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Yamabe Solitons on Some Conformal Almost Contact B-Metric Manifolds

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Abstract: A Yamabe soliton is defined on an arbitrary almost-contact B-metric manifold, which is obtained by a contact conformal transformation of the Reeb vector field, its dual contact 1-form, the B-metric, and its associated B-metric. The cases when the given manifold is cosymplectic or Sasaki-like are studied. In this manner, manifolds are obtained that belong to one of the main classes of the studied manifolds. The same class contains the conformally equivalent manifolds of cosymplectic manifolds by the usual conformal transformation of the B-metric on contact distribution. In both cases, explicit five-dimensional examples are given, which are characterized in relation to the results obtained.

Keywords: Yamabe soliton; almost contact B-metric manifold; almost contact complex Riemannian manifold; Sasaki-like manifold

MSC: 53C25; 53D15; 53C50; 53C44; 53D35; 70G45

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1. Introduction

Richard Hamilton introduced the notion of the Yamabe flow in 1988 (see [1]) as an apparatus for constructing metrics with constant scalar curvature. The significance of this problem from the point of view of mathematical physics is that Yamabe flow corresponds to the case of fast diffusion of the porous medium [2].

It is said that a time-dependent metric $g(t)$ on a (pseudo-)Riemannian manifold (M, g) is deformed by Yamabe flow if g satisfies the following equation:

$$\frac{\partial}{\partial t} g(t) = -\tau(t) g(t), \qquad g(0) = g_0,$$

where $\tau(t)$ is the scalar curvature of $g(t)$. Yamabe flow can be interpreted as a process that deforms the metric of M to a conformal metric of constant scalar curvature as this flow converges. The latter metric is one of the fixed points of the Yamabe flow in the given conformal class.

If M is 2-dimensional, the Yamabe flow is equivalent to Ricci flow [3], defined by $\frac{\partial}{\partial t} g(t) = -2\rho(t)$, where $\rho(t)$ denotes the Ricci tensor of $g(t)$. In contrast, when M has a dimension greater than 2, these two types of flows do not match because Yamabe flow preserves the conformal class of $g(t)$, but the Ricci flow generally does not.

A self-similar solution of the Yamabe flow, defined on a Riemannian manifold or a pseudo-Riemannian manifold, is called a Yamabe soliton and is determined by the following ([4]):

$$\frac{1}{2}\mathcal{L}_v g = (\tau - \sigma) g,$$

where $\mathcal{L}_v g$ stands for the Lie derivative of the metric g along the vector field v, called the soliton potential, τ denotes the scalar curvature of g and σ is a constant known as the soliton constant. The properties of the Yamabe soliton depend on the additional tensor structure of the studied manifold.

Many authors have studied Yamabe solitons on different types of manifolds since the introduction of this concept (see, e.g., [5–9]).

In the present paper, we begin the study of the mentioned Yamabe solitons on almost contact B-metric manifolds. The geometry of these manifolds is significantly influenced by the presence of two B-metrics, which are interconnected by the almost contact structure.

It is a well-known fact that the Yamabe flow preserves the conformal class of the metric. Thus, this gives us a reason to study Yamabe solitons and conformal transformations together. Contact conformal transformations were studied in [10], which transform not only the metric but also the Reeb vector field and its associated contact 1-form through the pair of B-metrics. According to this work, the class of almost contact B-metric manifolds, which are closed under the action of contact conformal transformations, is the direct sum of the four main classes among the eleven basic classes of these manifolds known from the classification of Ganchev–Mihova–Gribachev in [11]. Main classes are called those for which their manifolds are characterized by the fact that the covariant derivative of the structure tensors with respect to the Levi–Civita connection of some of the B-metrics is expressed only by a pair of B-metrics and the corresponding traces.

We study Yamabe solitons on two of the simplest types of manifolds among those studied, namely cosymplectic and Sasaki-like. The former have parallel structure tensors with respect to the Levi–Civita connections of the B-metrics. The latter are those for which their complex cone is a Kähler manifold with a pair of Norden metrics, i.e., again with parallel structure tensors with respect to the Levi–Civita connections of the metrics. Note that the class of Sasaki-like manifolds does not contain cosymplectic manifolds, although they are in each of the eleven basic classes of the classification used.

We find that the manifolds thus constructed in both cases belong to one of the main classes. This class is the only one that contains the conformally equivalent manifolds of the cosymplectic ones by the usual conformal transformations on contact distribution.

The present paper is organized as follows. After the present introductory words, Section 2 is devoted to the basic concepts of almost contact B-metric manifolds, contact conformal transformations of the structure tensors on them, and the introduction of the notion of a Yamabe soliton on a transformed almost contact B-metric manifold. In Section 3 and Section 4, we study the constructed manifolds when the initial manifold is cosymplectic and Sasaki-like, respectively. In Section 5, we provide two explicit examples for each of considered two cases as five-dimensional manifolds equipped with the structures studied.

2. Almost Contact B-Metric Manifolds, Contact Conformal Transformations, and Yamabe Solitons

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact manifold with B-metric or an almost contact B-metric manifold, i.e., M is a $(2n+1)$-dimensional differentiable pseudo-Riemannian manifold with metric g of signature $(n+1, n)$, endowed with an almost contact structure (φ, ξ, η) consisting of an endomorphism φ of the tangent bundle, a vector field ξ, its dual 1-form η such that the following algebraic relations are satisfied [11]:

$$\varphi\xi = 0, \quad \varphi^2 = -\iota + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \tag{1}$$

$$g(X, Y) = -g(\varphi X, \varphi Y) + \eta(X)\eta(Y) \tag{2}$$

for arbitrary X and Y of the algebra $\mathfrak{X}(M)$ on the smooth vector fields on M, where ι stands for the identity transformation on $\mathfrak{X}(M)$.

