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Random Perturbation of Invariant Manifolds for Non-Autonomous Dynamical Systems

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Abstract: Random invariant manifolds are geometric objects useful for understanding dynamics near the random fixed point under stochastic influences. Under the framework of a dynamical system, we compared perturbed random non-autonomous partial differential equations with original stochastic non-autonomous partial differential equations. Mainly, we derived some pathwise approximation results of random invariant manifolds when the Gaussian white noise was replaced by colored noise, which is a type of Wong–Zakai approximation.

Keywords: random invariant manifold; random non-autonomous partial differential equations; stochastic non-autonomous partial differential equation; invariant manifolds; Wong–Zakai approximation

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1. Introduction

Invariant manifolds play an important role in qualitative dynamical behaviors by providing geometric structures (i.e., to understand or reduce stochastic dynamics) [1–4]. The existence and regularity properties of invariant manifolds for stochastic partial differential equations with linear noise are reasonably understood. However, for general stochastic partial differential equations, random invariant manifolds consist of samples that are subsets of an infinite dimensional space. Thus, it is difficult to describe random invariant manifolds, let alone the dynamics of them. Many authors have attempted to describe and approximate such invariant manifolds.

If the Brownian motion has the form $W(t,\omega) = \omega(t)$, then for each $\delta \in \mathbb{R}$, define $\mathcal{G}_\circ : \Omega \to \mathbb{R}$ by

$$\mathcal{G}_\circ(\omega) = \frac{1}{\delta}\omega(\delta),$$

we have

$$\mathcal{G}_\circ(\theta_t\omega) = \frac{1}{\delta}(\omega(t+\delta) - \omega(t)),$$

where θ_t is the Wiener shift. $\mathcal{G}_\circ(\theta_t\omega)$ can be viewed as an approximation of white noise. It was proved in [5,6] that as $\delta \to 0$,

$$W_\delta(t,\omega) = \int_0^t \mathcal{G}_\circ(\theta_s\omega)ds$$

converges to $W(t,\omega)$ uniformly on any finite time interval, almost surely. In [6–8], the authors used the same approximation to study random invariant manifolds and invariant foliations of autonomous systems. In [9,10], the authors considered stochastic evolutionary equations driven by an integrated Ornstein–Uhlenbeck process by another Wong–Zakai-type approximation of random invariant stable manifolds. Blomker and Wang [11], Sun et al. [4] have also derived some meaningful results from approximating such invariant manifolds. The purpose of the present paper is to extend the result in [9] to a nonautonomous case. In other words, we approximated the invariant manifolds for a class of nonautonomous stochastic partial differential equations via the integrated Ornstein–Uhlenbeck process, which is colored (or correlated) noise. Similar results can be obtained if we replace the integrated Ornstein–Uhlenbeck process by a stationary stochastic process $\mathcal{G}_\circ(\theta_t\omega)$.

Wong and Zakai' [12,13] conducted the study of approximating a Wiener process by its piecewise linear counterpart. For many subjects, concerning the asymptotic behavior of stochastic differential equations, their results are of great importance. Their theorems have discovered the "correction term" in the limit equation.

The classical Wong–Zakai approximation [12,13] states that when piecewise linear approximations approach the Brownian motion, the corresponding ordinary integrals converge to the stochastic integral in Stratonovich sense. Moreover, the solution of a stochastic differential equation (SDE) with Gaussian white noise is approximated by differential equations with smooth noises, in the Stratonovich sense. In an infinite dimensional case, the situation becomes more complicated. For stochastic partial differential equations (PDEs), with space–time white noise, the authors in [14] derived a general Wong–Zakai correction term instead of the ô-Stratonovich correction term. The pathwise convergence between a family of random PDEs and the original stochastic PDE was finally derived. The result of [14] has been generalized to the generalized KPZ equations, where one has the Itô's isometry property for the solution, see [15]. A weak convergence result was derived by Tessitore and Zabczyk [16] for Wong–Zakai approximations for a stochastic PDE of multiplicative noise. Moreover, the Wong–Zakai approximation for stochastic evolutionary equations has also been considered by other authors—references [17–19] are included among the earliest investigations. Important contributions are also due to [20–22]. These authors mainly considered the case of one space variable for stochastic evolution equations. For stochastic differential equations driven by semi-martingales, an approximation theorem can be found in [23], where the authors use a sequence of processes with piecewise monotonic sample functions to approximate semi-martingales. When the state space is a domain in $\mathbb{R}^d$, the boundary conditions are indispensable. We refer to [24–27] for more interesting work.

In this paper, we focus on a Wong–Zakai type approximation of a random invariant manifold for random dynamical systems. The systems are generated by stochastic evolutionary equations, which are driven by the integrated Ornstein–Uhlenbeck process.

