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Fekete–Szegö Functional Problem for a Special Family of *m*-Fold Symmetric Bi-Univalent Functions

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Abstract: In the current work, we introduce a special family of the function family of analytic and m-fold symmetric bi-univalent functions and obtain estimates of the Taylor–Maclaurin coefficients $|d_{m+1}|$ and $|d_{2m+1}|$ for functions in the special family. For δ a real number, Fekete–Szegö functional $|d_{2m+1} - \delta d_{m+1}^2|$ for functions in the special family is also estimated. We indicate several cases of the defined family and connections to existing results are also discussed.

Keywords: bi-univalent functions; coefficient estimates; Fekete–Szegö functional; m-fold symmetric bi-univalent functions

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1. Introduction

Let $\mathcal{A}$ be the set of functions s that are holomorphic in $\mathfrak{D} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, normalized by $s(0) = s'(0) - 1 = 0$ having the form:

$$s(\varsigma) = \varsigma + \sum_{k=2}^{\infty} d_k \varsigma^k. \tag{1}$$

Let $\mathcal{S}$ stand for the subfamily of $\mathcal{A}$ which are univalent in $\mathfrak{D}$. The image of $\mathfrak{D}$ under every function $s \in \mathcal{S}$ contains a disc of radius $1/4$ is known as one-quarter theorem of Koebe [1]. According to this, every function $s \in \mathcal{S}$ has an inverse $g = s^{-1}$, satisfying $s^{-1}(s(\varsigma)) = \varsigma$, $\varsigma \in \mathfrak{D}$ and $s(s^{-1}(\omega)) = \omega$, $|\omega| < r_0(s)$, $r_0(s) \geq 1/4$ and is in fact given by:

$$g(\omega) = s^{-1}(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \cdots \tag{2}$$

If a function s and its inverse s^{-1} are both univalent in $\mathfrak{D}$, then a member s of $\mathcal{A}$ is called bi-univalent in $\mathfrak{D}$. We symbolize by σ the family of bi-univalent functions in $\mathfrak{D}$ given by (1). Some functions in the family σ are given by $\frac{\varsigma}{1-\varsigma}, -\log(1-\varsigma)$ and $\frac{1}{2}\log\left(\frac{1+\varsigma}{1-\varsigma}\right)$. However, the Koebe function does not belong to the set σ. Other functions $\in \mathcal{S}$ such as $\varsigma - \frac{\varsigma^2}{2}$ and $\frac{\varsigma}{1-\varsigma^2}$ are not members of σ.

Lewin [2] examined the family σ and proved that $|d_2| < 1.51$ for elements of the family σ. Later, Brannan and Clunie [3] claimed that $|d_2| < \sqrt{2}$ for $s \in \sigma$. Subsequently, Tan [4] obtained some initial coefficient estimates of functions in the class σ. Brannan and Taha, in [5], proposed bi-convex and bi-starlike functions which are similar to well-known subfamilies of $\mathcal{S}$. The research trend in the last decade was the study of subfamilies of σ. Generally, interest was shown in obtaining the initial coefficient bounds for the special subfamilies of σ. In 2010, Srivastava et al. [6] introduced two interesting subclasses of the function class σ and found bounds for $|d_2|$ and $|d_3|$ of functions belonging to these subclasses. In 2011, Frasin and Aouf [7] studied two new subclasses of the function class σ

and obtained bounds for $|d_2|$ and $|d_3|$ of functions belonging to these subclasses. Deniz [8], in 2013, introduced four subclasses of the class σ and investigated bounds for $|d_2|$ and $|d_3|$ of functions belonging to these four subclasses. Tang et al. [9] in 2013 determined the coefficient estimates for new subclasses of Ma–Minda bi-univalent functions. Frasin [10] in 2014 examined two more new subclasses of σ. The recent research trend is the study of functions in σ linked with certain polynomials such as Lucas polynomials, Legendrae polynomials, Fibonacci polynomials, Chebyshev polynomials, Horadam polynomials and Gegenbauer polynomials. Interesting results related to initial coefficient estimates and the Fekete–Szegö functional problem $|d_3 - \delta d_2^2|$ for some special subclasses of σ associated with any of the above mentioned polynomials appeared to be like the ones in [11–14].

Let $m \in \mathbb{N} := \{1, 2, 3, \dots\}$. If a rotation of the domain $\mathfrak{E}$ about the origin with an angle 2π maps $\mathfrak{E}$ on itself, then $\mathfrak{E}$ is known as m-fold symmetric. A holomorphic function s in $\mathfrak{D}$ is called m-fold symmetric if $s\left(e^{\frac{2\pi i}{m}}\varsigma\right) = e^{\frac{2\pi i}{m}} s(\varsigma)$. For each function $f \in \mathcal{S}$, $s(\varsigma) = \sqrt[m]{f(\varsigma^m)}$ is univalent and maps $\mathfrak{D}$ into a region with m-fold symmetry. We symbolize the family of m-fold symmetric univalent functions in $\mathfrak{D}$ by $\mathcal{S}_m$. Clearly, $\mathcal{S}_1 = \mathcal{S}$. A function $s \in \mathcal{S}_m$ has a series expansion given by:

$$s(\varsigma) = \varsigma + \sum_{k=1}^{\infty} d_{mk+1}\varsigma^{mk+1} \qquad (m \in \mathbb{N}; \varsigma \in \mathfrak{D}). \tag{3}$$

A natural extension of $\mathcal{S}_m$ was explored by Srivastava et al. [15] and they introduced the family σ_m of m-fold symmetric bi-univalent functions. The series expansion for $g = s^{-1}$ obtained by them is as below:

$$\begin{aligned} g(\omega) = s^{-1}(\omega) = \omega - d_{m+1}\omega^{m+1} + [(m+1)d_{m+1}^2 - d_{2m+1}]\omega^{2m+1} \\ -\left[\frac{(m+1)(3m+2)}{2}d_{m+1}^3 - (3m+2)d_{m+1}d_{2m+1} + d_{3m+1}\right]\omega^{3m+1} + \cdots. \end{aligned} \tag{4}$$

Some functions in the family σ_m are $\left(\frac{\varsigma^m}{1-\varsigma^m}\right)^{1/m}$, $\left[\frac{1}{2}\log\left(\frac{1+\varsigma^m}{1-\varsigma^m}\right)\right]^{1/m}$ and $[-\log(1-\varsigma^m)]^{1/m}$ and $\left(\frac{\omega^m}{1+\omega^m}\right)^{1/m}$, $\left(\frac{e^{2\omega^m}-1}{e^{2\omega^m}-1}\right)^{1/m}$ and $\left(\frac{e^{\omega^m}-1}{e^{\omega^m}}\right)^{1/m}$ are respective inverse functions. The momentum on the investigation of the family σ_m was gained in recent years, which is due to two papers [16,17] of Srivastava et al. and it has led to a large number of papers on the subfamilies of σ_m. Note that $\sigma_1 = \sigma$. In 2018, Srivastava et al. [18] addressed initial coefficient estimations of the Taylor–Maclaurin series of functions in a new subfamily of σ_m. Sakar and Tasar [19] introduced new subfamilies of σ_m and obtained initial coefficient bounds for functions belonging to these families, coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions were determined in [20], and a comprehensive subclass of σ_m using the subordination principle was examined in [21]. Interesting results related to the initial coefficient estimates and Fekete–Szegö functional problem $|d_{2m+1} - \delta d_{m+1}^2|$ for certain subfamilies of σ_m appeared like the ones in [22–24].

Inspired substantially by the works of Ma and Minda [25] and Tang et al. [26], we define a special subfamily $\mathfrak{M}_{\sigma_m}(\mu, \nu, \eta, \varphi)$ $(0 \le \nu \le 1,\ \mu \ge 0,\ \eta \in \mathbb{C}^* = \mathbb{C} - \{0\})$ of m-fold symmetric bi-univalent functions.

Definition 1. *A function $s \in \sigma_m$ is said to be in the class $\mathfrak{M}_{\sigma_m}(\mu, \nu, \eta, \varphi)$ $(0 \le \nu \le 1,\ \mu \ge 0,\ \eta \in \mathbb{C}^*)$, if*

$$\left[1 + \frac{1}{\eta}\left(\frac{\varsigma s'(\varsigma) + \mu\varsigma^2 s''(\varsigma)}{(1-\nu)\varsigma + \nu s(\varsigma)} - 1\right)\right] \prec \varphi(\varsigma),$$

and

$$\left[1 + \frac{1}{\eta}\left(\frac{\omega g'(\omega) + \mu\omega^2 g''(\omega)}{(1-\nu)\omega + \nu g(\omega)} - 1\right)\right] \prec \varphi(\omega).$$

where ς, $\omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$ is as stated in (4).