The manifolds $(M, \varphi, \xi, \eta, g)$ are also known as almost contact complex Riemannian manifolds (see, e.g., [12]).

Furthermore, X, Y, and Z will stand for arbitrary elements of $\mathfrak{X}(M)$ or vectors in the tangent space T_pM of M at an arbitrary point p in M.

The associated B-metric $\tilde{g}$ of g on M is defined by the following:

$$\tilde{g}(X,Y) = g(X,\varphi Y) + \eta(X)\eta(Y)$$

and it is also of signature $(n+1,n)$ as its counterpart g.

The fundamental tensor F of type (0,3) on (M,φ,ξ,η,g) is defined by the following:

$$F(X,Y,Z) = g\big((\nabla_X\varphi)Y, Z\big),$$

where ∇ is the Levi–Civita connection of g. The following properties of F are consequences of (1) and (2):

$$\begin{aligned} F(X,Y,Z) &= F(X,Z,Y) \\ &= F(X,\varphi Y,\varphi Z) + \eta(Y)F(X,\xi,Z) + \eta(Z)F(X,Y,\xi) \end{aligned}$$

and the relations of F with $\nabla\xi$ and $\nabla\eta$ are as follows.

$$(\nabla_X\eta)Y = g(\nabla_X\xi, Y) = F(X,\varphi Y,\xi). \tag{3}$$

The following 1-forms, known as Lee forms of the manifold, are associated with F:

$$\theta = g^{ij}F(E_i,E_j,\cdot), \qquad \theta^* = g^{ij}F(E_i,\varphi E_j,\cdot), \qquad \omega = F(\xi,\xi,\cdot),$$

where g^{ij} are the components of the inverse matrix of g with respect to a basis $\{E_i;\xi\}$ $(i = 1,2,\ldots,2n)$ of T_pM. The following general identities for the Lee forms of $(M,\varphi,\xi,\eta,$ $g)$ are known from the following [13].

$$\theta^*\circ\varphi = -\theta\circ\varphi^2, \qquad \omega(\xi) = 0. \tag{4}$$

A classification of the almost contact B-metric manifolds in terms of F is given in [11]. This classification includes eleven basic classes $\mathcal{F}_i$, $i\in\{1,2,\ldots,11\}$. Their intersection is the special class $\mathcal{F}_0$ defined by condition for the vanishing of F. Hence, $\mathcal{F}_0$ is the class of cosymplectic B-metric manifolds, where structures φ, ξ, η, g, and $\tilde{g}$ are ∇-parallel.

In the present work, we obtain manifolds from class $\mathcal{F}_1$ defined by the following [11].

$$\begin{aligned} F(X,Y,Z) = \frac{1}{2n}\Big\{&\theta(\varphi^2 Z)g(\varphi X,\varphi Y) + \theta(\varphi Z)g(X,\varphi Y) \\ &+ \theta(\varphi^2 Y)g(\varphi X,\varphi Z) + \theta(\varphi Y)g(X,\varphi Z)\Big\}. \end{aligned}$$

This class contains the conformally equivalent manifolds of the cosymplectic B-metric manifolds by usual conformal transformations of the B-metric $\bar{g} = e^{2u}g$, where u is a differentiable function on M with $\mathrm{d}u(\xi) = 0$ [13].

Using the pair of B-metrics g and $\tilde{g}$ as well as $\eta\otimes\eta$, in [13], the author and K. Gribachev introduced the so-called contact conformal transformation of the B-metric g into a new B-metric $\bar{g}$ for (M,φ,ξ,η,g). Later, in [10], this transformation is generalized as a contact conformal transformation that provides an almost contact B-metric structure $(\varphi,\bar{\xi},\bar{\eta},\bar{g})$ as follows:

$$\begin{aligned} &\bar{\xi} = e^{-w}\xi, \qquad \bar{\eta} = e^{w}\eta, \\ &\bar{g} = e^{2u}\cos 2v\, g + e^{2u}\sin 2v\,\tilde{g} + \big(e^{2w} - e^{2u}\cos 2v - e^{2u}\sin 2v\big)\eta\otimes\eta, \end{aligned} \tag{5}$$

where u, v, and w are differentiable functions on M.

