60}x^5 + \frac{11}{840}x^7, \\
B_7(x) &= \frac{1}{60}x^5 + \left(\pi - \frac{181}{60}\right)x^7; \\
A_9(x) &= \frac{1}{60}x^5 + \frac{11}{840}x^7 + \frac{67}{6720}x^9, \\
B_9(x) &= \frac{1}{60}x^5 + \frac{11}{840}x^7 + \left(\pi - \frac{509}{168}\right)x^9;
\end{aligned}$$

$$\begin{aligned}
A_{11}(x) &= \frac{1}{60}x^5 + \frac{11}{840}x^7 + \frac{67}{6720}x^9 + \frac{3461}{443,520}x^{11}, \\
B_{11}(x) &= \frac{1}{60}x^5 + \frac{11}{840}x^7 + \frac{67}{6720}x^9 + \left(\pi - \frac{6809}{2240}\right)x^{11};
\end{aligned}$$

$$\begin{aligned}
A_{13}(x) &= \frac{1}{60}x^5 + \frac{11}{840}x^7 + \frac{67}{6720}x^9 + \frac{3461}{443,520}x^{11} + \frac{29,011}{4,612,608}x^{13}, \\
B_{13}(x) &= \frac{1}{60}x^5 + \frac{11}{840}x^7 + \frac{67}{6720}x^9 + \frac{3461}{443,520}x^{11} + \left(\pi - \frac{1,351,643}{443,520}\right)x^{13};
\end{aligned}$$

$$\begin{aligned}
A_{15}(x) &= \frac{x^5}{60} + \frac{11x^7}{840} + \frac{67x^9}{6720} + \frac{3461x^{11}}{443,520} + \frac{29,011x^{13}}{4,612,608} + \frac{239,711x^{15}}{46,126,080}, \\
B_{15}(x) &= \frac{x^5}{60} + \frac{11x^7}{840} + \frac{67x^9}{6720} + \frac{3461x^{11}}{443,520} + \frac{29,011x^{13}}{4,612,608} + \left(\pi - \frac{70,430,491}{23,063,040}\right)x^{15}.
\end{aligned}$$

Then,

$$A_{2m+1}(x) \le \arcsin x - \frac{3x}{2 + \sqrt{1 - x^2}} \le B_{2m+1}(x),\ m = \overline{2,7}. \tag{18}$$

Corollary 2. *Let* $0 \le x \le 1$*,* a_n *defined by* (7)*,* $f(x)$ *showed in* (14)*, and*

$$S(x) = \arcsin x - \frac{3x}{2+\sqrt{1-x^2}} = \frac{f(x)-3x}{2+\sqrt{1-x^2}},$$

$$A_{2m+1}(x) = \sum_{n=2}^{m} a_n x^{2n+1}, B_{2m+1}(x) = \sum_{n=2}^{m-1} a_n x^{2n+1} + \beta_m x^{2m+1},\ m \ge 2.$$

Then, for $2 \le p < q$*,*

$$\frac{A_{2p+1}(x)}{2+\sqrt{1-x^2}} \le \frac{A_{2q+1}(x)}{2+\sqrt{1-x^2}} \le S(x) \le \frac{B_{2q+1}(x)}{2+\sqrt{1-x^2}} \le \frac{B_{2p+1}(x)}{2+\sqrt{1-x^2}}. \tag{19}$$

The left-hand side inequality of (19) holds for all $x \in [0,1]$ due to $a_n \ge 0$, and the light-hand side inequality of (19) holds just due to $0 \le x \le 1$.

Remark 2. *Taking* $m = 2$ *in* (18) *gives*

$$\arcsin x - \frac{3x}{2+\sqrt{1-x^2}} \le \frac{(\pi-3)x^5}{2+\sqrt{1-x^2}},$$

which is sharper than the light-hand side one of (2) *due to*

$$\frac{(\pi-3)x^5}{2+\sqrt{1-x^2}} \le \frac{\pi-3}{2+\sqrt{1-x^2}} \le \frac{\pi-3}{2}.$$

So, by (19) *we have*

$$\arcsin x - \frac{3x}{2+\sqrt{1-x^2}} \le \frac{B_{2m+1}(x)}{2+\sqrt{1-x^2}} \le \frac{\pi-3}{2},\ m \ge 2. \tag{20}$$

Remark 3. *Taking* $m = 3$ *in* (18) *gives* (3)*. We can find that this inequality is sharper than the left-hand side one of* (2)*:*

$$\begin{aligned} & \frac{x^5}{60} + \frac{11x^7}{840} - \left(\frac{x^5}{180} + \frac{x^7}{189}\right)\left(2+\sqrt{1-x^2}\right) \\ = & \frac{x^5}{7560}\left(19x^2 + 42 - \left(42+40x^2\right)\sqrt{1-x^2}\right) \ge 0 \\ \iff & 19x^2 + 42 \ge \left(42+40x^2\right)\sqrt{1-x^2}. \end{aligned}$$

In fact,

$$\left(19x^2+42\right)^2 - \left(42+40x^2\right)^2\left(1-x^2\right) = x^4\left(1600x^2+2121\right) > 0.$$

From Theorem 2, we can obtain the following results.