Consider the following non-autonomous stochastic PDE in a separable Hilbert space H with norm $|\cdot|$.

$$\frac{du(t)}{dt} = A(t)u(t) + F(t, u(t)) + g\dot{B}(t), \tag{1}$$

where g is an element in H. Precise definitions of $A(t), B(t)$, and F will be provided in the next section.

We approximate the system above by the following system:

$$\frac{du^\varepsilon(t)}{dt} = A(t)u^\varepsilon(t) + F(t, u^\varepsilon(t)) + g\dot{\Phi}^\varepsilon(t), . \tag{2}$$

Φ^ε is an integrated Ornstein–Uhlenbeck process whose definition will be provided in the following section. We first show that, as ε tends 0, the solution u can be approximated by u^ε. Then we prove the Wong–Zakai type approximation of the random invariant manifold.

The paper is organized as follows. After we recall some basic concepts for random dynamical systems in Section 2, we will show that hyperbolic equilibrium/solutions of perturbation random PDEs converge to that of the original stochastic PDEs, presented in Section 3.1. The existence theorem for stable manifold is presented in Section 3.2. Then, we derive some approximation results for random invariant manifolds for colored noise in Section 4.

2. Random Dynamical Systems

In this section, we first introduce some basic notations, assumptions, and concepts on stochastic evolutionary equations and random dynamical systems.

2.1. Stochastic Evolutionary Equations with Additive Noise

Consider the non-autonomous stochastic PDE in H,

$$\frac{du(t)}{dt} = A(t)u(t) + F(t, u(t)) + g\dot{B}(t). \tag{3}$$

Here, $B(t)$ is a standard scalar Brownian motion. Denote the domain of the operator $A(t)$ by $D(A(t))$. Under the following hypotheses, system (3) has a unique mild solution [28].

Hypothesis 1 (H1). *The linear operators $A(t) : D(A(t)) \to H$ generates a two-parameter family of a strongly continuous semigroup $T(t, \tau)$ on H, which satisfies*

$$\frac{\partial T(t,\tau)}{\partial t} = A(t)T(t,\tau) \tag{4}$$

$$\frac{\partial T(t,\tau)}{\partial \tau} = -T(t,\tau)A(\tau) \tag{5}$$

and there exists a continuous projection P_t^+ on H, such that, for any $x \in H$,

$$\begin{aligned} |T(t,\tau)P_\tau^+ x| &\le Ke^{\alpha(t-\tau)}|x|, \quad t \le \tau, \\ |T(t,\tau)P_\tau^- x| &\le Ke^{\beta(t-\tau)}|x|, \quad t \ge \tau, \end{aligned} \tag{6}$$

where $P_\tau^- = I - P_\tau^+$, $\alpha > 0 > \beta$ and K is positive. Denote $H_\tau^- = P_\tau^- H$ and $H_\tau^+ = P_\tau^+ H$. Then, $H = H_\tau^+ \oplus H_\tau^-$. We will call H_τ^- the stable subspace and H_τ^+ the unstable subspace, respectively.

Hypothesis 2 (H2). *The nonlinear term F is Lipschitz continuous on the second variable, i.e., for any $x_1, x_2 \in H$,*

$$|F(\tau, x_1) - F(\tau, x_2)| \le L|x_1 - x_2|,$$

with the Lipschitz constant $L > 0$.

Hypothesis 3 (H3). *The family of operators $A(t)$ is uniformly bounded in the operator norm: $|A(t)| \le C$ for all $t \in \mathbb{R}$.*

2.2. Random Cocycle

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ be an ergodic metric dynamical system (briefly MDS θ).

Remark 1. *A family of mappings $\varphi : \mathbb{R} \times \mathbb{R} \times \Omega \times H \to H, (t, \tau, \omega, x) \mapsto \varphi(t, \tau, \omega, x)$ is called a random cocycle on H over an MDS θ if for all $s, t \in \mathbb{R}, \tau \in \mathbb{R}$, and $\omega \in \Omega$, the following statements are satisfied:*

1. *$\varphi(., \tau, ., .) : \mathbb{R} \times \Omega \times H \to H$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H))-$measurable;*
2. *$\varphi(0, \tau, \omega, .)$ is the identity on H;*
3. *$\varphi(t, \tau, \omega, .) = \varphi(t, s, \theta_{s-\tau}\omega, \varphi(s, \tau, \omega, .))$.*

A random cocycle φ is said to be continuous in H if the mapping $\varphi(0, \tau, \omega.)$: $H \to H$ is continuous for each $t \in \mathbb{R}, \tau \in \mathbb{R}$, and $\omega \in \Omega$. Then φ together with the MDS θ form a non-autonomous random dynamical system.