The corresponding tensors $\bar{F}$ and F are related by [10] (see also ([14], (22))):

$$
\begin{aligned}
2\bar{F}(X,Y,Z) &= 2e^{2u}\cos 2v\, F(X,Y,Z) \\
&+ e^{2u}\sin 2v[F(\varphi Y,Z,X) - F(Y,\varphi Z,X) + F(X,\varphi Y,\xi)\eta(Z)] \\
&+ e^{2u}\sin 2v[F(\varphi Z,Y,X) - F(Z,\varphi Y,X) + F(X,\varphi Z,\xi)\eta(Y)] \\
&+ (e^{2w} - e^{2u}\cos 2v)[F(X,Y,\xi) + F(\varphi Y,\varphi X,\xi)]\eta(Z) \\
&+ (e^{2w} - e^{2u}\cos 2v)[F(X,Z,\xi) + F(\varphi Z,\varphi X,\xi)]\eta(Y) \\
&+ (e^{2w} - e^{2u}\cos 2v)[F(Y,Z,\xi) + F(\varphi Z,\varphi Y,\xi)]\eta(X) \\
&+ (e^{2w} - e^{2u}\cos 2v)[F(Z,Y,\xi) + F(\varphi Y,\varphi Z,\xi)]\eta(X) \\
&- 2e^{2u}[\cos 2v\,\alpha(Z) + \sin 2v\,\beta(Z)]g(\varphi X,\varphi Y) \\
&- 2e^{2u}[\cos 2v\,\alpha(Y) + \sin 2v\,\beta(Y)]g(\varphi X,\varphi Z) \\
&- 2e^{2u}[\cos 2v\,\beta(Z) - \sin 2v\,\alpha(Z)]g(X,\varphi Y) \\
&- 2e^{2u}[\cos 2v\,\beta(Y) - \sin 2v\,\alpha(Y)]g(X,\varphi Z) \\
&+ 2e^{2w}\eta(X)[\eta(Y)\mathrm{d}w(\varphi Z) + \eta(Z)\mathrm{d}w(\varphi Y)],
\end{aligned}
\tag{6}
$$

where we use the following notations for brevity.

$$
\alpha = \mathrm{d}u\circ\varphi + \mathrm{d}v, \qquad \beta = \mathrm{d}u - \mathrm{d}v\circ\varphi. \tag{7}
$$

In the general case, the relations between the Lee forms of the manifolds (M,φ,ξ,η,g) and $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ are the following (see [10]).

$$
\bar{\theta} = \theta + 2n\,\alpha, \qquad \bar{\theta}^* = \theta^* + 2n\,\beta, \qquad \bar{\omega} = \omega + \mathrm{d}w\circ\varphi. \tag{8}
$$

Definition 1. *We say that B-metric $\bar{g}$ generates a Yamabe soliton with the potential Reeb vector field $\bar{\xi}$ and soliton constant $\bar{\sigma}$ on a conformal almost contact B-metric manifold $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$, if the following condition is satisfied:*

$$
\frac{1}{2}\mathcal{L}_{\bar{\xi}}\bar{g} = (\bar{\tau} - \bar{\sigma})\bar{g}, \tag{9}
$$

where $\bar{\tau}$ is the scalar curvature of $\bar{g}$.

As is well known (e.g., [15]), a Yamabe soliton is called *shrinking*, *steady*, or *expanding* depending on whether its soliton constant is positive, zero, or negative, respectively.

It is well known the following expression of the Lie derivative in terms of the covariant derivative with respect to the Levi–Civita connection $\bar{\nabla}$ of $\bar{g}$.

$$
\left(\mathcal{L}_{\bar{\xi}}\bar{g}\right)(X,Y) = \bar{g}(\bar{\nabla}_X\bar{\xi},Y) + \bar{g}(X,\bar{\nabla}_Y\bar{\xi}). \tag{10}
$$

3. The Case When the Given Manifold Is Cosymplectic

In this section, we consider (M,φ,ξ,η,g) to be a cosymplectic manifold with almost contact B-metric structure, i.e., an $\mathcal{F}_0$-manifold, defined by covariant constant structure tensor φ (and, consequently, the same condition for ξ, η, and $\tilde{g}$ is valid) with respect to ∇ of g. Therefore, $F = 0$ and Equation (6) takes the following form.

$$\begin{aligned}\bar{F}(X,Y,Z) = &- e^{2u}[\cos 2v\,\alpha(Z) + \sin 2v\,\beta(Z)]g(\varphi X, \varphi Y)\\ &- e^{2u}[\cos 2v\,\alpha(Y) + \sin 2v\,\beta(Y)]g(\varphi X, \varphi Z)\\ &- e^{2u}[\cos 2v\,\beta(Z) - \sin 2v\,\alpha(Z)]g(X, \varphi Y)\\ &- e^{2u}[\cos 2v\,\beta(Y) - \sin 2v\,\alpha(Y)]g(X, \varphi Z)\\ &+ e^{2w}\eta(X)[\eta(Y)\mathrm{d}w(\varphi Z) + \eta(Z)\mathrm{d}w(\varphi Y)].\end{aligned} \tag{11}$$

From the latter equality, bearing in mind the general identity Equation (3) for the manifolds under consideration, we obtain the following.

$$\begin{aligned}\bar{g}(\bar{\nabla}_X\bar{\xi}, Y) = &- e^{2u-w}[\cos 2v\,\beta(\xi) - \sin 2v\,\alpha(\xi)]g(\varphi X, \varphi Y)\\ &+ e^{2u-w}[\cos 2v\,\alpha(\xi) + \sin 2v\,\beta(\xi)]g(X, \varphi Y)\\ &+ e^{w}\eta(X)\mathrm{d}w(\varphi^2 Y).\end{aligned}$$

Combining the last expression with (7) and (10) produces the following.

