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Continuous Dependence for the Boussinesq Equations under Reaction Boundary Conditions in R^2

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Abstract: In this paper, we studied the continuous dependence result for the Boussinesq equations. We considered the case where Ω was a bounded domain in R^2. Temperatures T and C satisfied reaction boundary conditions. A first-order inequality for the differences of energy could be derived. An integration of this inequality produced a continuous dependence result. The result told us that the continuous dependence type stability was also valid for the Boussinesq coefficient λ of the Boussinesq equations with reaction boundary conditions.

Keywords: structural stability; Boussinesq equations; Darcy equations; reaction boundary condition

MSC: 35B40; 35Q30; 76D05

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1. Introduction

Recently, many researchers have began to study the results of continuous dependence for different types of constructive coefficients. These stabilities are usually reffered to as structural stability. Unlike the studies of the traditional stability, they do not focus on the change of initial data, and they mainly focus on the change of the model itself. Let us introduce some papers explaining the nature of this structural stability. Ames and Straughan in their monograph [1] explained the the natural of the stability. In continuum mechanics, it is very important to establish the structural stability for different models. For example, in [2], the authors obtained some results about the structural stability for different equations and showed the importance of this stability. It is meaningful to discuss structural stability, for we wonder if a tiny variation in the coefficient may lead to a sharp change in the solution. What is more is that many errors exist in each step of establishing a model. We want to know whether the errors will affect the correctness of the model. The study of the structural stability can solve this problem.

Many papers in the literature have studied the behavior of solutions of fluid equations in porous media. In the book of [3], some models in porous media were introduced in detail and their properties were studied. In paper [4], the authors studied the spatial behavior of the solution for a class of Brinkman and Forchheimer-type equations in porous media, and they obtained results for the spatial decay estimates of the solutions. Some recent research results on the structural stability of fluid equations in porous media can be observed in [5–15]. For a more detailed understanding of the structural stability, readers should refer to [16–23]. In these papers, the fluid models were widely studied. Other studies about the stability for wave equations may be found in [24–33].

The studies of the structural stability are famous for the Brinkman, Forchheimer and Darcy equations in porous media. Little attention has paid to the study of the Boussinesq equations. The nonlinear term in the Boussineq equations is different from the Brinkman and Forchheimer-type equations. The methods used in previous cannot be applicable. The Boussinesq equations have many applications in reality. We want to know if a small error can cause a big perturbation of the solution. What is more is that the problem we studied in this paper has the same nonlinear item, $u_j u_{i,j}$,as the Navier–Stokes equation; thus, the method proposed in this paper can be useful in studying the properties of the solution of the Navier–Stokes equation. In this sense, our paper is interesting and involved. In [34], the authors studied the Boussinesq equations. They obtained structural stability results for the constructive coefficient. The region they discussed was a bounded region in R^2. In [22], Liu studied the same equations and obtained a result of structural stability. They considered the same problem as [34], but the region they discussed was a bounded region in R^3. The result of the the continuous dependence type was obtained for the Boussinesq coefficient λ. In this paper, we want to continue to study the continuous dependence for these equations in a bounded domain in R^2. We suppose that temperatures T and C satisfy the reaction boundary conditions. Under these conditions, we cannot obtain the bounds for T and C, and the methods proposed in [22,34], which are based on the maximum bounds of T and C, cannot be used in the present paper. The main difficulty is how to tackle the convection terms. Additionally, the boundary conditions are different from [22,34]. It is difficult to deal with the terms on the boundary. With the aid of the L_4 bounds for T and C and some useful inequalities, we can overcome this difficulty. In our opinion, we can overcome this difficulty and obtain the bound for $\int_0^t \int_\Omega u_{i,j} u_{i,j} dx d\eta$ by using the methods proposed by the paper in the literature. Using these methods, we cannot obtain the bound for $\int_\Omega u_{i,j} u_{i,j} dx$. The biggest innovation of this paper is that we follow a new method to obtain the bound for $\int_\Omega u_{i,j} u_{i,j} dx$. The energy method is widely used in these studies.

In this paper, the comma is used to denote partial differentiation. The symbol $u_{i,k}$ denotes $\frac{\partial u_i}{\partial x_k}$. The repeated Latin subscripts denote summation. Hence, $u_{i,i} = \sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i}$, and $dx = dx_1 dx_2$.

The fundamental model we study is based upon the equations of balance of momentum, balance of mass, conservation of energy and conservation of salt concentration (see [34]). Let (u_i, T, C, p) denote velocity, temperature, salt concentration and pressure in Ω, where Ω is a bounded star-shaped domain in R^2. We will study the following Boussinesq equations (see [22]).

$$\begin{aligned} &\frac{\partial u_i}{\partial t} - \Delta u_i + \lambda u_j \frac{\partial u_i}{\partial x_j} + g_i T - h_i C + \frac{\partial p}{\partial x_i} = 0, && \text{in } \Omega \times [0, \tau], \\ &\frac{\partial u_i}{\partial x_i} = 0, && \text{in } \Omega \times [0, \tau], \\ &\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \Delta T, && \text{in } \Omega \times [0, \tau], \\ &\frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = \Delta C + LT - kC, && \text{in } \Omega \times [0, \tau]. \end{aligned} \tag{1}$$

In Equation (1), g_i and h_i denote gravity functions. For the sake of simplicity, we provide bounds for g_i and h_i: $| g | = (g_i g_i)^{\frac{1}{2}} \leq 1$ and $| h | = (h_i h_i)^{\frac{1}{2}} \leq 1$. Δ denotes the Laplacian operator. L and k are non-negative coefficients. λ denotes the Boussinesq coefficient. τ is a nonnegative constant. Equation (1) follows in practice by employing a Boussinesq approximation, which accounts for variable C, allowing the incompressibility condition to hold (see [32,34]).