2.3. A Random Equation Approximating a Stochastic Equation

Let B_t be a two-sided Brownian motion with trajectories in the space $C_0(\mathbb{R}, \mathbb{R})$ of real continuous functions defined on $\mathbb{R}$, taking zero value at $t = 0$. This set is equipped with the compact open topology. A set Ω is called $\{\theta_t\}_{t\in\mathbb{R}}$-invariant if $\theta_t\Omega = \Omega$ for $t \in \mathbb{R}$. We will consider, instead of the whole $C_0(\mathbb{R}, \mathbb{R})$, a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\Omega \subset C_0(\mathbb{R}, \mathbb{R})$ of $\mathbb{P}$-measure one and the trace g-algebra $\mathcal{F}$ of $\mathcal{B}(C_0(\mathbb{R}, \mathbb{R}))$ with respect to Ω.

The driving system $\theta = (\theta_t, t \in \mathbb{R})$ on Ω is defined by the shifts

$$(\theta_t\omega)(s) := \omega(t+s) - \omega(t), \ \ \omega \in \Omega. \tag{7}$$

The mapping $(t, \omega) \to \theta_t\omega$ is continuous, thus measurable. The probability measure is ergodic and θ-invariant, i.e.,

$$\mathbb{P}(\theta_t^{-1}(A)) = \mathbb{P}(A)$$

for all $A \in \mathcal{F}$.

We consider a Langevin equation

$$\begin{cases} dz^\varepsilon = -\frac{1}{\varepsilon}z^\varepsilon dt + \frac{1}{\varepsilon}dB_t, \varepsilon > 0 \\ z^\varepsilon(0) = \frac{1}{\varepsilon}\int_{-\infty}^0 e^{\frac{s}{\varepsilon}}dB_s. \end{cases} \tag{8}$$

For simplicity, from now on, we will assume ε to be discrete, say, $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}^+$. So, $\varepsilon \to 0$ actually means $n \to \infty$. A solution of this equation is called an Ornstein–Uhlenbeck process.

Define

$$\Phi_t^\varepsilon \triangleq \int_0^t z^\varepsilon(s)ds.$$

We recall a result from [19]:

Lemma 1. *Let B_t be a scalar standard Brownian motion. Then,$\Phi_t^\varepsilon \to B_t$ uniformly in $[0, T]$ almost surely for every finite $T > 0$, as $\varepsilon \to 0$.*

Moreover, we have the following results:

Lemma 2. *Let $B(t)$ be a standard Brownian motion on $\mathbb{R}^d$. Then we have (I) There exists a $\{\theta_t : t \in \mathbb{R}\}$-invariant set Ω of full measure, such that the sample paths $\omega(t)$ of $B(t)$ satisfies*

$$\lim_{t\to\pm\infty} \frac{\omega(t)}{t} = 0, \quad \omega \in \Omega.$$

(II) The random variable

$$z^\varepsilon(\omega) = \int_{-\infty}^0 e^{\frac{s}{\varepsilon}}dB(s) = -\int_{-\infty}^0 \frac{1}{\varepsilon}e^{\frac{s}{\varepsilon}}\omega(s)ds, \ \omega \in \Omega,$$

is well-defined. Moreover,

$$z^\varepsilon(\theta_t\omega) = -\int_{-\infty}^0 \frac{1}{\varepsilon}e^{\frac{s}{\varepsilon}}\theta_t\omega(s)\, ds = \omega(t) - \int_{-\infty}^0 \frac{1}{\varepsilon}e^{\frac{s}{\varepsilon}}\omega(t+s)\, ds$$

is the unique stationary solution of (8)

(III)

$$\lim_{t\to\pm\infty} \frac{|z^\varepsilon(\theta_t\omega)|}{|t|} = 0 \ \textit{and} \ \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z^\varepsilon(\theta_s\omega)ds = 0.$$

(IV) For any fixed $\gamma > 0$,

$$\lim_{\varepsilon\to 0+} \sup_{0\le t\le T} \int_0^t e^{\gamma(t-s)}|\Phi^\varepsilon(s) - B(s)|ds = 0,$$

and for any $\eta \geq 0$,

$$\lim_{\varepsilon \to 0+} \sup_{0 \leq t < \infty} e^{-\eta t} \left| \int_t^\infty T(t,s) P_s^+ g d(B_s - \Phi_s^\varepsilon)(\omega) \right| = 0.$$

(V) Similarly, for any fixed $\gamma > 0$,

$$\lim_{\varepsilon \to 0+} \sup_{-T \leq t \leq 0} \int_t^0 e^{-\gamma(t-s)} |\Phi^\varepsilon(s) - B(s)| ds = 0,$$

and for any $\eta \geq 0$,

$$\lim_{\varepsilon \to 0+} \sup_{-\infty < t \leq 0} e^{\eta t} \int_{-\infty}^t T(t,s) P_s^- g d(B_s - \Phi_s^\varepsilon)(\omega) | = 0.$$