$$\begin{aligned}\left(\mathcal{L}_{\bar{\xi}}\bar{g}\right)(X,Y) = &- 2e^{2u-w}[\cos 2v\,\mathrm{d}u(\xi) - \sin 2v\,\mathrm{d}v(\xi)]g(\varphi X, \varphi Y)\\ &+ 2e^{2u-w}[\cos 2v\,\mathrm{d}v(\xi) + \sin 2v\,\mathrm{d}u(\xi)]g(X, \varphi Y)\\ &+ e^{w}\Big[\eta(X)\mathrm{d}w(\varphi^2 Y) + \eta(Y)\mathrm{d}w(\varphi^2 X)\Big].\end{aligned} \tag{12}$$

Theorem 2. *An almost contact B-metric manifold that is cosymplectic can be transformed by a contact conformal transformation of type (5) so that the transformed B-metric is a Yamabe soliton with potential the transformed Reeb vector field and a soliton constant $\bar{\sigma}$ if and only if the functions (u, v, w) of the used transformation satisfy the following conditions.*

$$\mathrm{d}u(\xi) = 0, \qquad \mathrm{d}v(\xi) = 0, \qquad \mathrm{d}w = \mathrm{d}w(\xi)\eta. \tag{13}$$

Moreover, the obtained Yamabe soliton has a constant scalar curvature with value $\bar{\tau} = \bar{\sigma}$ and the obtained almost contact B-metric manifold belongs to the subclass of the main class $\mathcal{F}_1$ determined by the following conditions.

$$\bar{\theta} = 2n\Big\{\mathrm{d}u \circ \varphi - \mathrm{d}v \circ \varphi^2\Big\}, \qquad \bar{\theta}^* = -2n\Big\{\mathrm{d}u \circ \varphi^2 + \mathrm{d}v \circ \varphi\Big\}. \tag{14}$$

Proof. If we assume that $\bar{g}$ generates a Yamabe soliton with potential $\bar{\xi}$ and a soliton constant $\bar{\sigma}$ on $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$, then due to (9) and (12), we have the following.

$$\begin{aligned}&- e^{2u-w}\{[\cos 2v\,\mathrm{d}u(\xi) - \sin 2v\,\mathrm{d}v(\xi)]g(\varphi X, \varphi Y)\\ &\qquad -[\sin 2v\,\mathrm{d}u(\xi) + \cos 2v\,\mathrm{d}v(\xi)]g(X, \varphi Y)\}\\ &+ \frac{1}{2}e^{w}\Big\{\mathrm{d}w(\varphi^2 X)\eta(Y) + \mathrm{d}w(\varphi^2 Y)\eta(X)\Big\}\\ &= (\bar{\tau} - \bar{\sigma})\Big\{-e^{2u}[\cos 2v\,g(\varphi X, \varphi Y) - \sin 2v\,g(X, \varphi Y)]\\ &\qquad +e^{2w}\eta(X)\eta(Y)\Big\}.\end{aligned} \tag{15}$$

The substitution ξ for Y in the latter equality gives the following:

$$\mathrm{d}w(\varphi^2 X) = 2e^w(\bar{\tau} - \bar{\sigma})\eta(X), \tag{16}$$

which is valid if and only if

$$\bar{\tau} = \bar{\sigma}. \tag{17}$$

Therefore, (16) implies condition $\mathrm{d}w \circ \varphi = 0$. In this case, because of (15), we obtain $\mathrm{d}u(\xi) = \mathrm{d}v(\xi) = 0$, which completes (13).

Let us remark that $\mathcal{L}_{\bar{\xi}}\bar{g}$ vanishes because of (17) and (9), i.e., $\bar{\xi}$ is a Killing vector field in the considered case.

After that, we apply the following relations to (11), which are equivalent to the last formula in (5).

$$\begin{aligned} \bar{g}(\varphi X, \varphi Y) &= e^{2u}\cos 2v\, g(\varphi X, \varphi Y) - e^{2u}\sin 2v\, g(X, \varphi Y), \\ \bar{g}(X, \varphi Y) &= e^{2u}\cos 2v\, g(X, \varphi Y) + e^{2u}\sin 2v\, g(\varphi X, \varphi Y). \end{aligned} \tag{18}$$

Then, we use (13) and the formula $\bar{\eta} = e^w\eta$ from (5) to obtain the following expression.

$$\begin{aligned} \bar{F}(X, Y, Z) = {} & \bar{g}(\varphi X, \varphi Y)\, \alpha(\varphi^2 Z) + \bar{g}(X, \varphi Y)\, \beta(\varphi^2 Z) \\ & + \bar{g}(\varphi X, \varphi Z)\, \alpha(\varphi^2 Y) + \bar{g}(X, \varphi Z)\, \beta(\varphi^2 Y). \end{aligned} \tag{19}$$

Now, we substitute the Lee forms of $\bar{F}$ from (8) in (19) and obtain the following.

$$\begin{aligned} \bar{F}(X, Y, Z) = \frac{1}{2n}\Big\{ & \bar{g}(\varphi X, \varphi Y)\, \bar{\theta}(\varphi^2 Z) + \bar{g}(X, \varphi Y)\, \bar{\theta}^*(\varphi^2 Z) \\ & + \bar{g}(\varphi X, \varphi Z)\, \bar{\theta}(\varphi^2 Y) + \bar{g}(X, \varphi Z)\, \bar{\theta}^*(\varphi^2 Y)\Big\}. \end{aligned}$$

The last expression of $\bar{F}$ means that the obtained manifold belongs to the main class $\mathcal{F}_1$, according to the classification in [11], and the corresponding Lee forms have the following properties.

$$\bar{\theta} = -\bar{\theta} \circ \varphi^2, \qquad \bar{\theta}^* = -\bar{\theta}^* \circ \varphi^2, \qquad \bar{\omega} = 0.$$

Finally, taking into account (8), (13), and the vanishing θ, θ^*, and ω for any cosymplectic B-metric manifold, we find the expressions of the Lee forms of the transformed manifold as in (14). $\square$

As a result from (13), we obtain the result that the case in the present section is possible when functions (u, v, w) of the contact conformal transformation in (5) satisfy the following conditions:

- u and v are constants on the vertical distribution $\mathcal{V} = \operatorname{span}\xi = \ker\varphi$;
- w is a constant on the horizontal (contact) distribution $\mathcal{H} = \ker\eta = \operatorname{im}\varphi$.