The following boundary conditions are satisfied:

$$u_i = 0, \quad \frac{\partial T}{\partial n} = k_1 T, \quad \frac{\partial C}{\partial n} = k_2 C, \quad \text{on } \partial\Omega \times [0, \tau], \tag{2}$$

for the prescribed positive reaction boundary coefficients k_1 and k_2, and n_i is the unit-outward normal. We impose the following conditions on $t = 0$:

$$u_i(x, 0) = f_i(x), \quad T(x, 0) = T_0(x), \quad C(x, 0) = C_0(x), \quad \text{in } \Omega, \tag{3}$$

where f_i, T_0 and C_0 are prescribed functions.

The structure of the article is as follows. In Section 2, some lemmas will be obtained. In Section 3, the main result of this paper will be obtained. In the last Section, some conclusions are provided.

2. Some Important Bounds

In this section, we will obtain some important bounds that will be used in proving our main results.

Lemma 1. *For an arbitrary differentiable function $H = H(x,t)$, we have the following estimates:*

$$\int_{\partial\Omega} H^2 dS \le \left[\frac{2}{m} + \frac{2d^2\varepsilon_0}{m^2}\right]\int_\Omega H^2 dx + \frac{1}{2\varepsilon_0}\int_\Omega H_{,i}H_{,i}dx, \tag{4}$$

where m and d are positive constants, and ε_0 is a positive constant.

Proof. From the divergence theorem, we have the following.

$$\int_{\partial\Omega} H^2 x\cdot \overrightarrow{n}\, dS = \int_\Omega div(H^2x)dx = 2\int_\Omega H^2 dx + 2\int_\Omega H(x\cdot\nabla H)dx.$$

Since Ω is a strictly convex domain, we set the following.

$$m = \min_{\partial\Omega} x_i n_i > 0, \quad d^2 = \max_{\Omega} x_i x_i.$$

We have the following.

$$m\int_{\partial\Omega} H^2 dS \le \int_{\partial\Omega} H^2 x\cdot\overrightarrow{n}\, dS \le 2\int_\Omega H^2 dx + 2\int_\Omega H(x\cdot\nabla H)dx.$$

We can easily obtain the following.

$$\int_{\partial\Omega} H^2 dS \le \left[\frac{2}{m} + \frac{2d^2\varepsilon_0}{m^2}\right]\int_\Omega H^2 dx + \frac{1}{2\varepsilon_0}\int_\Omega H_{,i}H_{,i}dx.$$

The proof is complete. □

Lemma 2. *For temperature T and concentration C, we have the following estimates:*

$$\int_0^t\int_\Omega (T^2 + C^2)dxd\eta \le k_1(t), \tag{5}$$

and the following is the case:

$$\int_0^t\int_\Omega (T_{,i}T_{,i} + C_{,i}C_{,i})dxd\eta \le k_2(t), \tag{6}$$

where $k_1(t)$ and $k_2(t)$ are positive functions defined later.

Proof. Multiplying $(1)_3$ by $2T$ and integrating over $\Omega\times[0,t]$, we have the following.

$$\int_\Omega T^2 dx + 2\int_0^t\int_\Omega T_{,i}T_{,i}dxd\eta \le \int_\Omega T_0^2 dx + 2k_1\int_0^t\oint_{\partial\Omega} T^2 dSd\eta. \tag{7}$$

We can also obtain the following.

$$\int_\Omega C^2 dx + 2\int_0^t\int_\Omega C_{,i}C_{,i}dxd\eta \le \int_\Omega C_0^2 dx + \frac{L^2}{2k}\int_0^t\int_\Omega T^2 dxd\eta + 2k_2\int_0^t\oint_{\partial\Omega} C^2 dSd\eta. \tag{8}$$

Combining (7) and (8), we obtain the following

$$\begin{aligned}\int_\Omega (T^2+C^2)dx + 2\int_0^t\int_\Omega (T_{,i}T_{,i}+C_{,i}C_{,i})dxd\eta \leq \int_\Omega (T_0^2+C_0^2)dx \\ +\frac{L^2}{2k}\int_0^t\int_\Omega T^2dxd\eta + 2\int_0^t\oint_{\partial\Omega}(k_1T^2+k_2C^2)dSd\eta.\end{aligned} \tag{9}$$

Using (4) and choosing $\varepsilon_0 = 2k_1$, we have the following.

$$2k_1\oint_{\partial\Omega}T^2dS \leq \left[\frac{2}{m}+\frac{4d^2k_1}{m^2}\right]\int_\Omega T^2dx + \frac{1}{2}\int_\Omega T_{,i}T_{,i}dx. \tag{10}$$

Similarly, we can obtain the following.

$$2k_2\oint_{\partial\Omega}C^2dS \leq \left[\frac{2}{m}+\frac{4d^2k_2}{m^2}\right]\int_\Omega C^2dx + \frac{1}{2}\int_\Omega C_{,i}C_{,i}dx. \tag{11}$$

Inserting (10) and (11) into (9), we have the following.

$$\begin{aligned}\int_\Omega (T^2+C^2)dx + \frac{1}{2}\int_0^t\int_\Omega (T_{,i}T_{,i}+C_{,i}C_{,i})dxd\eta \leq \int_\Omega (T_0^2+C_0^2)dx \\ + m_1\int_0^t\int_\Omega (T^2+C^2)dxd\eta,\end{aligned} \tag{12}$$

with $m_1 = \max\left\{\frac{L^2}{2k}+\frac{2}{m}+\frac{4d^2k_1}{m^2}, \frac{2}{m}+\frac{4d^2k_2}{m^2}\right\}$.