Proof. For the proof of part (I) to (III), see [1]. Since the proof of (IV) applies to (V), we only prove (IV). Denote

$$\sup_{0 \leq t < \infty} e^{-\eta t} \left| \int_t^\infty T(t,s) P_s^+ g d(B_s - \Phi_s^\varepsilon)(\omega) \right|$$

by I_1. By integration by parts for s, Equations (5) and (6), the trajectory property of $B(t)$ and Φ_t^ε in (III), and the property of the two-parameter operator $T(t,s)$, we can derive that

$$\int_\infty^t T(t,s) P_s^+ g d(B_s - \Phi_s^\varepsilon)(\omega) = P_t^+ g (B_t - \Phi_t^\varepsilon) - \int_\infty^t T(t,s) P_s^+ A(s) g (B_s - \Phi_s^\varepsilon)(\omega) ds.$$

Thus,

$$\begin{aligned} I_1 &\leq \sup_{t \in [0,\infty)} e^{-\eta t} |P_s^+ g| |B_t - \Phi_t^\varepsilon| + \sup_{t \in [0,\infty)} e^{-\eta t} \int_t^\infty |T(t,s) P_s^+ A(s) g| |(B_s - \Phi_s^\varepsilon)(\omega)| ds \\ &\leq \sup_{t \in [0,\infty)} e^{-\eta t} |P_s^+ g| |B_t - \Phi_t^\varepsilon| + \sup_{t \in [0,\infty)} e^{-\eta t} \int_t^\infty e^{-\alpha(s-t)} |A(s) g| |(B_s - \Phi_s^\varepsilon)(\omega)| ds. \end{aligned} \tag{9}$$

Denote the two terms of the last inequality above by I_{11} and I_{12}, respectively. It is easy to see that $I_{11} \to 0$ as $\varepsilon \to 0$. As to I_{12}, by the trajectory property of B_t and Φ_t^ε (I) and (III), it is not difficult to derive that: for $\forall \tilde{\varepsilon}$, there exists T, such that $e^{-\frac{\alpha t}{2}} |B_t - \Phi_t^\varepsilon| \leq \tilde{\varepsilon}$, if $t \geq T$. By Lemma 1, for every finite T, there exists a $\varepsilon_0 > 0$ such that $|B_t - \Phi_t^\varepsilon| \leq \tilde{\varepsilon}$ for $\varepsilon \leq \varepsilon_0, t \in [0,T]$.

If $t \geq T$, then, with the aid of (H3),

$$\begin{aligned} & e^{-\eta t} \int_t^\infty e^{-\alpha(s-t)} |A(s) g| |(B_s - \Phi_s^\varepsilon)(\omega)| ds \\ & \leq C |g| \tilde{\varepsilon} e^{-(\eta - \alpha) t} \int_t^\infty e^{-\frac{\alpha s}{2}} ds \\ & < \frac{2C |g| \tilde{\varepsilon} e^{-(\eta-\alpha)t}}{\alpha}. \end{aligned}$$

If $t \leq T$, with the aid of (H3), we have

$$
\begin{aligned}
& e^{-\eta t}\int_t^{\infty} e^{-\alpha(s-t)}|A(s)g||(B_s-\Phi_s^{\varepsilon})(\omega)|ds \\
& = e^{-\eta t}\int_t^{T} e^{-\alpha(s-t)}|A(s)g||(B_s-\Phi_s^{\varepsilon})(\omega)|ds \\
& + e^{-\eta t}\int_T^{\infty} e^{-\alpha(s-t)}|A(s)g||(B_s-\Phi_s^{\varepsilon})(\omega)|ds \\
& \le e^{-(\eta-\alpha)t}C|g|\tilde{\varepsilon}\int_t^{T} e^{-\alpha s}ds + e^{-(\eta-\alpha)t}C|g|\tilde{\varepsilon}\int_T^{\infty} e^{-\frac{\alpha s}{2}}ds.
\end{aligned}
$$

Hence,

$$
I_{12} < \frac{C|g|\tilde{\varepsilon}}{\alpha} + \frac{2C|g|\tilde{\varepsilon}}{\alpha},
$$

which means that $I_{12} \to 0$ as $\varepsilon \to 0$. □

We consider the following process

$$
\frac{du^{\varepsilon}(t)}{dt} = A(t)u^{\varepsilon}(t) + F(t,u^{\varepsilon}(t)) + g\dot{\Phi}^{\varepsilon}(t) \tag{10}
$$

to approximate the original system (3).

Equation (10) is a Wong–Zakai type approximation [12,13,16] for Equation (3). We will consider the dynamical behavior of (3) by random invariant manifold [1–3,29,30]. Random invariant manifolds are geometric objects useful for understanding dynamical behavior near the random fixed point under stochastic influences. In detail, under some gap conditions, we will show that the random invariant manifold of (10) converges to that of (3) as ε tends to 0.

3. Existence of Invariant Manifold

In order to model a non-autonomous system defined on Hilbert space H, we need the two-parameter flow setup (for example, see [31]) to define the dynamical system and invariant set.

$$
\phi(t,u,\omega)\phi(u,s,\omega)x = \phi(t,s,\omega)x,
$$

where ϕ depends measurably on its variables.