4. The Case When the Given Manifold Is Sasaki-like

In the present section, we suppose that the given almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$ is Sasaki-like, i.e., the complex cone $M \times \mathbb{R}^-$ is a Kähler manifold with Norden metric also known as a holomorphic complex Riemannian manifold [12].

If $(M, \varphi, \xi, \eta, g)$ is Sasaki-like, then the following condition is met.

$$\nabla_X\xi = -\varphi X.$$

The Sasaki-like condition in terms of F given in [12] is the following.

$$F(X, Y, Z) = g(\varphi X, \varphi Y)\eta(Z) + g(\varphi X, \varphi Z)\eta(Y). \tag{20}$$

Then, by virtue of (3), (6), and (10) for $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ and (20), we obtain the following.

$$\begin{aligned}\left(\mathcal{L}_{\bar{\xi}}\bar{g}\right)(X,Y) = 2e^{2u-w}\{&\langle \sin 2v\, \mathrm{d}u(\xi) - \cos 2v\,[1-\mathrm{d}v(\xi)]\rangle g(X,\varphi Y)\\ &-\langle \cos 2v\, \mathrm{d}u(\xi) + \sin 2v[1-\mathrm{d}v(\xi)]\rangle g(\varphi X,\varphi Y)\}\\ &+e^{w}\left\{\mathrm{d}w(\varphi^2 X)\eta(Y) + \mathrm{d}w(\varphi^2 Y)\eta(X)\right\}.\end{aligned} \tag{21}$$

Theorem 3. *An almost contact B-metric manifold that is Sasaki-like can be transformed by a contact conformal transformation of type (5) so that the transformed B-metric is a Yamabe soliton with potential the transformed Reeb vector field and a soliton constant $\bar{\sigma}$ if and only if the functions (u, v, w) of the used transformation satisfy the following conditions.*

$$\mathrm{d}u(\xi) = 0, \qquad \mathrm{d}v(\xi) = 1, \qquad \mathrm{d}w = \mathrm{d}w(\xi)\eta. \tag{22}$$

Moreover, the obtained Yamabe soliton has a constant scalar curvature with a value $\bar{\tau} = \bar{\sigma}$ and the transformed almost contact B-metric manifold belongs to a subclass of the main class $\mathcal{F}_1$ determined by the following.

$$\bar{\theta} = 2n\left\{\mathrm{d}u\circ\varphi - \mathrm{d}v\circ\varphi^2\right\}, \qquad \bar{\theta}^* = 2n\{\mathrm{d}u - \mathrm{d}v\circ\varphi\}. \tag{23}$$

Proof. The expression in (21) and the assumption that $\bar{g}$ generates a Yamabe soliton with potential $\bar{\xi}$ and a soliton constant $\bar{\sigma}$ on $(M, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ imply the following.

$$\begin{aligned}&e^{2u-w}\{\langle \sin 2v\, \mathrm{d}u(\xi) - \cos 2v\,[1-\mathrm{d}v(\xi)]\rangle g(X,\varphi Y)\\ &\qquad-\langle \cos 2v\, \mathrm{d}u(\xi) + \sin 2v[1-\mathrm{d}v(\xi)]\rangle g(\varphi X,\varphi Y)\}\\ &+\frac{1}{2}e^{w}\left\{\mathrm{d}w(\varphi^2 X)\eta(Y) + \mathrm{d}w(\varphi^2 Y)\eta(X)\right\}\\ &=(\bar{\tau}-\bar{\sigma})\left\{e^{2u}[\sin 2v\, g(X,\varphi Y) - \cos 2v\, g(\varphi X,\varphi Y)]\right.\\ &\qquad\left.+e^{2w}\eta(X)\eta(Y)\right\}.\end{aligned} \tag{24}$$

An obvious consequence for $Y = \xi$ is the following:

$$\mathrm{d}w(\varphi^2 X) = 2e^{w}(\bar{\tau}-\bar{\sigma})\eta(X),$$

which is satisfied if and only it the following conditions are fulfilled.

$$\bar{\tau} = \bar{\sigma}, \tag{25}$$

$$\mathrm{d}w = \mathrm{d}w(\xi)\eta. \tag{26}$$

Due to (25) and (9), we have a vanishing $\mathcal{L}_{\bar{\xi}}\bar{g}$, which means that $\bar{\xi}$ is a Killing vector field in this case as well.