An integration of (12) provides the following.

$$\int_0^t\int_\Omega (T^2+C^2)dxd\eta \leq e^{m_1t}t\int_\Omega (T_0^2+C_0^2)dx = k_1(t). \tag{13}$$

An insertion of (13) into (12) provides the following.

$$\begin{aligned}\int_0^t\int_\Omega (T_{,i}T_{,i}+C_{,i}C_{,i})dxd\eta \leq 2\int_\Omega (T_0^2+C_0^2)dx + 2m_1k_1(t) \\ = k_2(t).\end{aligned} \tag{14}$$

□

Lemma 3. *We suggest that u_i is a solution of Equation (1) and satisfies the initial boundary conditions (2) and (3). The following estimates can be obtained:*

$$\int_{\Omega_1} u_iu_idx \leq k_3(t), \tag{15}$$

where $k_3(t)$ is a positive function defined later.

Proof. By multiplying $(1)_1$ by u_i and integrating, we obtain the following.

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_iu_idx + \int_\Omega u_{i,j}u_{i,j}dx \leq \frac{1}{2}\int_\Omega T^2dx + \frac{1}{2}\int_\Omega C^2dx + \int_\Omega u_iu_idx. \tag{16}$$

An integration of (16) provides the following.

$$\int_\Omega u_iu_idx \leq e^{2t}\left(\int_\Omega u_{i0}u_{i0}dx + k_1(t)\right) = k_3(t). \tag{17}$$

The proof is complete. □

Lemma 4. *We suggest that T and C are the solutions of Equation (1) and satisfy boundary conditions (2) and (3). The following estimates can be obtained:*

$$\int_\Omega T^4 dx + \int_\Omega C^4 dx \le k_4(t), \tag{18}$$

where $k_4(t) = (\gamma e^{\gamma t} t + 1) \int_\Omega (T_0^4 + C_0^4) dx.$

Proof. Multiplying $(1)_3$ by $4T^3$ and integrating over $\Omega \times [0, t]$, we have the following.

$$\int_\Omega T^4 dx + 3\int_0^t \int_\Omega (T^2)_{,i}(T^2)_{,i} dx d\eta \le \int_\Omega T_0^4 dx + 4k_1 \int_0^t \oint_{\partial\Omega} T^4 dS d\eta. \tag{19}$$

Multiplying $(1)_4$ by $4C^3$ and integrating over $\Omega \times [0, t]$, we have the following.

$$\begin{aligned}\int_\Omega C^4 dx + 3\int_0^t \int_\Omega (C^2)_{,i}(C^2)_{,i} dx d\eta \le{}& \int_\Omega C_0^4 dx + 4k_2 \int_0^t \oint_{\partial\Omega} C^4 dS d\eta \\ &+ 4L \int_0^t \int_\Omega C^3 T dx d\eta \\ \le{}& \int_\Omega C_0^4 dx + 4k_2 \int_0^t \oint_{\partial\Omega} C^4 dS d\eta \\ &+ 3L \int_0^t \int_\Omega C^4 dx d\eta + L \int_0^t \int_\Omega T^4 dx d\eta.\end{aligned} \tag{20}$$

A combination of (19) and (20) provides the following.

$$\begin{aligned}\int_\Omega (T^4 + C^4) dx + 3\int_0^t \int_\Omega ((T^2)_{,i}(T^2)_{,i} + (C^2)_{,i}(C^2)_{,i}) dx d\eta \le{}& \int_\Omega (T_0^4 + C_0^4) dx \\ + (4k_1 + 4k_2) \int_0^t \oint_{\partial\Omega} (T^4 + C^4) dS d\eta + 3L \int_0^t \int_\Omega (T^4 + C^4) dx d\eta.&\end{aligned} \tag{21}$$

Using (4), we obtain the following:

$$\int_{\partial\Omega} T^4 dS \le \left[\frac{2}{m} + \frac{2d^2\varepsilon_0}{m^2}\right] \int_\Omega T^4 dx + \frac{1}{2\varepsilon_0} \int_\Omega T_{,i}^2 T_{,i}^2 dx, \tag{22}$$

and the following is the case.

$$\int_{\partial\Omega} C^4 dS \le \left[\frac{2}{m} + \frac{2d^2\varepsilon_0}{m^2}\right] \int_\Omega C^4 dx + \frac{1}{2\varepsilon_0} \int_\Omega C_{,i}^2 C_{,i}^2 dx. \tag{23}$$

Inserting (22) and (14) into (21) and choosing $\varepsilon_0 = k_1 + k_2$, we have the following:

$$\int_\Omega (T^4 + C^4) dx \le \int_\Omega (T_0^4 + C_0^4) dx + \gamma \int_0^t \int_\Omega (T^4 + C^4) dx d\eta, \tag{24}$$

with $\gamma = (4k_1 + 4k_2)\left[\frac{2}{m} + \frac{2d^2(k_1+k_2)}{m^2}\right] + 3L.$

Integrating (24), we obtain the following.

$$\int_0^t \int_\Omega (T^4 + C^4) dx d\eta \le e^{\gamma t} t \int_\Omega (T_0^4 + C_0^4) dx. \tag{25}$$

Inserting (25) into (24), we obtain the following.

$$\int_\Omega (T^4 + C^4) dx \le (\gamma e^{\gamma t} t + 1) \int_\Omega (T_0^4 + C_0^4) dx. \tag{26}$$

The proof is finished. □

Lemma 5. *We suggest that u_i is a solution of Equation (1) and that it satisfies the initial boundary conditions (2) and (3). The following estimates can be obtained.*

$$\int_\Omega u_{i,j}u_{i,j}dx \leq (k_3(t))^{\frac{1}{2}}\left(\int_\Omega u_{i,t}u_{i,t}dx\right)^{\frac{1}{2}} + 2(k_1(t))^{\frac{1}{2}}(k_3(t))^{\frac{1}{2}}. \tag{27}$$

Proof. Multiplying $(1)_1$ by u_i and integrating over Ω, we have the following.