Remark 2. *A collection of nonempty closed sets $M = \{M(s,\omega)\}_{\omega\in\Omega}$, in a complete separable metric space (H, d_H), is called a* random set *if for any $x \in H$*

$$
\omega \to \inf_{y\in M(s,\omega)} d_H(x,y)
$$

is a random variable.

Remark 3. *A random set $M(s,\omega)$ is called* positive invariant set *of ϕ, if, for all $s \le t$,*

$$
\phi(t,s,\omega)M(s,\omega) \subset M(t,\omega).
$$

If an invariant set $M^{\mp}(\tau,\omega)$ can be represented by a graph of a (Lipschitz) mapping

$$
h^{\mp}(\tau,\cdot,\omega) : H_{\tau}^{\mp} \to H_{\tau}^{\pm}
$$

such that

$$
M^{-}(\tau,\omega) = \{(\tau,\xi,h^{-}(\tau,\xi,\omega))|\xi \in H_{\tau}^{-}\} \tag{11}
$$

and

$$
M^{+}(\tau,\omega) = \{(h^{+}(\tau,\xi,\omega),\tau,\xi)|\xi \in H_{\tau}^{+}\}, \tag{12}
$$

then $M^-(\tau,\omega)$ *is called a* (Lipschitz) stable manifold *and* $M^+(\tau,\omega)$ *is called a* (Lipschitz) unstable manifold. *Here,* H_τ^- *is the stable subspace and* H_τ^+ *is the unstable subspace.*

3.1. Existence Theorem for Hyperbolic Equilibrium of Cocycle

Here, we briefly introduce a sufficient condition to ensure the existence of hyperbolic equilibriums for Equations (3) and (10). For details, we refer to Section 2.4 in [3]. First, suppose the nonlinear term is Lipschitz continuous with the Lipschitz constant L and $KL(\frac{1}{\alpha-\eta}+\frac{1}{\eta-\beta})<1$. By contraction mapping arguments, the fixed point defines a measurable mapping $u^*(\tau,\tau,\omega)$ as below,

$$u^*(\tau,\tau,\omega)=\int_{-\infty}^{\tau}T(\tau,s)P_s^-F(s,u^*(s,\tau,\omega))ds+\int_{\infty}^{\tau}T(\tau,s)P_s^+F(s,u^*(s,\tau,\omega))ds \\ +\int_{-\infty}^{\tau}T(\tau,s)P_s^-gdB(s)(\omega)+\int_{\infty}^{\tau}T(\tau,s)P_s^+gdB(s)(\omega).$$

Then, similar to Proposition 2.4.2 of [3], it can be checked that the above $u^*(\tau,\tau,\omega)$ is the desired unique hyperbolic equilibrium. $u^*(t,\tau,\omega)$ is the hyperbolic solution with initial value $u^*(\tau,\tau,\omega)$.

For each positive η, with $\eta<a$, define Banach space

$$C_\eta^+=\{\phi:[\tau,\infty)\to H\mid \phi \text{ is continuous and } \sup_{t\in[\tau,\infty)}e^{-\eta(t-\tau)}|\phi(t)|<\infty\}$$

with norm

$$|\phi(\cdot)|_{C_\eta^+}=\sup_{t\in[\tau,\infty)}e^{-\eta(t-\tau)}|\phi(t)|,$$

and

$$C_\eta^-=\{\phi:(-\infty,\tau]\to H\mid \phi \text{ is continuous and } \sup_{t\in(-\infty,\tau]}e^{-\eta(t-\tau)}|\phi(t)|<\infty\}$$

with norm

$$|\phi(\cdot)|_{C_\eta^-}=\sup_{t\in(-\infty,\tau]}e^{-\eta(t-\tau)}|\phi(t)|.$$

Denote $C_\eta=C_\eta^-\cap C_\eta^+$ and define the norm as

$$|\phi(\cdot)|_{C_\eta}=|\phi(\cdot)|_{C_\eta^-}+|\phi(\cdot)|_{C_\eta^+}.$$

Taking the above arguments, we give the following result on the hyperbolic equilibrium.

Theorem 1. Let $u^*(t),u^{*,\varepsilon}(t)$ be the hyperbolic solutions of (3) and (10), respectively. Assume that the exponential dichotomy parameters K,α,β and the Lipschitz constant L satisfy the gap condition

$$K\,L\,(\frac{1}{\eta-\beta}+\frac{1}{\alpha-\eta})<1,$$

and there exists a positive constant C, such that, $\sup_t|A(t)g|\le C$. Then the hyperbolic solutions of (10) converge to that of (3); that is, $\lim_{\varepsilon\to0+}|u^{*,\varepsilon}(\cdot)-u^*(\cdot)|_{C_\eta}=0$, i.e.,

$$\lim_{\varepsilon\to0+}[\sup_{t\in[\tau,+\infty)}e^{-\eta t}|u^{*,\varepsilon}(t)-u^*(t)|+\sup_{t\in(-\infty,\tau]}e^{\eta t}|u^{*,\varepsilon}(t)-u^*(t)|]=0.$$