We observe that the certain choice of ν and μ leads the class $\mathfrak{M}_{\sigma_m}(\mu,\nu,\eta,\varphi)$ to the following few subfamilies:

(i). $\mathscr{G}_{\sigma_m}(\nu,\eta,\varphi) \equiv \mathfrak{M}_{\sigma_m}(0,\nu,\eta,\varphi)$ $(0 \le \nu \le 1,\ \eta \in \mathbb{C}^*)$ is the family of $s \in \sigma_m$ of the form (1) satisfying:

$$\left[1+\frac{1}{\eta}\left(\frac{\varsigma s'(\varsigma)}{(1-\nu)\varsigma+\nu s(\varsigma)}-1\right)\right] \prec \varphi(\varsigma),$$

and

$$\left[1+\frac{1}{\eta}\left(\frac{\omega g'(\omega)}{(1-\nu)\omega+\nu g(\omega)}-1\right)\right] \prec \varphi(\varsigma),$$

where $\varsigma,\ \omega \in \mathfrak{D},\ g(\omega) = s^{-1}(\omega)$ is as stated in (4).

(ii) $\mathscr{H}_{\sigma_m}(\mu,\eta,\varphi) \equiv \mathfrak{M}_{\sigma_m}(\mu,0,\eta,\varphi)$ $(\mu \ge 0,\ \eta \in \mathbb{C}^*)$ is the family of $s \in \sigma_m$ of the form (1) satisfying:

$$\left[1+\frac{1}{\eta}\left(s'(\varsigma)+\mu\varsigma s''(\varsigma)-1\right)\right] \prec \varphi(\varsigma),$$

and

$$\left[1+\frac{1}{\eta}\left(g'(\omega)+\mu\omega g''(\omega)-1\right)\right] \prec \varphi(\varsigma),$$

where $\varsigma,\ \omega \in \mathfrak{D},\ g(\omega) = s^{-1}(\omega)$ is as stated in (4).

(iii) $\mathscr{I}_{\sigma_m}(\mu,\eta,\varphi) \equiv \mathfrak{M}_{\sigma_m}(\mu,1,\eta,\varphi)(\mu \ge 0,\ \eta \in \mathbb{C}^*)$ is the family of $s \in \sigma_m$ of the form (1) satisfying:

$$\left[1+\frac{1}{\eta}\left(\left(\frac{\varsigma s'(\varsigma)}{s(\varsigma)}\right)\left(1+\mu\frac{\varsigma s''(\varsigma)}{s'(\varsigma)}\right)-1\right)\right] \prec \varphi(\varsigma),$$

and

$$\left[1+\frac{1}{\eta}\left(\left(\frac{\omega g'(\omega)}{g(\omega)}\right)\left(1+\mu\frac{\omega g''(\omega)}{g'(\omega)}\right)-1\right)\right] \prec \varphi(\varsigma),$$

where $\varsigma,\ \omega \in \mathfrak{D},\ g(\omega) = s^{-1}(\omega)$ is as stated in (4).

Remark 1. *We note that* $\mathscr{G}_{\sigma_m}(1,\eta,\varphi) \equiv \mathscr{I}_{\sigma_m}(0,\eta,\varphi)$ *and* $\mathscr{G}_{\sigma_m}(0,\eta,\varphi) \equiv \mathscr{H}_{\sigma_m}(0,\eta,\varphi)$.

In Section 2, we find bounds on $|d_{m+1}|$ and $|d_{2m+1}|$ in the Taylor–Maclaurin's expansion and Fekete–Szegö [27] functional problem for functions in the class $\mathfrak{M}_{\sigma_m}(\mu,\nu,\eta,\varphi)$. We also indicate interesting cases of the main results. In Section 3, we obtain bounds on $|d_{m+1}|$ and $|d_{2m+1}|$ in the Taylor–Maclaurin's expansion and Fekete–Szegö functional problem for functions in the class $\mathfrak{W}^{\varrho}_{\sigma_m}(\mu,\nu,\eta)$. In Section 4, we determine bounds on $|d_{m+1}|$ and $|d_{2m+1}|$ in the Taylor–Maclaurin's expansion and Fekete–Szegö functional problem for functions in the class $\mathfrak{X}^{\tau}_{\sigma_m}(\mu,\nu,\eta)$, respectively. We also indicate interesting cases of the main results. Relevant connections to the existing results are also mentioned.

2. Coefficient Bounds for Function Family $\mathfrak{M}_{\sigma_m}(\mu,\nu,\eta,\varphi)$

We denote by $\mathscr{P}$ the family of holomorphic functions of the form: $p(\varsigma) = 1 + p_1\varsigma + p_2\varsigma^2 + p_3\varsigma^3 + \cdots$ with $\Re(\mathscr{P}(\varsigma)) > 0$ $(\varsigma \in \mathfrak{D})$. In view of the study of Pommerenke [28], the m-fold symmetric function p in the family $\mathscr{P}$ is of the form: $p(\varsigma) = 1 + p_m\varsigma + p_{2m}\varsigma^{2m} + p_{3m}\varsigma^{3m} + \cdots$. In the sequel, it is assumed that $\varphi(\varsigma)$ is a holomorphic function, having a positive real part in $\mathfrak{D}$ satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(\mathfrak{D})$ is symmetric with respect to the real axis. Such a function has an infinite series expansion of the form: $\varphi(\varsigma) = 1 + B_1\varsigma + B_2\varsigma^2 + B_3\varsigma^3 + \cdots (B_1 > 0)$. Let $\mathfrak{h}(\varsigma)$ and $\mathfrak{p}(\omega)$ be two holomorphic functions in $\mathfrak{D}$ with $\mathfrak{h}(0) = \mathfrak{p}(0) = 0$ and $\max\{|\mathfrak{h}(\varsigma)|;|\mathfrak{p}(\omega)|\} < 1$. We suppose that

$\mathfrak{h}(\varsigma) = h_m\varsigma^m + h_{2m}\varsigma^{2m} + h_{3m}\varsigma^{3m} + \cdots$ and $\mathfrak{p}(\omega) = p_m\omega^m + p_{2m}\omega^{2m} + p_{3m}\omega^{3m} + \cdots$. We also know that:

$$|h_m| < 1; |h_{2m}| \le 1 - |h_m|^2; |p_m| < 1; |p_{2m}| \le 1 - |p_m|^2. \tag{5}$$

By simple calculations, we obtain:

$$\varphi(\mathfrak{h}(\varsigma)) = 1 + B_1 h_m \varsigma^m + (B_1 h_{2m} + B_2 h_m^2)\varsigma^{2m} + ...(|\varsigma| < 1) \tag{6}$$

and

$$\varphi(\mathfrak{p}(\omega)) = 1 + B_1 p_m \omega^m + (B_1 p_{2m} + B_2 p_m^2)\omega^{2m} + ...(|\omega| < 1). \tag{7}$$

Theorem 1. *Let $\mu \ge 0$, $0 \le \nu \le 1$ and $\eta \in \mathbb{C}^*$. If a function s in $\mathcal{A}$ belongs to the class $\mathfrak{M}_{\sigma_m}(\mu, \nu, \eta, \varphi)$, then:*

$$|d_{m+1}| \le \frac{|\eta| B_1 \sqrt{2B_1}}{\sqrt{|((m+1)(N-\nu) - 2\nu(L-\nu))\eta B_1^2 - 2(L-\nu)^2 B_2| + 2(L-\nu)^2 B_1}}, \tag{8}$$

$$|d_{2m+1}| \le \tag{9}$$

$$\begin{cases} \frac{|\eta| B_1}{N-\nu} & ; B_1 < \frac{2(L-\nu)^2}{|\eta|(m+1)(N-\nu)} \\ \frac{|\eta| B_1}{N-\nu} + \left(\frac{m+1}{2} - \frac{(L-\nu)^2}{|\eta| B_1 (N-\nu)}\right) \frac{2\eta^2 B_1^3}{|((m+1)(N-\nu)-2\nu(L-\nu))\eta B_1^2 - 2(L-\nu)^2 B_2| + 2(L-\nu)^2 B_1} & \\ & ; B_1 \ge \frac{2(L-\nu)^2}{|\eta|(m+1)(N-\nu)}, \end{cases}$$

and for δ a real number:

$$|d_{2m+1} - \delta d_{m+1}^2| \le \begin{cases} \frac{|\eta| B_1}{N-\nu} & ; |m+1-2\delta| < J \\ \frac{|\eta|^2 B_1^3 |m+1-2\delta|}{|((m+1)(N-\nu)-2\nu(L-\nu))\eta B_1^2 - 2(L-\nu)^2 B_2|} & ; |m+1-2\delta| \ge J, \end{cases} \tag{10}$$

where

$$L = (m+1)(\mu m + 1), \tag{11}$$

$$N = (2m+1)(2\mu m + 1). \tag{12}$$

and

$$J = \left| \frac{((m+1)(N-\nu) - 2\nu(L-\nu))\eta B_1^2 - 2(L-\nu)^2 B_2}{\eta(N-\nu) B_1^2} \right|. \tag{13}$$