$$\int_\Omega u_{i,t}u_i dx = -\int_\Omega u_{i,j}u_{i,j}dx + \int_\Omega g_i u_i T dx - \int_\Omega h_i u_i C dx.$$

Using the Schwarz's inequality, we have the following.

$$\begin{aligned}\int_\Omega u_{i,j}u_{i,j}dx \leq &\left(\int_\Omega u_{i,t}u_{i,t}dx\right)^{\frac{1}{2}}\left(\int_\Omega u_i u_i dx\right)^{\frac{1}{2}} + \left(\int_\Omega T^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega u_i u_i dx\right)^{\frac{1}{2}} \\ &+ \left(\int_\Omega C^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega u_i u_i dx\right)^{\frac{1}{2}}.\end{aligned} \tag{28}$$

Inserting (5) and (15) into (28), we have the following.

$$\int_\Omega u_{i,j}u_{i,j}dx \leq (k_3(t))^{\frac{1}{2}}\left(\int_\Omega u_{i,t}u_{i,t}dx\right)^{\frac{1}{2}} + 2(k_1(t))^{\frac{1}{2}}(k_3(t))^{\frac{1}{2}}. \tag{29}$$

Thus, the proof is complete. □

Lemma 6. *We suggest that u_i is a solution of Equation (1) and that it satisfies the initial boundary conditions (2) and (3). The following estimates can be obtained:*

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_{i,t}u_{i,t}dx \leq &\frac{\lambda^2 \nu k_3^{\frac{1}{2}}(t)}{2}\left(\int_\Omega u_{i,t}u_{i,t}dx\right)^{\frac{3}{2}} + k_5(t)\int_\Omega u_{i,t}u_{i,t}dx \\ &+ \int_\Omega T_{,t}^2 dx + \int_\Omega C_{,t}^2 dx - \int_\Omega u_{i,jt}u_{i,jt}dx,\end{aligned} \tag{30}$$

where $k_5(t)$ is a positive function determined later.

Proof. By multiplying $(1)_1$ by $u_{i,t}$ and integrating over Ω, we have the following.

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_{i,t}u_{i,t}dx &= 2\int_\Omega u_{i,t}u_{i,tt}dx \\ &= 2\int_\Omega u_{i,t}(-\lambda u_j u_{i,j} - p_{,i} + u_{i,jj} - g_i T + h_i C)_{,t}dx \\ &= -2\int_\Omega u_{i,jt}u_{i,jt}dx - 2\lambda\int_\Omega u_{i,t}u_{j,t}u_{i,j}dx - 2\int_\Omega g_i u_{i,t}T_{,t}dx \\ &+ 2\int_\Omega h_i u_{i,t}C_{,t}dx \\ &\leq -2\int_\Omega u_{i,jt}u_{i,jt}dx + 2\lambda\left(\int_\Omega (u_{i,t}u_{i,t})^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega u_{i,j}u_{i,j}dx\right)^{\frac{1}{2}} \\ &+ 2\int_\Omega u_{i,t}u_{i,t}dx + \int_\Omega T_{,t}^2 dx + \int_\Omega C_{,t}^2 dx.\end{aligned} \tag{31}$$

The following Sobolev inequality in R^2(see [34] (41)) is valid:

$$\int_\Omega |F|^4 dx \leq \nu\int_\Omega |F|^2 dx\int_\Omega |\nabla F|^2 dx, \tag{32}$$

with ν as a positive constant.

If we choose $F = u_{i,t}$ in (32), we have the following.

$$\int_\Omega (u_{i,t}u_{i,t})^2 dx \le \nu \int_\Omega u_{i,t}u_{i,t}dx \int_\Omega u_{i,jt}u_{i,jt}dx. \tag{33}$$

A combination of (31) and (33) provides the following.

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_{i,t}u_{i,t}dx + \int_\Omega u_{i,jt}u_{i,jt}dx \le \frac{\lambda^2\nu}{2}\int_\Omega u_{i,t}u_{i,t}dx \int_\Omega u_{i,j}u_{i,j}dx + 2\int_\Omega u_{i,t}u_{i,t}dx \\ + \int_\Omega T_{,t}^2 dx + \int_\Omega C_{,t}^2 dx.\end{aligned} \tag{34}$$

Inserting (27) into (34), we obtain the following:

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_{i,t}u_{i,t}dx + \int_\Omega u_{i,jt}u_{i,jt}dx \le \frac{\lambda^2\nu k_3^{\frac{1}{2}}(t)}{2}\left(\int_\Omega u_{i,t}u_{i,t}dx\right)^{\frac{3}{2}} + k_5(t)\int_\Omega u_{i,t}u_{i,t}dx \\ + \int_\Omega T_{,t}^2 dx + \int_\Omega C_{,t}^2 dx,\end{aligned} \tag{35}$$

with $k_5(t) = \lambda^2\nu k_1^{\frac{1}{2}}(t)k_3^{\frac{1}{2}}(t) + 2$.

Thus, the proof is complete. □

We suggest that T and C are the solutions of Equation (1) and that they satisfy boundary conditions (2) and (3). The following estimates can be obtained:

$$\begin{aligned}\frac{d}{dt}\int_\Omega T_{,t}^2 dx \le \left(2k_1\left[\frac{2}{m} + \frac{2d^2k_1}{m^2}\right] + 1\right)\int_\Omega T_{,t}T_{,t}dx \\ + 2k_4(t)\nu^2\int_\Omega u_{i,t}u_{i,t}dx + \frac{1}{2}\int_\Omega u_{i,jt}u_{i,jt}dx,\end{aligned} \tag{36}$$

and the following is the case.