Proof. By the expression of the mapping for the fix point $u^*(\tau,\tau,\omega)$, we obtain that,

$$\begin{aligned}|u^{*,\varepsilon}(t)-u^*(t)| \leq &\,|\int_{-\infty}^{t} T(t,s)P_s^-[F(s,u^{*,\varepsilon}(s,\tau,\omega))-F(s,u^*(s,\tau,\omega))]ds| \\ &+|\int_{\infty}^{t} T(t,s)P_s^+[F(s,u^{*,\varepsilon}(s,\tau,\omega))-F(s,u^*(s,\tau,\omega))]ds| \\ &+|\int_{\infty}^{t} T(t,s)P_s^+ gd(B(s)-\Phi^\varepsilon(s))(\omega)| \\ &+|\int_{-\infty}^{t} T(t,s)P_s^- gd(B(s)-\Phi^\varepsilon(s))(\omega)|.\end{aligned}$$

Denote the sum of the first two terms on the right side by I. By the Hypothesis 1(H1), we can derive that

$$\begin{aligned} I \leq &\int_{-\infty}^{t} e^{-\beta(s-t)}P_s^-|F(s,u^{*,\varepsilon}(s,\tau,\omega))-F(s,u^*(s,\tau,\omega))|ds \\ &+\int_{t}^{\infty} e^{-\alpha(s-t)}P_s^+|F(s,u^{*,\varepsilon}(s,\tau,\omega))-F(s,u^*(s,\tau,\omega))|ds \\ \leq &\int_{-\infty}^{t} e^{-\beta(s-t)}KL|u^{*,\varepsilon}(s,\tau,\omega)-u^*(s,\tau,\omega)|ds \\ &+\int_{t}^{\infty} e^{-\alpha(s-t)}KL|u^{*,\varepsilon}(s,\tau,\omega)-u^*(s,\tau,\omega)|ds \end{aligned}$$

Thus, we can obtain that

$$\begin{aligned} &[1-KL(\frac{1}{\alpha-\eta}+\frac{1}{\eta-\beta})][|u^{*,\varepsilon}(s,\tau,\omega)-u^*(s,\tau,\omega)|_{C_\eta^+}+|u^{*,\varepsilon}(s,\tau,\omega)-u^*(s,\tau,\omega)|_{C_\eta^-}] \\ &\leq |\int_{\infty}^{\cdot} T(\cdot,s)P_s^+ gd(B(s)-\Phi^\varepsilon(s))(\omega)|_{C_\eta}+|\int_{-\infty}^{\cdot} T(\cdot,s)P_s^- gd(B(s)-\Phi^\varepsilon(s))(\omega)|_{C_\eta} \end{aligned}$$

Then by (IV), (V) of Lemma 2, this result follows. This completes the proof. □

Remark 4. *Here, we use the spectral gap condition $KL(\frac{1}{\alpha-\eta}+\frac{1}{\eta-\beta})<1$, the same as [2], which is sufficient to ensure the existence of the hyperbolic stationary solution and the invariant manifold by the contraction mapping argument. We should note that this strategy has been utilized by others. For example, in [29], a different gap condition (3.4) was formulated to ensure the strict contraction of a mapping, of which the unique fixed point was the stationary solution. We also present a rough interpretation of the form of the gap condition here. It is easy to deduce from the gap condition that the larger L signifies a larger gap between exponents α and β, i.e., $\alpha-\beta$. While a large Lipschitz constant means a big fluctuation of the nonlinear function F, the large gap $\alpha-\beta$ implies hyperbolicity of the linear operator A. Therefore, in some sense, the gap condition actually requires that the fluctuations of nonlinear function not destroy the hyperbolicity of the linear term A. We should note that this condition cannot always be satisfied. However, quite a large class of equations satisfy this condition locally. If the condition is satisfied locally, then the stationary solution and invariant manifolds exist locally.*

3.2. Existence Theorem for Invariant Manifold

Denote the solutions of Equations (3) and (10) by $\phi(t,\omega,u_0)$ and $\phi^\varepsilon(t,\omega,u_0^\varepsilon)$ with the initial data $\phi(0,\omega,u_0)=(P_0^-u_0,P_0^+u_0)$ and $\phi^\varepsilon(0,\omega,u_0^\varepsilon)=(P_0^-u_0^\varepsilon,P_0^+u_0^\varepsilon)$, respectively.