Proof. Let the function s given by (1) be in the family $\mathfrak{M}_{\sigma_m}(\mu, \nu, \eta, \varphi)$. Then there are holomorphic functions $\mathfrak{h} : \mathfrak{D} \longrightarrow \mathfrak{D}$ and $\mathfrak{p} : \mathfrak{D} \longrightarrow \mathfrak{D}$ with $\mathfrak{h}(0) = \mathfrak{p}(0) = 0$ satisfying:

$$1 + \frac{1}{\eta}\left(\frac{\varsigma s'(\varsigma) + \mu\varsigma^2 s''(\varsigma)}{(1-\nu)\varsigma + \nu s(\varsigma)} - 1\right) = \varphi(\mathfrak{h}(\varsigma)), \tag{14}$$

and

$$1 + \frac{1}{\eta}\left(\frac{\omega g'(\omega) + \mu\omega^2 g''(\omega)}{(1-\nu)\omega + \nu g(\omega)} - 1\right) = \varphi(\mathfrak{p}(\omega)), \tag{15}$$

where ς, $\omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$.

Taylor–Maclaurin series expansions of the left hand side of Equations (14) and (15) are, respectively:

$$\begin{aligned} 1 + \frac{1}{\eta}\{((m+1)(\mu m+1) - \nu)d_{m+1}\varsigma^m + [((2m+1)(2\mu m+1) - \nu)d_{2m+1} \\ - ((m+1)(\mu m+1) - \nu)\nu d_{m+1}^2\Big]\varsigma^{2m} + \cdots\Big\} \end{aligned} \tag{16}$$

and

$$1+\frac{1}{\eta}\{-((m+1)(\mu m+1)-\nu)d_{m+1}\omega^{m}+[((2m+1)(2\mu m+1)-\nu) \\ ((m+1)d_{m+1}^{2}-d_{2m+1})\ -\ ((m+1)(\mu m+1)-\nu)\nu d_{m+1}^{2}\Big]\omega^{2m}+\cdots\Big\} \tag{17}$$

Comparing the coefficients in (6) and (16), (7) and (17), we get:

$$(L-\nu)d_{m+1}=\eta B_1 h_m\,, \tag{18}$$

$$(N-\nu)d_{2m+1}-\nu(L-\nu)d_{m+1}^{2}=\eta[B_1 h_{2m}+B_2 h_m^{2}]\,, \tag{19}$$

$$-(L-\nu)d_{m+1}=\eta B_1 p_m\,, \tag{20}$$

and

$$\Big[(N-\nu)((m+1)d_{m+1}^{2}-d_{2m+1})-\nu(L-\nu)d_{m+1}^{2}\Big]=\eta[B_1 p_{2m}+B_2 p_m^{2}]\,, \tag{21}$$

where L is given by (11) and N is as in (12).

From (18) and (20), we get:

$$h_m=-p_m \tag{22}$$

and

$$2(L-\nu)^2 d_{m+1}^{2}=\eta^2 B_1^2(h_m^2+p_m^2). \tag{23}$$

Using (23) in the addition of (19) and (21), we obtain:

$$[((m+1)(N-\nu)-2\nu(L-\nu))\eta B_1^2-2(L-\nu)^2B_2]d_{m+1}^{2}=\eta^2 B_1^3(h_{2m}+p_{2m}). \tag{24}$$

By using (5) and (18) in (24) for the coefficients h_{2m} and p_{2m}, we obtain:

$$[|((m+1)(N-\nu)-2\nu(L-\nu))\eta B_1^2-2(L-\nu)^2B_2|+2(L-\nu)^2B_1]|d_{m+1}|^2\le 2\eta^2B_1^3, \tag{25}$$

which achieves the desired estimate (8).

Subtracting (21) from (19), we get:

$$d_{2m+1}=\frac{\eta B_1(h_{2m}-p_{2m})}{2(N-\nu)}+\left(\frac{m+1}{2}\right)d_{m+1}^{2}. \tag{26}$$

In view of (18), (22), (26) and applying inequalities (5), it follows that:

$$|d_{2m+1}|\le\frac{|\eta|B_1}{N-\nu}+\left(\frac{m+1}{2}-\frac{(L-\nu)^2}{|\eta|B_1(N-\nu)}\right) \tag{27}$$

$$\frac{2\eta^2B_1^3}{|((m+1)(N-\nu)-2\nu(L-\nu))\eta B_1^2-2(L-\nu)^2B_2|+2(L-\nu)^2B_1},$$

which implies the assertion (9).

It follows from (24) and (26) that:

$$d_{2m+1}-\delta d_{m+1}^{2}=\frac{\eta B_1}{2}\left[\left(T(\delta)+\frac{1}{N-\nu}\right)h_{2m}+\left(T(\delta)-\frac{1}{N-\nu}\right)p_{2m}\right],$$

where

$$T(\delta)=\frac{\eta B_1^2(m+1-\delta)}{((m+1)(N-\nu)-2\nu(L-\nu))\eta B_1^2-2(L-\nu)^2B_2}.$$

In view of (5), we conclude that:

$$|d_{2m+1}-\delta d_{m+1}^{2}|\le\begin{cases}\frac{|\eta|B_1}{N-\nu} & ;0\le|T(\delta)|<\frac{1}{N-\nu}\\ |\eta|B_1|T(\delta)| & ;|T(\delta)|\ge\frac{1}{N-\nu},\end{cases}$$

which implies the assertion (10) with J as in (13). This completes the proof. □

We note that, for specializing the parameters, as mentioned in special cases (i)–(iii) of Definition 1, we deduce the following new results.

Corollary 1. *Let $0 \le \nu \le 1$ and $\eta \in \mathbb{C}^*$. If a function s in $\mathcal{A}$ belongs to the family $\mathcal{G}_{\sigma_m}(\nu, \eta, \varphi)$, then:*

$$|d_{m+1}| \le$$

$$\frac{|\eta| B_1 \sqrt{2B_1}}{\sqrt{|((m+1)(2m+1-\nu) - 2\nu(m+1-\nu))\eta B_1^2 - 2(m+1-\nu)^2 B_2| + 2(m+1-\nu)^2 B_1}},$$

$$|d_{2m+1}| \le$$

$$\begin{cases} \frac{|\eta| B_1}{2m+1-\nu} & ;\ B_1 < \frac{2(m+1-\nu)^2}{|\eta|(m+1)(2m+1-\nu)} \\ \frac{|\eta| B_1}{2m+1-\nu} + \left(\frac{m+1}{2} - \frac{(m+1-\nu)^2}{|\eta| B_1 (2m+1-\nu)}\right) \frac{2\eta^2 B_1^3}{|((m+1)(2m+1-\nu)-2\nu(m+1-\nu))\eta B_1^2 - 2(m+1-\nu)^2 B_2| + 2(m+1-\nu)^2 B_1} & \\ & ;\ B_1 \ge \frac{2(m+1-\nu)^2}{|\eta|(m+1)(2m+1-\nu)}, \end{cases}$$

and for δ a real number:

$$|d_{2m+1} - \delta d_{m+1}^2| \le \begin{cases} \frac{|\eta| B_1}{2m+1-\nu} & ;\ |m+1-2\delta| < J_1 \\ \frac{|\eta|^2 B_1^3 |m+1-2\delta|}{|((m+1)(2m+1-\nu)-2\nu(m+1-\nu))\eta B_1^2) - 2(m+1-\nu)^2 B_2)|} & ;\ |m+1-2\delta| \ge J_1, \end{cases}$$

where

$$J_1 = \left| \frac{((m+1)(2m+1-\nu) - 2\nu(m+1-\nu))\eta B_1^2 - 2(m+1-\nu)^2 B_2}{\eta(2m+1-\nu) B_1^2} \right|.$$