$$\begin{aligned}\frac{d}{dt}\int_\Omega C_{,t}^2 dx \le \left(2k_2\left[\frac{2}{m} + \frac{2d^2k_2}{m^2}\right] + 1\right)\int_\Omega C_{,t}C_{,t}dx \\ + 2k_4(t)\nu^2\int_\Omega u_{i,t}u_{i,t}dx + \frac{1}{2}\int_\Omega u_{i,jt}u_{i,jt}dx + \frac{L^2}{k}\int_\Omega T_{,t}T_{,t}dx.\end{aligned} \tag{37}$$

Proof. Multiplying $(1)_4$ by $C_{,t}$ and integrating over $\Omega \times [0,t]$, we have the following.

$$\begin{aligned}\frac{d}{dt}\int_\Omega C_{,t}^2 dx &= 2\int_\Omega C_{,t}C_{,tt}dx \\ &= 2\int_\Omega C_{,t}(-u_iC_{,i} + C_{,jj} + LT - kC)_{,t}dx \\ &= -2\int_\Omega C_{j,t}C_{j,t}dx + 2\oint_{\partial\Omega} C_{,t}\frac{\partial C_{,t}}{\partial n}dS + 2\int_\Omega C_{i,t}u_{i,t}Cdx \\ &+ 2L\int_\Omega C_{,t}T_{,t}dx - 2k\int_\Omega C_{,t}C_{,t}dx \\ &\le -2\int_\Omega C_{j,t}C_{j,t}dx + 2k_2\oint_{\partial\Omega} C_{,t}C_{,t}dS + \frac{L^2}{k}\int_\Omega T_{,t}T_{,t}dx - k\int_\Omega C_{,t}C_{,t}dx \\ &+ 2\left(\int_\Omega C^4 dx\right)^{\frac{1}{4}}\left(\int_\Omega (u_{i,t}u_{i,t})^2 dx\right)^{\frac{1}{4}}\left(\int_\Omega C_{,it}C_{,it}dx\right)^{\frac{1}{2}}.\end{aligned}$$

Using (4) and (18), we obtain the following.

$$\frac{d}{dt}\int_\Omega C_{,t}^2 dx \le -\left(2-\frac{k_2}{\varepsilon_0}\right)\int_\Omega C_{j,t}C_{j,t}dx + 2k_2\left[\frac{2}{m}+\frac{2d^2\varepsilon_0}{m^2}\right]\int_\Omega C_{,t}C_{,t}dx$$
$$+2(k_4(t))^{\frac{1}{2}}\left(\int_\Omega (u_{i,t}u_{i,t})^2 dx\right)^{\frac{1}{2}} + \frac{L^2}{k}\int_\Omega T_{,t}T_{,t}dx + \int_\Omega C_{,t}C_{,t}dx. \tag{38}$$

If we choose $\varepsilon_0 = k_2$, we obtain the following.

$$\frac{d}{dt}\int_\Omega C_{,t}^2 dx \le -\int_\Omega C_{,jt}C_{,jt}dx + 2k_2\left[\frac{2}{m}+\frac{2d^2k_2}{m^2}\right]\int_\Omega C_{,t}C_{,t}dx$$
$$+2(k_4(t))^{\frac{1}{2}}\left(\int_\Omega (u_{i,t}u_{i,t})^2 dx\right)^{\frac{1}{2}} + \frac{L^2}{k}\int_\Omega T_{,t}T_{,t}dx + \frac{1}{2}\int_\Omega C_{,it}C_{,it}dx. \tag{39}$$

Using (33) again, we obtain the following.

$$\frac{d}{dt}\int_\Omega C_{,t}^2 dx \le \left(2k_2\left[\frac{2}{m}+\frac{2d^2k_2}{m^2}\right]+1\right)\int_\Omega C_{,t}C_{,t}dx$$
$$+2k_4(t)\nu^2\int_\Omega u_{i,t}u_{i,t}dx + \frac{1}{2}\int_\Omega u_{i,jt}u_{i,jt}dx + \frac{L^2}{k}\int_\Omega T_{,t}T_{,t}dx. \tag{40}$$

Following the same procedures, we can obtain the following.

$$\frac{d}{dt}\int_\Omega T_{,t}^2 dx \le \left(2k_1\left[\frac{2}{m}+\frac{2d^2k_1}{m^2}\right]+1\right)\int_\Omega T_{,t}T_{,t}dx$$
$$+2k_4(t)\nu^2\int_\Omega u_{i,t}u_{i,t}dx + \frac{1}{2}\int_\Omega u_{i,jt}u_{i,jt}dx. \tag{41}$$

Thus, the proof is complete. □

Lemma 7. *We suggest that u_i, T and C are the solutions of Equation (1) and satisfy boundary conditions (2) and (3). The following estimates can be obtained:*

$$\int_\Omega u_{i,t}u_{i,t}dx + \int_\Omega T_{,t}T_{,t}dx + \int_\Omega C_{,t}C_{,t}dx \le k_8(t), \tag{42}$$

where $k_8(t)$ is a positive function .

Proof. We define a new function $G(t)$ by the following.

$$G(t) = \int_\Omega u_{i,t}u_{i,t}dx + \int_\Omega T_{,t}T_{,t}dx + \int_\Omega C_{,t}C_{,t}dx.$$

A combination of (30), (36) and (37) provides the following:

$$\frac{d}{dt}G(t) \le k_6(t)G^{\frac{3}{2}}(t) + k_7(t)G(t), \tag{43}$$

with $k_6(t) = \frac{\lambda^2\nu k_3^{\frac{1}{2}}(t)}{2}$ and $k_7(t) = k_4(t)(1+4\nu^2) + \left(2k_1\left[\frac{2}{m}+\frac{2d^2k_1}{m^2}\right]\right)$ $+\left(2k_2\left[\frac{2}{m}+\frac{2d^2k_2}{m^2}\right]+1\right)$.
We can also obtain the following.

$$\frac{d}{dt}G(t) \le k_6(\tau)G^{\frac{3}{2}}(t) + k_7(\tau)G(t). \tag{44}$$

Integrating (44) from 0 to t, we obtain the following.

$$\begin{aligned} G &\leq \left(\frac{k_6(\tau)\sqrt{G(0)}}{k_6(\tau)\sqrt{G(0)}\left(e^{-\frac{1}{2}k_7(\tau)t} - 1\right) + k_7(\tau)e^{-\frac{1}{2}k_7(\tau)t}} \right)^2 \\ &= k_8(t). \end{aligned} \tag{45}$$

Inequality (45) is valid when $t \leq \frac{2\ln \frac{k_6(\tau)\sqrt{G(0)}+k_7(\tau)}{k_6(\tau)\sqrt{G(0)}}}{k_7(\tau)}$.