Applying (25) and (26) in (24), we obtain the following.

$$\begin{aligned}&e^{2u-w}\{\langle \sin 2v\, \mathrm{d}u(\xi) - \cos 2v\,[1-\mathrm{d}v(\xi)]\rangle g(X,\varphi Y)\\ &\qquad-\langle \cos 2v\, \mathrm{d}u(\xi) + \sin 2v[1-\mathrm{d}v(\xi)]\rangle g(\varphi X,\varphi Y)\} = 0.\end{aligned}$$

The latter equality is valid for arbitrary vector fields if and only if the following conditions are satisfied.

$$\mathrm{d}u(\xi) = 0, \qquad \mathrm{d}v(\xi) = 1.$$

Substitute (20) and (22) into (6) to obtain the following.

$$\begin{aligned}\bar{F}(X,Y,Z) = \frac{e^{2u}}{2n}\Big\{&[\cos 2v\{\alpha(Z)-\eta(Z)\}+\sin 2v\,\beta(Z)]g(\varphi X,\varphi Y)\\ &+[\cos 2v\,\beta(Z)-\sin 2v\{\alpha(Z)-\eta(Z)\}]g(X,\varphi Y)\\ &+[\cos 2v\{\alpha(Y)-\eta(Y)\}+\sin 2v\,\beta(Y)]g(\varphi X,\varphi Z)\\ &+[\cos 2v\,\beta(Y)-\sin 2v\{\alpha(Y)-\eta(Y)\}]g(X,\varphi Z)\Big\}.\end{aligned} \tag{27}$$

Using our assumption that (M,φ,ξ,η,g) is Sasaki-like, we have the following [12].

$$\theta = -2n\,\eta, \qquad \theta^* = \omega = 0.$$

Then, by notations (7) and conditions (22), formulae (8) yield the the following.

$$\bar{\theta} = 2n[\alpha-\eta], \qquad \bar{\theta}^* = 2n\,\beta, \qquad \bar{\omega} = 0. \tag{28}$$

Equalities (7) and (22) imply the following relation: $\beta = -\alpha\circ\varphi$, which, together with the first identity in (4), helps to obtain the following expressions.

$$\alpha = -\frac{\bar{\theta}\circ\varphi^2}{2n} + \eta, \qquad \beta = -\frac{\bar{\theta}\circ\varphi}{2n}. \tag{29}$$

Substitute (29) into (27) and obtain the following formula.

$$\begin{aligned}\bar{F}(X,Y,Z) = \frac{e^{2u}}{2n}\Big\{&\Big[\cos 2v\,\bar{\theta}(\varphi^2 Z)+\sin 2v\,\bar{\theta}(\varphi Z)\Big]g(\varphi X,\varphi Y)\\ &+\Big[\cos 2v\,\bar{\theta}(\varphi Z)-\sin 2v\,\bar{\theta}(\varphi^2 Z)\Big]g(X,\varphi Y)\\ &+\Big[\cos 2v\,\bar{\theta}(\varphi^2 Y)+\sin 2v\,\bar{\theta}(\varphi Y)\Big]g(\varphi X,\varphi Z)\\ &+\Big[\cos 2v\,\bar{\theta}(\varphi Y)-\sin 2v\,\bar{\theta}(\varphi^2 Y)\Big]g(X,\varphi Z)\Big\}.\end{aligned}$$

Then, we apply (18) in the last equality and obtain the following expression.

$$\begin{aligned}\bar{F}(X,Y,Z) = \frac{1}{2n}\Big\{&\bar{\theta}(\varphi^2 Z)\bar{g}(\varphi X,\varphi Y)+\bar{\theta}(\varphi Z)\bar{g}(X,\varphi Y)\\ &+\bar{\theta}(\varphi^2 Y)\bar{g}(\varphi X,\varphi Z)+\bar{\theta}(\varphi Y)\bar{g}(X,\varphi Z)\Big\}.\end{aligned}$$

The obtained form of $\bar{F}$ coincides with definition $(M,\varphi,\bar{\xi},\bar{\eta},\bar{g})$ to belong to the basic class $\mathcal{F}_1$, according to the classification of Ganchev–Mihova–Gribachev in [11].

Finally, the expression of the Lee forms of the transformed manifold given in (23) follows from (28) and (7). □

5. Examples

5.1. Example in the Case of Cosymplectic Manifold

Let us consider a real space $\mathbb{R}^5 = \{(x^0,x^1,x^2,x^3,x^4)\,|\,x^i\in\mathbb{R}\}$ equipped with an almost contact B-metric structure defined with respect to the local basis as follows:

$$\varphi\frac{\partial}{\partial x^1} = \frac{\partial}{\partial x^3}, \quad \varphi\frac{\partial}{\partial x^2} = \frac{\partial}{\partial x^4}, \quad \varphi\frac{\partial}{\partial x^3} = -\frac{\partial}{\partial x^1}, \quad \varphi\frac{\partial}{\partial x^4} = -\frac{\partial}{\partial x^2}, \quad \varphi\frac{\partial}{\partial x^0} = 0,$$
$$\xi = \frac{\partial}{\partial x^0}, \quad \eta = \mathrm{d}x^0, \quad g(X,Y) = X^0Y^0 + X^1Y^1 + X^2Y^2 - X^3Y^3 - X^4Y^4$$

for $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$, $i \in \{0,1,2,3,4\}$. Then, $(\mathbb{R}^5, \varphi, \xi, \eta, g)$ is a cosymplectic B-metric manifold according to [11].