Let

$$M^-(\omega)=\{u_0\in H \mid \phi(\cdot,\omega,u_0)\in C_\eta^+\}; M^{\varepsilon,-}(\omega)=\{u_0^\varepsilon\in H \mid \phi^\varepsilon(\cdot,\omega,u_0^\varepsilon)\in C_\eta^+\}.$$

Now we formulate the existence theorem of stable manifolds for the random systems (3) and (10). In fact, by the method of the standard Banach fixed point argument, we have the following result [2]:

Lemma 3 (Random stable manifold)**.** *Suppose the gap condition*

$$KL(\frac{1}{\eta-\beta}+\frac{1}{\alpha-\eta})<1$$

is satisfied by the parameters K,α,β, and the Lipschitz constant L in Hypotheses 1(H1) and 2(H2). Then the stochastic systems of (3) and random systems (10) possess random stable manifolds. Moreover, the random stable manifolds for (3) are represented by

$$M^-(\tau,\omega)=\{(\xi,h^-(\tau,\xi,\omega))\big|\xi\in H_\tau^-\},$$

where $h^-(\tau,\xi,\omega):H_\tau^-\to H_\tau^+$ is Lipschitz continuous mapping, as below,

$$h^-(\tau,\xi,\omega)=\int_\infty^\tau T(\tau,s)P_s^+F(s,u_s)ds+\int_\infty^\tau T(\tau,s)P_s^+gdB(s).$$

The random stable manifolds for (10) are represented by

$$M^{\varepsilon,-}(\tau,\omega)=\{(\xi,h^{\varepsilon,-}(\tau,\xi,\omega))\big|\xi\in H_\tau^-\},$$

where $h^{\varepsilon,-}(\tau,\xi,\omega):H_\tau^-\to H_\tau^+$ is Lipschitz continuous mapping as below,

$$h^{\varepsilon,-}(\tau,\xi,\omega)=\int_\infty^\tau T(\tau,s)P_s^+F(s,u^\varepsilon(s))ds+\int_\infty^\tau T(\tau,s)P_s^+gd\Phi_s^\varepsilon.$$

Proof. The first part of this lemma is by the method of the standard Banach fixed point argument, as [2]. As for the second part, for (10), we have the graph of the random manifold as follows,

$$h^{\varepsilon,-}(\tau,\xi,\omega)=\int_\infty^\tau T(\tau,s)P_s^+F(s,u^\varepsilon(s))ds+\int_\infty^\tau T(\tau,s)P_s^+gd\Phi_s^\varepsilon.$$

For (3), by random transformation $v(t)=u(t)-u^*(t)$, $v(t)$ satisfies a random PDE, as below,

$$\frac{dv(t)}{dt}=A(t)v(t)+F(t,u(t))-F(t,u^*(t)).$$

Then, similar to [2], we can find the graph of random manifold for $v(t)$ given by

$$\tilde{h}^-(\tau,\xi,\omega)=\int_\infty^\tau T(\tau,s)P_s^+[F(s,v(s)+u^*(s))-F(s,u^*(s))]ds.$$

By the fact that $v(t)=u(t)-u^*(t)$ and following expression for hyperbolic solutions of (3),

$$\begin{aligned}u^*(t,\tau,\omega)=&\int_{-\infty}^t T(t,s)P_s^-F(s,u^*(s,\tau,\omega))ds+\int_\infty^t T(t,s)P_s^+F(s,u^*(s,\tau,\omega))ds\\&+\int_{-\infty}^t T(t,s)P_s^-gdB(s)(\omega)+\int_\infty^t T(t,s)P_s^+gdB(s)(\omega),\end{aligned}$$

we have the graph of random manifold for u given by

$$h^-(\tau,\xi,\omega)=\int_\infty^\tau T(\tau,s)P_s^+F(s,u(s))ds+\int_\infty^\tau T(\tau,s)P_s^+gdB(s).$$

□

4. Approximation of Invariant Manifold

In this section, we will consider a type of Wong–Zakai approximation of invariant manifolds for (3), i.e., the Gaussian white noise is approached by color noises.

We formulate the main result of this paper:

Theorem 2. Assume that the assumptions (H1–H3) and the gap condition in Theorem 1 hold. Then the invariant manifolds of (10) converge to that of (3); that is,

$$\lim_{\varepsilon\to 0+} |h^{\varepsilon,-}(\tau,\xi,\omega) - h^{-}(\tau,\xi,\omega)| = 0.$$

Proof. By Theorem 3, we have,

$$h^{\varepsilon,-}(\tau,\xi,\omega) = \int_{\infty}^{\tau} T(\tau,s)P_s^{+}F(s,u^{\varepsilon}(s))ds + \int_{\infty}^{\tau} T(\tau,s)P_s^{+}gd\Phi_s^{\varepsilon},$$
$$h^{-}(\tau,\xi,\omega) = \int_{\infty}^{\tau} T(\tau,s)P_s^{+}F(s,u(s))ds + \int_{\infty}^{\tau} T(\tau,s)P_s^{+}gdB(s).$$