If the initial data $G(0)$ are small enough, we can obtain the result where inequality (45) is valid for all t.

Thus, the proof is complete. □

We suggest that u_i is a solution of Equation (1) and satisfies the initial boundary conditions (2) and (3). The following estimates can be obtained:

$$\int_\Omega u_{i,j}u_{i,j}dx \leq k_9(t), \tag{46}$$

where $k_9(t)$ is a positive function defined later.

Proof. By inserting (42) into (27), we obtain the following.

$$\begin{aligned} \int_\Omega u_{i,j}u_{i,j}dx &\leq (k_3(t))^{\frac{1}{2}}(k_8(t))^{\frac{1}{2}} + 2(k_1(t))^{\frac{1}{2}}(k_3(t))^{\frac{1}{2}} \\ &= k_9(t). \end{aligned} \tag{47}$$

Thus, the proof is complete. □

3. Main Result

We suggest that (u_i, p, T, C) is the solution of the initial boundary problems for the Boussinesq equations.

$$\begin{cases} \frac{\partial u_i}{\partial t} + \lambda_1 u_j \frac{\partial u_i}{\partial x_j} = -p_{,i} + \Delta u_i + g_i T - h_i C \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial u_i}{\partial x_i} = 0 \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \Delta T \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = \Delta C + LT - kC \quad (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{48}$$

$$u_i = 0, \frac{\partial T}{\partial n} = k_1 T, \frac{\partial C}{\partial n} = k_2 C \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{49}$$

$$u_i(x,0) = u_{i0}(x), T(x,0) = T_0(x), C(x,0) = C_0(x) \quad x \in \Omega. \tag{50}$$

Furthermore, we suggest that (u_i^*, p^*, T^*, C^*) satisfies the following problems.

$$\begin{cases} \frac{\partial u_i^*}{\partial t} + \lambda_2 u_j^* \frac{\partial u_i^*}{\partial x_j} = -p_{,i}^* + \Delta u_i^* + g_i T^* - h_i C^* \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial u_i^*}{\partial x_i} = 0 \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial T^*}{\partial t} + u_i^* \frac{\partial T^*}{\partial x_i} = \Delta T^* \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial C^*}{\partial t} + u_i^* \frac{\partial C^*}{\partial x_i} = \Delta C^* + LT^* - kC^* \quad (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{51}$$

$$u_i^* = 0, \frac{\partial T^*}{\partial n} = k_1 T^*, \frac{\partial C^*}{\partial n} = k_2 C^* \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{52}$$

$$u_i^*(x,0) = u_{i0}(x), T^*(x,0) = T_0(x), C^*(x,0) = C_0(x) \quad x \in \Omega. \tag{53}$$

We introduce some new functions—ω_i, π, θ , S and λ:

$$\omega_i = u_i - u_i^*, \pi = p - p^*, \theta = T - T^*, S = C - C^*, \lambda = \lambda_1 - \lambda_2. \tag{54}$$

Then, we can obtain the result where $(\omega_i, \pi, \theta, S, \lambda)$ satisfies the following equations.

$$\begin{cases} \frac{\partial \omega_i}{\partial t} + \lambda u_j \frac{\partial u_i}{\partial x_j} + \lambda_2 u_j \frac{\partial \omega_i}{\partial x_j} + \lambda_2 \omega_j \frac{\partial u_i^*}{\partial x_j} = -\pi_{,i} + \Delta\omega_i + g_i\theta - h_i S \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial \omega_i}{\partial x_i} = 0 \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial \theta}{\partial t} + \omega_i \frac{\partial T}{\partial x_i} + u_i^* \frac{\partial \theta}{\partial x_i} = \Delta\theta \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial S}{\partial t} + u_i \frac{\partial S}{\partial x_i} + \omega_i \frac{\partial C^*}{\partial x_i} = \Delta S + L\theta - kS \quad (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{55}$$

$$\omega_i = 0, \frac{\partial \theta}{\partial n} = k_1\theta, \frac{\partial S}{\partial n} = k_2 S \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{56}$$

$$\omega_i(x,0) = 0, \theta(x,0) = 0, S(x,0) = 0, x \in \Omega \ . \tag{57}$$

In this paper, we want to obtain the following main result.

Theorem 1. *We suggest that (u_i, T, C, P) is the classical solution that satisfies conditions (48)–(50). We also suggest that (u_i^*, T^*, C^*, p^*) is the classical solution that satisfies conditions (51)–(53). The difference $(\omega_i, \theta, S, \pi)$ satisfies (55)–(57). We obtain the result where solution (u_i, T, C, p) can converge to solution (u_i^*, T^*, C^*, P^*) when the Boussinesq coefficient λ_1 tends to λ_2. Additionally, the norm of $(\omega_i, \theta, S, \pi)$ satisfies the following estimates:*

$$\int_\Omega \omega_i\omega_i dx + \int_\Omega \theta^2 dx + \int_\Omega S^2 dx \le 4\lambda^2 e^{\int_0^t k_{12}(s)ds} \int_0^t e^{-\int_0^s k_{12}(\eta)d\eta} k_{10}(s)ds. \tag{58}$$

where $k_{10}(t)$ and $k_{12}(t)$ are positive functions.