Suppose that a contact conformal transformation of the form in (5) is determined by the following functions u, v, and w on the manifold under consideration:

$$u = -\ln\left\{\left(x^1 + x^4\right)^2 + \left(x^2 - x^3\right)^2\right\}, \quad v = \mathrm{const}, \quad w = 0, \tag{30}$$

where $x^1 \neq -x^4$ or $x^2 \neq x^3$. A more general form of u is given in [16]. As a consequence of (30), we obtain the result that such a function u has the property $\mathrm{d}u(\xi) = 0$, i.e., $\mathrm{d}u = -\mathrm{d}u \circ \varphi^2$. Furthermore, given (7) and (8), the Lee forms of the transformed manifold take the following form.

$$\bar{\theta} = 2n\, \mathrm{d}u \circ \varphi, \qquad \bar{\theta}^* = -2n\, \mathrm{d}u \circ \varphi^2, \qquad \bar{\omega} = 0. \tag{31}$$

The first two are not zero due to the following calculations starting from (30).

$$\begin{aligned} \frac{\partial u}{\partial x^1} &= \frac{\partial u}{\partial x^4} = -\frac{2(x^1 + x^4)}{(x^1 + x^4)^2 + (x^2 - x^3)^2}, \\ \frac{\partial u}{\partial x^2} &= -\frac{\partial u}{\partial x^3} = -\frac{2(x^2 - x^3)}{(x^1 + x^4)^2 + (x^2 - x^3)^2}, \end{aligned} \tag{32}$$

$$\mathrm{d}u(\varphi X) = -2\frac{(x^1 + x^4)(X^2 - X^3) - (x^2 - x^3)(X^1 + X^4)}{(x^1 + x^4)^2 + (x^2 - x^3)^2},$$

$$\mathrm{d}u(\varphi^2 X) = 2\frac{(x^1 + x^4)(X^1 + X^4) + (x^2 - x^3)(X^2 - X^3)}{(x^1 + x^4)^2 + (x^2 - x^3)^2}.$$

Therefore, following [16], the transformed manifold $(\mathbb{R}^5, \varphi, \xi, \eta, \bar{g})$ belongs to the class $\mathcal{F}_1$, but not to its subclass $\mathcal{F}_0$. According to Theorem 2, $(\mathbb{R}^5, \varphi, \xi, \eta, \bar{g})$ admits a Yamabe soliton with potential $\bar{\xi} = \xi$ and a scalar curvature $\bar{\tau} = \bar{\sigma}$.

In [16], the expression of the curvature tensor $\bar{R}$ for $\bar{g}$ is given in the following form:

$$\bar{R} = -(\bar{\psi}_1 + \bar{\psi}_2 - \bar{\psi}_4)(\mathrm{d}u \otimes \mathrm{d}u), \tag{33}$$

where $\bar{\psi}_1(S)$, $\bar{\psi}_2(S)$, and $\bar{\psi}_4(S)$ are the following tensors for the B-metric $\bar{g}$ and an arbitrary tensor S of type (0,2), which has form $S = \mathrm{d}u \otimes \mathrm{d}u$ in (33).

$$\begin{aligned} \bar{\psi}_1(X,Y,Z,W) &= \bar{g}(Y,Z)S(X,W) - \bar{g}(X,Z)S(Y,W) \\ &\quad + g(X,W)S(Y,Z) - g(Y,W)S(X,Z), \\ \bar{\psi}_2(X,Y,Z,W) &= \bar{g}(Y,\varphi Z)S(X,\varphi W) - \bar{g}(X,\varphi Z)S(Y,\varphi W) \\ &\quad + \bar{g}(X,\varphi W)S(Y,\varphi Z) - \bar{g}(Y,\varphi W)S(X,\varphi Z), \\ \bar{\psi}_4(X,Y,Z,W) &= \bar{\eta}(Y)\bar{\eta}(Z)S(X,W) - \bar{\eta}(X)\bar{\eta}(Z)S(Y,W) \\ &\quad + \bar{\eta}(X)\bar{\eta}(W)S(Y,Z) - \bar{\eta}(Y)\bar{\eta}(W)S(X,Z). \end{aligned} \tag{34}$$

By calculating the scalar curvature $\bar{\tau}$ using (33) and (34), we obtain the following: $\bar{\tau} = -8\,\mathrm{d}u(\mathrm{grad}\, u)$ and due to (32), we find that $(\mathbb{R}^5, \varphi, \xi, \eta, \bar{g})$ is scalar flat. $\bar{\tau} = 0$.

Hence, because of (17), the Yamabe soliton constant is also zero, i.e., $\bar{\sigma} = 0$, and then the obtained Yamabe soliton for $\bar{g}$ is steady. The constructed example in this subsection supports Theorem 2.

5.2. Example in the Case of Sasaki-like Manifold

We recall a known example of a Sasaki-like manifold given in [12] as Example 2. A Lie group G of dimension 5 is considered to have a basis of left-invariant vector fields $\{E_0, \dots, E_4\}$ defined by the following commutators for $\lambda, \mu \in \mathbb{R}$.

$$\begin{aligned} &[E_0, E_1] = \lambda E_2 + E_3 + \mu E_4, &&[E_0, E_2] = -\lambda E_1 - \mu E_3 + E_4, \\ &[E_0, E_3] = -E_1 - \mu E_2 + \lambda E_4, &&[E_0, E_4] = \mu E_1 - E_2 - \lambda E_3. \end{aligned}$$

An invariant almost contact B-metric structure is then defined on G by the following.

$$\begin{aligned} &g(E_0, E_0) = g(E_1, E_1) = g(E_2, E_2) = 1 \\ &g(E_3, E_3) = g(E_4, E_4) = -1, \\ &g(E_i, E_j) = 0, \quad i, j \in \{0, 1, 2, 3, 4\},\ i \neq j, \\ &\xi = E_0, \quad \varphi E_1 = E_3, \quad \varphi E_2 = E_4. \end{aligned}$$

It is verified that the constructed manifold $(G, \varphi, \xi, \eta, g)$ is an almost contact B-metric manifold that is Sasaki-like.