Remark 2. *For $\eta = 1$ and $\nu = 0$ in Corollary 1, we get Theorems 1 and 2 of Tang et al. [26]. Further, for $m = 1$ the case of one-fold symmetric functions, we obtain a result of Peng et al. [29] (which is recalled as Corollary 1 by Tang et al. in [26]) and Corollary 4 of Tang et al. [26].*

Corollary 2. *Let $\mu \ge$ and $\eta \in \mathbb{C}^*$. If a function $s \in \mathcal{A}$ belongs to the family $\mathcal{H}_{\sigma_m}(\mu, \eta, \varphi)$, then*

$$|d_{m+1}| \le \frac{|\eta| B_1 \sqrt{2B_1}}{\sqrt{|((m+1)N\eta B_1^2 - 2L^2 B_2| + 2L^2 B_1}},$$

$$|d_{2m+1}| \le \begin{cases} \frac{|\eta| B_1}{N} & ;\ B_1 < \frac{2L^2}{N|\eta|(m+1)} \\ \frac{|\eta| B_1}{N} + \left(\frac{m+1}{2} - \frac{L^2}{N|\eta| B_1}\right) \frac{2\eta^2 B_1^3}{|((m+1)N\eta B_1^2 - 2L^2 B_2| + 2L^2 B_1} & ;\ B_1 \ge \frac{2L^2}{N|\eta|(m+1)}, \end{cases}$$

and for δ a real number

$$|d_{2m+1} - \delta d_{m+1}^2| \le \begin{cases} \frac{|\eta| B_1}{N} & ;\ |m+1-2\delta| < \left| \frac{((m+1)N\eta B_1^2 - 2L^2 B_2}{N\eta B_1^2} \right| \\ \frac{|\eta|^2 B_1^3 |m+1-2\delta|}{|(m+1)N\eta B_1^2 - 2L^2 B_2|} & ;\ |m+1-2\delta| \ge \left| \frac{((m+1)N\eta B_1^2 - 2L^2 B_2}{N\eta B_1^2} \right|, \end{cases}$$

where L is as in (11) and N is as in (12).

Remark 3. *For $\eta = 1$ and $\mu = 0$ in Corollary 2, we get Theorems 1 and 2 of Tang et al. [26]. Further, for $m = 1$ the case of a one-fold symmetric function, we obtain a result of Peng et al. [29] (which is recalled as Corollary 1 by Tang et al. in [26]) and Corollary 4 of Tang et al. [26].*

Corollary 3. *Let $\mu \geq 0$ and $\eta \in \mathbb{C}^*$. If the function s in $\mathcal{A}$ belongs to the family $\mathscr{I}_{\sigma_m}(\mu, \eta, \varphi)$, then:*

$$|d_{m+1}| \leq \frac{|\eta| B_1 \sqrt{2B_1}}{\sqrt{|((m+1)N_2 - 2L_2)\eta B_1^2 - 2L_2^2 B_2| + 2L_2^2 B_1}},$$

$$|d_{2m+1}| \leq$$

$$\begin{cases} \frac{|\eta| B_1}{N_2} & ;\ B_1 < \frac{2L_2^2}{|\eta|(m+1)N_2} \\ \frac{|\eta| B_1}{N_2} + \left(\frac{m+1}{2} - \frac{L_2^2}{|\eta| B_1 N_2}\right) \frac{2\eta^2 B_1^3}{|((m+1)N_2 - 2L_2)\eta B_1^2 - 2L_2^2 B_2| + 2L_2^2 B_1} & ;\ B_1 \geq \frac{2L_2^2}{|\eta|(m+1)N_2}, \end{cases}$$

and for δ a real number:

$$|d_{2m+1} - \delta d_{m+1}^2| \leq \begin{cases} \frac{|\eta| B_1}{N_2} & ;\ |m+1-2\delta| < \left|\frac{((m+1)N_2 - 2L_2)\eta B_1^2 - 2L_2^2 B_2}{\eta N_2 B_1^2}\right| \\ \frac{|\eta|^2 B_1^3 |m+1-2\delta|}{|((m+1)N_2 - 2L_2)\eta B_1^2 - 2L_2^2 B_2|} & ;\ |m+1-2\delta| \geq \left|\frac{((m+1)N_2 - 2L_2)\eta B_1^2 - 2L_2^2 B_2}{\eta N_2 B_1^2}\right|, \end{cases}$$

where

$$L_2 = m(\mu m + \mu + 1), \tag{28}$$

and

$$N_2 = 2m(2\mu m + \mu + 1). \tag{29}$$

Remark 4. *For $\eta = 1$ in Corollary 3, we get Theorems 3 and 4 of Tang et al. [26]. Further, for $m = 1$ we obtain Corollary 12 of Tang et al. [26].*

3. Coefficient Bounds for Function Family $\mathfrak{W}^{\varrho}_{\sigma_m}(\mu, \nu, \eta)$

If $\varphi(\varsigma) = \left(\frac{1+\varsigma^m}{1-\varsigma^m}\right)^{\varrho}$ $(0 < \varrho \leq 1)$, in the Definition 1, then we get $\mathfrak{M}_{\sigma_m}(\mu, \nu, \eta, \left(\frac{1+\varsigma^m}{1-\varsigma^m}\right)^{\varrho}) = \mathfrak{W}^{\varrho}_{\sigma_m}(\mu, \nu, \eta)$, the subclass of functions $s \in \sigma_m$ satisfying the conditions

$$\left|\arg\left[1 + \frac{1}{\eta}\left(\frac{\varsigma s'(\varsigma) + \mu\varsigma^2 s''(\varsigma)}{(1-\nu)\varsigma + \nu s(\varsigma)} - 1\right)\right]\right| < \frac{\varrho\pi}{2},$$

and

$$\left|\arg\left[1 + \frac{1}{\eta}\left(\frac{\omega g'(\omega) + \mu\omega^2 g''(\omega)}{(1-\nu)\omega + \nu g(\omega)} - 1\right)\right]\right| < \frac{\varrho\pi}{2}.$$

where $\varsigma, \omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$ is as stated in (4).

We observe that the certain choice of ν and μ leads the class $\mathfrak{W}^{\varrho}_{\sigma_m}(\mu, \nu, \eta)$ to the following few subfamilies:

(i) $\mathscr{A}^{\varrho}_{\sigma_m}(\nu, \eta) \equiv \mathfrak{W}^{\varrho}_{\sigma_m}(0, \nu, \eta)$ $(0 \leq \nu \leq 1, 0 < \varrho \leq 1,\ \eta \in \mathbb{C}^*)$ is the family of $s \in \sigma_m$ of the form (1) satisfying:

$$\left|\arg\left[1 + \frac{1}{\eta}\left(\frac{\varsigma s'(\varsigma)}{(1-\nu)\varsigma + \nu s(\varsigma)} - 1\right)\right]\right| < \frac{\varrho\pi}{2},$$

and

$$\left|\arg\left[1 + \frac{1}{\eta}\left(\frac{\omega g'(\omega)}{(1-\nu)\omega + \nu g(\omega)} - 1\right)\right]\right| < \frac{\varrho\pi}{2},$$

where $\varsigma, \omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$ is as stated in (4).

(ii) $\mathscr{B}^{\varrho}_{\sigma_m}(\mu, \eta) \equiv \mathfrak{W}^{\varrho}_{\sigma_m}(\mu, 0, \eta)$ $(0 < \varrho \leq 1,\ \mu \geq 0,\ \eta \in \mathbb{C}^*)$ is the family of $s \in \sigma_m$ of the form (1) satisfying:

$$\left|\arg\left[1 + \frac{1}{\eta}\left(s'(\varsigma) + \mu\varsigma s''(\varsigma) - 1\right)\right]\right| < \frac{\varrho\pi}{2},$$

and
$$\left|\arg\left[1+\frac{1}{\eta}\left(g'(\omega)+\mu\omega g''(\omega)-1\right)\right]\right|<\frac{\varrho\pi}{2},$$
where $\varsigma,\ \omega\in\mathfrak{D}$, $g(\omega)=s^{-1}(\omega)$ is as stated in (4).