The proof of the theorem will be divided into the following Lemmas.

Lemma 8. *Let u_i be solution of Equation (55) satisfying boundary conditions (56) and (57); we can obtain the following estimates:*

$$\frac{d}{dt}\|\omega\|^2 \quad \le 4\lambda^2 k_{10}(t) + k_{11}(t)\|\omega\|^2 + \|\theta\|^2 + \|S\|^2 - \frac{1}{2}\|\nabla\omega\|^2, \tag{59}$$

where $k_{10}(t)$ and $k_{11}(t)$ are positive functions defined later.

Proof. We multiply both sides of $(55)_1$ by w_i and integrate over Ω; we can obtain the following:

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|\omega\|^2 \quad &= -\lambda \int_\Omega u_j u_{i,j}\omega_i dx - \lambda_2 \int_\Omega u_j \omega_{i,j}\omega_i dx - \lambda_2 \int_\Omega \omega_j u_{i,j}^* \omega_i dx - \|\nabla\omega\|^2 \\ &+ \int_\Omega g_i\theta\omega_i dx - \int_\Omega h_i S\omega_i dx \\ &= \lambda \int_\Omega u_j u_i \omega_{i,j} dx - \lambda_2 \int_\Omega \omega_j\omega_i u_{i,j}^* dx - \|\nabla\omega\|^2 + \int_\Omega g_i\theta\omega_i dx - \int_\Omega h_i S\omega_i dx \\ &\le \lambda\|u\|_4^2\|\nabla\omega\| + \lambda_2\|\omega\|_4^2\|\nabla u^*\| - \|\nabla\omega\|^2 + \|\theta\|\|\omega\| + \|S\|\|\omega\| \\ &\le \lambda^2\varepsilon_1\|u\|_4^4 + \frac{1}{\varepsilon_1}\|\nabla\omega\|^2 + \lambda_2\nu^{\frac{1}{2}}\|\nabla u^*\|\|\omega\|\|\nabla\omega\| - \|\nabla\omega\|^2 \\ &+ \|\theta\|\|\omega\| + \|S\|\|\omega\| \\ &\le \lambda^2\varepsilon_1\nu[\|u\|^2 + \|\nabla u\|^2]^2 + \frac{1}{\varepsilon_1}\|\nabla\omega\|^2 + \frac{\lambda_2^2\varepsilon_2\nu}{2}\|\nabla u^*\|^2\|\omega\|^2 + \frac{1}{2\varepsilon_2}\|\nabla\omega\|^2 \\ &+ \|\theta\|\|\omega\| + \|S\|\|\omega\| - \|\nabla\omega\|^2, \end{aligned} \tag{60}$$

where ε_1 and ε_2 are constants no lesser than zero.

If we choose $\varepsilon_1 = 4$, $\varepsilon_2 = 2$, we obtain the following.

$$\begin{aligned} \frac{d}{dt}\|\omega\|^2 \quad &\le 8\lambda^2\nu[\|u\|^2 + \|\nabla u\|^2]^2 + (2\lambda_2^2\nu\|\nabla u^*\|^2 + 2)\|\omega\|^2 \\ &+ \|\theta\|^2 + \|S\|^2 - \frac{1}{2}\|\nabla\omega\|^2. \end{aligned} \tag{61}$$

Following the same procedures in deriving (46), we can also obtain the following:

$$\int_\Omega u_{i,j}^* u_{i,j}^* dx \le \tilde{k_9}(t), \tag{62}$$

where $\tilde{k_9}(t)$ is a positive function.

Inserting (15), (46) and (62) into (61), we obtain the following:

$$\frac{d}{dt}\|\omega\|^2 \quad \leq 4\lambda^2 k_{10}(t) + k_{11}(t)\|\omega\|^2 + \|\theta\|^2 + \|S\|^2 - \frac{1}{2}\|\nabla\omega\|^2, \tag{63}$$

where $k_{10}(t) = 8k_1^2(k_3(t) + k_9(t))^2$ and $k_{11}(t) = 2\lambda_2^2\tilde{k_9}(t) + 2$.

Thus, the proof is complete. □

Lemma 9. *We suggest that θ and S are solutions of Equation (55) and satisfy boundary conditions (2) and (3). The following estimates can be obtained:*

$$\begin{aligned}\frac{d}{dt}(\|\theta\|^2 + \|S\|^2) \quad &\leq -\frac{1}{2}\int_\Omega(|\nabla S|^2 + |\nabla\theta|^2)dx + M\int_\Omega(\theta^2 + S^2)dx \\ &+2\left(\int_\Omega(\omega_i\omega_i)^2\right)^{\frac{1}{2}}(\tilde{k_4}(t))^{\frac{1}{2}},\end{aligned} \tag{64}$$

where M is a positive constant, and $\tilde{k_4}(t)$ is a positive function.

Proof. We multiply both sides of $(55)_3$ by 2θ and integrate over Ω; we have the following.

$$\begin{aligned}\frac{d}{dt}\|\theta\|^2 \quad &= 2\int_\Omega \theta(\triangle\theta - u_i\theta_{,i} - \omega_i T^*_{,i})dx \\ &= -2\int_\Omega|\nabla\theta|^2dx + 2k_1\oint_{\partial\Omega}\theta^2 dS + 2\int_\Omega \theta_{,i}\omega_i T^* dx \\ &\leq -2\int_\Omega|\nabla\theta|^2dx + 2k_1\oint_{\partial\Omega}\theta^2 dS + \int_\Omega\theta_{,i}\theta_{,i}dx \\ &+\left(\int_\Omega(\omega_i\omega_i)^2\right)^{\frac{1}{2}}\left(\int_\Omega T^{*4}\right)^{\frac{1}{2}}.\end{aligned} \tag{65}$$

Following the same procedures in deriving (18), we can also obtain the following:

$$\int_\Omega T^{*4}dx + \int_\Omega C^{*4}dx \leq \tilde{k_4}(t), \tag{66}$$

where $\tilde{k_4}(t)$ is a positive function.