In [17], the components $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ of its curvature tensor are calculated and the type of the corresponding Ricci tensor ρ is found. Namely, it is $\rho = 4\,\eta \otimes \eta$ and the scalar curvature is $\tau = 4$.

Using the non-zero ones of R_{ijkl} determined by the following equalities:

$$\begin{aligned} &R_{0110} = R_{0220} = -R_{0330} = -R_{0440} = 1, \\ &R_{1234} = R_{1432} = R_{2341} = R_{3412} = 1, \qquad R_{1331} = R_{2442} = 1 \end{aligned}$$

and the properties $R_{ijkl} = -R_{jikl} = -R_{ijlk}$, we compute that the associated quantity τ^* of τ is zero.

$$\tau^* = g^{ij} \rho(E_i, \varphi E_j) = 0.$$

Now, we define the following functions on $\mathbb{R}^5 = \{(x^0, x^1, x^2, x^3, x^4)\}$.

$$\begin{aligned} u &= \frac{1}{2} \ln\Big\{ \Big[(x^1)^2 + (x^3)^2\Big] \Big[(x^2)^2 + (x^4)^2\Big] \Big\}, \\ v &= \arctan \frac{x^1 x^4 + x^2 x^3}{x^3 x^4 - x^1 x^2} + x^0, \\ w &= x^0. \end{aligned} \tag{35}$$

The non-zero ones between their partial derivatives are the following.

$$\begin{aligned} &\frac{\partial u}{\partial x^1} = -\frac{\partial v}{\partial x^3} = \frac{x^1}{(x^1)^2 + (x^3)^2}, \quad &&\frac{\partial u}{\partial x^2} = -\frac{\partial v}{\partial x^4} = \frac{x^2}{(x^2)^2 + (x^4)^2}, \\ &\frac{\partial u}{\partial x^3} = \frac{\partial v}{\partial x^1} = \frac{x^3}{(x^1)^2 + (x^3)^2}, &&\frac{\partial u}{\partial x^4} = \frac{\partial v}{\partial x^2} = \frac{x^4}{(x^2)^2 + (x^4)^2}, \\ &\frac{\partial v}{\partial x^0} = \frac{\partial w}{\partial x^0} = 1. \end{aligned}$$

Then, for an arbitrary vector field $X = X^i \frac{\partial}{\partial x^i}$, $i \in \{0, 1, 2, 3, 4\}$, we have the following.

$$\varphi X = -X^3 \frac{\partial}{\partial x^1} - X^4 \frac{\partial}{\partial x^2} + X^1 \frac{\partial}{\partial x^3} + X^2 \frac{\partial}{\partial x^4}, \quad X^0 = \eta(X), \quad \xi = \frac{\partial}{\partial x^0}.$$

We verify immediately that the functions defined by (35) satisfy the following properties.

$$\mathrm{d}u = -\mathrm{d}v \circ \varphi, \qquad \mathrm{d}u(\xi) = 0, \qquad \mathrm{d}v(\xi) = 1, \qquad \mathrm{d}w = \eta.$$

Let us consider a contact conformal transformation defined by (5), where functions (u, v, w) are determined as in (35).

Then, the transformed manifold $(G, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ is an $\mathcal{F}_1$-manifold with a Yamabe soliton with potential $\bar{\xi}$ and a constant scalar curvature $\bar{\tau} = \bar{\sigma}$, according to Theorem 3. Moreover, taking into account (23), we obtain the corresponding Lee forms as follows.

$$\bar{\theta} = 4n\,\mathrm{d}u \circ \varphi, \qquad \bar{\theta}^* = 4n\,\mathrm{d}u, \qquad \bar{\omega} = 0.$$

According to Proposition 8 in [12], the Ricci tensor of an almost contact B-metric manifold is invariant under a contact homothetic transformation (i.e., when u, v, and w are constants); therefore, for the corresponding scalar curvatures, we have the following.

$$\begin{aligned} \bar{\tau} &= e^{-2u}\cos 2v\,\tau - e^{-2u}\sin 2v\,\tau^* + \{e^{-2w} - e^{-2u}\cos 2v\}\rho(\xi, \xi), \\ \bar{\tau}^* &= e^{-2u}\sin 2v\,\tau + e^{-2u}\cos 2v\,\tau^* - e^{-2u}\sin 2v\,\rho(\xi, \xi). \end{aligned}$$

Then, bearing in mind the last results, we obtain, for our example, the following.

$$\bar{\tau} = 4\,e^{-2w}, \qquad \bar{\tau}^* = 0.$$

Hence, $(G, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$ is scalar flat and the Yamabe soliton constant for $\bar{g}$ is $\bar{\sigma} = 4$; thus, the obtained Yamabe soliton is shrinking.

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