(iii) $\mathscr{C}^{\varrho}_{\sigma_m}(\mu,\eta)\equiv\mathfrak{W}^{\varrho}_{\sigma_m}(\mu,1,\eta)(0<\varrho\leq1,\ \mu\geq0,\ \eta\in\mathbb{C}^*)$ is the family of $s\in\sigma_m$ of the form (1) satisfying:
$$\left|\arg\left[1+\frac{1}{\eta}\left(\left(\frac{\varsigma s'(\varsigma)}{s(\varsigma)}\right)\left(1+\mu\frac{\varsigma s''(\varsigma)}{s'(\varsigma)}\right)-1\right)\right]\right|<\frac{\varrho\pi}{2},$$
and
$$\left|\arg\left[1+\frac{1}{\eta}\left(\left(\frac{\omega g'(\omega)}{g(\omega)}\right)\left(1+\mu\frac{\omega g''(\omega)}{g'(\omega)}\right)-1\right)\right]\right|<\frac{\varrho\pi}{2},$$
where $\varsigma,\ \omega\in\mathfrak{D}$, $g(\omega)=s^{-1}(\omega)$ is as stated in (4).

Remark 5. *We note that* $\mathscr{A}^{\varrho}_{\sigma_m}(1,\eta)\equiv\mathscr{C}^{\varrho}_{\sigma_m}(0,\eta)$ *and* $\mathscr{A}^{\varrho}_{\sigma_m}(0,\eta)\equiv\mathscr{B}^{\varrho}_{\sigma_m}(0,\eta)$.

If we take $\varphi(\varsigma)=\left(\frac{1+\varsigma^m}{1-\varsigma^m}\right)^{\varrho}$ in Theorem 1, we get:

Corollary 4. *Let* $\mu\geq0$, $0\leq\nu\leq1$, $0<\varrho\leq1$ *and* $\eta\in\mathbb{C}^*$. *If a function s in* $\mathcal{A}$ *belongs to the class* $\mathfrak{W}^{\varrho}_{\sigma_m}(\mu,\nu,\eta)$ *, then*
$$|d_{m+1}|\leq\frac{2|\eta|\varrho}{\sqrt{\varrho|((m+1)(N-\nu)-2\nu(L-\nu))\eta-(L-\nu)^2|+(L-\nu)^2}},$$
$$|d_{2m+1}|\leq$$
$$\begin{cases}\frac{2|\eta|\varrho}{N-\nu} & ;\ \varrho<\frac{(L-\nu)^2}{|\eta|(m+1)(N-\nu)}\\ \frac{2|\eta|\varrho}{N-\nu}+\left(m+1-\frac{(L-\nu)^2}{|\eta|\varrho(N-\nu)}\right)\frac{2\eta^2\varrho^2}{\varrho|((m+1)(N-\nu)-2\nu(L-\nu))\eta-(L-\nu)^2|+(L-\nu)^2} & ;\ \varrho\geq\frac{(L-\nu)^2}{|\eta|(m+1)(N-\nu)},\end{cases}$$
and for δ a real number:
$$|d_{2m+1}-\delta d^2_{m+1}|\leq\begin{cases}\frac{2|\eta|\varrho}{N-\nu} & ;\ |m+1-2\delta|<J_2\\ \frac{2|\eta|^2\varrho|m+1-2\delta|}{|((m+1)(N-\nu)-2\nu(L-\nu))\eta-(L-\nu)^2|} & ;\ |m+1-2\delta|\geq J_2,\end{cases}$$
where L is as in (11), N is as in (12) and
$$J_2=\left|\frac{((m+1)(N-\nu)-2\nu(L-\nu))\eta-(L-\nu)^2}{\eta(N-\nu)}\right|.$$

We note that for specializing the parameters, as mentioned in special cases (i)–(iii) of the class $\mathfrak{W}^{\varrho}_{\sigma_m}(\mu,\nu,\eta)$, we deduce the following new results.

Corollary 5. *Let* $0\leq\nu\leq1$, $0<\varrho\leq1$ *and* $\eta\in\mathbb{C}^*$. *If a function s in* $\mathcal{A}$ *belongs to the class* $\mathscr{A}^{\varrho}_{\sigma_m}(\nu,\eta)$, *then:*
$$|d_{m+1}|\leq\frac{2|\eta|\varrho}{\sqrt{\varrho|((m+1)(2m+1-\nu)-2\nu(m+1-\nu))\eta-(m+1-\nu)^2|+(m+1-\nu)^2}},$$
$$|d_{2m+1}|\leq$$
$$\begin{cases}\frac{2|\eta|\varrho}{2m+1-\nu} & ;\ \varrho<\frac{(m+1-\nu)^2}{|\eta|(m+1)(2m+1-\nu)}\\ \frac{2|\eta|\varrho}{2m+1-\nu}+\left(m+1-\frac{(m+1-\nu)^2}{|\eta|\varrho(2m+1-\nu)}\right)\frac{2\eta^2\varrho^2}{\varrho|((m+1)(2m+1-\nu)-2\nu(m+1-\nu))\eta-(m+1-\nu)^2|+(m+1-\nu)^2} & \\ & ;\ \varrho\geq\frac{(m+1-\nu)^2}{|\eta|(m+1)(2m+1-\nu)},\end{cases}$$

and for δ any real number:

$$|d_{2m+1}-\delta d_{m+1}^2| \leq \begin{cases} \frac{2|\eta|\varrho}{2m+1-\nu} & ; |m+1-2\delta| < J_3 \\ \frac{2|\eta|^2\varrho|m+1-2\delta|}{|((m+1)(2m+1-\nu)-2\nu(m+1-\nu))\eta-(m+1-\nu)^2|} & ; |m+1-2\delta| \geq J_3, \end{cases}$$

where

$$J_3 = \left|\frac{((m+1)(2m+1-\nu)-2\nu(m+1-\nu))\eta-(m+1-\nu)^2}{\eta(2m+1-\nu)}\right|.$$

Remark 6. *(i) For $\eta=1$ and $\nu=0$ in Corollary 5, we get Corollary 2 of Tang et al. [26]; (ii) For $\eta=\nu=1$ in Corollary 4, bound on $|d_{m+1}|$ reduce to the bound given in [30]. Further, for $m=1$ we obtain a result of [31]; (iii) For $\eta=\nu=1$ in Corollary 5, the result shown on $|d_{2m+1}|$ is better than the bound given in [30], in terms of ranges of ϱ as well as the bounds.*

Corollary 6. *Let $\mu \geq 0$, $0<\varrho\leq 1$ and $\eta \in \mathbb{C}^*$. If a function s in $\mathcal{A}$ belongs to the class $\mathscr{B}_{\sigma_m}^{\varrho}(\mu,\eta)$, then*

$$|d_{m+1}| \leq \frac{2|\eta|\varrho}{\sqrt{\varrho|((m+1)N\eta-L^2|+L^2}},$$

$$|d_{2m+1}| \leq \begin{cases} \frac{2|\eta|\varrho}{N} & ; \varrho < \frac{L^2}{N|\eta|(m+1)} \\ \frac{2|\eta|\varrho}{N} + \left(m+1-\frac{L^2}{N|\eta|\varrho}\right)\frac{2\eta^2\varrho^2}{\varrho|((m+1)N\eta-L^2|+L^2} & ; \varrho \geq \frac{L^2}{N|\eta|(m+1)}, \end{cases}$$

and for δ a real number:

$$|d_{2m+1}-\delta d_{m+1}^2| \leq \begin{cases} \frac{2|\eta|\varrho}{N} & ; |m+1-2\delta| < \left|\frac{((m+1)N\eta-L^2}{\eta N}\right| \\ \frac{2|\eta|^2\varrho|m+1-2\delta|}{|(m+1)N\eta-L^2|} & ; |m+1-2\delta| \geq \left|\frac{((m+1)N\eta-L^2}{\eta N}\right|, \end{cases}$$

where L is as in (11) and N is as in (12).