Inserting (66) into (65), we obtain the following.

$$\begin{aligned}\frac{d}{dt}\|\theta\|^2 \quad &\leq -\int_\Omega|\nabla\theta|^2dx + 2k_1\oint_{\partial\Omega}\theta^2 dS \\ &+\left(\int_\Omega(\omega_i\omega_i)^2\right)^{\frac{1}{2}}(\tilde{k_4}(t))^{\frac{1}{2}}.\end{aligned} \tag{67}$$

We multiply both sides of $(55)_4$ by $2S$ and integrate over Ω; we have the following.

$$\begin{aligned}\frac{d}{dt}\|S\|^2 \quad &= 2\int_\Omega S(\triangle S - u_iS_{,i} - \omega_i C^*_{,i} - kS + L\theta)dx \\ &\leq -\int_\Omega|\nabla S|^2dx + 2k_2\oint_{\partial\Omega}S^2dS + L\int_\Omega\theta^2dx + L\int_\Omega S^2dx \\ &+\left(\int_\Omega(\omega_i\omega_i)^2\right)^{\frac{1}{2}}(\tilde{k_4}(t))^{\frac{1}{2}}.\end{aligned} \tag{68}$$

Using (4) and combining (67) and (68), we obtain the following.

$$\begin{aligned}\frac{d}{dt}(\|\theta\|^2 + \|S\|^2) \quad &\leq -\frac{1}{2}\int_\Omega(|\nabla S|^2 + |\nabla\theta|^2)dx + M\int_\Omega(\theta^2 + S^2)dx \\ &+2\left(\int_\Omega(\omega_i\omega_i)^2\right)^{\frac{1}{2}}(\tilde{k_4}(t))^{\frac{1}{2}},\end{aligned} \tag{69}$$

with $M = \max\{2k_1\left[\frac{2}{m} + \frac{2d^2k_1}{m^2}\right] + L, 2k_2\left[\frac{2}{m} + \frac{2d^2k_2}{m^2} + L\right]\}$.

Thus, the proof is complete. □

Lemma 10. *We suggest that ω_i, θ and S are solutions of Equation (55), and boundary conditions (56) and (57) are also satisfied. We can obtain the following estimates:*

$$\int_\Omega\omega_i\omega_i dx + \int_\Omega\theta^2dx + \int_\Omega S^2dx \leq 4\lambda^2 e^{\int_0^t k_{12}(s)ds}\int_0^t e^{-\int_0^s k_{12}(\eta)d\eta}k_{10}(s)ds. \tag{70}$$

with $k_{12}(t)$ as a positive function .

Proof. We define a function

$$\zeta(t) = \int_\Omega \omega_i\omega_i dx + \int_\Omega \theta^2 dx + \int_\Omega S^2 dx.$$

Combining (59) and (64); we thus have the following.

$$\begin{aligned} \tfrac{d}{dt}\zeta(t) &\leq 4\lambda^2 k_{10}(t) + (k_{11}(t) + M + 1)\zeta(t) - \tfrac{1}{2}\|\nabla\omega\|^2 \\ &+2\big(\int_\Omega(\omega_i\omega_i)^2 dx\big)^{\frac{1}{2}}(\tilde{k_4}(t))^{\frac{1}{2}}. \end{aligned} \tag{71}$$

Using (32), we have the following.

$$2\big(\int_\Omega(\omega_i\omega_i)^2 dx\big)^{\frac{1}{2}}(\tilde{k_4}(t))^{\frac{1}{2}} \leq 8\tilde{k_4}(t)\nu\|\omega\|^2 + \tfrac{1}{2}\|\nabla\omega\|^2. \tag{72}$$

Inserting (72) into (71), we obtain the following:

$$\tfrac{d}{dt}\zeta(t) \leq k_{12}(t)\zeta(t) + 4\lambda^2 k_{10}(t), \tag{73}$$

with $k_{12}(t) = k_{11}(t) + M + 1 + 8\tilde{k_4}(t)\nu$.

Equation (73) can be rewritten as follows.

$$\tfrac{d}{dt}\Big(e^{-\int_0^t k_{12}(s)ds}\zeta(t)\Big) \leq 4e^{-\int_0^t k_{12}(s)ds}\lambda^2 k_{10}(t). \tag{74}$$

An integration of (74) provides the following.

$$\zeta(t) \leq 4\lambda^2 e^{\int_0^t k_{12}(s)ds}\int_0^t e^{-\int_0^s k_{12}(\eta)d\eta}k_{10}(s)ds. \tag{75}$$

Inequality (75) shows that when λ tends to zero, the differences of the solutions tend to zero as the indicated norm.

Thus, the proof is complete. □

Inequality (75) is the result we want to prove in the Theorem. Thus, we complete the proof of the theorem.

4. Conclusions

In the present paper, the result of the continuous dependence type could be obtained for the Boussinesq coefficient λ. An energy method was used. For the case when $\Omega \in R^3$, the *Poincaré* inequality used in this paper was no longer applicable. There were difficulties in obtaining the key bound for $\int_\Omega u_{i,j}u_{i,j}dx$. We thought it would be an interesting topic to study the case when $\Omega \in R^3$ in future. Additionally, if we changed the bounded domain, Ω, by an unbounded domain, we thought the terms containing pressure p were difficult to tackle. These terms could not be bounded by the prescribed data by using the same method proposed in this paper. Some new methods might be developed in order to overcome these difficulties. We are sure that some good results would be obtained if we studied the above two problems. These studies would be new and interesting.

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