Thus, we obtain that

$$\begin{aligned}
&|h^{\varepsilon,-}(\tau,\xi,\omega) - h^{-}(\tau,\xi,\omega)| \\
&\le |\int_{\infty}^{\tau} T(\tau,s)P_s^{+}[F(s,u^{\varepsilon}(s)) - F(s,u(s))]ds| \\
&+ |\int_{\infty}^{\tau} T(\tau,s)P_s^{+}g[d\Phi^{\varepsilon}(s) - dB(s)] \\
&\le \int_{\tau}^{\infty} KLe^{\alpha(\tau-s)}|u^{\varepsilon}(s) - u(s)|ds \\
&+ \int_{\tau}^{\infty} KLe^{\alpha(\tau-s)}|A(s)g||\Phi^{\varepsilon}(s) - B(s)|ds.
\end{aligned}$$

The solution $u^{\varepsilon}(t)$ on the manifold $M^{\varepsilon,-}(\tau,\omega)$ is as follows,

$$\begin{aligned}
u^{\varepsilon}(t) = T(t,\tau)\xi &+ \int_{\tau}^{t} T(t,s)P_s^{-}F(s,u^{\varepsilon}(s))ds + \int_{\tau}^{t} T(t,s)P_s^{-}gd\Phi^{\varepsilon}(s) \\
&+ \int_{\infty}^{t} T(t,s)P_s^{+}F(s,u^{\varepsilon}(s))ds + \int_{\infty}^{t} T(t,s)P_s^{+}gd\Phi_s^{\varepsilon}.
\end{aligned}$$

Similarly, on the manifold $M^{-}(\tau,\omega)$, $u(t)$ is as follows,

$$\begin{aligned}
u(t) = T(t,\tau)\xi &+ \int_{\tau}^{t} T(t,s)P_s^{-}F(s,u(s))ds + \int_{\tau}^{t} T(t,s)P_s^{-}gdB(s) \\
&+ \int_{\infty}^{t} T(t,s)P_s^{+}F(s,u(s))ds + \int_{\infty}^{t} T(t,s)P_s^{+}gdB(s).
\end{aligned}$$

Thus,

$$\begin{aligned}
u^{\varepsilon}(t) - u(t) &= \int_{\tau}^{t} T(t,s)P_s^{-}[F(s,u^{\varepsilon}(s)) - F(s,u(s))]ds \\
&+ \int_{\infty}^{t} T(t,s)P_s^{+}[F(s,u^{\varepsilon}(s)) - F(s,u(s))]ds \\
&+ \int_{\tau}^{t} T(t,s)P_s^{-}g[d\Phi^{\varepsilon}(s) - dB(s)] \\
&+ \int_{\infty}^{t} T(t,s)P_s^{+}g[d\Phi^{\varepsilon}(s) - dB(s)].
\end{aligned}$$

We have

$$|u^{\varepsilon}(\cdot) - u(\cdot)|_{C_{\eta}^{+}} \leq KL(\frac{1}{\eta - \beta} + \frac{1}{\alpha - \eta})|u^{\varepsilon}(\cdot) - u(\cdot)|_{C_{\eta}^{+}}$$
$$+ \sup_{t\in[\tau,\infty)} e^{-\eta(t-\tau)}|\int_{\tau}^{t} T(t,s)P_s^{-}g[d\Phi^{\varepsilon}(s) - dB(s)]$$
$$+ \int_{\infty}^{t} T(t,s)P_s^{+}g[d\Phi^{\varepsilon}(s) - dB(s)]|.$$

Thus,

$$|u^{\varepsilon}(\cdot) - u(\cdot)|_{C_{\eta}^{+}} \leq C \sup_{t\in[\tau,\infty)} e^{-\eta(t-\tau)}|\int_{\tau}^{t} T(t,s)P_s^{-}g[d\Phi^{\varepsilon}(s) - dB(s)]$$
$$+ \int_{\infty}^{t} T(t,s)P_s^{+}g[d\Phi^{\varepsilon}(s) - dB(s)]|.$$

Then, by (IV), (V) of Lemma 2, we can deduce that,

$$|h^{\varepsilon,-}(\tau,\xi,\omega) - h^{-}(\tau,\xi,\omega)| = o(1).$$

This completes the proof. □

5. Conclusions

The Wong–Zakai approximation for the solution of stochastic differential equations has been studied by many authors since 1965. However, most differential equations that are investigated are autonomous. Moreover, to the best of our knowledge, few researchers have investigated the approximation of Gaussian white noise via the Ornstein–Uhlenbeck process, which is more suitable for calculating and programming. In view of this, we approximated a stochastic nonautonomous dynamical system by random nonautonomous dynamical systems driven by the integrated Ornstein–Uhlenbeck process. First, in Theorem 1, we obtained the convergence of the hyperbolic solutions of (10) to that of (3) in a suitable space. Then, by the method of the standard Banach fixed point argument, we prove in Lemma 3 the existence of the random stable manifolds for systems (10) and (3) and present the mapping for the graph of the random manifolds, respectively. Finally, we derive the main result of this paper in Theorem 2, i.e., the approximation of the random stable manifolds of (10) to that of (3).

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