Remark 7. *For $\eta=1$ and $\mu=0$ in Corollary 6, we get Corollary 2 of Tang et al. [26].*

Corollary 7. *Let $\mu \geq 0, 0<\varrho\leq 1$ and $\eta \in \mathbb{C}^*$. If a function s in $\mathcal{A}$ belongs to the class $\mathscr{C}_{\sigma_m}^{\varrho}(\mu,\eta,\varphi)$, then:*

$$|d_{m+1}| \leq \frac{2|\eta|\varrho}{\sqrt{\varrho|((m+1)N_2-2L_2)\eta-L_2^2|+L_2^2}},$$

$$|d_{2m+1}| \leq \begin{cases} \frac{2|\eta|\varrho}{N_2} & ; \varrho < \frac{L_2^2}{|\eta|(m+1)N_2} \\ \frac{2|\eta|\varrho}{N_2} + \left(m+1-\frac{L_2^2}{|\eta|\varrho N_2}\right)\frac{2\eta^2\varrho^2}{\varrho|((m+1)N_2-2L_2)\eta-L_2^2|+L_2^2} & ; \varrho \geq \frac{L_2^2}{|\eta|(m+1)N_2}, \end{cases}$$

and for δ a real number:

$$|d_{2m+1}-\delta d_{m+1}^2| \leq \begin{cases} \frac{2|\eta|\varrho}{N_2} & ; |m+1-2\delta| < \left|\frac{((m+1)N_2-2L_2)\eta-L_2^2}{\eta N_2}\right| \\ \frac{2|\eta|^2\varrho|m+1-2\delta|}{|((m+1)N_2-2L_2)\eta-L_2^2|} & ; |m+1-2\delta| \geq \left|\frac{((m+1)N_2-2L_2)\eta-L_2^2}{\eta N_2}\right|, \end{cases}$$

where L_2 is as in (28) and N_2 is as in (29).

4. Coefficient Bounds for Function Family $\mathfrak{X}^{\tau}_{\sigma_m}(\mu,\nu,\eta)$

If $\varphi(\varsigma) = \frac{1+(1-2\tau)\varsigma^m}{1-\varsigma^m}$ $(0 \leq \tau < 1)$ in the Definition 1, then we obtain $\mathfrak{M}_{\sigma_m}\left(\mu,\nu,\eta, \left(\frac{1+(1-2\tau)\varsigma^m}{1-\varsigma^m}\right)\right) = \mathfrak{X}^{\tau}_{\sigma_m}(\mu,\nu,\eta)$, a subclass of functions $s \in \sigma_m$ satisfying the conditions:

$$\Re\left[1+\frac{1}{\eta}\left(\frac{\varsigma s'(\varsigma)+\mu\varsigma^2 s''(\varsigma)}{(1-\nu)\varsigma+\nu s(\varsigma)}-1\right)\right] > \tau,$$

and

$$\Re\left[1+\frac{1}{\eta}\left(\frac{\omega g'(\omega)+\mu\omega^2 g''(\omega)}{(1-\nu)\omega+\nu g(\omega)}-1\right)\right] > \tau.$$

where ς, $\omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$ is as stated in (4).

We observe that the certain values of ν and μ lead the class $\mathfrak{X}^{\tau}_{\sigma_m}(\mu,\nu,\eta)$ to the following few subfamilies:

(i) $\mathscr{D}^{\tau}_{\sigma_m}(\nu,\eta) \equiv \mathfrak{X}^{\tau}_{\sigma_m}(0,\nu,\eta)$ $(0 \leq \nu \leq 1, 0 \leq \tau < 1,\ \eta \in \mathbb{C}^*)$, is the family of functions $s \in \sigma_m$ of the form (1) satisfying:

$$\Re\left[1+\frac{1}{\eta}\left(\frac{\varsigma s'(\varsigma)}{(1-\nu)\varsigma+\nu s(\varsigma)}-1\right)\right] > \tau,$$

and

$$\Re\left[1+\frac{1}{\eta}\left(\frac{\omega g'(\omega)}{(1-\nu)\omega+\nu g(\omega)}-1\right)\right] > \tau,$$

where ς, $\omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$ is as stated in (4).

(ii) $\mathscr{E}^{\tau}_{\sigma_m}(\mu,\eta) \equiv \mathfrak{X}^{\tau}_{\sigma_m}(\mu,0,\eta)$ $(0 \leq \tau < 1,\ \mu \geq 0,\ \eta \in \mathbb{C}^*)$ is the class of functions $s \in \sigma_m$ of the form (1) satisfying:

$$\Re\left[1+\frac{1}{\eta}(s'(\varsigma)+\mu\varsigma s''(\varsigma)-1)\right] > \tau,$$

and

$$\Re\left[1+\frac{1}{\eta}(g'(\omega)+\mu\omega g''(\omega)-1)\right] > \tau,$$

where ς, $\omega \in \mathfrak{D}$, $g(\omega) = s^{-1}(\omega)$ is as stated in (4).

(iii) $\mathscr{F}^{\tau}_{\sigma_m}(\mu,\eta) \equiv \mathfrak{X}^{\beta}_{\sigma_m}(\mu,1,\eta)$ $(0 \leq \tau < 1,\ \mu \geq 0,\ \eta \in \mathbb{C}^*)$, is the family of functions $s \in \sigma_m$ of the form (1) satisfying:

$$\Re\left[1+\frac{1}{\eta}\left(\left(\frac{\varsigma s'(\varsigma)}{s(\varsigma)}\right)\left(1+\mu\frac{\varsigma s''(\varsigma)}{s'(\varsigma)}\right)-1\right)\right] > \tau,$$

and

$$\Re\left[1+\frac{1}{\eta}\left(\left(\frac{\omega g'(\omega)}{g(\omega)}\right)\left(1+\mu\frac{\omega g''(\omega)}{g'(\omega)}\right)-1\right)\right] > \tau.$$

Remark 8. *We note that $\mathscr{D}^{\tau}_{\sigma_m}(1,\eta) \equiv \mathscr{F}^{\tau}_{\sigma_m}(0,\eta)$ and $\mathscr{D}^{\tau}_{\sigma_m}(0,\eta) \equiv \mathscr{E}^{\tau}_{\sigma_m}(0,\eta)$.*

If we take $\varphi(\varsigma) = \frac{1+(1-2\tau)\varsigma^m}{1-\varsigma^m}$ in Theorem 1, we get:

Corollary 8. *Let $\mu \geq 0$, $0 \leq \nu \leq 1$, $0 \leq \tau < 1$ and $\eta \in \mathbb{C}^*$. If a function s in $\mathcal{A}$ belongs to the class $\mathfrak{X}^{\tau}_{\sigma_m}(\mu,\nu,\eta)$, then:*

$$|d_{m+1}| \leq \frac{2|\eta|(1-\tau)}{\sqrt{|((m+1)(N-\nu)-2\nu(L-\nu))\eta(1-\tau)-(L-\nu)^2|+(L-\nu)^2}},$$

$$|d_{2m+1}| \leq$$

$$\begin{cases} \frac{2(1-\tau)|\eta|}{N-\nu} & ; (1-\tau) < \frac{(L-\nu)^2}{|\eta|(m+1)(N-\nu)} \\ \frac{2(1-\tau)|\eta|}{N-\nu} + \left(m+1-\frac{(L-\nu)^2}{|\eta|(1-\tau)(N-\nu)}\right)\frac{2|\eta|^2(1-\tau)^2}{|((m+1)(N-\nu)-2\nu(L-\nu))(1-\tau)\eta-(L-\nu)^2|+(L-\nu)^2} & \\ & ; (1-\tau) \geq \frac{(L-\nu)^2}{|\eta|(m+1)(N-\nu)} \end{cases}$$

and for δ a real number:

$$|d_{2m+1}-\delta d_{m+1}^2| \leq \begin{cases} \frac{2|\eta|(1-\tau)}{N-\nu} & ; |m+1-2\delta| < J_4 \\ \frac{2|\eta|^2(1-\tau)^2|m+1-2\delta|}{|((m+1)(N-\nu)-2\nu(L-\nu))(1-\tau)\eta-(L-\nu)^2|} & ; |m+1-2\delta| \geq J_4, \end{cases}$$

where L is as in (11) and N is as in (12) and

$$J_4 = \left|\frac{((m+1)(N-\nu)-2\nu(L-\nu))\eta(1-\tau)-(L-\nu)^2}{\eta(N-\nu)(1-\tau)}\right|.$$

We note that for specializing μ and ν, as mentioned in special cases (i)–(iii) of the class $\mathfrak{X}^{\tau}_{\sigma_m}(\mu,\nu,\eta)$, we deduce the following new results.