Corollary 3. *Let* $0 \le x \le 1$*,*

$$\begin{aligned} C_6(x) &= \frac{1}{5}x^6, D_6(x) = \left(\frac{\pi^2}{2} - 3\right)x^6; \\ C_8(x) &= \frac{1}{5}x^6 + \frac{52}{315}x^8, D_8(x) = \frac{1}{5}x^6 + \left(\frac{\pi^2}{2} - \frac{16}{5}\right)x^8; \\ C_{10}(x) &= \frac{1}{5}x^6 + \frac{52}{315}x^8 + \frac{68}{525}x^{10}, D_{10}(x) = \frac{1}{5}x^6 + \frac{52}{315}x^8 + \left(\frac{\pi^2}{2} - \frac{212}{63}\right)x^{10}, \end{aligned}$$

$$\begin{aligned} C_{12}(x) &= \frac{1}{5}x^6 + \frac{52}{315}x^8 + \frac{68}{525}x^{10} + \frac{256}{2475}x^{12}, \\ D_{12}(x) &= \frac{1}{5}x^6 + \frac{52}{315}x^8 + \frac{68}{525}x^{10} + \left(\frac{\pi^2}{2} - \frac{5504}{1575}\right)x^{12}, \end{aligned}$$

$$\begin{aligned} C_{14}(x) &= \frac{1}{5}x^6 + \frac{52}{315}x^8 + \frac{68}{525}x^{10} + \frac{256}{2475}x^{12} + \frac{16,000}{189,189}x^{14}, \\ D_{14}(x) &= \frac{1}{5}x^6 + \frac{52}{315}x^8 + \frac{68}{525}x^{10} + \frac{256}{2475}x^{12} + \left(\frac{\pi^2}{2} - \frac{62,336}{17,325}\right)x^{14}, \end{aligned}$$

$$\begin{aligned} C_{16}(x) &= \frac{x^6}{5} + \frac{52x^8}{315} + \frac{68x^{10}}{525} + \frac{256x^{12}}{2475} + \frac{16,000x^{14}}{189,189} + \frac{7424x^{16}}{105,105}, \\ D_{16}(x) &= \frac{x^6}{5} + \frac{52x^8}{315} + \frac{68x^{10}}{525} + \frac{256x^{12}}{2475} + \frac{16,000x^{14}}{189,189} + \left(\frac{\pi^2}{2} - \frac{1640,418,688}{468,242,775}\right)x^{16}. \end{aligned}$$

Then,

$$C_{2m}(x) \le (\arcsin x)^2 - \frac{3x^2}{3-x^2} \le D_{2m}(x),\ m = \overline{3,8}. \tag{21}$$

Corollary 4. *Let $0 \le x \le 1$, b_n defined by (9), $g(x)$ showed in (17), and*

$$T(x) = (\arcsin x)^2 - \frac{3x^2}{3-x^2} = \frac{g(x) - 3x^2}{3-x^2},$$

$$C_{2m}(x) = \sum_{n=3}^{m} b_n x^{2n}, D_{2m}(x) = \sum_{n=3}^{m-1} b_n x^{2n} + \mu_m x^{2m},\ m \ge 3.$$

Then, for $3 \le p < q$,

$$\frac{C_{2p}(x)}{3-x^2} \le \frac{C_{2q}(x)}{3-x^2} \le T(x) \le \frac{D_{2q}(x)}{3-x^2} \le \frac{D_{2q}(x)}{3-x^2}. \tag{22}$$

The left-hand side inequality of (22) holds for all $x \in [0,1]$ due to $b_n \ge 0$, and the light-hand side inequality of (22) holds just due to $0 \le x \le 1$.

Remark 4. *Taking $m = 3$ in (21) gives*

$$(\arcsin x)^2 - \frac{3x^2}{3-x^2} \le \left(\frac{\pi^2}{2} - 3\right)x^6.$$

So, by (22) we have

$$(\arcsin x)^2 - \frac{3x^2}{3-x^2} \le \frac{D_{2m}(x)}{3-x^2} \le \left(\frac{\pi^2}{2} - 3\right)x^6 \le \frac{\pi^2}{2} - 3,\ m \ge 3. \tag{23}$$

Remark 5. *In the process of proving Theorems 1 and 2, we prove that $a_n, b_n > 0$, which just meet a condition in a theorem called "Theorem on double-sided TAYLOR's approximations" (see [21] (Theorem 4), [22] (Theorem 2), [23] (Theorem 22)). Therefore, the proofs of Theorems 1 and 2 can be completed by "Theorem on double-sided TAYLOR's approximations".*

6. Conclusions

Throughout the history of mathematics, function estimation is widely used in various fields of mathematics, including engineering mathematics. In this paper, we have given the power series truncation of the correlation functions of the ones $\arcsin x$ and $(\arcsin x)^2$ as their upper and lower bounds. Based on these basic conclusions, we have drawn a large number of practical estimates about $\arcsin x$ and $(\arcsin x)^2$.

Funding: This research received no external funding.

Conflicts of Interest: The author declares that he has no conflict of interest.

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