Corollary 9. *Let $0 \leq \nu \leq 1, 0 \leq \tau < 1$ and $\eta \in \mathbb{C}^*$. If a function s in $\mathcal{A}$ belongs to the class $\mathscr{D}^{\tau}_{\sigma_m}(\nu,\eta)$, then:*

$$|d_{m+1}| \leq \frac{2|\eta|(1-\tau)}{\sqrt{|((m+1)(2m+1-\nu)-2\nu(m+1-\nu))\eta(1-\tau)-(m+1-\nu)^2|+(m+1-\nu)^2}},$$

$$|d_{2m+1}| \leq$$

$$\begin{cases} \frac{2(1-\tau)|\eta|}{2m+1-\nu} & ; (1-\tau) < \frac{(m+1-\nu)^2}{|\eta|(m+1)(2m+1-\nu)} \\ \frac{2|\eta|(1-\tau)}{N_1-\nu} + \left(m+1-\frac{(m+1-\nu)^2}{|\eta|(1-\tau)(2m+1-\nu)}\right)\frac{2|\eta|(1-\tau)}{|((m+1)(2m+1-\nu)-2\nu(m+1-\nu)\eta(1-\tau)-(m+1-\nu)^2)|+(m+1-\nu)^2} & \\ & ;(1-\tau) \geq \frac{(m+1-\nu)^2}{|\eta|(m+1)(2m+1-\nu)}, \end{cases}$$

and for δ a real number:

$$|d_{2m+1}-\delta d_{m+1}^2| \leq \begin{cases} \frac{2|\eta|(1-\tau)}{2m+1-\nu} & ; |m+1-2\delta| < J_5 \\ \frac{4|\eta|^2(1-\tau)^2|m+1-2\delta|}{|((m+1)(2m+1-\nu)-2\nu(m+1-\nu))\eta(1-\tau)-(m+1-\nu)^2|} & ; |m+1-2\delta| \geq J_5, \end{cases}$$

where

$$J_5 = \left|\frac{((m+1)(2m+1-\nu)-2\nu(m+1-\nu))\eta(1-\tau)-(m+1-\nu)^2}{\eta(2m+1-\nu)(1-\tau)}\right|.$$

Remark 9. *$\eta = 1$ and $\nu = 0$ in Corollary 4, we get Corollary 2 of Tang et al. (i) For $\eta = \nu = 1$ in Corollary 9, bound on $|d_{m+1}|$ reduce to the bound given in [30]. Further, for $m = 1$ we obtain a result of [31]; (ii) For $\eta = \nu = 1$ in Corollary 9, the result proved on $|d_{2m+1}|$ is better than the bound given in [30], in terms of ranges of τ as well as the bounds .*

Corollary 10. *Let $\mu \geq 0$, $0 \leq \tau < 1$ and $\eta \in \mathbb{C}^*$. If a function s in $\mathcal{A}$ belongs to the class $\mathscr{E}^{\tau}_{\sigma_m}(\mu,\eta)$, then:*

$$|d_{m+1}| \leq \frac{2|\eta|(1-\tau)}{\sqrt{|(m+1)\eta(1-\tau)N-L^2|+L^2}},$$

$$|d_{2m+1}| \leq \begin{cases} \frac{2(1-\tau)|\eta|}{N} & ; (1-\tau) < \frac{L^2}{|\eta|(m+1)N} \\ \frac{2|\eta|(1-\tau)}{N} + \left(m+1-\frac{L^2}{|\eta|(1-\tau)N}\right)\frac{2\eta^2(1-\tau)^2}{|(m+1)\eta(1-\tau)N-L^2|+L^2} & ; (1-\tau) \geq \frac{L^2}{|\eta|(m+1)N}, \end{cases}$$

and for δ a real number:

$$|d_{2m+1} - \delta d_{m+1}^2| \le \begin{cases} \frac{2|\eta|(1-\tau)}{N} & ;\ |m+1-2\delta| < \left|\frac{((m+1)\eta(1-\tau)N - L^2}{\eta(1-\tau)N}\right| \\ \frac{2|\eta|^2(1-\tau)^2|m+1-2\delta|}{|((m+1)\eta(1-\tau)N-L^2|} & ;\ |m+1-2\delta| \ge \left|\frac{((m+1)\eta(1-\tau)N - L^2}{\eta(1-\tau)N}\right|, \end{cases}$$

where L is as in (11) and N is as in (12).

$\mu = 0$ and $\eta = 1$ in Corollary 10, we obtain Corollary 11.

Corollary 11. *Let $0 \le \tau < 1$. If a function s in $\mathcal{A}$ belongs to the class $\mathscr{E}^{\tau}_{\sigma_m}(0,1)$, then:*

$$|d_{m+1}| \le \frac{2(1-\tau)}{\sqrt{(m+1)[|(2m+1)(1-\tau)-(m+1)|+(m+1)]}},$$

$$|d_{2m+1}| \le \begin{cases} \frac{2(1-\tau)}{2m+1} & ;\ (1-\tau) < \frac{L^2}{|\eta|(m+1)N} \\ \frac{2(1-\tau)}{2m+1} + \left(1 - \frac{m+1}{(1-\tau)(2m+1)}\right)\frac{2(1-\tau)^2}{|(2m+1)(1-\tau)-(m+1)|+(m+1)} & ;\ (1-\tau) \ge \frac{L^2}{|\eta|(m+1)N}, \end{cases}$$

and for δ a real number

$$|d_{2m+1} - \delta d_{m+1}^2| \le \begin{cases} \frac{2(1-\tau)}{2m+1} & ;\ |m+1-2\delta| < \left|\frac{((m+1)(2m+1)(1-\tau)-(m+1)^2}{(1-\tau)N}\right| \\ \frac{2(1-\tau)^2|m+1-2\delta|}{|((m+1)(2m+1)(1-\tau)-(m+1)^2|} & ;|m+1-2\delta| \ge \left|\frac{((m+1)(2m+1)(1-\tau)-(m+1)^2}{(1-\tau)(2m+1)}\right|. \end{cases}$$

Corollary 11 would lead us to the following result, when $m = 1$.

Corollary 12. *Let $0 \le \tau < 1$. If a function s in $\mathcal{A}$ belongs to the class $\mathscr{E}^{\tau}_{\sigma_1}(0,1)$, then:*

$$|d_2| \le \frac{\sqrt{2}\,(1-\tau)}{\sqrt{2+|1-3\tau|}},$$

$$|d_3| \le \begin{cases} \frac{2(1-\tau)}{3} & for\ \frac{1}{3} < \tau < 1 \\ \frac{4}{9}(2-3\tau) & for\ \ 0 \le \tau \le 1, \end{cases}$$

and for δ a real number:

$$|d_3 - \delta d_2^2| \le \begin{cases} \frac{2(1-\tau)}{3} & ;\ |1-\delta| < \frac{1}{3(1-\tau)}|1-3\tau| \\ \frac{2(1-\tau)^2|1-\delta|}{|1-3\tau|} & ;|1-\delta| \ge \frac{1}{3(1-\tau)}|1-3\tau|. \end{cases}$$

Remark 10. *(i) The bounds of $|d_2|$ and $|d_3|$ stated in Corollary 12 were obtained in Corollary 3 by Tang et al. [26]; (ii) Taking $\delta = 0$ in Corollary 12, we have $|d_3 - d_2^2| < \frac{2}{3}(1-\tau)$, provided $|1-3\tau| > 0$.*

Corollary 13. *Let $\mu \ge 0$, $0 \le \tau < 1$ and $\eta \in \mathbb{C}^*$. If a function s in $\mathcal{A}$ belongs to the class $\mathscr{F}^{\tau}_{\sigma_m}(\mu,\eta)$, then:*

$$|d_{m+1}| \le \frac{2|\eta|(1-\tau)}{\sqrt{|((m+1)N_2 - 2L_2)(1-\tau)\eta - L_2^2| + L_2^2}},$$

$$|d_{2m+1}| \le \begin{cases} \frac{2(1-\tau)|\eta|}{N_2} & ;\ (1-\tau) < \frac{L_1^2}{|\eta|(m+1)N_2} \\ \frac{2|\eta|(1-\tau)}{N_2} + \left(m+1-\frac{L_2^2}{|\eta|(1-\tau)N_2}\right)\frac{2\eta^2(1-\tau)^2}{|((m+1)N_2-2L_2)(1-\tau)\eta - L_2^2|+L_2^2} & \\ & ;\ (1-\tau) \ge \frac{L_2^2}{|\eta|(m+1)N_2}, \end{cases}$$

and for δ a real number:

$$|d_{2m+1}-\delta d_{m+1}^2| \leq \begin{cases} \frac{2|\eta|(1-\tau)}{N_2} & ; |m+1-2\delta| < \left|\frac{((m+1)N_2-2L_2)\eta(1-\tau)-L_2^2}{\eta N_2(1-\tau)}\right| \\ \frac{2|\eta|^2(1-\tau)^2|m+1-2\delta|}{|((m+1)N_2-2L_2)\eta(1-\tau)-L_2^2|} & ; |m+1-2\delta| \geq \left|\frac{((m+1)N_2-2L_2)\eta(1-\tau)-L_2^2}{\eta N_2(1-\tau)}\right|, \end{cases}$$

where L_2 is as in (28) and N_2 is as in (29).

5. Conclusions

In this study, we have introduced a special family $\mathfrak{M}_{\sigma_m}(\mu,\nu,\eta,\varphi)$ of m-fold symmetric bi-univalent functions in the disc $\{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$ and studied coefficient problems associated with the defined family. For functions belonging to this family, we have determined the upper bounds for $|d_{m+1}|$ and $|d_{2m+1}|$. The Fekete–Szegö functional problem for functions in this family is also considered. Various cases of the special family $\mathfrak{M}_{\sigma_m}(\mu,\nu,\eta,\varphi)$ are discussed. Our results generalize many results of Tang et al. [26].

A special family examined in this paper could inspire further research related to some aspects such as a special family of bi-univalent functions using a Hohlov operator associated with Legendrae polynomial [32], a special family using an integro-differential operator [33], meromorphic univalent function family [34], a special family using q-derivative operator [35], and so on.

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References

1. Duren, P.L. *Univalent Functions*; Springer: New York, NY, USA, 1983.
2. Lewin, M. On a coefficient problem for bi-univalent functions . *Proc. Am. Math. Soc.* **1967**, *18*, 63–68. [CrossRef]
3. Brannan, D.A.; Clunie, J.; Kirwan, W.E. Coefficient estimates for a class of starlike functions. *Can. J. Math.* **1970**, *22*, 476–485. [CrossRef]
4. Tan, D.L. Coefficient estimates for bi-univalent functions . *Chin. Ann. Math. Ser. A* **1984**, *5*, 559–568.
5. Brannan, D.A.; Taha, T.S. On some classes of bi-univalent functions. *Math. Anal. Appl.* **1985**, *3*, 18–21.
6. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses analytic and bi-univalent functions. *Appl. Math. Lett.* **2010**, *23*, 1188–1192. [CrossRef]
7. Frasin, B.A.; Aouf, M.K. New subclass of bi-univalent functons. *Appl. Math. Lett.* **2011**, *24*, 1569–1573. [CrossRef]
8. Deniz, E. Certain subclasses of bi-univalent functions satisfying subordinate conditions. *J. Class. Anal.* **2013**, *2*, 49–60. [CrossRef]
9. Tang, H.; Deng, G.; Li, S. Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions. *J. Inequal. Appl.* **2013**, *2013*, 317. [CrossRef]
10. Frasin, B.A. Coefficient bounds for certain classes of bi-univalent functions. *Hacet. J. Math. Stat.* **2014**, *43*, 383–389. [CrossRef]
11. Frasin, B.A.; Swamy, S.R.; Nirmala, J. Some special families of holomorphic and Al-Oboudi type bi-univalent functions related to *k*-Fibonacci numbers involving modified sigmoid activated function. *Afr. Mat.* **2021**, *32*, 631–643. [CrossRef]
12. Shammaky, A.E.; Frasin, B.A.; Swamy, S.R. Fekete–Szegö inequality for bi-univalent functions subordinate to Horadam polynomials. *J. Funct. Spaces.* **2022**, *2022*, 9422945. [CrossRef]

13. Swamy, S.R. Coefficient bounds for Al-Oboudi type bi-univalent functions based on a modified sigmoid activation function and Horadam polynomials. *Earthline J. Math. Sci.* **2021**, *7*, 251–270. [CrossRef]
14. Swamy, S.R.; Bulut, S.; Sailaja, Y. Some special families of holomorphic and Sălăgean type bi-univalent functions associated with Horadam polynomials involving modified sigmoid activation function. *Hacet. J. Math. Stat.* **2021**, *50*, 710–720. [CrossRef]
15. Srivastava, H.M.; Sivasubramanian, S.; Sivakumar, R. Initial coefficients estimate for some subclasses of m-fold symmetric bi-univalent functions. *Tbilisi Math. J.* **2014**, *7*, 1–10. [CrossRef]
16. Srivastava, H.M.; Gaboury, S.; Ghanim, F. Coefficients estimate for some subclasses of m-fold symmetric bi-univalent functions. *Acta Univ. Apulensis Math. Inform.* **2015**, *41*, 153–164.
17. Srivastava, H.M.; Gaboury, S.; Ghanim, F. Initial coefficients estimate for some subclasses of m-fold symmetric bi-univalent functions. *Acta Math. Sci. Ser. B* **2016**, *36*, 863–971. [CrossRef]
18. Srivastava, H.M.; Zireh, A.; Hajiparvaneh, S. Coefficients estimate for some subclasses of m-fold symmetric bi-univalent functions. *Filomat* **2018**, *32*, 3143–3153. [CrossRef]
19. Sakar, F.M.; Tasar, N. Coefficients bounds for certain subclasses of m-fold symmetric bi-univalent functions. *New Trends Math. Sci.* **2019**, *7*, 62–70. [CrossRef]
20. Wanas, A.K.; Páll-Szabó, A.O. Coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions. *Stud. Univ. Babes-Bolai Math.* **2021**, *66*, 659–666. [CrossRef]
21. Bulut, S.; Salehian, S.; Motamednezhad, A. Comprehensive subclass of m-fold symmetric biunivalent functions defined by subordination. *Afr. Mat.* **2021**, *32*, 531–541. [CrossRef]
22. Shehab, N.H.; Juma, A.R.S. Coefficient bounds of m-fold symmetric bi-univalent functions for certain subclasses. *Int. J. Nonlinear Anal. Appl.* **2021**, *12*, 71–82. [CrossRef]
23. Breaz, D.; Cotîrlă, L.-I. The study of the new classes of m-fold symmetric bi-univalent Functions. *Mathematics* **2022**, *10*, 75. [CrossRef]
24. Oros, G.I.; Cotîrlă, L.-I. Coefficient Estimates and the Fekete–Szegö problem for new classes of m-fold symmetric bi-univalent functions. *Mathematics* **2022**, *10*, 129. [CrossRef]
25. Minda, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–23 June 1992; Li, T.J., Ren, F., Yang, L., Zhang, S., Eds.; Conference Proceedings and Lecture Notes in Analysis; International Press: Cambridge, MA, USA, 1994; Volume I, pp. 157–169.
26. Tang, H.; Srivastava, H.M.; Sivasubramanian, S.; Gurusamy, P. Fekete–Szegö functional problems of m-fold symmetric bi-univalent functions. *J. Math. Ineq.* **2016**, *10*, 1063–1092. [CrossRef]
27. Fekete, M.; Szegö, G. Eine Bemerkung Über Ungerade Schlichte Funktionen. *J. Lond. Math. Soc.* **1933**, *89*, 85–89. [CrossRef]
28. Pommerenke, C. *Univalent Functions*; Vandenhoeck and Ruprecht: Gottingen, Germany, 1975.
29. Peng, Z.-G.; Han, Q.-Q. On the coefficients of several classes of bi-univalent functions. *Acta. Math. Sci. Ser. B Engl. Ed.* **2014**, *34*, 228–240. [CrossRef]
30. Altınkaya, Ş.; Yalçın, S. Coefficients bounds for certain subclasses of m-fold symmetric bi-univalent functions. *J. Math.* **2015**, *2015*, 241683. [CrossRef]
31. Mururugusundaramoorthy, G.; Magesh, N.; Prameela, V. Coefficient bounds for certain subclasses of bi-univalent functions. *Abst. Appl. Anal.* **2013**, *2013*, 573017.
32. Murugusundaramoorthy, G.; Cotîrlă, L.-I. Bi-univalent functions of complex order defined by Hohlov operator associated with Legendrae polynomial. *AIMS Math.* **2022**, *7*, 8733–8750. [CrossRef]
33. Páll-Szabó, Á.O.; Oros, G.I. Coefficient related studies for new classes of bi-univalent functions. *Mathematics* **2020**, *8*, 1110. [CrossRef]
34. Zayed, H.M.; Irmak, H. Some inequalities in relation with Fekete–Szegö problems specified by the Hadamard products of certain meromorphically analytic functions in the punctured unit disc. *Afr. Mat.* **2019**, *30*, 715–724. [CrossRef]
35. Srivastava, H.M.; Mostafa, A.O.; Aouf, M.K.; Zayed, H.M. Basic and fractional q-calculus and associated Fekete–Szegö problem for p-valently q-starlike functions and p-valently q-convex functions of complex order. *Miskolc Math. Notes.* **2019**, *20*, 489–509. [CrossRef]